Random Walk in the Quadrant
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Chapter IV

A system of two linear equations in two unknown functions and two unknown constants

IV.0 Summary of Chapter IV

This chapter is independent of the other ones; in it only the \( \theta_4 \)-function is used, which has been introduced in Subsec. III.2.1.

The chapter deals with a system (L) of two linear equations in two unknown functions and two unknown constants. Systems of this type are studied extensively in Groeneveld [27]. We follow his method closely. A difference is the use here of “generalized strips”.

The system (L) is described in Sec. IV.1; the solution, obtained by Fourier analysis, is given in Sec. IV.5. A modification (\( \tilde{L} \)) of this system arises in Sec. V.5 as a result of transformations of the backward conditions I.3.(B 1′′), and the solution of this system (\( \tilde{L} \)) is used in Sec. V.7 to determine the solution of the backward conditions, viz. the generating function \( P(x, y, s) \) of the transition probabilities in the case \( 0 < s < 1 \).

The contents of this chapter are summarized as follows.

IV.0.1 Summary of Sec. IV.1

In Sec. IV.1 the system (L) is described. Each function occurring in that system is a complex valued function of one complex variable with period 1. Its domain, however, is assumed to be a “generalized open horizontal strip”. Such strips are introduced at the beginning of the section. Three propositions are formulated containing some intuitively obvious properties of these strips; the proofs of these properties, however, need some nontrivial topology and are postponed, therefore, to the end of the chapter (viz. Subsecs. IV.6.2 and IV.6.3). In particular, every “generalized open horizontal strip” is the intersection of a unique “gener-
alized open upper half plane" and a unique "generalized open lower half plane" (Proposition 3). The basic strips occurring in the system \( L \) are \( S_1 \) and \( S_2 \). The strip \( S_1 \) resp. \( S_2 \) arises, in Chapter V, as the inverse image of the open unit disc under the mapping \( f_1 \) resp. \( f_2 \) (modulo complex periods), where \( f_1 \) and \( f_2 \) are the uniformizing elliptic functions to be defined in Subsec. V.4.1. These strips are the domains of the unknown functions in the system \( L \).

For the reader’s convenience we list here the assumptions on the strips, the known functions and unknown constants in the system \( L \):

- the strips \( S_i \) \((i = 1, 2)\) satisfy: (S0), (S1), (S2), (S3), and (1),
- the known functions \( F_{i\lambda} \) \((i = 1, 2; \lambda = \pm)\) satisfy: (F0), (F1), and (F2),
- the unknown constants \( v_{0i} \) \((i = 1, 2)\) satisfy: (8).

Also, we remark that modification \( \tilde{L} \) of the system \( L \) arises if, in addition, the assumption (F7) is made.

In the remainder of this section we derive with elementary means some properties of the solutions of the system \( L \), notably the unicity (not the existence), without actually solving the system (the solution is obtained in Sec. IV.5).

**IV.0.2 Summary of Sec. IV.2**

In Sec. IV.2 we review, for the reader’s convenience, the definitions of some well-known concepts from complex function theory, viz. paths and chains, the index function and related concepts, with applications to generalized strips.

**IV.0.3 Summary of Sec. IV.3**

In Sec. IV.3 we review some well-known formulae from the Fourier analysis of functions with unit period. Because the domains of our functions are "generalized open horizontal strips" (rather than ordinary horizontal strips) we need to check the validity of the usual formulae, including convolution product formulae.

**IV.0.4 Summary of Sec. IV.4**

In Sec. IV.4 we analyze certain convolution products which play a role in the solution of the backward conditions, viz. the products \( f \ast \alpha, f \ast \beta, \) and \( f \ast \eta \); here \( f \) is a general function, and \( \alpha, \beta, \) and \( \eta \) are special ones (introduced in Subsec. IV.4.1) related to the logarithmic derivative of the \( \theta_4 \)-function.

In Subsec. IV.4.2 these convolution products are introduced in the form of convolution integrals. The existence of these integrals depends on the existence of the integration paths. These paths must be contained in the domain of \( f \) (which is a “generalized open horizontal strip”), and, in addition, pass in the right way between the poles associated with \( \alpha, \beta, \) and \( \eta \). The existence of such integration paths is established in Proposition 1; the regularity of the resulting integral is established in Property 2 and Corollary 3.
After having introduced these integrals we compute in Subsec. IV.4.3 the Fourier coefficients of these integrals, and so prove they indeed are convolution products (Proposition 4, Corollary 5).

**IV.0.5 Summary of Sec. IV.5**

In Sec. IV.5 we show that a suitable linear combination of the convolution integrals of the kind described in the previous section solves the system (L). The central theorem here is Theorem 4. Using the foregoing, the solution is found in the usual way, as follows. 
First, the linear equations (within the system (L)) in the unknown functions are transformed into linear equations in the Fourier coefficients of the unknown functions. 
Next, these Fourier coefficients are computed. 
Finally, the unknown functions are determined from their Fourier coefficients using results of the previous section.
The section can be summarized as follows. 
First, in Subsec. IV.5.1, we introduce the functions by means of which we will solve the system (L). 
Next, in Subsec. IV.5.2, the system (L) is solved. 
In the subsequent Subsec. IV.5.3 we check that the expressions from Theorem 4 indeed do solve the system (L); together with the proof of the uniqueness of the solution in Sec. IV.1 (see Corollary IV.1.9) this amounts to a different proof of the central Theorem 4.
Finally, in Subsec. IV.5.4, the obtained result is formulated in a different way, viz. with the aid of the function \(SF\); this formulation facilitates the computation of the limit in Chapter VI.

**IV.0.6 Summary of Sec. IV.6**
The final Sec. IV.6 is devoted to some topological tools used in the foregoing and contains deferred proofs. 
First, in Subsec. IV.6.1, we recall some well-known topological properties used in the foregoing. 
Next, in Subsec. IV.6.2, we recall some topological theorems in the plane needed in the final Subsec. IV.6.3, which is devoted to the proof of the three propositions on “generalized open horizontal strips” from Subsec. IV.6.1.
IV.1 A system of two linear equations: definitions and properties of solutions

IV.1.1 Generalized strips and half planes

As usual, a subset of $\mathbb{C}$ is a region if it is nonempty, open, and connected.

A subset $S$ of $\mathbb{C}$ is called a generalized open (horizontal) strip if:
- $S$ is a region,
- $S$ is simply connected,
- $S + 1 = S$ (i.e. $S$ is invariant under the translation over the unit).

A subset $A$ of $\mathbb{C}$ is called a generalized open upper half plane if:
- $A$ is a generalized open horizontal strip,
- $A$ contains some upper half plane.

Similarly a subset $B$ of $\mathbb{C}$ is called a generalized open lower half
IV.1

plane if:
- B is a generalized open strip,
- B contains some lower half plane.

The following property asserts that a subset of \( \mathbb{C} \) cannot be both a generalized open upper half plane and a generalized open lower half plane, unless it is the full plane.

**Proposition 1.** Let \( S \) be a generalized open horizontal strip. If \( S \) contains some upper half plane and some lower half plane as well, then \( S = \mathbb{C} \).

**Proof.** Postponed to Section 6.

\[ \square \]

The next propositions characterize generalized open horizontal strips in terms of generalized open half planes.

**Proposition 2.** Let \( A \) be a generalized open upper half plane, and \( B \) a generalized open lower half plane. Then the following assertions are equivalent:
(i) \( A \cap B \) is a generalized open strip,
(ii) \( A \cap B \) is nonempty and connected,
(iii) \( A \cup B = \mathbb{C} \).

**Proof.** Postponed to Section 6.

\[ \square \]

**Proposition 3.** Let \( S \) be a generalized open strip. Then there exists a unique generalized open upper half plane \( A \), and a unique generalized open lower half plane \( B \), such that \( S = A \cap B \).

**Proof.** Postponed to Section 6.

\[ \square \]

IV.1.2 The conditions (L): two linear equations in two unknown functions and two unknown constants

This section is devoted to an examination of the conditions (L) stated below. The occurring functions are defined on certain generalized open horizontal strips, and we start with the assumptions on the generalized half planes determining these strips.

Assumptions. We assume that \( \sigma \) is a complex constant satisfying
Also we assume that $A := (A_1, A_2)$ and $B := (B_1, B_2)$ are pairs of subsets of $\mathbb{C}$ satisfying:

(S1) $A_\xi$ is a generalized open upper half plane, 
$B_\xi$ is a generalized open lower half plane 
(for $\xi = 1, 2$)

(S2) $A_j + \sigma \tau \subset A_\xi$ and $B_j + \sigma \tau \subset B_\xi$ 
(for $\xi = 1, 2$ with $j := 3 - \xi$)

(S3) $(A_j + \sigma \tau) \cup B_\xi = \mathbb{C}$ (for $\xi = 1, 2$ with $j := 3 - \xi$).

The above conditions are quite similar to the conditions (S0) - (S4) from Subsection II.3.1.

The pair of sets $S := (S_1, S_2)$ is defined as follows:

(1) $S_\xi := A_\xi \cap B_\xi$
for $\xi = 1, 2$. The sets $S_1$ and $S_2$ are generalized open horizontal strips 
because of $A_\xi \cup B_\xi = \mathbb{C}$ by (S2) and (S3), cf. Proposition 2. Also we put for $\xi = 1, 2$ (with $j = 3 - \xi$) and $\lambda = \pm$

(2) $F^\lambda_\xi := S_\xi \cap (S_j + \lambda \sigma \tau),$

and have by (S2)

(3) $\begin{cases} 
F^+_\xi = (A_j + \sigma \tau) \cap B_\xi, \\
F^-_\xi = A_\xi \cap (B_j - \sigma \tau), 
\end{cases}$

so $F^\lambda_\xi$ is a generalized open horizontal strip because of (S3) and Proposition 2. For later use we also introduce (for $\xi = 1, 2$ and $\lambda = \pm$)

(4) $\tilde{F}^\lambda_\xi := S_\xi \cup (S_j + \lambda \sigma \tau),$

and have

(5) $\begin{cases} 
\tilde{F}^+_\xi = A_j \cap (B_\xi + \sigma \tau), \\
\tilde{F}^-_\xi = (A_\xi - \sigma \tau) \cap B_j.
\end{cases}$

We leave the proof of (5) to the reader. The set $\tilde{F}^\lambda_\xi$ is a generalized open
horizontal strip because of
\[ A'_j \cup (B'_i + \sigma t) = \mathbb{A} = (A'_i - \sigma t) \cup B'_j \]
(by (S2) and (S3)) and Proposition 2.

Obviously one has
\[ F^\lambda_j = F^\lambda_j^{-1} + \lambda \sigma t, \]
\[ \widetilde{F}^\lambda_j = \widetilde{F}^\lambda_j^{-1} + \lambda \sigma t. \]

We intend to introduce for \( i = 1,2 \) and \( \lambda = \pm \) the function \( F^\lambda_i (\cdot) \)
(which is supposed to be known) with the properties:

\[(F1)\quad F^\lambda_i \quad \text{is regular in} \quad F^\lambda_i, \]
\[(F2)\quad F^\lambda_i \quad \text{has the period} \quad 1, \]
and, moreover,
\[(F0)\quad F^\lambda_i (v) = F^{-\lambda}_i (v - \lambda \sigma t) \]
for \( i = 1,2 \) and \( \lambda = \pm \). So two of the four functions can be chosen freely
(viz. \( F^\lambda_i \) for \( i = 1,2 \) and fixed \( \lambda \), or \( F^\lambda_i \) for fixed \( i \) and \( \lambda = \pm \)),
and then the remaining two are determined. Also,
\[ \text{(8) for} \quad i = 1,2 \quad \text{let} \quad v^\lambda_i \quad \text{be a given number from} \quad S^\lambda_i. \]

Consider the following conditions on the two (unknown) functions \( Z^\lambda_i (\cdot) \)
and the four (unknown) constants \( Z^\lambda_0 (i = 1,2 \quad \text{and} \quad \lambda = \pm) \):

\[\begin{align*}
\text{Regularity} & : Z^\lambda_i (\cdot) \text{ is regular in } S^\lambda_i \text{ and has period } 1 \text{ for } i = 1,2 \\
\text{Initial Value} & : Z^\lambda_i (v^\lambda_i) = 0 \text{ for } i = 1,2 \\
\text{Equality} & : Z^\lambda_0 (i = 1,2 \text{ and } \lambda = \pm) \\
\text{Equations} & : \text{For } i = 1,2 \text{ and } \lambda = \pm \text{ the equation } (E^\lambda_i) \text{ is satisfied:} \\
\text{(E^\lambda_i)} & \quad \text{if} \quad v \in F^\lambda_i \quad \text{then} \quad Z^\lambda_0 (1,2) + Z^\lambda_i (v) + Z^\lambda_j (v - \lambda \sigma t) = F^\lambda_i (v). \\
\end{align*}\]

We note that, in fact, there are only two unknown constants to be determined, due to the equality condition. Also we note that the condition (F0) is necessary for the existence of constants \( Z^\lambda_0 (i = 1,2 \text{ and } \lambda = \pm) \) and functions \( Z^\lambda_i \) satisfying (L), and that (F0) implies that, in fact, the number of equations to be satisfied is two, cf. Property 4 to come.

We will show that the conditions (L) determine the unknown constants and the unknown functions uniquely. In this section we will prove the uniqueness,
see Corollary 9.

The unknown functions and constants will be determined in Section 5, see Theorem IV.5.4, by means of Fourier analysis. This analysis will also show that the conditions (L) determine the unknown functions and constants uniquely.

Also we consider the following modification \( \tilde{(L)} \) of the conditions (L).

In this modification, \( Z_0 \) is an (unknown) constant and \( Z_\zeta(\cdot) \) is an (unknown) function for \( \zeta = 1, 2 \). 

\[
\begin{align*}
\text{Regularity} & : Z_\zeta(\cdot) \text{ is regular in } S_\zeta \text{ and has period } 1 \text{ for } \zeta = 1, 2 \\
\text{Symmetry} & : Z_\zeta(\cdot) \text{ is even for } \zeta = 1, 2 \\
\text{Initial Value:} & \quad Z_\zeta(v_0) = 0 \text{ for } \zeta = 1, 2 \\
\text{Equations} & : \text{For } \zeta = 1, 2 \text{ and } \lambda = \pm: \\
& \quad \text{if } v \in F_\zeta^\lambda \text{ then } Z_0 + Z_\zeta(v) + Z_\zeta(v - \lambda \sigma \tau) = F_\zeta^\lambda(v) .
\end{align*}
\]

We remark that the conditions

\[
(F7) \quad F_\zeta^\lambda(v) = F_\zeta^{-\lambda}(-v)
\]

for \( \zeta = 1, 2 \) and \( \lambda = \pm \) are necessary for the existence of a constant \( Z_0 \) and functions \( Z_\zeta \) satisfying \( \tilde{(L)} \) (see Corollary 14) and that the assumption \( (F7) \) implies that all four equations coincide. In fact, if the condition \( (F7) \) is satisfied, then conditions (L) and \( \tilde{(L)} \) coincide, cf. Remark 7. The modification \( \tilde{(L)} \) rather than (L) will be encountered in Chapter V.

IV.1.3 Examination of the conditions (L)

We now assume that \( F_\zeta^\lambda \) is a (known) function with domain \( F_\zeta^\lambda \) for \( \zeta \in \{1, 2\} \) and \( \lambda \in \{-, +\} \). Consider the following conditions on these functions (as always \( j = 3 - \zeta \)):

\[
(F0) \quad F_\zeta^\lambda(v) = F_j^{-\lambda}(v - \lambda \sigma \tau) \quad \text{for } \zeta = 1, 2 \text{ and } \lambda \in \{-, +\} ,
\]

\[
(F1) \quad F_\zeta^\lambda \text{ is regular in } F_\zeta^\lambda \text{ for } \zeta = 1, 2 \text{ and } \lambda \in \{-, +\} ,
\]

\[
(F1M) \quad F_\zeta^\lambda \text{ can be continued analytically as a meromorphic function in the plane for } \zeta = 1, 2 \text{ and } \lambda \in \{-, +\} ,
\]

\[
(F1R) \quad F_\zeta^\lambda \text{ can be continued analytically as a regular function in the plane for } \zeta = 1, 2 \text{ and } \lambda \in \{-, +\} .
\]
(F2) \[ P_{\xi}^\lambda \text{ has period 1 for } i = 1, 2 \text{ and } \lambda \in \{-, +\}. \]

Also let \((v_{01}, v_{02})\) be a pair of (known) constants satisfying

(8) \[ v_{0i} \in S_{\xi} \text{ for } i = 1, 2. \]

Let \(Z_{0i}^\lambda\) be an (unknown) constant for \(i = 1, 2\) and \(\lambda \in \{-, +\}\), and consider the following condition:

(Z0) \[ Z_{0i}^\lambda = Z_{0j}^{-\lambda} \text{ for } i = 1, 2 \text{ and } \lambda \in \{-, +\}. \]

Let \(Z_\xi(\cdot)\) be an (unknown) function with domain \(S_{\xi}\) for \(i = 1, 2\). Consider the following conditions on these functions:

(Z) \[ Z_\xi(v_{0i}) = 0 \text{ for } i = 1, 2, \]

(Z1) \[ Z_\xi(\cdot) \text{ is regular in } S_{\xi} \text{ for } i = 1, 2, \]

(Z1N) \[ Z_\xi(\cdot) \text{ can be continued analytically as a meromorphic function in the plane for } i = 1, 2, \]

(Z1R) \[ Z_\xi(\cdot) \text{ can be continued analytically as a regular function in the plane for } i = 1, 2, \]

(Z2) \[ Z_\xi(\cdot) \text{ has period 1 for } i = 1, 2. \]

Finally consider the following equation for \(i = 1, 2\) and \(\lambda \in \{-, +\}\):

(E\(\xi\)) \[ \nu \in F_{\xi}^\lambda = Z_{0i}^\lambda + Z_\xi(\nu) + Z_j(\nu - \lambda \sigma \tau) = F_{\xi}^\lambda(\nu). \]

**PROPERTY 4.** If (F0), (F1), (Z0), and (Z1) are satisfied then the equation (E\(\xi\)) is equivalent to the equation \((E_j^\lambda)\) for \(i = 1, 2\) and \(\lambda \in \{-, +\}\. \)

**PROOF.** In \((E_j^\lambda)\) substitute \(\nu' := \nu - \lambda \sigma \tau\), and apply (6), (F0) and (Z0).

Clearly, two of the four equations \((E_j^\lambda)\) are redundant under the assumptions (F0), (F1), (Z0), and (Z1).

**THEOREM 5.** Let \(F_{\xi}^\lambda(\cdot), Z_\xi(\cdot)\) be complex functions, and \(Z_{0i}^\lambda\) complex constants \((i = 1, 2\) with \(j = 3 - i\) and \(\lambda = \pm\)) satisfying the conditions (F0), (F1), (Z0), (Z1).
Assume that the equation \((E_i^\lambda)\) is satisfied for \(i = 1, 2\) and \(\lambda = \pm\). Then the following holds.

I. The functions \(Z_i\) satisfy \((Z1M)\) iff the functions \(F_i^\lambda\) satisfy \((F1M)\).

II. The functions \(Z_i\) satisfy \((Z1R)\) iff the functions \(F_i^\lambda\) satisfy \((F1R)\).

Assume that in addition \((F1M)\) or \((Z1M)\) is satisfied. Then:

III. For \(i = 1, 2\) and \(\lambda = \pm\) the equation

\[
Z_0^\lambda + Z_i(v) + Z_j(v - \lambda \sigma) = F_i^\lambda(v)
\]

holds for all \(v \in \mathbb{C}\).

**Proof.** Parts I and II, "only if". Trivial.

Parts I and II, "if", and Part III. The proof is quite similar to the proof of Theorem II.3.6 (Parts I and II) and left to the reader.

\(\Box\)

**Remark 1.** Notice that the equalities (9) for \(v \in \mathbb{C}\) imply (subtract the equation with \(\lambda = +\) from the corresponding equation with \(\lambda = -\), and interchange \(i\) and \(j\)) the equality

\[
Z_i^\lambda(v + \sigma) - Z_i^\lambda(v - \sigma) = Z_0^\lambda - Z_j^\lambda + F_i^\lambda(v) - F_j^\lambda(v)
\]

for \(v \in \mathbb{C}\), \(i = 1, 2\), and \(\lambda = \pm\).

**Lemma 6.** Let \(f(\cdot)\) be an entire function with period 1. Let \(\omega\) and \(c\) be complex constants, with \(\omega \not\in \mathbb{R}\). Assume

\[
f(v + \omega) = c + f(v)
\]

for all \(v \in \mathbb{C}\). Then \(c = 0\) and \(f(\cdot)\) is constant.

**Proof.** Integrate the functions in (*) on the left and on the right over \([0, 1]\). Due to the period 1 we have (by Cauchy's theorem)

\[
\int_0^1 f(v)dv = \int_0^1 f(v + \omega)dv = c + \int_0^1 f(v)dv,
\]

so \(c = 0\). Hence, \(f\) has the period \(\omega\). Since \(f(\cdot)\) is doubly periodic and entire, \(f(\cdot)\) is constant.

\(\Box\)

**Proposition 7.** Assume:

(i) the functions \(F_i^\lambda\) satisfy \((F0)\), \((F1)\), and \((F2)\);

(ii) the constants \(Z_0^\lambda\) satisfy \((Z0)\);

(iii) the functions \(Z_i\) satisfy \((Z1)\) and \((Z2)\);
(iv) the four equations (E^λ_ι) are satisfied. Then the following equivalence holds:

\[(F3)\quad \text{if and only if } F^λ_ι(\cdot) \text{ is a constant function for } i = 1, 2 \text{ and } λ = ±\]

\[(Z3)\quad Z^λ_ι(\cdot) \text{ is a constant function for } i = 1, 2 .\]

**Proof.** "If". Assume (Z3). Then, obviously, (Z1R) is satisfied; by Theorem 5 (Parts II and III) the condition (F1R) is satisfied and (9) holds for all \( v \in C \). This implies (F3) in a trivial way.

"Only if". Assume (F3). So (F1R) is satisfied, hence (by Theorem 5, Part II) the condition (Z1R) is satisfied, and (10) holds for all \( v \in C \) (by Theorem 5 (Part III) and Remark 1). Now apply Lemma 6 (with \( w := 2\sigma r \)) to conclude that (Z3) is true.

\[\Box\]

**Theorem 8.** Assume that (i) through (iv) from Proposition 7 are satisfied, and assume in addition:

\[(v)\quad \text{the functions } Z_ι \text{ satisfy (Z).}\]

Then the following holds:

\[(F4)\quad F^λ_ι \text{ vanishes in } F^λ_ι \text{ for } i = 1, 2 \text{ and } λ = ±\]

\[(Z4)\quad Z^λ_{0i} = 0 \text{ and } Z^λ_ι(\cdot) \text{ vanishes in } S_ι \text{ for } i = 1, 2 \text{ and } λ = ±.\]

**Proof.** "If". Trivial from the equations (E^λ_ι).

"Only if". Assume that (F4) holds. Then (F3) holds, hence (Z3) is true by Proposition 7. This implies that the functions \( Z_ι \) vanish identically, because of the assumption (v). Consequently the constants \( Z^λ_{0i} \) vanish, because of (F4) and (E^λ_ι). So (Z4) is true.

\[\Box\]

**Corollary 9.** Assume that the four functions \( F^λ_ι(\cdot) \) satisfy (F0), (F1) and (F2). Then the conditions (L) are satisfied by at most one set of functions \( Z^λ_ι(\cdot) \) and constants \( Z^λ_{0i} \).

**Proof.** By Theorem 8 the claim is true if the functions \( F^λ_ι(\cdot) \) vanish identically; in other words, if the linear equations (E^λ_ι) are homogeneous, then
the only solution is the trivial one. It follows by linearity that in the
general (non-homogeneous) case there is at most one solution, q.e.d.

Next we investigate the relationship between the conditions (L) and the
conditions ( Lif), see Subsec. IV.1.2. Observe that one obtains ( Lif) from (L)
by imposing the extra conditions (Z5) and (Z6) below (following (23)).
Corollary 14 (to come) shows that these extra conditions can be replaced by
the assumption (F7) below (following (22)).

Notations. We put

\[ \tilde{A}_\ell := -B_\ell, \quad \tilde{B}_\ell := -A_\ell \]

for \( \ell = 1,2 \). Clearly these sets are generalized open half planes satisfying
the conditions (S1) and (S2), and we put

\[ \tilde{S}_\ell := \tilde{A}_\ell \cap \tilde{B}_\ell = -S_\ell, \]

\[ \tilde{F}_\ell^\lambda := \tilde{S}_\ell \cap (\tilde{S}_j + \lambda \sigma \tau) = -F_\ell^{-\lambda}, \]

(for \( \ell = 1,2 \) with \( j = 3-\ell \), and \( \lambda = \pm \)), which are generalized open
horizontal strips.

**Remark 2.** Proposition 3 can be used to deduce (cf. (1), (12), and (13),
IV.5.(2)-(1A)-(1B))

\[ S_\ell = \tilde{S}_\ell \iff \tilde{A}_\ell = -B_\ell, \]

\[ F_\ell^\lambda = \tilde{F}_\ell^\lambda \iff \tilde{A}_\ell = -B_\ell \quad \text{for } k = 1,2. \]

Also the following notations will be used:

\[ \tilde{E}_\ell^\lambda(v) := F_\ell^{-\lambda}(-v), \]

\[ \tilde{Z}_0^\lambda := Z_0^{-\lambda}, \]

\[ \tilde{Z}_\ell(v) := Z_\ell(-v). \]

**Remark 3.** We note that the condition \((\tilde{E}_\ell^{-\lambda})\) is equivalent to \((\tilde{E}_\ell^\lambda)\) where

\[ v \in \tilde{F}_\ell^\lambda \Rightarrow \tilde{Z}_0^\lambda + \tilde{Z}_\ell(v) + \tilde{Z}_j(v - \lambda \sigma \tau) = \tilde{F}_\ell^\lambda(v). \]

In the sequel we use for integrals the notation introduced at the end of
Subsec.II.4.1. Assuming that the functions \( F_\ell^\lambda \) satisfy (F1) and (F2) we put
\( F^\lambda_{\hat{i}}(0) := \int_a^{a+1} F^\lambda_{\hat{i}}(w) \, dw \),

where \( a \) is an arbitrary number from \( F^\lambda_{\hat{i}} \). Due to the period 1 the value of the above integral does not depend on \( a \).

Analogously, if the functions \( Z^\lambda_{\hat{i}}(\cdot) \) satisfy (21) and (22) then

\( Z^\lambda_{\hat{i}}(0) := \int_a^{a+1} Z^\lambda_{\hat{i}}(w) \, dw \),

where \( a \) is an arbitrary number from \( S^\lambda_{\hat{i}} \).

In a similar way the constants \( \tilde{F}^\lambda_{\hat{i}}(0) \) and \( \tilde{Z}^\lambda_{\hat{i}}(0) \) are defined.

**PROPERTY 10.** I. Assume that the functions \( F^\lambda_{\hat{i}} \) satisfy (F1) and (F2), and that the functions \( Z^\lambda_{\hat{i}} \) satisfy (21) and (22). Then the following holds for \( \hat{i} = 1, 2 \) and \( \lambda = \pm \cdot \)

\( \tilde{F}^\lambda_{\hat{i}}(0) = F^{-\lambda}_{\hat{i}}(0) \),

\( \tilde{Z}^\lambda_{\hat{i}}(0) = Z^{-\lambda}_{\hat{i}}(0) \).

II. If, in addition, the condition (F0) is satisfied, then

\( F^\lambda_{\hat{i}}(0) = F^{-\lambda}_{\hat{j}}(0) \).

**PROOF.** Left to the reader.

Consider the following conditions on the functions \( F^\lambda_{\hat{i}} \) (we assume \( F^\lambda_{\hat{i}} = F^\lambda_{\hat{i}} \)), cf. (15):

\( F^\lambda_{\hat{i}}(0) = \tilde{F}^\lambda_{\hat{i}}(0) \) for \( \hat{i} = 1, 2 \) and \( \lambda = \pm \cdot \)

\( F^\lambda_{\hat{i}}(\cdot) - \tilde{F}^\lambda_{\hat{i}}(\cdot) \) is a constant for \( \hat{i} = 1, 2 \) and \( \lambda = \pm \cdot \)

\( F^\lambda_{\hat{i}}(\cdot) \equiv \tilde{F}^\lambda_{\hat{i}}(\cdot) \) for \( \hat{i} = 1, 2 \) and \( \lambda = \pm \cdot \).

**PROPERTY 11.** Under the assumptions (F1) and (F2) the equivalence

\( (F7) \Leftrightarrow (F5) \& (F6) \)

holds.

**PROOF.** This is obvious.
REMARK 4. In combination with (20) and (22) the condition (F5) implies that \( F^\lambda_{i}(0) \) does not depend on \( i \) or \( \lambda \).

Consider the following condition on the constants \( Z^\lambda_{0i} \), cf. (16):

\[
Z^\lambda_{0i} = \tilde{Z}^\lambda_{0i} \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad \lambda = \pm,
\]

and the following condition on the functions \( Z^\lambda_{i} \) (we assume \( S^\lambda_{i} = \tilde{S}^\lambda_{i} \)), cf. (17):

\[
Z^\lambda_{i}(\cdot) \equiv \tilde{Z}^\lambda_{i}(\cdot) \quad \text{(i.e.} \quad Z^\lambda_{i}(\cdot) \text{is even) for} \quad i = 1, 2.
\]

REMARK 5. In combination with (16) and (20) the condition (Z5) implies that \( Z^\lambda_{0i} \) doesn't depend on \( i \) or \( \lambda \).

PROPERTY 12. Assume \( S^\lambda_{i} = \tilde{S}^\lambda_{i} \) for \( i = 1, 2 \). If the conditions (F0), (F1), (F7), and the conditions (20), (Z1), (Z5), and (Z6) are satisfied, then all four equations \( (E^\lambda_{i}) \) are equivalent.

PROOF. One has \( (E^\lambda_{i}) \iff (E^{-\lambda}_{i}) \) by Property 4, and by our assumptions the equation \( (E^\lambda_{i}) \) is equivalent to the equation \( (\tilde{E}^\lambda_{i}) \) which in turn is equivalent (see Remark 3) to the equation \( (E^{-\lambda}_{i}) \). This implies the claim. \( \Box \)

THEOREM 13. Under the assumptions of Proposition 7 the following holds:

I. For \( i = 1, 2 \) and \( \lambda = \pm \)

\[
Z^\lambda_{0i} + Z^\lambda_{1}(0) + Z^\lambda_{2}(0) = F^\lambda_{i}(0).
\]

II. The functions \( F^\lambda_{i} \) satisfy the condition (F5) iff the constants \( Z^\lambda_{0i} \) satisfy the condition (Z5).

III. Assume in addition \( S^\lambda_{i} = \tilde{S}^\lambda_{i} \) for \( i = 1, 2 \). Then the functions \( F^\lambda_{i} \) satisfy the condition (F6) iff the functions \( Z^\lambda_{i} \) satisfy the condition (Z6).

PROOF. Part I. Choose an \( a \in F^\lambda_{i} \) and apply \( \int_{a}^{a+1} dv \ldots \) to both sides of the equation (9). Observe:

\[
\int_{a}^{a+1} dv \cdot Z^\lambda_{i}(v) = Z^\lambda_{i}(0),
\]

because of \( F^\lambda_{i} \subseteq S^\lambda_{i} \) (cf. (2)), and
\[ \int_a^{a+1} dv \cdot Z_j(v - \lambda \sigma t) = \int_{a-\lambda \sigma t}^{a+1-\lambda \sigma t} d\tilde{v} \cdot Z_j(\tilde{v}) = Z_j(0), \]

where \( \tilde{v} := v - \lambda \sigma t \). The claim follows from (25), (26), and (18).

Part II. This is a trivial consequence of Part I, cf. (16) and (20).

Part III. From the equations \((E^\lambda_0)\) and \((E^\lambda_i)\) it follows (by subtraction)

\[ (E^\lambda_0) \quad v \in F_i^\lambda = \tilde{Z}_0^\lambda + \tilde{Z}^\lambda_0(v) + \tilde{Z}_j^\lambda(v - \lambda \sigma t) = F_i^\lambda(v), \]

where

\[
\begin{align*}
\tilde{Z}_0^\lambda & := Z_0^\lambda - \tilde{Z}_0^\lambda = \tilde{Z}_0^\lambda \\
\tilde{Z}_k^\lambda(v) & := Z_k^\lambda(v) - \tilde{Z}_k^\lambda(v) \\
\tilde{F}_i^\lambda(v) & := F_i^\lambda(v) - \tilde{F}_i^\lambda(v) = \tilde{F}_i^\lambda(v - \lambda \sigma t) \quad \text{by (F0), cf. (15).}
\end{align*}
\]

Now apply Proposition 7 to conclude that the condition (F6) is satisfied iff \( Z_k^\lambda(\cdot) \) is a constant function for \( k = 1, 2 \); the latter condition is equivalent to (26) because \( \tilde{Z}_k^\lambda(\cdot) \) is odd.

\[ \square \]

**REMARK 6.** We note that under the hypotheses of Property 4 all four equations \((E^\lambda_i)\) are equivalent, as we will show now.

First, \((E^\lambda_i)\) changes into \((E^\tilde{\lambda}_j)\) through the substitution \( v' := v - \lambda \sigma t \), because of (F0) and (20).

Secondly, note that the constants and functions defined in (27) satisfy

\[ (28) \quad \begin{align*}
\tilde{F}_i^\lambda(v) & = -F_i^{\tilde{\lambda}}(-v), \\
\tilde{Z}^\lambda_0 & = -\tilde{Z}^{\tilde{\lambda}}_0, \\
\tilde{Z}_i^\lambda(v) & = -\tilde{Z}_i^{\tilde{\lambda}}(-v).
\end{align*} \]

So \((E^\lambda_i)\) transforms into \((E^\tilde{\lambda}_j)\) through the substitution \( v' := -v \).

**COROLLARY 14.** Under the hypotheses of Proposition 7 the combination of the conditions (25) and (26) is equivalent to the condition (F7).

**PROOF.** From Theorem 13 (Parts II and III) and Property 11.

\[ \square \]

**REMARK 7.** From Corollary 14 it follows that the conditions (L) coincide with the conditions \((\tilde{L})\) if the functions \( F_i^\lambda(\cdot) \) satisfy (F7). Observe that (F7) coincides with \((**)\) (following (7)), cf. (15). Also compare (23) and Remark 4.
IV.2 Preliminaries. Paths and chains. Closed paths and cycles

As usual (cf. RUDIN [1966], Sec. 10.8), a path with parameter interval \([a, b]\) is a continuous, piecewise differentiable mapping from \([a, b]\) into \(\mathbb{C}\). If \(\gamma(\cdot)\) is a path with parameter interval \([a, b]\), then \(\gamma(a)\) is the initial point, \(\gamma(b)\) the end point; the path is closed if \(\gamma(a) = \gamma(b)\). We denote the range of \(\gamma\) by \(\gamma^*\); so

\[
\gamma^* := \gamma([a, b]).
\]

If \(\gamma\) is a path with parameter interval \([a, b]\) then the opposite path \(\tilde{\gamma}(\cdot)\) is defined by

\[
\tilde{\gamma}(t) := \gamma(a + b - t).
\]

So \(\gamma\) and \(\tilde{\gamma}\) have the same range, but different orientation, and one has, if \(f(\cdot)\) is continuous on \(\gamma^*\),

\[
\int_{\tilde{\gamma}} f(z) \, dz = -\int_\gamma f(z) \, dz.
\]

As usual (cf. RUDIN [1966], Sec. 10.34), a chain is a finite collection of paths, and a cycle is a finite collection of closed paths. If

\[
\Gamma = \{\gamma_1, \ldots, \gamma_n\}
\]

is a chain, then the range of \(\Gamma\) is

\[
\Gamma^* := \bigcup_{k=1}^{n} \gamma_k^*.
\]

and if \(f(\cdot)\) is a continuous function \(\Gamma^* \to \mathbb{C}\) then one puts

\[
\int_{\Gamma} f(z) \, dz := \sum_{k=1}^{n} \int_{\gamma_k} f(z) \, dz.
\]

Let \(O\) be an open subset of \(\mathbb{C}\), and \(T\) a nonconstant regular function \(O \to \mathbb{C}\). If \(\gamma(\cdot)\) is a path in \(O\), then \(T \circ \gamma\) is a path in the open set \(T(O)\); more general, if \(\Gamma\) (see (4)) is a chain in \(O\), then

\[
T \circ \Gamma := \{T \circ \gamma_1, T \circ \gamma_2, \ldots, T \circ \gamma_n\}
\]

is a chain in \(T(O)\), and one has the following change-of-variable formula
for, if $\Gamma$ consists of a single path $\gamma(\cdot)$ with parameter interval $[a,b]$, then both sides equal
\[ \int_a^b f \circ T \circ \gamma(t) \cdot (T \circ \gamma)'(t) \, dt. \]
Also one has, if $\phi(\cdot)$ is regular in $T(0)$,
\[ \int \phi(z) \, dz = \int f \circ T \circ \gamma(t) \cdot (\phi \circ T)'(t) \, dt, \]
for, if $\Gamma$ consists of a single path $\gamma(\cdot)$ with parameter interval $[a,b]$, then both sides equal
\[ \int_a^b f \circ T \circ \gamma(t) \cdot (\phi \circ T \circ \gamma)'(t) \, dt. \]

The exponential transformation $E(\cdot)$

We put
\[ E(w) := e^{2\pi i w} \]
for all $w \in \mathbb{C}$.

It is well-known that the function $w \to E(w)$ maps the set $\mathbb{C}/\mathbb{Z}$ (the complex plane modulo 1) regularly and bijectively onto $\mathbb{C} \setminus \{0\}$ (the punctured plane).

**PROPERTY 1.** Let $0$ be an open subset of $\mathbb{C}$ satisfying
\[ 0 + 1 = 0 \quad \text{(or $0 = E^{-1} \circ E(0)$).} \]
Assume that $f(\cdot)$ is regular in $0$ and has period 1. Then there exists a function $\hat{f}(\cdot)$, defined and regular in $E(0)$, satisfying
\[ \hat{f} \circ E = f. \]

**PROOF.** Obviously, $\hat{f}(\cdot)$ is well defined in $E(0)$ by requirement (*). The claim now follows from Lemma II.1.6.

Let $\mathcal{O}$ be an open subset of $\mathbb{C}$ and $\Gamma$ a chain in $\mathcal{O}$. If $\hat{f}(\cdot)$ is regular in $E(0)$ then one has from (8)
\[ \int_{E \circ \Gamma} \frac{\hat{f}(t) \, dt}{2\pi i t} = \int \hat{f} \circ E(w) \, dw. \]
(in (8) put \( f(t) := \frac{\hat{f}(t)}{2\pi i t} \), so \( f \circ E(w) = \frac{\hat{f} \circ E(w)}{E'(w)} \).

The index function \( \text{Ind}(\cdot|\cdot) \)

As usual, the index of a point \( z \) with respect to a cycle \( \Gamma \) is defined as follows:

\[
\text{Ind}(z|\Gamma) := \frac{1}{2\pi i} \int_{\Gamma} \frac{dw}{w-z} \quad \text{for} \quad z \in (\Gamma^*)^C,
\]

cf. RUDIN [1966], 10.10 and 10.34 ((7) and (8)).

Obviously, if (4) holds and \( \gamma_1, \ldots, \gamma_n \) are closed paths, then

\[
\text{Ind}(z|\Gamma) = \sum_{k=1}^{n} \text{Ind}(z|\gamma_k).
\]

It is well-known that \( \text{Ind}(z|\Gamma) \) can be interpreted as "the number of times \( \Gamma \) winds around \( z \)."

Also it is well-known, that the function \( z + \text{Ind}(z|\Gamma) \)
- is integer valued,
- is continuous, hence, is constant in the components of \((\Gamma^*)^C\),
- vanishes in the unbounded component of \((\Gamma^*)^C\)
(RUDIN, Theorem 10.10).

In the following property we use some notations. Let \( \Gamma \) be the chain (4).

Then we put

\[ a_{\Gamma} := \{a\gamma_1, \ldots, a\gamma_n\} \quad \text{if} \quad a \in \mathbb{C} \setminus \{0\}, \]
\[ \frac{1}{\Gamma} := \left(\frac{1}{\gamma_1}, \ldots, \frac{1}{\gamma_n}\right) \quad \text{if} \quad 0 \notin \Gamma^*. \]

PROPERTY 2. Let \( \Gamma \) be a cycle. Then the following holds.

I. If \( a \neq 0 \) and \( \frac{z}{a} \in (\Gamma^*)^C \) then

\[
\text{Ind}(z|a\Gamma) = \text{Ind}(\frac{z}{a}|\Gamma).
\]

II. If both \( z \) and the origin are in \((\Gamma^*)^C\) then

\[
\text{Ind}(z|\Gamma) = \text{Ind}(0|\Gamma) + \text{Ind}(\frac{1}{\Gamma} | \Gamma).
\]

PROOF. Part I. Trivial.

Part II. If \( \gamma(\cdot) \) is a closed path then

\[
\int_{\gamma} \left(\frac{1}{w-z} - \frac{1}{w}\right) dw = \int_{\gamma} \frac{dw}{\frac{\sim w}{w} - \frac{1}{z}}, \quad \text{where} \quad \sim w = \frac{1}{w}.
\]

Part II. If \( \gamma(\cdot) \) is a closed path then

\[
\int_{\gamma} \left(\frac{1}{w-z} - \frac{1}{w}\right) dw = \int_{\gamma} \frac{dw}{\frac{\sim w}{w} - \frac{1}{z}}, \quad \text{where} \quad \sim w = \frac{1}{w}.
\]
This easily implies the claim.

\[ \square \]

**The index function $\text{Ind}^*(\cdot | \cdot)$**

Let $\gamma(\cdot)$ be a path with parameter interval $[0,1]$. The path $E \circ \gamma$ is closed iff

\[ (16) \quad \gamma(1) - \gamma(0) \in \mathbb{Z}; \]

if (16) holds then

\[ (17) \quad \text{Ind}(0 | E \circ \gamma) = \gamma(1) - \gamma(0). \]

(apply (11) with $f(\cdot) = 1$).

Conversely, if $\tilde{\gamma}(\cdot)$ is a path in $E \setminus \{0\}$, then there exists a path $\gamma(\cdot)$ such that $\tilde{\gamma} = E \circ \gamma$. Let $T$ be the chain (4). If the path $E \circ \gamma_k$ is closed for all $k$, then the chain $\tilde{T} := E \circ T$ is a cycle in $E \setminus \{0\}$.

Conversely, let $\tilde{T}$ be a cycle in $E \setminus \{0\}$. Then there exist paths $\gamma_1, \ldots, \gamma_n$ such that the paths $E \circ \gamma_k$ are closed and $\tilde{T} = E \circ \Gamma$ with $\Gamma$ given by (4).

Let $\Gamma$ be a chain such that $E \circ \Gamma$ is a cycle. Then we put:

\[ (18) \quad \text{Ind}^*(\omega_i \mid \Gamma) := \text{Ind}(0 | E \circ \Gamma), \]

\[ (19) \quad \text{Ind}^*(v \mid \Gamma) := \text{Ind}(E(v) | E \circ \Gamma) \quad \text{if} \quad v \in (\Gamma^* + \mathbb{Z})^C. \]

From (12) and (11) it follows easily

\[ (20) \quad \text{Ind}^*(v \mid \Gamma) = \int_{\Gamma} \frac{dw}{1 - E(v - w)} \quad \text{if} \quad v \in (\Gamma^* + \mathbb{Z})^C. \]

Property 2 has the following trivial corollary.

**COROLLARY 3.** Let $\Gamma$ be a chain and $E \circ \Gamma$ a cycle.

I. If $v - w \in (\Gamma^* + \mathbb{Z})^C$ then

\[ (21) \quad \text{Ind}^*(v \mid w + \Gamma) = \text{Ind}^*(v - w \mid \Gamma). \]

II. If $v \in (\Gamma^* + \mathbb{Z})^C$ then

\[ (22) \quad \text{Ind}^*(v \mid \Gamma) = \text{Ind}^*(\omega_i \mid \Gamma) + \text{Ind}^*(-v \mid -\Gamma). \]

\[ \square \]
PROPERTY 4. Let \( V \) be a generalized open upper or lower half plane. Let \( \Gamma \) be a chain in \( V \) such that \( E \circ \Gamma \) is a cycle. If \( v \in V^C \) then

\[
\text{Ind}^*(v|\Gamma) = \begin{cases} 
0 & \text{if } V \text{ is a generalized open upper half plane}, \\
\text{Ind}^*(\omega i|\Gamma) & \text{if } V \text{ is a generalized open lower half plane}.
\end{cases}
\]

PROOF. Assume that \( E \circ \Gamma \) consists of a single closed path. So let \( \gamma(\cdot) \) be a path satisfying

\[
\gamma(1) - \gamma(0) = N \in \mathbb{Z}.
\]

Then one has

\[
\text{Ind}^*(v|\gamma) = \int_{\gamma} \frac{dw}{\left(1 - \bar{E}(v-w)\right)} = \int_{a}^{a+N} \frac{dw}{\left(1 - \bar{E}(v-w)\right)} \quad \text{for every } a \in V;
\]

here we used the integral notation introduced at the end of Subsection II.4.1. First assume that \( V \) is a generalized open upper half plane. If \( \Im a \) is positive and sufficiently large then let the integration path be a straight line segment; perform the translation \( t := w - a \) to find

(\#)

\[
\text{Ind}^*(v|\gamma) = \int_{0}^{N} \frac{dt}{1 - \bar{E}(-a) E(v-t)},
\]

which has limit 0 for \( \Im a \to +\infty \). Hence \( \text{Ind}^*(v|\gamma) = 0 \).

Secondly assume that \( V \) is a generalized open lower half plane. In this case (\#) holds if \( \Im a \) is negative and sufficiently large; the limit of the integral for \( \Im a \to -\infty \) is \( N \), hence \( \text{Ind}^*(v|\gamma) = N \). So the claim is proved if \( E \circ \Gamma \) consists of a single closed path. This easily implies the general case.

We note that the second part can be derived also from the first part, as follows. By (22),

\[
\text{Ind}^*(v|\Gamma) = \text{Ind}^*(\omega i|\Gamma) + \text{Ind}^*(-v|\Gamma),
\]

where \( -\Gamma \) is a chain in \( -V \). Now apply the first part to conclude that \( \text{Ind}^*(-v|\Gamma) = N \), q.e.d.

\( \square \)

PROPERTY 5. If \( v \in E \) and \( \Gamma \) is a cycle such that \( v \in (\Gamma^* + \mathbb{Z})^C \), then

(23)

\[
\text{Ind}^*(v|\Gamma) = \sum_{k \in \mathbb{Z}} \text{Ind}(v+k|\Gamma),
\]

where the sum on the right is finite.
Proof. From (20) one has

\[ \text{Ind}^*(v|\Gamma) = \oint_{\Gamma} \frac{dw}{2\pi i} \ g(w-v), \]

where

\[ g(z) := \frac{2\pi i}{1 - E(-z)} = \frac{E'(z)}{E(z) - 1}. \]

The set of poles of \( g(\cdot) \) is \( \mathbb{Z} \), and all residues are 1. So the claims follow from the Residue Theorem (RUDIN [1966], Theorem 10.42).
IV.3 Fourier analysis

IV.3.1 Notations

In this section it is assumed that

\[
\begin{cases}
A & \text{is a generalized open upper half plane,} \\
B & \text{is a generalized open lower half plane,}
\end{cases}
\]

cf. Subsection IV.1.1. It is not excluded that \( A \) or \( B \) is (or both are) the full plane. Also it is assumed that

\( A \cup B = \mathbb{C}, \)

or, equivalently (by Proposition IV.1.2),

\( A \cap B \) is a generalized open horizontal strip.

We write \( A' \) for the union of all open upper half planes contained in \( A \), and \( B' \) for the union of all open lower half planes contained in \( B \); in other words,

\[
\begin{cases}
A' & \text{is the largest open upper half plane contained in} \ A, \\
B' & \text{is the largest open lower half plane contained in} \ B.
\end{cases}
\]

We want to stress that the case \( A' \cap B' = \emptyset \) is not excluded, so \( A \cap B \) needs not contain a rectilinear strip.

As before we put

\[
E(\omega) = e^{2\pi i \omega}
\]

for \( \omega \in \mathbb{C} \). Usually the set \( E(A' \cap B') \) is called an open annulus (provided it is nonempty). We might call the nonempty set \( E(A \cap B) \) a generalized open annulus.

IV.3.2 Fourier coefficients and Fourier series

Notations. If \( \Omega \) is an open subset of \( \mathbb{C} \) satisfying \( \Omega + 1 = \Omega \), then we put
(6) \( HP(\Omega) \): the set of complex functions which are Holomorphic on \( \Omega \) and Periodic with period 1.

If \( f \in HP(A \cap B) \) then its Fourier coefficients \( f(n), n \in \mathbb{Z} \), are defined as follows:

(7) \[
    f(n) = \int \limits_{\Lambda} dw \, f(w) \, E(w)^{-n},
\]

where \( \Lambda \) is a chain satisfying

(8) \[
    \begin{cases}
    E \circ \Lambda \text{ is a cycle with } \text{Ind} (0 \mid E \circ \Lambda) = 1, \\
    \Lambda^* \subset A \cap B.
    \end{cases}
\]

Assume \( f \in HP(A \cap B) \). We put

(9A) \[
    Af(v) = \int \limits_{\Lambda} dw \, \frac{f(w)}{E(w-v)-1},
\]

where \( \Lambda \) is a chain satisfying (8) and (10A):

(10A) \[
    \begin{cases}
    \Lambda^* \text{ disjoint from } v + \mathbb{Z}, \\
    \text{Ind}^* (v \mid \Lambda) = 1.
    \end{cases}
\]

Similarly we put

(9B) \[
    Bf(v) = \int \limits_{\Lambda} dw \, \frac{f(w)}{E(v-w)-1},
\]

where \( \Lambda \) is a chain satisfying (8) and (10B):

(10B) \[
    \begin{cases}
    \Lambda^* \text{ disjoint from } v + \mathbb{Z}, \\
    \text{Ind}^* (v \mid \Lambda) = 0.
    \end{cases}
\]

**Proposition 1.**

I. A chain \( \Lambda \) satisfying (8) and (10A) exists iff \( v \in A \);

II. If \( f \in HP(A \cap B) \) then \( Af \in HP(A) \) and \( Bf \in HP(B) \).
REMARK. If $\Lambda$ is a chain satisfying (8), then it follows from Proposition IV.2.4

$$\text{Ind}^* (w \mid \Lambda) = \begin{cases} 
0 & \text{if } w \in A^c, \\
1 & \text{if } w \in B^c.
\end{cases}$$

PROOF of Proposition 1. Part I. Let $\Lambda$ satisfy (8) and (10A). Then necessarily $v \in A$ because of (11).

Conversely assume $v \in A$. Choose an arbitrary $a \in (A \cap B) \setminus (v + Z)$, and an arbitrary path $\Lambda_1$ from $a$ to $a + 1$, such that $(\Lambda_1)^* \subset (A \cap B) \setminus (v + Z)$.

Let $\Lambda_2$ be a cycle in $(A \cap B) \setminus (v + Z)$ satisfying

$$\begin{align*}
\text{Ind} (v \mid \Lambda_2) &= 1 - \text{Ind}^* (v \mid \Lambda_1), \\
\text{Ind} (w \mid \Lambda_2) &= 0 \quad \text{if } w \in v + Z \text{ and } w \neq v.
\end{align*}$$

Obviously such a cycle exists.

Because of IV.2.(17) and Property IV.2.5 the chain $\Lambda = \{\Lambda_1\} \cup \Lambda_2$ satisfies (8) and (10A).

The remaining claims are shown similarly.

Part II. Left to the reader.

□

PROPOSITION 2. If $f \in HP(A \cap B)$ then $f(v) = f(0) + Af(v) + Bf(v)$ for $v \in A \cap B$.

PROOF. Assume $f \in HP(A \cap B)$ and $v \in A \cap B$.

Let $\Lambda_\pm (\cdot)$ be chains so that $\Lambda = \Lambda_+$ satisfies (8) and (10A), and $\Lambda = \Lambda_-$ satisfies (8) and (10B). We define the function $\hat{f}(\cdot)$ by means of

$$f(w) = \hat{f} \circ E(w)$$

(cf. Property IV.2.1), and have
\[ f(0) + Af(v) + Bf(v) = \]
\[ = \int_{\Lambda_+} \text{df}(w) \left[ 1 + \frac{1}{E(w-v)-1} \right] + \int_{\Lambda_-} \text{df}(w) \frac{1}{E(v-w)-1} = \]
\[ = \sum_{\mu \in \mathbb{F}_k} \int_{\Lambda_\mu} \text{df}(w) \frac{1}{1-E(v-w)} = \]
(\text{change of variable } t = E(w), \text{ see IV.2.(11)})
\[ = \sum_{\mu \in \mathbb{F}_k} \frac{\mu \cdot 1}{2\pi i} \int \frac{\hat{f}(t)}{t-E(v)} dt. \]

The residue of the integrand at the pole \( t = E(v) \) is \( \hat{f}(E(v)) = f(v) \).

Now apply the residue theorem (RUDIN [1966], Theorem 10.42) to arrive at the claim.

\[ \square \]

**Proposition 3.** Assume \( f \in H^p(A \cap \mathbb{B}) \).

I. If \( v \in A' \) then the series \( \sum \text{f(n)} E(v)^n \) converges absolutely, and its sum is \( Af(v) \).

II. If \( v \in B' \) then the series \( \sum \text{f(n)} E(v)^n \) converges absolutely, and its sum is \( BF(v) \).

(For \( A' \) and \( B' \) see (4).)

**Proof.** Part I. Fix \( v \in A' \). Choose an \( a \in A' \) such that \( \text{Im} a < \text{Im} v \), and put
\[ r := |E(v-a)|, \]
so \( r < 1 \). Let \( \Lambda(\cdot) \) be the path from \( a \) to \( a + 1 \) whose range is the straight line segment \([a, a+1]\). Then \( \text{Ind}^*(v | \Lambda) = 1 \) (cf. Property IV.2.4 with \( \psi := \{w: \text{Im} w < \text{Im} v\} \)). Observe

\[ f(n) E(v)^n = \int_{\Lambda} \text{df}(w) E(v-w)^n, \]

where \( |E(v-w)| = r \), so
\[ |f(w) E(v-w)^n| \leq r^n \max_{w \in \Lambda^*} |f(w)|. \]

This proves the convergence of the series \( \sum |f(n) E(v)^n| \).
To compute its sum, apply (*) and interchange the order of summation and integration to find

$$\sum_{n \geq 1} f(n) E(v)^n = \int dw f(w) \left[ \sum_{n \geq 1} E(v-w)^n \right];$$

hence, the sum equals $Af(v)$, q.e.d.

Part II. Left to the reader.

COROLLARY 4. Assume $f \in HP(A \cap B)$,

If $v \in A' \cap B'$, then the series $\sum_{n \in \mathbb{Z}} f(n) E(v)^n$ converges absolutely, and its sum is $f(v)$.

PROOF. This is a trivial consequence of Proposition 2 and Proposition 3.

PROPOSITION 5. Let $A$, $B$ be (proper) open half planes satisfying (2) (it is not excluded that $A$ or $B$ is the full plane). Assume that the series

$$\sum_{n \in \mathbb{Z}} c_n E(v)^n$$

converges if $v \in A \cap B$, and call its sum $f(v)$.

Then $f \in HP(A \cap B)$ and $f(n) = c_n$ for $n \in \mathbb{Z}$.

PROOF. The first claim being trivial, we show the second claim. By definition (see (7)) we have

$$f(n) = \int dw \sum_{k \in \mathbb{Z}} c_k E(w)^{k-n},$$

where $\Lambda(\cdot)$ is a chain satisfying (8). Observe that the series $\sum_{k} c_k E(w)^{k-n}$ converges absolutely for $w \in A \cap B$; this justifies (by Fubini's theorem) permutation of the order of summation and integration. Hence,

$$f(n) = \sum_{k \in \mathbb{Z}} c_k \int_A E(w)^{k-n}.$$  

This implies the claim because of the equality

(14)
\[ (15) \quad \int \frac{dw}{E(w)^k} = \delta(k,n) \quad \text{(Kronecker delta)} \]

for integers \( k, n \).
\[ \square \]

**Corollary 6.** If \( f \in HP(A \cap B) \) then

\[ (16A) \quad (Af)(n) = \begin{cases} 
    f(n) & \text{for} \quad n \geq 1, \\
    0 & \text{for} \quad n \leq 0,
\end{cases} \]

\[ (16B) \quad (Bf)(n) = \begin{cases} 
    0 & \text{for} \quad n \geq 0, \\
    f(n) & \text{for} \quad n \leq -1.
\end{cases} \]

**Proof.** Because of Proposition 5, the claim \((16A)\) follows from Proposition 3 (Part I), and the claim \((16B)\) from Proposition 3 (Part II).
\[ \square \]

**Corollary 7.** If \( f \in HP(A \cap B) \) then

\[ (17) \quad f(\cdot) \equiv 0 \quad \text{iff} \quad f(n) = 0 \quad \text{for all} \quad n, \]

\[ (17A) \quad Af(\cdot) \equiv 0 \quad \text{iff} \quad f(n) = 0 \quad \text{for} \quad n \geq 1, \]

\[ (17B) \quad Bf(\cdot) \equiv 0 \quad \text{iff} \quad f(n) = 0 \quad \text{for} \quad n \leq -1. \]

**Proof.** We show \((17)\).

"Only if". Trivial consequence of \((7)\).

"If". Assume that \( f(n) = 0 \) for all \( n \).

Then \( f(0) = 0 \). Also \( Af(v) = 0 \) for \( v \in A' \), and \( Bf(v) = 0 \) for \( v \in B' \) because of Proposition 3; hence \( Af(v) = 0 \) for all \( v \in A \), and \( Bf(v) = 0 \) for all \( v \in B \) (cf. Proposition 1 (Part II)). Now apply Proposition 2 to conclude \( f(\cdot) \equiv 0 \).

To see \((17A)\), replace \( f \) by \( Af \) in \((17)\), and apply \((16A)\).

Analogously one shows \((17B)\) by replacing \( f \) by \( Bf \) in \((17)\), and applying \((16B)\).
\[ \square \]

Observe that, by \((17)\), a function from \( HP(A \cap B) \) is uniquely determined by its Fourier coefficients (even if \( A \cap B \) does not contain a proper horizontal strip).
IV.3.3 Convolution products

In the following it is assumed that \((A_1, B_1)\) and \((A_2, B_2)\) are fixed pairs of sets satisfying (1) and (2), and that \(f_k \in \text{HP}(A_k \cap B_k)\) for \(k = 1, 2\). If there exists a pair of sets \((A, B)\) satisfying (1) and (2), and a function \(\varphi \in \text{HP}(A \cap B)\) with Fourier coefficients \(\varphi(n) = f_1(n) f_2(n)\) for \(n \in \mathbb{Z}\), then \(\varphi\) is called the convolution product of \(f_1\) and \(f_2\) and one writes \(\varphi = f_1 * f_2\). So the convolution product (if it exists) is uniquely (cf. (17)) determined by the requirement

\[
f_1 * f_2(n) = f_1(n) f_2(n) \quad \text{for} \quad n \in \mathbb{Z},
\]

and is commutative:

\[
f_1 * f_2 = f_2 * f_1.
\]

The product also is linear: if both \(f_1 * g\) and \(f_2 * g\) exist as elements of some \(\text{HP}(A \cap B)\) and \(\lambda_1, \lambda_2 \in \mathbb{T}\), then \((\lambda_1 f_1 + \lambda_2 f_2) * g\) exists as an element of \(\text{HP}(A \cap B)\), and one has

\[
(\lambda_1 f_1 + \lambda_2 f_2) * g = \lambda_1 (f_1 * g) + \lambda_2 (f_2 * g).
\]

In view of Corollary 6 the convolution product \((Af_1) * (Bf_2)\) always exists, and one has

\[
(Af_1) * (Bf_2) (\cdot) = 0.
\]

Denoting the function which is 1 identically by \(1(\cdot)\) one has that \(f_1 * 1\) exists and

\[
(f_1 * 1) (\cdot) = f_1(0).
\]

The following proposition shows the existence of the convolution product in two cases, and exhibits the well-known integral representation for the product.
IV.3

PROPOSITION 8.
I. Let $A_1, A_2$ be generalized open upper half planes.
Assume $f_k \in HP(A_k)$ for $k = 1, 2$. Then $f_1 \ast f_2$ exists and

$$f_1 \ast f_2 \in HP(A_1' + A_2'),$$

and if $v \in A_1' + A_2'$ then

$$(22) \quad f_1 \ast f_2(v) = \int_a^{a+1} dw \, f_1(w) \, f_2(v-w)$$

for arbitrary $a \in A_1' \cap (v - A_2')$ (where $A_k'$ is the largest open upper half plane contained in $A_k$).

II. Let $B_1, B_2$ be generalized open lower half planes.
Assume $f_k \in HP(B_k)$ for $k = 1, 2$. Then $f_1 \ast f_2$ exists and

$$f_1 \ast f_2 \in HP(B_1' + B_2'),$$

and if $v \in B_1' + B_2'$ then

$$(23) \quad f_1 \ast f_2(v) = \int_b^{b+1} dw \, f_1(w) \, f_2(v-w)$$

for arbitrary $b \in B_1' \cap (v - B_2')$ (where $B_k'$ is the largest open lower half plane contained in $B_k$).

PROOF. Part I. For $k = 1, 2$ the series $\sum_{n \in \mathbb{Z}} f_k(n) \, E(v)^n$ converges absolutely if $v \in A_k'$
(see Corollary 4 with $B_k = \emptyset$). We wish to show that the series

$$\sum_{n \in \mathbb{Z}} f_1(n) \, f_2(n) \, E(v)^n$$

converges if $v \in A_1' + A_2'$. Choose $v = v_1 + v_2$ with $v_k \in A_k'$ for $k = 1, 2$. Then
\[
\sum_n |f_1(n) f_2(n) E(v_1 + v_2)^n| = \\
\sum_n \prod_{k=1,2} |f_k(n) E(v_k)^n| \leq \\
\prod_{k=1,2} \sum_n |f_k(n) E(v_k)^n| < \infty,
\]

which proves \(f_1 \ast f_2 \in HP(A_1' + A_2').\)

To show the second claim, choose \(v \in A_1' + A_2'.\)

Consider the integral on the right in (22). Because \(A_1' \cap (v - A_2')\) is a nonempty horizontal open strip where the integrand is regular, the integral is well-defined. Calling this integral \(I(v)\) we have

\[
I(v) = \int_a^{a+1} dw \sum_{n_1, n_2} f_1(n_1) f_2(n_2) E(v)^{n_2} E(w)^{n_1 - n_2}.
\]

Because the series in the integrand converges absolutely, permutation of the order of integration and summation is justified by Fubini's theorem; so

\[
I(v) = \sum_{n_1, n_2} f_1(n_1) f_2(n_2) E(v)^{n_2} \int_a^{a+1} dw E(w)^{n_1 - n_2}.
\]

Hence, by (15),

\[
I(v) = \sum_n f_1(n) f_2(n) E(v)^n.
\]

This proves \(I(n) = f_1 \ast f_2(n)\) for all \(n\) (by Proposition 5), q.e.d.

Part II. Left to the reader.

The above implies the following. If \(f_k \in HP(A_k \cap B_k)\) for \(k = 1, 2\) then 
\((Af_1) \ast (Af_2)\) and \((Bf_1) \ast (Bf_2)\) always exist, and one has

\[
(24A) \quad (Af_1) \ast (Af_2) \in HP(A_1' + A_2'),
\]

\[
(24B) \quad (Bf_1) \ast (Bf_2) \in HP(B_1' + B_2').
\]

If, furthermore, there exists a pair \((A, B)\) satisfying (1) and (2), such that \((Af_1) \ast (Af_2)\) can be continued in \(A\) as a regular function, and 
\((Bf_1) \ast (Bf_2)\) can be continued in \(B\) as a regular function, then \(f_1 \ast f_2\) exists as an element of \(HP(A \cap B)\), and one has (use (18) and Corollary 6):
(25A) \[ A(f_1 \ast f_2) = (Af_1) \ast (Af_2) \in HP(A), \]

(25B) \[ B(f_1 \ast f_2) = (Bf_1) \ast (Bf_2) \in HP(B), \]

and so (use Proposition 2 and (18) with \( n = 0 \))

(26) \[ f_1 \ast f_2 = (Af_1) \ast (Af_2) + f_1(q) f_2(q) + (Bf_1) \ast (Bf_2). \]

COROLLARY 9.

I. Let \((A, B)\) be a pair of sets satisfying (1).

Assume \( f \in HP(A) \) and \( g \in HP(B) \). Then \( f \ast g \) exists and

\[ f \ast g \in HP(\emptyset), \]

and if \( v \in \emptyset \) then

(27) \[ f \ast g(v) = \int_{c}^{c+1} \text{dw} \ f(w) \ g(v-w) \]

for arbitrary \( c \in A' \cap (v - B') \).

II. Let \((A, B)\) be a pair of sets satisfying (1) and (2).

Assume \( f \in HP(A \cap B) \) and \( g \in HP(\emptyset) \).

Then \( f \ast g \in HP(\emptyset) \), and if \( v \in \emptyset \) then

(28) \[ f \ast g(v) = \int_{\Lambda} \text{dw} \ f(w) \ g(v-w) \]

where \( \Lambda \) is a chain satisfying (8).

PROOF. Part I. From Property 1 (Part II) it follows

\[ Af \in HP(A) \text{ and } Bf \in HP(\emptyset), \]

\[ Ag \in HP(\emptyset) \text{ and } Bg \in HP(B). \]

Hence

\[ (Af) \ast (Ag) \in HP(\emptyset) \text{ by Property 8 (Part I),} \]

\[ (Bf) \ast (Bg) \in HP(\emptyset) \text{ by Property 8 (Part II).} \]
It follows that \( f \ast g \) exists and is element of \( HP(\mathbb{R}) \), see (26).  
The proof of (27) is left to the reader. 
Part II. Left to the reader (apply Proposition 2 and Proposition 8).
\( \square \)

**Remark 2.** For \( n \in \mathbb{Z} \) put

\[
E_n(w) := E(w)^n,
\]

so \( E_n \in HP(\mathbb{R}) \). If \( f \in HP(\mathbb{R} \cap B) \) then \( f \ast E_n \in HP(\mathbb{R}) \) by Corollary 9 
(Part II), and (from (7))

\[
f \ast E_n = f(n) E_n.
\]

In particular, because of \( E_m(n) = \delta(m,n) \) (Kronecker delta), cf. (15),

\[
E_m \ast E_n = \delta(m,n) E_n.
\]

**Remark 3.** Put

\[
p(w) := \frac{E(w)}{1-E(w)} = \frac{1}{E(-w)-1} \text{ for } w \notin \mathbb{Z},
\]

and set

\[
p_+(w) := p(w) \text{ for } w \in \mathbb{C}_+,
\]

\[
p_-(w) := p(-w) \text{ for } w \in \mathbb{C}_-,
\]

where \( \mathbb{C}_+ \) is the open upper half plane, and \( \mathbb{C}_- \) is the open lower half plane. So

\[
p_+ \in HP(\mathbb{C}_+) \text{ and } p_- \in HP(\mathbb{C}_-);
\]

one has

\[
p_+(n) = \begin{cases} 1 & \text{for } n \geq 1 \\ 0 & \text{for } n \leq 0 \end{cases}, \quad p_-(n) = \begin{cases} 0 & \text{for } n \geq 0 \\ 1 & \text{for } n \leq -1 \end{cases}.
\]
It follows (in agreement with (9A) and (9B))

\[ f \ast p_+ = Af \text{ and } f \ast p_- = Bf \]

if \( f \in HP(A \cap B) \). Also put

\[ \tilde{p}_+(w) := -p(-w) = p_+(w) + 1 \text{ for } w \in \mathbb{C}_+, \]

\[ \tilde{p}_-(w) := -p(w) = p_-(w) + 1 \text{ for } w \in \mathbb{C}_-. \]

So \( \tilde{p}_+ \in HP(\mathbb{C}_+) \) and \( \tilde{p}_- \in HP(\mathbb{C}_-) \) with

\[ \tilde{p}_+(n) = \begin{cases} 1 & \text{for } n \geq 0, \\ 0 & \text{for } n \leq -1 \end{cases}, \quad \tilde{p}_-(n) = \begin{cases} 0 & \text{for } n \geq 1, \\ 1 & \text{for } n \leq 0. \end{cases} \]

If \( f \in HP(A \cap B) \) then

\[ f \ast \tilde{p}_+ = \tilde{A}f \text{ and } f \ast \tilde{p}_- = \tilde{B}f, \]

where

\[ \begin{cases} \tilde{A}f(v) := f(0) + Af(v) \in HP(A), \\ \tilde{B}f(v) := f(0) + Bf(v) \in HP(B), \end{cases} \]

(29)

cf. Proposition 1 (Part II).
IV.4 The integrals \( I_\alpha f, I_\beta f, \) and \( I_\eta f \)

IV.4.1 The functions \( \alpha, \beta, \) and \( \eta \)

In the following it is assumed:

\[ \sigma t \text{ is a complex constant satisfying } \Im(\sigma t) > 0. \]

In the integrals to be defined the functions \( \alpha(\cdot | 2\sigma t), \beta(\cdot | 2\sigma t) \) and \( \eta(\cdot | 2\sigma t) \) occur, as introduced (with parameter \( \tau \) rather than \( 2\sigma t \)) in III.2.(23), III.2.(25), and Subsection III.2.1, respectively, with \( u \) and \( q \) given in III.2.(2) and III.2.(3), respectively. As before we put

\[ E(\omega) = e^{2\pi i \omega}. \]

In addition we put

\[ p = E(\sigma t), \]

so \( |p| < 1, \) and have

\[ \alpha(\omega | 2\sigma t) = 2\pi i \sum_{k \geq 0} \frac{-E(\omega)p^{2k+1}}{1-E(\omega)p^{2k+1}} (=: \alpha_{2\sigma t}(\omega)), \]

\[ \beta(\omega | 2\sigma t) = 2\pi i \sum_{k \geq 0} \frac{E(-\omega)p^{2k+1}}{1-E(-\omega)p^{2k+1}} (=: \beta_{2\sigma t}(\omega)). \]

From III.2.(27) one has

\[ \alpha(\omega | 2\sigma t) + \beta(\omega | 2\sigma t) = \eta(\omega | 2\sigma t) (=: \eta_{2\sigma t}(\omega)). \]

One has

\[ \alpha(\cdot | 2\sigma t) \in \mathcal{H}_p(-\sigma t + C_+), \]

\[ \beta(\cdot | 2\sigma t) \in \mathcal{H}_p(\sigma t - C_+), \]
(7) \[ \eta(\cdot \mid 2r) \in \mathcal{HP}((-\tau + \mathcal{C}_+ \cap (\tau - \mathcal{C}_+)). \]

**Remark 1.** One has from III.2.(20)

(8) \[ \eta(\cdot \mid 2r) = \frac{\theta_4'(\cdot \mid 2r)}{\theta_4(\cdot \mid 2r)}, \]

where \( \theta_4(\cdot \mid \cdot) \) is one of Jacobi's \( \theta \)-functions, which can be defined as follows (cf. Section III.2, (16) - (19)):

(9) \[ \theta_4(w \mid 2r) = q_0(2r) \prod_{k \geq 0} \frac{(1-E(w)p^{2k+1})(1-E(-w)p^{2k+1})}{1-tp^{2k+1}}, \]

with

(10) \[ q_0(2r) = \prod_{k \geq 0} (1 - p^{2k}). \]

The above functions depend on \( w \) through \( E(w) \) only. We write

(11) \[ t := E(w), \]

and define the functions \( \hat{a}, \hat{b}, \text{ and } \hat{n} \) as follows:

(12A) \[ \hat{a}(t \mid 2r) := \frac{\alpha(w \mid 2r)}{2\pi i E(w)} = \sum_{k \geq 0} \frac{-p^{2k+1}}{1-tp^{2k+1}}, \]

(12B) \[ \hat{b}(t \mid 2r) := \frac{\bar{\theta}(w \mid 2r)}{2\pi i E(w)} = \sum_{k \geq 0} \frac{t^{-2k+1}}{1-t^{-1}p^{2k+1}}, \]

(13) \[ \hat{n}(t \mid 2r) := \frac{n(w \mid 2r)}{2\pi i E(w)}. \]

They are related by

(14) \[ \hat{a}(t \mid 2r) + \hat{b}(t \mid 2r) = \hat{n}(t \mid 2r). \]

The functions \( a, b, \text{ and } \hat{a}, \hat{b} \) are the logarithmic derivatives of certain entire functions with simple zeros, etc.
\[
\begin{align*}
\alpha(w | 2\sigma t) &= \frac{\partial}{\partial w} \log \prod_{k \geq 0} (1 - E(w)p^{2k+1}), \\
\beta(w | 2\sigma t) &= \frac{\partial}{\partial w} \log \prod_{k \geq 0} (1 - E(-w)p^{2k+1}),
\end{align*}
\]

and

\[
\begin{align*}
\tilde{\alpha}(t | 2\sigma t) &= \frac{\partial}{\partial t} \log \prod_{k \geq 0} (1 - tp^{2k+1}), \\
\tilde{\beta}(t | 2\sigma t) &= \frac{\partial}{\partial t} \log \prod_{k \geq 0} (1 - t^{-1}p^{2k+1}).
\end{align*}
\]

So these functions are meromorphic in the plane and have simple poles with residues +1. We consider the sets of poles.

**The point sets** $P^\pm(v)$ and $P(v)$

We put for $v \in \mathcal{E}$

(15A) $P^+(v) = v + \sigma t + \mathbb{Z}^+ \cdot 2\sigma t + \mathbb{Z},$

(15B) $P^-(v) = v - \sigma t - \mathbb{Z}^+ \cdot 2\sigma t + \mathbb{Z},$

(16) $P(v) = v + \sigma t + \mathbb{Z} \cdot 2\sigma t + \mathbb{Z} = P^+(v) \cup P^-(v),$

where $\mathbb{Z}^+$ is the set of nonnegative integers. One has (cf. III.2.(24) and III.2.(26))

(17A) $P^+(v)$ is the set of poles of $w \to \alpha(v-w | 2\sigma t),$

(17B) $P^-(v)$ is the set of poles of $w \to \beta(v-w | 2\sigma t),$

(18) $P(v)$ is the set of poles of $w \to \eta(v-w | 2\sigma t).$

**IV.4.2 Construction of the integrals** $\frac{I_f}{\alpha}, \frac{I_f}{\beta},$ and $\frac{I_f}{\eta}$

In the following we assume, as in the previous section,

(19) \[
\begin{align*}
A & \text{ is a generalized open upper half plane,} \\
B & \text{ is a generalized open lower half plane}
\end{align*}
\]
satisfying

\[(20) \quad A \cup B = \emptyset, \]

so \(A \cap B\) is a generalized open horizontal strip. In addition we will assume that \(A\) and \(B\) satisfy

\[(21) \quad A \subset A - 2\sigma \tau \quad \text{and} \quad B \subset B + 2\sigma \tau. \]

These assumptions have the following obvious consequences:

\[(22A) \quad v \in A - \sigma \tau \quad \text{iff} \quad P^+(v) \subset A, \]
\[(22B) \quad v \in B + \sigma \tau \quad \text{iff} \quad P^-(v) \subset B. \]

The integration paths

We intend to impose for \(v \in \emptyset\) the following conditions on the chain \(\Lambda:\)

\[(23) \quad E \circ \Lambda \quad \text{is a cycle with} \quad \text{Ind}(0 \mid E \circ \Lambda) = 1, \]
\[(24) \quad \Lambda^* \subset A \cap \emptyset, \]
\[(25A) \quad \begin{cases} 
\Lambda^* \subset P^+(v)^C, \\
\text{Ind}^*(w \mid \Lambda) = 1 \quad \text{for} \quad w \in P^+(v), 
\end{cases} \]
\[(25B) \quad \begin{cases} 
\Lambda^* \subset P^-(v)^C, \\
\text{Ind}^*(w \mid \Lambda) = 0 \quad \text{for} \quad w \in P^-(v). 
\end{cases} \]

For the function \(\text{Ind}^*(\cdot \mid \cdot)\) see IV.2.(19) and IV.2.(20); and for the sets \(P^+(v)\) see (15A) and (15B).

**REMARK 2.** Let \(\Lambda\) be a chain satisfying (23) and (24). From Property IV.2.4 conclude

\[(26A) \quad \text{Ind}^*(w \mid \Lambda) = 0 \quad \text{if} \quad w \in A^C, \]
(26B) \[ \text{Ind}_A^* (w \mid \Lambda) = 1 \text{ if } w \in \mathcal{B}^c. \]

So \( \text{Ind}_A^* (w \mid \Lambda) \) is known for all \( w \in (A \cap \mathcal{B})^c. \)

The following proposition deals with the existence of a chain \( \Lambda \) satisfying (23) - (25B).

**Proposition 1.** Let \( A, \mathcal{B} \) satisfy (19) through (21). Then the following holds.

I. A chain \( \Lambda \) satisfying (23), (24), and (25A) exists iff

\[ (27A) \quad v \in A - \sigma t. \]

II. A chain \( \Lambda \) satisfying (23), (24), and (25B) exists iff

\[ (27B) \quad v \in \mathcal{B} + \sigma t. \]

III. A chain \( \Lambda \) satisfying (23), (24), (25A) and (25B) exists iff

\[ (28) \quad v \in (A - \sigma t) \cap (\mathcal{B} + \sigma t). \]

**Remark 3.** Under the assumptions of Proposition 1 the intersection \( (A - \sigma t) \cap (\mathcal{B} + \sigma t) \) is a generalized open horizontal strip, because (cf. Proposition IV.1.2)

\[ (A - \sigma t) \cup (\mathcal{B} + \sigma t) = [A \cup (\mathcal{B} + 2\sigma t)] - \sigma t \supset (A \cup \mathcal{B}) - \sigma t = \mathcal{E}. \]

**Proof** of Proposition 1. Part I, "only if".

Assume that \( \Lambda \) satisfies (23), (24) and (25A). Then (26A) holds, hence \( A^c \cap P_t^+ (v) = \emptyset \) because of (25A). This implies (27A) because of (22A).

Part II, "only if". Assuming that (23), (24) and (25B) holds, conclude that (26B) holds, hence \( \mathcal{B}^c \cap P_t^- (v) = \emptyset \) because of (25B). This implies (27B) because of (22B).

Part III, "only if". This is a combination of the "only if"-statements from the Parts I and II.

We prove Part III, "if" (Parts I and II, "if", go similar). Let \( \Lambda (\cdot) \) be an arbitrary path with parameter interval \([0,1]\) and satisfying
\[\begin{align*}
\Lambda(1) - \Lambda(0) &= 1, \\
\Lambda^* &\subset (A \cap B) \setminus P(v).
\end{align*}\]

So \(\Lambda\) satisfies (23) and (24).

Because \(\Lambda^*\) is compact, one has \(|\text{Im } w| < M\) for some positive \(M\) and all \(w \in \Lambda^*\). Consequently,

\[\text{Ind}^* (w | \Lambda) = \begin{cases}
1 & \text{if } \text{Im } w \geq M, \\
0 & \text{if } \text{Im } w \leq -M,
\end{cases}\]

by Property IV.2.4. Observe:

\[P(v) \cap \{w: \mu \text{Im } w \geq M\} \subset P^\mu(v)\]

for \(\mu = \pm\). So \(\text{Ind}^* (w | \Lambda)\) has the required value if

(a) \quad w \in P(v) \text{ and } |\text{Im } w| \geq M.

By Remark 2,

\[\text{Ind}^* (w | \Lambda) = \begin{cases}
1 & \text{if } w \in B^c, \\
0 & \text{if } w \in A^c.
\end{cases}\]

Assuming (28) one has by (22A) and (22B)

\[P(v) \cap (B^c) \subset P^+(v) \text{ and } P(v) \cap (A^c) \subset P^-(v).\]

So \(\text{Ind}^* (w | \Lambda)\) has the required value if

(b) \quad w \in P(v) \cap (A \cap B)^c.

Hence, \(\text{Ind}^* (w | \Lambda)\) possibly has not the required value only if \(w \in \mathcal{P}\), where

\[\mathcal{P} = P(v) \cap A \cap B \cap \{w: |\text{Im } w| < M\}.
\]

Observe that \(\mathcal{P}/\mathbb{Z}\) is a finite set. In order to change \(\text{Ind}^* (w | \Lambda)\) for \(w \in \mathcal{P}\) into the required value, one can add to \(\Lambda\) a finite number of suitable chosen closed paths, \(\Lambda_1, \ldots, \Lambda_n\) say, contained in \((A \cap B) \setminus P(v)\);
e.g. each \( \Lambda_k \) might be a small circle centered at some point of \( \mathcal{F} \).

In view of Property IV.2.5 this can be done in such a way that for \( w \in \mathcal{F} \)

\[
\text{Ind}^* (w \mid \Lambda) + \sum_{k=1}^{n} \text{Ind}^* (w \mid \Lambda_k)
\]

has the desired value. Then the chain \( \{\Lambda, \Lambda_1, \ldots, \Lambda_n\} \) satisfies all requirements.

\[\Box\]

The integrals

In the following assume that \( f \in H^p(\Lambda \cap B) \). The functions \( \alpha(\cdot \mid \cdot) \) and \( \beta(\cdot \mid \cdot) \) are defined in (4A) and (4B), respectively.

For \( v \in \Lambda - \sigma \tau \) we define

\[
\begin{aligned}
\text{I}_\alpha f(v) &= \int \text{d}w \alpha(v-w \mid 2\sigma \tau) \cdot f(w), \\
\text{where } \Lambda &\text{ is a chain satisfying (23), (24), and (25A).}
\end{aligned}
\]

(29A)

and for \( v \in B + \sigma \tau \)

\[
\begin{aligned}
\text{I}_\beta f(v) &= \int \text{d}w \beta(v-w \mid 2\sigma \tau) \cdot f(w), \\
\text{where } \Lambda &\text{ is a chain satisfying (23), (24), and (25B).}
\end{aligned}
\]

(29B)

The existence of such chains is guaranteed by Proposition 1 (Parts I and II); we will show below (see Property 2 (Part I)) that these integrals are well-defined, and that their values do not depend on the choice of \( \Lambda \).

We rewrite the above integrals by introducing the new integration variable \( t = E(w) \). For that purpose define the function \( \hat{f}(\cdot) \) by means of

\[
f = \hat{f} \circ E
\]

(30)

(such a function exists by Property IV.2.1). Also put

\[
u = E(v)
\]

(31)

and apply the change-of-variable formula IV.2.(11) to find (for \( \hat{\alpha}(\cdot \mid \cdot) \))
see (12A), and for $\hat{\beta}(\cdot \mid \cdot)$ see (12B))

\begin{equation}
(32A) \quad I_\alpha f(v) = \int_{\text{EoA}} \frac{\mu}{\tau^2} \hat{\alpha}(\frac{u}{\tau} \mid 2\sigma^2) \cdot \hat{f}(t),
\end{equation}

where $\Lambda$ is a chain satisfying (23), (24), and (25A),

\begin{equation}
(32B) \quad I_\beta f(v) = \int_{\text{EoA}} \frac{\mu}{\tau^2} \hat{\beta}(\frac{u}{\tau} \mid 2\sigma^2) \cdot \hat{f}(t),
\end{equation}

where $\Lambda$ is a chain satisfying (23), (24), and (25B).

PROPERTY 2. Let $A, B$ satisfy (19) through (21), and assume $f \in H^p(A \cap B)$.

I. The integrals in (29A) and (29B) are well-defined, and their values do not depend on the choice of $\Lambda$.

II. \[ I_\alpha f(\cdot) \in H^p(A - \sigma t), \]
\[ I_\beta f(\cdot) \in H^p(B + \sigma t). \]

PROOF. Part I. In stead of the integrals (29A) and (29B) consider the integrals (32A) and (32B).

First consider (32A) for fixed $v \in A - \sigma t$. The integrand is regular in the open set

\begin{equation}
(33) \quad E(A \cap B \setminus E(p^+(v)) = E((A \setminus B) \setminus p^+(v)),
\end{equation}

which contains the set $(E \circ \Lambda)^* = E(\Lambda^*)$ by the assumptions (24) and (25A). So the integral is well-defined.

Apply Cauchy's Theorem (RUDIN [1966], Theorem 10.35) to show that the value does not depend on $\Lambda$. The complement of the set (33) is

\begin{equation}
(34) \quad E(A^C) \cup E(B^C) \cup E(p^+(v)) \cup \{0\}.
\end{equation}

The assumptions (23), (24) and (25A) imply that for every element $t$ of the set (34) the value $\text{Ind}(t \mid E \circ \Lambda)$ does not depend on $\Lambda$ (cf. Remark 2). By Cauchy's Theorem this proves the assertion.

Similarly one shows the assertions with respect to (29B).

Part II. The regularity is an application of the standard theorem II.4.74. The periodicity follows from the periodicity of $\alpha$ and $\beta$.
For \( v \in (A - \sigma t) \cap (B + \sigma t) \), we define

\[
I_\eta f(v) = \int_A \frac{dw}{2} \eta(v-w, 2\sigma t) \cdot f(w),
\]

where \( A \) is a chain satisfying (23), (24), (25A) and (25B). Such a chain exists by Proposition 1 (Part III). By the identity (5) this definition is tantamount to

\[
I_\eta f(v) = I_\alpha f(v) + I_B f(v).
\]

The integral (35) can be rewritten by the introduction of the new integration variable \( t = E(w) \). Using the notations (31), (30) and (13) one finds

\[
I_\eta f(v) = \int_{EoA} \frac{dt}{2} \frac{u}{t^2} \hat{\eta}(\frac{u}{t} | 2\sigma t) \cdot \hat{f}(t),
\]

where \( A \) is a chain satisfying (23), (24), (25A) and (25B).

Property 2 has the following corollary.

**COROLLARY 3.** Under the assumptions of Property 2:

I. The integral in (35) is well-defined and its value does not depend on the choice of \( A \).

II. \( I_\eta f \in HP((A - \sigma t) \cap (B + \sigma t)) \).

**\( \square \)**

IV.4.3 The Fourier coefficients of the functions \( I_\alpha f, I_\beta f, \) and \( I_\eta f \) first we identify the integrals \( I_\alpha f, I_\beta f, \) and \( I_\eta f \) with certain convolution products, and next we determine their Fourier coefficients.

For \( f \in HP(A \cap B) \) the functions \( Af \) and \( Bf \) are defined in IV.3.(9A) and IV.3.(9B), respectively, and the functions \( \tilde{Af}, \tilde{Bf} \) in IV.3.(29). We recall (cf. Proposition IV.3.1) that \( Af, \tilde{Af} \in HP(A) \) and \( Bf, \tilde{Bf} \in HP(B) \). Observe

\[
Af + \tilde{Bf} = f = \tilde{Af} + Bf.
\]

By Proposition IV.3.8 (Part I) and Corollary IV.3.9 (Part I) one has from (6A) - (6B) (with a change in the notation, cf. (4A) - (4B))

\[
(Af) * a_{2\sigma t} \text{ exists and } \in HP(A' - \sigma t),
\]

\[
(\tilde{Bf}) * a_{2\sigma t} \text{ exists and } \in HP(B).
\]
Hence

\[(40A) \quad f \ast \alpha_{2\sigma_T} \text{ exists and } \in \text{HP}(A' - \sigma_T).\]

Similarly one has

\[(40B) \quad f \ast \beta_{2\sigma_T} \text{ exists and } \in \text{HP}(B' + \sigma_T).\]

The following proposition shows that \(\mathcal{I}_a f\) is the analytic continuation of \(f \ast \alpha_{2\sigma_T}\), and \(\mathcal{I}_b f\) that of \(f \ast \beta_{2\sigma_T}\).

**PROPOSITION 4.** Let \(A, B\) satisfy (19) through (21). Assume \(f \in \text{HP}(A \cap B)\). Then

\[(41A) \quad f \ast \alpha_{2\sigma_T} (\cdot) = \mathcal{I}_a f (\cdot) \text{ on } A' - \sigma_T,\]

\[(41B) \quad f \ast \beta_{2\sigma_T} (\cdot) = \mathcal{I}_b f (\cdot) \text{ on } B' + \sigma_T.\]

**PROOF.** We show (41A), and leave (41B) to the reader.

Because of (38) and \(f, Af, Bf \in \text{HP}(A \cap B)\) one has

\[(i) \quad \mathcal{I}_a f = \mathcal{I}_a (Af) + \mathcal{I}_a (Bf).\]

Next we show

\[(ii) \quad (Af) \ast \alpha_{2\sigma_T} (v) = \mathcal{I}_a (Af)(v) \text{ for } v \in A' - \sigma_T.\]

The integral \(\mathcal{I}_a (Af)\) is defined in (29A), with conditions (23), (24), and (25A) on the chain \(A\), where \(B = \mathcal{C}\) in condition (24) because of \(Af \in \text{HP}(A)\). For \((Af) \ast \alpha_{2\sigma_T}\) we have the following expression (see Proposition IV.3.8 (Part I)): if \(v \in A' - \sigma_T\) then

\[(Af) \ast \alpha_{2\sigma_T} (v) = \int_{\Lambda} dw (Af)(w) \alpha_{2\sigma_T}(v-w),\]

where \(\Lambda\) is a path from \(a\) to \(a + 1\) with range \([a, a + 1]\), and \(a \in A' \cap (v + \sigma_T - \mathcal{C} +) \subset A \setminus P^+(v)\). So \(\Lambda\) satisfies (23) and (24) (with \(B = \mathcal{C}\)); \(\Lambda(\cdot)\) also satisfies (25A), cf. Property IV.2.4. Hence (ii) is true.
Finally we show

(iii) \((\tilde{\Omega}f) * {a}_{2\sigma^t}(v) = I^\alpha_\lambda (\tilde{\Omega}f)(v)\) for \(v \in \mathbb{E}\).

For the integral \(I^\alpha_\lambda (\tilde{\Omega}f)\) see (29A), with the conditions (23), (24), and (25A) on the chain \(\Lambda\), where \(\Lambda = \mathbb{E}\) in (24) because of \(\tilde{\Omega}f \in HP(B)\).

For the convolution product \((\tilde{\Omega}f) * {a}_{2\sigma^t}\) we have the following expression (see Corollary IV.3.9 (Part I)):

if \(v \in \mathbb{E}\) then

\[(\tilde{\Omega}f) * {a}_{2\sigma^t}(v) = \int_A dw (\tilde{\Omega}f)(w) {a}_{2\sigma^t}(v-w),\]

where \(A(\cdot)\) is a path from \(c\) to \(c+1\) with range \([c, c+1]\), and \(c \in B' \cap (v + \sigma^t - \mathbb{E} +) \subset B - p^t(v)\).

So \(A\) satisfies (23), (24) (with \(A = \mathbb{E}\)), and (25A), which implies (iii).

The relationships (i), (ii) and (iii) imply (41A).

\(\Box\)

By the above proposition the function \(f * {a}_{2\sigma^t}\) can be identified with the function \(I^\alpha_\lambda f\), and \(f * {\beta}_{2\sigma^t}\) with \(I^\beta_\lambda f\); so one has by Proposition 2 (Part II)

\[(42) \quad f * {a}_{2\sigma^t} \in HP(A - \sigma^t), \quad f * {\beta}_{2\sigma^t} \in HP(B + \sigma^t).\]

Next we show that \(f * {\eta}_{2\sigma^t}\) exists and equals \(I^\eta f\).

\[\text{COROLLARY 5. Under the assumptions of Proposition 4 the convolution product}\]

\(f * {\eta}_{2\sigma^t}\) \[\text{exists, and one has}\]

\[f * {\eta}_{2\sigma^t} = I^\eta f \in HP((A - \sigma^t) \cap (B + \sigma^t)).\]

\[\text{PROOF. By (42) the products } f * {a}_{2\sigma^t}(\cdot) \text{ and } f * {\beta}_{2\sigma^t}(\cdot), \text{ exist as elements of } HP((A - \sigma^t) \cap (B + \sigma^t)).\]

By linearity (see IV.3.(19)) the product \(f * {\eta}_{2\sigma^t} = f * ( {a}_{2\sigma^t} + {\beta}_{2\sigma^t} )\) (cf. (5)) exists as an element of \(HP((A - \sigma^t) \cap (B + \sigma^t))\), and equals the sum

\[f * {a}_{2\sigma^t} + f * {\beta}_{2\sigma^t} = I^\alpha f + I^\beta f.\]
By (36) this implies the claim.

The Fourier coefficients

The Fourier coefficients of the functions \( \alpha_{2\sigma\tau}(\cdot) \) and \( \beta_{2\sigma\tau}(\cdot) \) follow from III.2.(29) and III.2.(32) (replace \( \tau \) by \( 2\sigma\tau \), and \( q \) by \( p \)); in view of Proposition IV.3.5 one finds:

\[ \alpha_{2\sigma\tau}(n) = \begin{cases} \frac{2\pi i}{p^n - p^{-n}} & \text{if } n \geq 1, \\ 0 & \text{if } n \leq 0, \end{cases} \]

(43A)

\[ \beta_{2\sigma\tau}(n) = \begin{cases} 0 & \text{if } n \geq 0, \\ \frac{2\pi i}{p^n - p^{-n}} & \text{if } n \leq -1. \end{cases} \]

(43B)

From III.3.(34) (or from the equality (5)) one finds

\[ \eta_{2\sigma\tau}(n) = \alpha_{2\sigma\tau}(n) + \beta_{2\sigma\tau}(n) = \begin{cases} 0 & \text{if } n = 0, \\ \frac{2\pi i}{p^n - p^{-n}} & \text{if } n \neq 0. \end{cases} \]

(44)

**REMARK 4.** In view of Corollary IV.3.6 it follows from the above:

\[ A_{2\sigma\tau} = \alpha_{2\sigma\tau} \quad \text{and} \quad B_{2\sigma\tau} = \beta_{2\sigma\tau}. \]

**Proposition 6.** Under the assumptions of Proposition 4 the following holds for \( n \in \mathbb{Z} \):

\[ I_{\alpha} f(n) = f(n) \cdot \alpha_{2\sigma\tau}(n), \]

(46A)

\[ I_{\beta} f(n) = f(n) \cdot \beta_{2\sigma\tau}(n), \]

(46B)

\[ I_{\eta} f(n) = f(n) \cdot \eta_{2\sigma\tau}(n). \]

(47)

**Proof.** Immediately from Proposition 4 and Corollary 5, in view of the definition of convolution product.

\[ \square \]
REMARK 5. It follows from (43A) (cf. Corollary IV.3.6)

\[(Af) * \alpha_{2\sigma_T} = f * \alpha_{2\sigma_T},\]

\[(Bf) * \alpha_{2\sigma_T} = 0,\]

in other words (cf. (41A))

\[
\begin{align*}
I_{\alpha}(Af) &= I_{\alpha}f, \\
I_{\alpha}(Bf) &= 0.
\end{align*}
\]

(48)

Similarly one has

\[
\begin{align*}
I_{\beta}(Af) &= 0, \\
I_{\beta}(Bf) &= I_{\beta}f.
\end{align*}
\]

(49)

Consequently (cf. (36)),

\[
I_{\eta}f = I_{\alpha}(Af) + I_{\beta}(Bf),
\]

(50)

which corresponds with the equality (cf. IV.3.(26))

\[f * \eta = (Af) * (A\eta) + (Bf) * (B\eta).\]

Finally we notice

\[
\begin{align*}
A(I_{\eta}f) &= (Af) * (A\eta_{2\sigma_T}) = I_{\alpha}f, \\
B(I_{\eta}f) &= (Bf) * (B\eta_{2\sigma_T}) = I_{\beta}f.
\end{align*}
\]

(51)
IV.5  The solution of the linear equations by means of Fourier analysis

We assume that $\sigma_T$ is a constant satisfying the condition (S0) from Subsection IV.1.2, and that $(A_1, A_2)$ and $(B_1, B_2)$ are pairs of generalized open half planes satisfying (S1) through (S3) from Subsection IV.1.2. The generalized open horizontal strips $S_\lambda^\iota$ and $F_\lambda^\iota$ are defined in IV.1.1(1) and IV.1.1(2), respectively (for $\iota = 1, 2$ and $\lambda = \pm$). Also, we assume that the functions $F_\lambda^\iota(\cdot), \iota = 1, 2$ and $\lambda = \pm$, satisfy the conditions (P0), (P1) and (P2) from Subsection IV.1.3. (We do not assume (F1M) or (F1R).)

Finally, we assume that the pair of constants $(v_{01}^\iota, v_{02}^\iota)$ satisfies IV.1.1(3).

In this section we will construct functions $Z_\lambda^\iota(\cdot)$ and constants $Z_{0j}^\lambda (\iota = 1, 2$ and $\lambda = \pm)$ satisfying the conditions (L) from Subsection IV.1.2, which are repeated here ($j = 3 - \iota$):

\[
\begin{align*}
\text{Regularity:} & \quad Z_\lambda^\iota(\cdot) \text{ is regular in } S_\lambda^\iota \text{ and has period 1 for } \iota = 1, 2 \\
\text{Initial Value:} & \quad Z_\lambda^\iota(v_{0j}^\iota) = 0 \text{ for } \iota = 1, 2 \\
\text{Equality:} & \quad Z_0^\iota + Z_\lambda^\iota(v_{0j}^\lambda) = Z_{0j}^\lambda \text{ for } \iota = 1, 2 \text{ and } \lambda = \pm \\
\text{Equations:} & \quad \text{for } \iota = 1, 2 \text{ and } \lambda = \pm \text{ the equation } (E_{\iota}^\lambda) \text{ is satisfied:
}\end{align*}
\]

\[
(E_{\iota}^\lambda) \quad \text{if } v \in F_\lambda^\iota \text{ then } Z_{0j}^\lambda + Z_\lambda^\iota(v) + Z_\lambda^\iota(v - \lambda \sigma_T) = F_\lambda^\iota(v).
\]

IV.5.1 The functions IF_j^\lambda(\cdot) and $(\Omega F)_\iota^\lambda(\cdot)$ ($\iota, j = 1, 2$ and $\lambda = \pm$)

As usual, $\iota$ is an index from the index set \{1, 2\}, $j = 3 - \iota$, and $\lambda \in \{-, +\}$.

In the sequel, the following abbreviations will be used:

\[
(1A) \quad A_\lambda^\iota = A_\iota \cap (A_j^\iota + \lambda \sigma_T) = \begin{cases} A_j^\iota + \lambda \sigma_T & \text{if } \lambda = +, \\ A_\iota & \text{if } \lambda = -, \end{cases}
\]

\[
(1B) \quad B_\lambda^\iota = B_\iota \cap (B_j^\iota + \lambda \sigma_T) = \begin{cases} B_j^\iota + \lambda \sigma_T & \text{if } \lambda = +, \\ B_\iota - \sigma_T & \text{if } \lambda = -. \end{cases}
\]

From IV.1.2 and IV.1.3 one has

\[
(2) \quad F_\lambda^\iota = S_\iota \cap (S_j^\iota + \lambda \sigma_T) = A_\lambda^\iota \cap B_\iota^\lambda,
\]
and from IV.1.(4) and IV.1.(5)

\[(3) \quad F^\lambda_{i} = S^\lambda_{i} \cup (S^\lambda_{i} + \lambda \sigma t) = (A^\lambda_{i} - \sigma t) \cap (B^\lambda_{i} + \sigma t).\]

The following property (which is the analogue of Property II.4.1) will be used.

**PROPERTY 1.** For \(i = 1,2\) and \(j = 3 - i\) the following holds:

\[(4) \quad F^\lambda_{i} = \bigcap_{\mu = \pm} (F^\lambda_{i} + \mu \sigma t) \quad \text{for} \quad \lambda = \pm,\]

\[(5) \quad S^\lambda_{i} = \bigcap_{\lambda = \pm} F^\lambda_{j},\]

\[(6) \quad F^\lambda_{i} = F^\lambda_{j} \cap (F^\lambda_{i} + \lambda \sigma t) \quad \text{for} \quad \lambda = \pm.\]

**PROOF.** The equality (4) follows from the inclusions

\[(7) \quad A^\lambda_{i} \subset A^\lambda_{i} - 2\sigma t \quad \text{and} \quad B^\lambda_{i} \subset B^\lambda_{i} + 2\sigma t\]

for \(i \in \{1,2\}\) and \(\lambda \in \{-,+,0\}\). These inclusions are consequences of the assumption (S2).

The equality (5) follows from the equality \(S^\lambda_{i} = A^\lambda_{i} \cap B^\lambda_{i}\) (cf. IV.1.(1)) and the inclusions

\[(8) \quad A^{+}_{j} \subset A^{-}_{j} \quad \text{and} \quad B^{-}_{j} \subset B^{+}_{j}\]

for \(j \in \{1,2\}\), which follow from (S2).

To see (6), rewrite (4) in the form

\[F^\lambda_{i} = (F^\lambda_{i} - \lambda \sigma t) \cap (F^\lambda_{i} + \lambda \sigma t)\]

and apply IV.1.(7).

\[\Box\]

The function \(I_{j}^{\lambda}(\cdot)\) for \(j = 1,2\) and \(\lambda = \pm\)

We put
(9) \[ IF_j^\lambda(v) = \frac{1}{2\pi i} \int_{\eta} F_j^\lambda(w), \]

where the definition IV.4.(35) of the integral operator \( I_{\eta} \) is used with
\( A_j^\lambda = A_j \) and \( B_j^\lambda = B_j \).

In other words, because of IV.4.(8),

(10) \[ IF_j^\lambda(v) = \int_{\Lambda} \frac{1}{2\pi i} \cdot \frac{\delta_j^\lambda}{\delta_4} \cdot (v-w | 2\sigma \tau) \cdot F_j^\lambda(w), \]

where \( \Lambda \) is a chain satisfying (for the \( P \)-sets see IV.4.(15A) et seq.)

\[
\begin{align*}
E \circ \Lambda \text{ is a cycle with } \text{Ind} (0 | E \circ \Lambda) &= 1, \\
\Lambda^* \subset F_j^\lambda \setminus P(v), \\
\text{Ind}^* (w | \Lambda) &= \begin{cases} 1 & \text{if } w \in P^+(v), \\ 0 & \text{if } w \in P^-(v). \end{cases}
\end{align*}
\]

By Corollary IV.4.3 one has (cf. (3) and IV.3.(6))

(12) \[ IF_j^\lambda \in HP(F_j^\lambda). \]

Also one has \( I_{\eta} F_j^\lambda = F_j^\lambda \star \eta \) \( 2\sigma \tau \) by Corollary IV.4.5; from IV.4.(47) and IV.4.(44) one has

(13) \[ IF_j^\lambda(n) = \begin{cases} \frac{F_j^\lambda(n)}{p^{-n}} & \text{if } n \neq 0, \\ 0 & \text{if } n = 0, \end{cases} \]

with \( p \) given by

(14) \[ p^\lambda = E(\sigma \tau) \]

where

(15) \[ E(w) = e^{2\pi i w} \]

for \( w \in \mathbb{C} \) (in agreement with IV.4.(3) and IV.4.(2)).
The pair of functions $\Omega F$

The pair of functions

\[(16)\quad ((\Omega F)_1(z), (\Omega F)_2(z_2))\]

is defined by

\[(17)\quad (\Omega F)_2(\cdot) = \sum_{\mu = \pm} \mu \Omega \mathcal{F}^\mu_{\cdot} \]

for $i = 1, 2$ and $j = 3 - i$.

From (12) and (5) it follows

\[(18)\quad (\Omega F)_2(\cdot) \in HP(S),\]

and by (13) the Fourier coefficients are

\[(19)\quad (\Omega F)_2(n) = \begin{cases} 
\frac{1}{p^n - p^{-n}} \sum_{\mu = \pm} \mu \mathcal{F}^\mu_{\cdot}(n) & \text{if } n \neq 0, \\
0 & \text{if } n = 0.
\end{cases}\]

IV.5.2 The solution of the conditions (L)

**Property 2.** Assume $f \in HP(S)$ with $S$ a generalized open horizontal strip, and $c \in \mathbb{C}$. Put

$\tilde{f}(\cdot): = f(\cdot + c)$.

Then $\tilde{f} \in HP(S - c)$ and $\tilde{f}(n) = E(c)^n f(n)$ for all $n \in \mathbb{Z}$.

**Proof.** Left to the reader.

\[\Box\]

**Proposition 3.** Fix $i, j: = 3 - i$, and $\lambda$.

If $Z_{i} \in HP(S)$ and $F_{\lambda}^i \in HP(F_{\lambda})$, then the equation $(F_{\lambda})$ (see the conditions (L)) is equivalent to (20) & (21):

\[(20)\quad Z_{i}^{\lambda} + Z_{1}(0) + Z_{2}(0) = F_{\lambda}^i(0),\]
IV.5

(21) \[ Z^\lambda_z(n) + p^{-\lambda n} Z^\lambda_j(n) = F^\lambda_z(n) \quad \text{for} \quad n \in \mathbb{Z} \setminus \{0\}. \]

**Proof.** Observe that both sides of the equation \((E^\lambda_z)\) are elements of \(\text{HP}(F^\lambda_z)\). Also observe that the \(n\)th Fourier coefficient of the function \(v \to Z^\lambda_j(v - \lambda \sigma t)\) is \(p^{-\lambda n} Z^\lambda_j(n)\) by Property 2 and (14).

Compute the \(n\)th Fourier coefficient of the sum on the left, and of the function on the right in the equation \((E^\lambda_z)\) (apply IV.1.(7) using a chain \(\Lambda\) with range \(\Lambda^*\) contained in \(F^\lambda_z\)).

Equating the two Fourier coefficients one finds (20) if \(n = 0\), and (21) if \(n \neq 0\). So \((E^\lambda_z)\) implies (20) and (21). The converse is true because every function of \(\text{HP}(F^\lambda_z)\) is uniquely determined by its Fourier coefficients, cf. IV.3.(17).

\[ \Box \]

**Remark 1.** Observe that the relationship

\[ F^\lambda_z(v) = F^{-\lambda}_z(v - \lambda \sigma t) \quad \text{for} \quad v \in F^\lambda_z \]

is equivalent to

(22) \[ F^\lambda_z(n) = p^{-\lambda n} F^{-\lambda}_z(n) \quad \text{for} \quad n \in \mathbb{Z} \]

by Property 2 and (14). So (22) holds for \(i = 1, 2\) and \(\lambda = \pm\) iff the conditions IV.1.(F0) are satisfied.

**Theorem 4.** Assume that the conditions (S0) through (S3), (F0), (F1), (F2), and (S) from Section IV.1 are satisfied. Then the conditions (L) are equivalent to (23) and (24):

(23) \[ Z^\lambda_z(v) = (\Omega F)_z(v) - (\Omega F)_z(0), \]

for \(v \in S_z\) and \(i = 1, 2,\)

(24) \[ Z^\lambda_{0z} = F^\lambda_z(0) + \sum_{k=1,2} (\Omega F)_k(v_{0k}) \]

for \(i = 1, 2\) and \(\lambda = \pm\).
PROOF. We show that the conditions (L) imply (23) and (24). In the conditions (L) the equation $E^λ_ż$ may be replaced by (20) and (21), by Proposition 3.

For fixed $i$, $j$, and $n \neq 0$ solve the equations (21) for $Z_ż(n)$ (subtract the equation (21) from the corresponding equation with $λ$ replaced by $-λ$, and interchange the indices $i$ and $j$) to find

$$Z_ż(n) = \frac{F^-_γ(n) - F^+_γ(n)}{p^λ n - p^{-λ} n} = \frac{F^-_δ(n) - F^+_δ(n)}{p^λ n - p^{-λ} n} = \frac{Σ - µF^γ_δ(n)}{p^n - p^{-n}} = (ΩF)_ż(n)$$

for $n \neq 0$, by (19). This is equivalent to (cf. (18))

$$Z_ż(v) = Z_ż(0) + (ΩF)_ż(v)$$

for $v ∈ S_ż$. In (*) substitute $v = v_{0_ż}$ and apply the initial value condition to find

$$Z_ż(0) = -(ΩF)_ż(v_{0_ż}).$$

The equalities (*) and (***) imply (23).

Finally substitute (**) for $ż = 1, 2$ in (20) to find (24).

Next we check that, conversely, the functions $Z_ż(•)$ and constants $Z^λ_0$ defined in (23) and (24) do satisfy the conditions (L).

The regularity condition (including the periodicity) is satisfied because of (18).

The initial value condition is trivially satisfied.

The equality condition is satisfied because (22) holds in particular for $n = 0$.

In order to show that the equations $E^λ_ż$ are satisfied, it suffices to show that (20) and (21) hold for $i = 1, 2$ and $λ = ±$.

Equate the Fourier coefficients of the functions in (23) on the left and on the right, and apply (19) to arrive at (25) and (**); observe that (25) (first equality) and (22) imply (21), and that (24) and (***) imply (20).

It also follows from Proposition 5 below that the equations $E^λ_ż$ are satisfied by the functions (23) and constants (24). The next subsection is devoted to a proof by substitution of Proposition 5.
PROPOSITION 5. Fix \( i, j : = 3 - i \) and \( \lambda \).

If \( v \in F^\lambda_i \) then

\[
(\Omega F)_i^z(v) + (\Omega F)_j(v - \lambda \sigma t) = F^\lambda_i(v) - F^\lambda_z(0).
\]

(REMARK 2. Because of (2) the sum on the left is well-defined for \( v \in F^\lambda_i \).

IV.5.3 Proof of Proposition 5

The following proof by substitution of Proposition 5 is analogous to the proof of Proposition II.4.10. We need Property 6 and Lemma 7 below (which are the analogues of Property II.4.7 and Lemma II.4.8).

PROPERTY 6. Fix \( i, j : = 3 - i \) and \( \lambda \).

If \( v \in F^\lambda_j \) then

\[
-\lambda IP_{j}^\lambda(v) + \lambda IP_{j}^{-\lambda}(v - \lambda \sigma t) = 0.
\]

(REMARK 3. Because of IV.1.(7) the second term on the left is well-defined for \( v \in F^\lambda_j \), cf. (12).

PROOF of Property 6. By definition one has for \( v \in F^\lambda_z + \lambda \sigma t = F^\lambda_j \)

(cf. IV.1.(7))

\[
IP^\lambda_z(v - \lambda \sigma t) = \int_{\tilde{\alpha}} \frac{d\tilde{w}}{2\pi i} \frac{\partial'}{\partial t'} (v - \lambda \sigma t - \tilde{w} | 2\sigma t) F^\lambda_z(\tilde{w})
\]

where \( \tilde{\alpha}(\cdot) \) is a path satisfying

\[
\begin{cases}
E \circ \tilde{\alpha} is a cycle with \text{Ind} (0 | E \circ \tilde{\alpha}) = 1, \\
(\tilde{\alpha})* \in F^{-\lambda}_z \prec P(v - \lambda \sigma t), \\
\text{Ind}*(\tilde{w} | \tilde{\alpha}) = \begin{cases} 
1 & \text{for } \tilde{w} \in P^+(v - \lambda \sigma t), \\
0 & \text{for } \tilde{w} \in P^-(v - \lambda \sigma t).
\end{cases}
\end{cases}
\]

In (*) introduce the new integration variable \( w = \tilde{w} + \lambda \sigma t \) and use

IV.1.(F0), IV.1.(6), and IV.2.(21) to conclude from (*)

\[
IF^{-\lambda}_z(v - \lambda \sigma t) = IP^\lambda_j(v)
\]

q.e.d.
**Lemma 7.** Fix \( \xi \) and \( \lambda \). If \( v \in F^\lambda_{\xi} \) then

\[
IF^\lambda_{\xi}(v + \sigma t) - IF^\lambda_{\xi}(v - \sigma t) = F^\lambda_{\xi}(v) - F^\lambda_{\xi}(0).
\]

**Remark 4.** Due to (4) the difference on the left in (29) is well-defined for \( v \in F^\lambda_{\xi} \), as is the difference on the right (because of the assumption IV.1.(F1)).

**Remark 5.** From (29) one has

\[
F^\lambda_{\xi}(v) - F^\lambda_{\xi}(0) = \sum_{\mu = \pm} \mu IF^\lambda_{\xi}(v + \mu \sigma t) = \sum_{\mu = \pm \lambda} \mu IF^\lambda_{\xi}(v + \mu \sigma t),
\]

so

\[
\lambda IF^\lambda_{\xi}(v + \lambda \sigma t) - \lambda IF^\lambda_{\xi}(v - \lambda \sigma t) = F^\lambda_{\xi}(v) - F^\lambda_{\xi}(0).
\]

**Proof of Lemma 7.** We have, cf. (10), using that the function \( \frac{\theta^I_4}{\theta^I_4}(\cdot \mid 2\sigma t) \) is odd, \(-IF^\lambda_{\xi}(v - \sigma t) = \int_{A_+} \frac{dw}{2\pi i} \frac{\theta^I_4}{\theta^I_4}(w - v + \sigma t \mid 2\sigma t)F^\lambda_{\xi}(w) \) where \( A_+(\cdot) \) is a path satisfying (cf. (11))

\[
E \circ A_+(\cdot) \text{ is a cycle with } \text{Ind } (0 \mid E \circ A_+) = 1,
\]

\[
\{ A_+ \} \subseteq F^\lambda_{\xi} \setminus (v + \mathbb{Z} \cdot 2\sigma t + \mathbb{Z}),
\]

\[
\text{Ind}^*(w \mid A_+) = \begin{cases} 1 & \text{if } w \in v + \mathbb{Z}^+ \cdot 2\sigma t, \\ 0 & \text{if } w \in v - \mathbb{N} \cdot 2\sigma t \end{cases}
\]

(where \( \mathbb{Z}^+ \) is the set of the nonnegative integers).

Now use the identity (cf. III.2.(7) and III.2.(20))

\[
\frac{1}{2\pi i} \frac{\theta^I_4}{\theta^I_4} (\cdot + \sigma t \mid 2\sigma t) = -1 + \frac{1}{2\pi i} \frac{\theta^I_4}{\theta^I_4} (\cdot - \sigma t \mid 2\sigma t)
\]

to conclude

(i)

\[
-IF^\lambda_{\xi}(v - \sigma t) = -F^\lambda_{\xi}(0) + \\
+ \int_{A_+} \frac{dw}{2\pi i} \frac{\theta^I_4}{\theta^I_4}(w - v - \sigma t \mid 2\sigma t)F^\lambda_{\xi}(w).
\]

Also we have (again using that \( \frac{\theta^I_4}{\theta^I_4} \) is odd)
(ii) \[ \text{IF}_t^\lambda (v + \sigma t) = \int_{\Lambda_-} dw \frac{\delta^t_4}{2\pi i} \frac{\delta^t_4}{\theta_4} (w - v - \sigma t \mid 2\sigma t) F_t^\lambda(w) \]

with \( \Lambda_-(\cdot) \) a path satisfying

\[
\begin{cases}
E \circ \Lambda_-(\cdot) \text{ is a cycle with } \text{Ind} (0 \mid E \circ \Lambda_-) = 1, \\
(\Lambda_-)^* \subset F_t^\lambda \sim (v + \mathbb{Z}, 2\sigma t + \mathbb{Z}), \\
\text{Ind}^*(w \mid \Lambda_-) = \begin{cases} 1 \text{ if } w \in v + \mathbb{N}, 2\sigma t, \\
0 \text{ if } w \in v - \mathbb{N}, 2\sigma t. 
\end{cases}
\end{cases}
\]

From (i) and (ii) we have

\[ \sum_{\mu = \pm} -\mu \text{IF}_t^\lambda (v - \mu \sigma t) = -F_t^\lambda(0) + \text{sum,} \]

with

\[ \text{sum} = \sum_{\mu = \pm} \mu \int_{\Lambda_\mu} dw \frac{\delta^t_4}{2\pi i} \frac{\delta^t_4}{\theta_4} (w - v - \sigma t \mid 2\sigma t) F_t^\lambda(w). \]

In order to show

\[ (*) \quad \text{sum} = F_t^\lambda(v), \]

rewrite the integrals by introducing the new integration variable \( t := E(w); \) for that purpose define the function \( \hat{F}_d^\lambda \) by means of

\[ (31) \quad \hat{F}_d^\lambda \circ E = F_d^\lambda, \]

and use the notations (14), \( u := E(v), \) and the formulas IV.4.(8) - (13) to find

\[ \text{sum} = \sum_{\mu = \pm} \mu \int_{E_0 \Lambda_\mu} dt \frac{\eta(t)}{2\pi i} \frac{\delta^t_4}{\theta_4} (\frac{t}{2\sigma t} \hat{F}_d^\lambda(t)) \frac{1}{2\sigma t}. \]

Notice that the only difference between the two cycles \( E \circ \Lambda_\pm \) is the index of \( u = E(v). \) Observe \( \sum_{\mu = \pm} \mu \text{Ind} (u \mid E \circ \Lambda_\mu) = 1. \)

The residue of the integrand at the pole \( t = u \) is \( F_d^\lambda(u) = F_d^\lambda(v). \) Now apply the residue theorem (RUDIN [1966], Theorem 10.42) to conclude that \((*)\) is true. This proves (29). □
COROLLARY 8. Fix $i$, $j = 3 - i$ and $\lambda$.

If $v \in F_{\bar{i}}^\lambda$ then

$$
\lambda IF_{\bar{j}}^{-\lambda}(v) - \lambda IF_{\bar{i}}^\lambda(v - \lambda\sigma) = F_{\bar{i}}^\lambda(v) - F_{\bar{i}}^\lambda(0).
$$

REMARK 6. Due to (6) the sum on the left in (32) is well-defined for $v \in F_{\bar{i}}^\lambda$, as is the sum on the right.

PROOF of Corollary 8. In (27) replace $\lambda$ by $-\lambda$, and add the result to (30) (note that $F_{\bar{i}}^\lambda = F_{\bar{j}}^{-\lambda}$ by (6)).

PROOF of Proposition 5. Add (27) and (32) (note that $F_{\bar{i}}^\lambda = S_{\bar{i}} \subset F_{\bar{j}}^\lambda$ by (2) and (5)).

IV.5.4 The function $(ZF)(z_1, z_2)$

In Theorem 4 for $Z_{\bar{i}}(\cdot)$ and $Z_{0\bar{i}}^\lambda$ expressions are given, which will be applied in Section V.7, see Theorem V.7.3 (Part I).

However, we are interested not so much in the quantities $Z_{0\bar{i}}^\lambda$, $Z_{\bar{i}}(z_{\bar{i}})$ separately as in the sum

$$
\frac{1}{4} \sum_{i=1,2} \sum_{\lambda=\pm} Z_{0\bar{i}}^\lambda + \sum_{i=1,2} Z_{\bar{i}}(z_{\bar{i}}) = \frac{1}{4} \sum_{i=1,2} \sum_{\lambda=\pm} F_{\bar{i}}^\lambda(0) + \sum_{i=1,2} (ZF)(z_{\bar{i}}).
$$

We now derive a different expression for this sum (see (40) and (37) below), to be applied in Corollary 11 and Theorem V.7.3 (Part II).

The logarithmic derivative of the $\theta_2$-function

We put

$$
\tilde{\eta}(v|\sigma) := \frac{1}{-2\sigma} \frac{\theta_2'}{\theta_2} \left( \frac{v}{-2\sigma} \right) = \frac{3}{2\sigma} \log \theta_2 \left( \frac{v}{-2\sigma} \right) = \frac{1}{-2\sigma \sigma}.
$$

for the $\theta_2$-function see III.2.(38).
The function $\tilde{\eta}(\cdot|2\sigma)$ can be expressed in a simple way in the function $\eta(\cdot)$ (for $\eta$ see IV.4.(8)), viz. (cf. Corollary III.2.6)

\begin{equation}
\tilde{\eta}(v|2\sigma) = \frac{\pi i}{\sigma T} v + \eta(v|2\sigma),
\end{equation}

and so one has, using III.2.(4), ..., (8), that the function $\tilde{\eta}(\cdot|2\sigma)$ satisfies the following requirements:

(36A) $\tilde{\eta}(\cdot|2\sigma)$ is meromorphic in the plane,

(36B) the set of poles is $\sigma T + \mathbb{Z}, 2\sigma T + \mathbb{Z}$; all poles are simple, and the residue at each pole is $+1$,

(36C) $\tilde{\eta}(\cdot + 1|2\sigma) = \frac{\pi i}{\sigma T} + \tilde{\eta}(\cdot|2\sigma)$,

(36D) $\tilde{\eta}(\cdot|2\sigma)$ has the period $2\sigma T$,

(36E) $\tilde{\eta}(0|2\sigma) = 0$.

One easily checks that the above requirements characterize the function $\tilde{\eta}(\cdot|2\sigma)$. Because of III.2.(9) this function has the additional property

(36F) $\tilde{\eta}(\cdot|2\sigma)$ is odd.

The $S$-operator

Under the assumptions of Theorem 4 we define for $z = (z_1, z_2) \in S_1 \times S_2$

\begin{equation}
(SF)(z) := \sum_{i=1,2} \sum_{\lambda=\pm} \int_{\Omega_{i}^{(2)}} \frac{dw}{2\pi i} \cdot \tilde{\eta}(z_i' - w|2\sigma T) \cdot F_j^\lambda(w),
\end{equation}

where $j = 3 - i$, and $A_j^\lambda$ is a chain to be specified presently.

Let $f_j^\lambda, j = 1, 2$ and $\lambda = \pm$, be arbitrary numbers satisfying

(38A) $f_j^\lambda \in F_j^\lambda$ ( = $S_j \cap (S_i + \lambda \sigma T)$, cf. IV.1.(2))

for $j = 1, 2$ (with $i = 3 - j$) and $\lambda = \pm$,

(38B) $f_j^\lambda = f_i^\lambda + \lambda \sigma T$ for $j = 1, 2$ and $\lambda = \pm$. 
Observe that such numbers do exist: choose $t^\lambda_2 \in F^\lambda_2$ for $\lambda=\pm$ arbitrarily, and put $t^\lambda_1 := t^\lambda_2 + \lambda \sigma t$ for $\lambda=\pm$, cf. IV.1.(6).

The chain $A^\lambda_j$ is required to satisfy the conditions (39A) and (39B):

(39A) the chain $A^\lambda_j$ consists of a path in $F^\lambda_j$ from $f^\lambda_j$ to $f^\lambda_j + 1$ and a finite number of closed paths,

(39B) the chain $A^\lambda_j$ satisfies the conditions for $\lambda$ in IV.4.(25A)-(25B), with $v := z^\lambda_j$.

Notice that by (39A) and (39B) the chain $A^\lambda_j$ satisfies the conditions for $\lambda$ in (11) with $v := z^\lambda_j$.

Obviously, $A^\lambda_j$ may be a suitable chosen single path from $f^\lambda_j$ to $f^\lambda_j + 1$.

We show now that (SF)(z) is well-defined, i.e. well-defined for a particular chain $A^\lambda_j$, and independent of the choice of $A^\lambda_j$ given the restrictions. Although a proof can be based on (35) and Corollary IV.4.3 (cf. the proof of Proposition 10 below), we prefer to give a straight-forward proof.

**PROPERTY 9.** Under the assumptions of Theorem 4 the function $z \sim (SF)(z)$ is well-defined on $S_1 \times S_2$.

**PROOF.** It is clear that each one of the integrals in (37) is well-defined and independent of the path given the numbers $(f^\lambda_j)$, cf. Corollary IV.4.3.

It remains to be shown

(+) the sum on the right in (37) does not depend on the numbers $(f^\lambda_j)$ given (38A) and (38B).

So let $(f^\lambda_j)$ be a set of numbers satisfying (38A) and (38B), and let $(\tilde{f}^\lambda_j)$ be another set of numbers with the same properties. For $j=1,2$ and $\lambda=\pm$ let $A^\lambda_j$ be a chain satisfying (39A) and (39B), and let $\tilde{A}^\lambda_j$ be a chain satisfying (39A) and (39B) with $f^\lambda_j$ replaced by $\tilde{f}^\lambda_j$.

Also, let $\gamma^\lambda_j$ be a path from $f^\lambda_j$ to $\tilde{f}^\lambda_j$ such that the range of $\gamma^\lambda_j$ is contained in $F^\lambda_j$, and is disjoint from the set of poles of $\tilde{\gamma}(z^\lambda_j - |2\sigma t|)$, viz. the set $P(z^\lambda_j)$ (see IV.4.(16)), and such that

(i) $\gamma^\lambda_j(\cdot) = \gamma^{-\lambda}_j(\cdot) + \lambda \sigma t$ for $j=1,2$ and $\lambda=\pm$.

Obviously such paths do exist: first choose $\gamma^\pm_1(\cdot)$, and next define $\gamma^\pm_2(\cdot)$ using (i).


We claim:

\[
(ii) \quad [ \int - \int ] \, dw \, \widetilde{n}(z_{\lambda} - w \vert 2 \sigma \tau) \, F_{j}^{\lambda}(w) = \\
= \left[ \int - \int \right] \, dw \, \tilde{n}(z_{\lambda} - w \vert 2 \sigma \tau) \, F_{j}^{\lambda}(w) = K_{j}^{\lambda},
\]

where

\[
(iii) \quad K_{j}^{\lambda} := \frac{\pi i}{\sigma \tau} \int_{Y_{j}^{\lambda}} dw \, F_{j}^{\lambda}(w). \]

Observe that the integrand in (ii) is regular on \((Y_{j}^{\lambda})^*\), and also on \((Y_{j}^{\lambda} + 1)^*\) by (36c) and IV.1.(F2).

We first derive (†) from (ii), and next prove (ii).

In (iii) substitute \(\tilde{w} := w - \lambda \sigma \tau\), and use (i) and IV.1.(F0) to conclude \(K_{j}^{\lambda} = K_{-j}^{-\lambda}\), hence \(\Sigma_{i=1,2} \Sigma_{\lambda=\pm} \lambda K_{j}^{\lambda} = 0\). So

\[
\Sigma_{i=1,2} \Sigma_{\lambda=\pm} \lambda \left[ \int - \int \right] \, dw \, \tilde{n}(z_{\lambda} - w \vert 2 \sigma \tau) \, F_{j}^{\lambda}(w) = 0,
\]

which proves (†).

Proof of (ii). Fix \(i\), \(j\), and \(\lambda\). Let \(\Gamma\) be the chain consisting of the paths \(\Lambda_{j}^{\lambda}\), \(\gamma_{j}^{\lambda} + 1\), and the opposites of \(\Lambda_{j}^{\lambda}\) and \(\gamma_{j}^{\lambda}\). Then \(\Gamma\) is, in fact, a closed path in the (see IV.1.(3)) generalized open horizontal strip \(F_{j}^{\lambda}\) (which is the domain of regularity of \(F_{j}^{\lambda}\)), and is disjoint from the point set \(P(z_{\lambda})\) (which is the set of poles of \(\tilde{n}(z_{\lambda} - \cdot \vert 2 \sigma \tau)\), see IV.4.(16)). Apply the residue theorem (RUDIN, Th. 10.42) to conclude (cf. (36b))

\[
(iv) \quad \frac{1}{2 \pi i} \int_{\Gamma} dw \, \tilde{n}(z_{\lambda} - w \vert 2 \sigma \tau) \, F_{j}^{\lambda}(w) = \Sigma_{w \in P(z_{\lambda}) \cap F_{j}^{\lambda}} \text{Ind}(w \mid \Gamma) \, F_{j}^{\lambda}(w).
\]

We claim: if \(w \in P(z_{\lambda})\) then

\[
(v) \quad \Sigma_{k \in \mathbb{Z}} \text{Ind}(w + k \mid \Gamma) = 0.
\]

To prove this, take \(w \in P(z_{\lambda})\). First observe

\[
\Sigma_{k \in \mathbb{Z}} \text{Ind}(w + k \mid \Gamma) = \text{Ind}^*(w \mid \Gamma)
\]
by Property IV.2.5. Next observe from IV.2.(20)

\[
\text{Ind}^*(\omega|\Gamma) = \left[ \int_{\gamma_{d}^{\lambda}} - \int_{\gamma_{d}^{\lambda}} + \int_{\lambda_{d}^{\lambda}} - \int_{\hat{\lambda}_{d}^{\lambda}} \right] \frac{dt}{1 - E(\omega - t)}
\]

where the first two integrals cancel each other because of the period 1. Hence (again by IV.2.(20))

\[
\text{Ind}^*(\omega|\Gamma) = \text{Ind}^*(\omega|\lambda_{d}^{\lambda}) - \text{Ind}^*(\omega|\hat{\lambda}_{d}^{\lambda}),
\]

which vanishes by the assumption (39 B). This proves (v).

From (v) it follows, because both the set \( P(z_{d}) \cap F_{d}^{\lambda}(\cdot) \) and the function \( P_{d}^{\lambda}(\cdot) \) have the period 1, that the sum on the right in (iv) vanishes. This proves the first equality in (ii). The second equality in (ii) is an easy consequence of (36 C) and the fact that \( P_{d}^{\lambda}(\cdot) \) has period 1.

\(\Box\)

**PROPOSITION 10.** Under the assumptions of Theorem 4 the following equality holds if \( z \in S_{1} \times S_{2} \):

\[
(40) \quad (SF)(z) + \sum_{i=1,2} \sum_{\lambda=\pm} \frac{z_{i}}{2\sigma_{i}} F_{d}^{\lambda}(0) = \frac{1}{4} \sum_{i=1,2} \sum_{\lambda=\pm} F_{d}^{\lambda}(0) + \sum_{i=1,2} (\text{RF})_{i}(z_{i}),
\]

where \( j = 3 - i \).

**PROOF.** In (37) substitute (35), and use (10) (cf. IV.4.(8)) to conclude

(i) \[
(SF)(z) = \sum_{i=1,2} \sum_{\lambda=\pm} \{ -\lambda \left[ L_{d}^{\lambda} \right. + \frac{z_{i}}{2\sigma_{i}} F_{d}^{\lambda}(0) \} + \text{IF}_{d}^{\lambda}(z_{i}) \}
\]

where the Fourier coefficient \( F_{d}^{\lambda}(0) \) is defined in IV.1.(18) and

(ii) \[
L_{j}^{\lambda} := -\frac{1}{2\sigma_{i}} \int_{\gamma_{j}^{\lambda}} \text{d}w \cdot w F_{d}^{\lambda}(w)
\]

(\( F_{d}^{\lambda}(\cdot) \))

(where we used a notation introduced at the end of subsection II.4.1). In (ii) substitute \( \hat{w} := w - \lambda \sigma \) and use (38 B), IV.1.(6) and IV.1.(FO) to
find $L_j^\lambda = L_j^{-\lambda} - \lambda \frac{1}{2} F_j^{-\lambda}(0)$. Hence,

$$
\sum_{i=1,2} \sum_{\lambda = \pm} [-\lambda L_j^\lambda] = \sum_{i=1,2} \sum_{\lambda = \pm} [-\lambda L_i^{-\lambda} + \frac{1}{2} F_i^{-\lambda}(0)]
$$

which proves

(iii) $$\sum_{i=1,2} \sum_{\lambda = \pm} [-\lambda L_i^\lambda] = \frac{1}{4} \sum_{i=1,2} \sum_{\lambda = \pm} F_i^\lambda(0).$$

The equality (40) now follows from (i), (iii) and (17).

\[\square\]

**COROLLARY 11.** Under the assumptions of Theorem 4 and the additional assumption IV.1.(F5) (following IV.1.(22)) the following equality holds:

(41) $$(SF)(z) = F_k^\lambda(0) + \sum_{i=1,2} \sum_{\lambda = \pm} (\Omega F)_i(z, \bar{z})$$

(which does not depend on $k \in \{1,2\}$ and $\lambda \in \{+, -\}$).

**PROOF.** From (40) and Remark IV.1.4.

\[\square\]
IV.6 Some topological tools

This section is devoted to properties and theorems which are used elsewhere in this chapter and in other ones. It also contains proofs of the first three propositions of this chapter.

IV.6.1 Some elementary topological properties

In the following $X$ is a fixed topological space, and $A$ and $B$ are subsets of $X$. For convenience we mention some well-known simple facts, to be used below. A bar denotes closure.

\begin{align*}
(1) \quad \overline{A \cap B} & = \overline{A} \cap \overline{B} \quad \text{(equality if $A$ and $B$ are closed)}, \\
(2) \quad \overline{A \cup B} & = \overline{A} \cup \overline{B}
\end{align*}

and the dual properties (take complements)

\begin{align*}
(3) \quad \text{Int}(A) \cup \text{Int}(B) & \subseteq \text{Int}(A \cup B) \quad \text{(equality if $A$, $B$ are open)}, \\
(4) \quad \text{Int}(A) \cap \text{Int}(B) & = \text{Int}(A \cap B)
\end{align*}

As usual the boundary $\partial A$ of $A$ is defined by

\begin{align*}
(5) \quad \partial A := \overline{A} \cap \overline{A^c} & = \overline{A} \cap \overline{\text{Int}(A)^c}.
\end{align*}

Clearly,

\begin{align*}
(6) \quad \partial(A^c) & = \partial A.
\end{align*}

**Property 1.** The following inclusions hold.

\begin{align*}
(7) \quad \partial(A \cup B) & \subseteq \overline{B^c \cap \partial A} \cup \overline{A^c \cap \partial B} \\
(8) \quad \partial(A \cup B) & \supseteq \overline{\text{Int}(B^c) \cap \partial A} \cup \overline{\text{Int}(A^c) \cap \partial B}
\end{align*}

**Proof.** To see (7) calculate $\partial(A \cup B) = \overline{A \cup B} \cap \overline{A^c \cup B^c} \subseteq \overline{A \cup B} \cap \overline{A^c \cap B^c}$ which is the union on the right.

Proof of (8): We only consider the first intersection on the right. The inclusion $\text{Int}(B^c) \cap \partial A \subseteq \overline{A \cup B}$ follows from $\partial A \subseteq \overline{A} \subseteq \overline{A \cup B}$.
The inclusion \( \text{Int}(B^c) \cap \exists A \subseteq [\text{Int}(A \cup B)]^c \) follows from
\( \text{Int}(A \cup B) \cap \text{Int}(B^c) \cap \exists A = \text{Int}(A \cap B^c) \cap \exists A \subseteq \text{Int}(A) \cap \exists A = \emptyset \).
Now use (5).

\[ \square \]

**PROPERTY 2.** The following three statements are equivalent.

\[ (9) \quad \text{Int}(A \cup B) = \text{Int}(A) \cup \text{Int}(B) \quad (\text{i.e. equality in (3))} \]

\[ (10) \quad \exists (A \cup B) = [B^c \cap \exists A] \cup [A^c \cap \exists B] \quad (\text{i.e. equality in (7))} \]

\[ (11) \quad \exists A \cap \exists B \subseteq \exists (A \cup B) \]

**PROOF.** (9) \( \iff \) (10): Because of \( [A \cup B]^c \subseteq [\text{Int}(A \cup B)]^c \subseteq [\text{Int}(A) \cup \text{Int}(B)]^c \)
one always has \( [A \cup B]^c \cap [\text{Int}(A \cup B)]^c = [A \cup B]^c \cap [\text{Int}(A) \cup \text{Int}(B)]^c \).
Hence, (9) is equivalent to
\[ A \cup B \cap [\text{Int}(A \cup B)]^c = A \cup B \cap [\text{Int}(A) \cup \text{Int}(B)]^c , \]
which is (10).

(10) \( \iff \) (11): Trivial (use (6)).

(10) \( \iff \) (11): By (7) one always has the inclusion "\( \subseteq \)". For the reversed
inclusion combine (8) and (11) using (6).

\[ \square \]

**PROPERTY 3.** The following two statements are equivalent.

\[ (12) \quad \exists (A \cup B) = [\text{Int}(B^c) \cap \exists A] \cup [\text{Int}(A^c) \cap \exists B] \quad (\text{i.e. equality in (8))} \]

\[ (13) \quad \exists A \cap \exists B \cap \exists (A \cup B) = \emptyset \]

**PROOF.** (12) \( \iff \) (13): Obvious.

(12) \( \iff \) (13): By (8) one always has the inclusion "\( \supseteq \)". For the reversed
inclusion combine (7) and (13) using (6).

\[ \square \]

**Notation.** If \( A \subseteq Y \subseteq X \) then

\[ (14) \quad \exists_Y (A) := \text{the boundary in } Y \text{ of } A . \]

**PROPERTY 4.** Assume \( A \subseteq Y \subseteq X \). If \( A \) is open then \( \exists_Y (A) = Y \cap \exists A \).

**PROOF.** The closure in \( Y \) of \( A \) is \( Y \cap \overline{A} \) (ENGELKING [1977], Proposition 2.1.1, p.93), so
\[ \exists_Y (A) = [Y \cap \overline{A}] \cap [Y \cap A^c] = Y \cap \exists A \] by (5).

\[ \square \]
The following property is used in the proof of Proposition V.6.8. As usual (e.g. cf. DUGUNDJI [1966], Ch. V, p.107) a set \( X \) is connected if it is not the union of two nonempty disjoint open subsets.

**PROPERTY 5.** Let \( A \) and \( B \) be open subsets of \( X \) satisfying:

(i) \( A \cup B \neq \emptyset \),

(ii) \( X \setminus (\partial A \cup \partial B) \) is connected,

(iii) \( \partial A \subset B \) and \( \partial B \subset A \).

Then \( A \cup B \cup (\partial A \cup \partial B) = X \).

**PROOF.** Put \( Y := X \setminus (\partial A \cup \partial B) \). Then \( A \cup B \subset Y \) and (use Property 4)

\[
\partial_y(\overline{A}) = \partial_y(B) \subset A \subset B, 
\]

and similarly \( \partial_y(B) \subset A \) . Hence (by (7)) \( \partial_y(A \cup B) \subset \partial_y(A) \cup \partial_y(B) \subset B \cup A \). Because also \( (A \cup B) \cap \partial_y(A \cup B) = \emptyset \) (as \( A \cup B \) is open in \( Y \)) one has

\[
\partial_y(A \cup B) = \emptyset. 
\]

Because \( Y \) is connected and \( A \cup B \neq \emptyset \) this implies \( A \cup B = Y \).

\( \Box \)

The following property has been used in the proof of Proposition III.1.30.

**PROPERTY 6.** Let \( A \) and \( B \) be subsets of \( X \) with nonempty intersection. If \( \partial B \subset \partial A \) and \( A \) is open and connected, then \( A \subset B \).

**PROOF.** Observe \( A \cap \partial B \subset A \cap \partial A = \emptyset \) because \( A \) is open, hence \( A \) is the disjoint union of two open subsets, viz. \( A \cap \text{Int}(B) \) and \( A \cap (\overline{B}^c) \). The first one is nonempty (because \( A \cap B \neq \emptyset \) and \( A \) is open); hence, the second one is empty (because \( A \) is connected), which proves the claim.

\( \Box \)

The following property is used tacitly in this section and elsewhere. For a proof see e.g. FRANZ [1973, Theorem 8.2, p.47].

**PROPERTY.** Let \( O \) be an open subset of \( \mathbb{R}^d \). Then the following are equivalent.

1) \( O \) is connected

2) every two points of \( O \) can be connected by a path in \( O \)

3) every two points of \( O \) can be connected by a polygonal (i.e. piecewise linear) path in \( O \)

\( \Box \)
IV.6.2 Some topological theorems on the plane

The following theorems refer to the topology of the Euclidean plane. As usual (cf. e.g. DUGUNDJI [1966, sec. V.3]) a subset $C$ of $X$ is called a component of $X$ if $C$ is nonempty and maximal connected: there is no connected subset of $X$ that properly contains $C$.

For the concept of simple connectedness see e.g. RUDIN [1966, sec.10.38, p. 239].

**THEOREM 7.** Let $A$ be an open and connected subset of $\mathbb{C}$. Then $A$ is simply connected iff every component of the complement of $A$ is unbounded.

**PROOF.** "If": From MOORE [1925, Theorem 25] it follows straightforwardly that $A$ is homeomorphic to the plane, and so simply connected.

"Only if": Omitted.

**THEOREM 8.** Let $A$ and $B$ be open and simply connected subsets of $\mathbb{C}$. If $A \cap B$ is nonempty and connected then $A \cup B$ is simply connected.

**PROOF.** See SPANIER [1966, Ch.1, Ex.G.3, p.58].

**THEOREM 9.** Let $A$ and $B$ be open and connected subsets of $\mathbb{C}$. If $A \cup B$ is simply connected then $A \cap B$ is connected.

**PROOF.** This follows immediately from the Components of Intersections Theorem, see SANDERSON [1980, p.85].

**THEOREM 10.** Let $A$ and $B$ be nonempty, open, simply connected subsets of $\mathbb{C}$. Then $A \cup B$ is simply connected iff $A \cap B$ is nonempty and simply connected.

**PROOF.** "If": See Theorem 8.

"Only if": Assume that $A \cup B$ is simply connected. Then, first, $A \cap B \neq \emptyset$ because $A \cup B$ is connected. Secondly, $A \cap B$ is connected by Theorem 9. Thirdly, consider $(A \cap B)^c = A^c \cup B^c$. By Theorem 7 the components of $A^c$
and those of $E^c$ are unbounded. This implies that $A^c \cup B^c$ is the union of unbounded connected sets; hence, the components of $A^c \cup B^c$ are unbounded. Consequently, $A \cap B$ is simply connected by Theorem 7.

We notice that (by Theorem 10) in Theorem 8 the conclusion "$A \cap B$ is simply connected" can be added.

IV.6.3 Proofs of the Propositions IV.1.1, IV.1.2, and IV.1.3

**Proof** of Proposition IV.1.1. In order to show $S = \emptyset$ we assume, contrarily, $c \in \mathbb{E} \setminus S$ and derive a contradiction. We may and will assume that $c$ is the origin. Choose a sufficiently large positive $M$, viz. such that $M$ is in an upper half plane contained in $S$ and $-M$ is in a lower half plane contained in $S$. Because $S$ is connected there exists a path $\gamma(*)$ from $-M$ to $+M$ (with range $\gamma^*$) contained in $S$. Also, choose a sufficiently large positive integer $r$, viz. so that
\[ \gamma^* + r \text{ is contained in the open right half plane}, \]
\[ \gamma^* - r \text{ is contained in the open left half plane}; \]
so $\gamma^* \pm r \subset S$ because of the period 1. Put
\[ \Lambda_+(*) := \gamma(*) + r, \quad \Lambda_-(*) := \gamma(*) - r, \]
where $\gamma(*)$ is the path opposite to $\gamma(*)$ (see IV.2.(2)). Let $L(*)$ be the path from $-M - r$ to $-M + r$ whose range $L^*$ is a straight line segment; so $\pm L^* \subset S$. Let $\Gamma$ be the chain (cf. the definition preceding IV.2.(2)) consisting of the collection \{ $\Lambda_\pm(*)$, $\pm L(*)$ \}. So $\Gamma$ is, in fact, a closed path in $S$, with the property (cf. IV.2.(12))
\[ \text{Ind}(O|\Gamma) = +1. \]

Consequently (Rudin [1966, Th. 10.10]), the origin is in a bounded component, say $K$, of the complement of the range of $\Gamma$.

Let $K$ be the component of $S^c$ which contains the origin. Then $K \subset \overline{K}$, so $K$ is bounded as well. This implies (by Theorem 7) that $S$ is not simply connected, which contradicts one of the hypotheses.

**Proof** of Proposition IV.1.2. Consider the following conditions:

(ii') $A \cap B$ is nonempty and simply connected,

(iii') $A \cup B$ is simply connected.

We show that the five requirements (i), (ii'), (ii), (iii'), (iii) are equivalent.
(i) $\Leftrightarrow$ (ii') : Trivial.
(ii') $\Rightarrow$ (ii) : Trivial.
(ii) $\Rightarrow$ (iii') : By Theorem 8.
(iii') $\Rightarrow$ (iii) : The assumptions imply that $A \cup B$ is a generalized open horizontal strip, and the claim follows from Proposition IV.1.1.
(iii') $\Rightarrow$ (iii) : Trivial.
(ii') $\Rightarrow$ (iii') : By Theorem 10.

PROOF of Proposition IV.1.3.

Uniqueness: Assume that $A, \hbar$ are generalized open upper half planes and $B, \hbar$ generalized open lower half planes satisfying $A \cap B = S = \hbar \cap \overline{B}$. It is enough to show $A \subseteq \hbar$ and $B \subseteq \overline{B}$. Assume, contrarily, $a \in A \setminus \hbar$.

Then $a \in A \setminus S$, so $a \in B^c$, and $a \in \hbar^c$. Choose an $s \in S$, and let $\Lambda(\cdot)$ be a path in $S \subseteq \hbar \cap \overline{B}$ from $s$ to $s + 1$. Then (by Property IV.2.4)

$$\text{Ind}^*(a|\Lambda) = 1 \text{ because of } a \in B^c,$$
$$\text{Ind}^*(a|\Lambda) = 0 \text{ because of } a \in \hbar^c,$$

which is a contradiction. Hence $A \setminus \hbar = \emptyset$, or $A \subseteq \hbar$.

Similarly one shows $B \subseteq \overline{B}$. This proves uniqueness.

Existence: Let $S$ be a generalized open horizontal strip. Choose an $s \in S$, and a path $\Lambda$ from $s$ to $s + 1$ with parameter interval $[0,1]$ and range $\Lambda^*$ satisfying $\Lambda^* \subseteq S$. Define $\gamma(\cdot) : [0,1] \mapsto S$ by

$$\gamma(t+k) := \Lambda(t) + k \text{ if } t \in [0,1) \text{ and } k \in \mathbb{Z}.$$ 

Obviously, $\gamma(\cdot)$ is continuous. Let $\gamma^*$ be the range of $\gamma(\cdot)$. Then $\gamma^*$ is closed, and $\gamma = \Lambda^* + \mathbb{Z} \subseteq S$.

Let $H_+$ (resp. $H_-$) be a proper open upper (resp. lower) half plane containing $\gamma^*$. Let $C_+$ (resp. $C_-$) be the component of $(\gamma^*)^c$ containing the upper half plane $H_+^c$ (resp. the lower half plane $H_-^c$). We claim

$$(*) \quad C_+ \cap C_- = \emptyset.$$ 

We prove $(*)$. From Property IV.2.4 we know

$$\text{Ind}^*(w|\Lambda) = \begin{cases} 1 & \text{if } w \in H_+^c, \\ 0 & \text{if } w \in H_-^c, \end{cases}$$

hence $C_+ \neq C_-$. This implies $(*)$ because the sets $C_{\pm}$ are components.

We define

$$(**) \quad A := S \cup C_+, \quad B := S \cup C_-,$$
and have because of (*)

(a) \[ A \cap B = S. \]

We complete the proof by showing that \( A \) (resp. \( B \)) is a generalized open upper (resp. lower) half plane.

First notice that, obviously,

(b) \[ A \text{ contains the open upper half plane } \text{Int}(H^-_C), \]

\[ B \text{ contains the open lower half plane } \text{Int}(H^+_C). \]

Next we show

(c) \[ A + 1 = A \quad \text{and} \quad B + 1 = B. \]

For \( \lambda = \pm \) the set \( C^\lambda \) is a component of the set \( (\gamma^*)^C \), furthermore, the intersection \( (C^\lambda + 1) \cap C^\lambda \) is nonempty (it contains the half plane \( (H^-_\lambda)^C \)). Hence, \( C^\lambda + 1 = C^\lambda \). By hypothesis \( S + 1 = S \).

This proves (c).

Observe that the sets \( C^\pm \) (being components of an open set) are open, cf. DUGUNDJI [1966, p.113, V.4.1 and V.4.2]. Hence,

(d) \[ A \text{ and } B \text{ are open.} \]

We proceed to show

(e) \[ A \text{ and } B \text{ are connected.} \]

Because \( S \) and \( C^\pm \) are connected it suffices to show

(e') \[ S \cap C^\lambda \neq \emptyset \]

for \( \lambda = \pm \). To see (e'), first observe \( \partial(C^\lambda) \neq \emptyset \) (because of \( \emptyset \neq C^\lambda \neq \emptyset \)).

Next observe (cf. DUGUNDJI, p.118, problem V.4.3, b. for the first inclusion)

\[ \partial(C^\lambda) \subset \partial(\gamma^*) = \partial(\gamma^*) \subset \gamma^* \]

(because \( \gamma^* \) is closed), where \( \gamma^* \subset S \). Hence \( \emptyset \neq \partial(C^\lambda) \subset S \). This implies (e') because \( S \) is open.

Finally we claim

(f) \[ A \text{ and } B \text{ are simply connected.} \]

We prove (f). By Theorem 7 it suffices to prove:

(f') every component of \( A^C \) is unbounded,

every component of \( B^C \) is unbounded.

Fix \( \lambda \in \{+, -\} \), and let \( K \) be a component of \((S \cup C^\lambda)^C = S^C \cap C^\lambda^C\)

so in particular
(α) \( K \subset \mathcal{C}_\lambda^C \).

In order to prove \((f')\) we assume, contrarily,

(β) \( K \) is bounded,

and derive a contradiction. Let \( \mathcal{D} \) be the component of \( S^C \) which contains \( K \), so

(γ) \( K \subset \mathcal{D} \).

Notice that \( \mathcal{D} \) is unbounded (because \( S \) is simply connected, cf. Theorem 7), so

(δ) \( K \neq \mathcal{D} \).

Also notice that

(ε) \( \mathcal{D} \) is a connected subset of \((\gamma^*)^C\).

We next show

(ζ) \( \mathcal{D} \cap \mathcal{C}_\lambda \neq \emptyset \).

Assume, contrarily, \( \mathcal{D} \cap \mathcal{C}_\lambda = \emptyset \), so \( \mathcal{D} \subset \mathcal{C}_\lambda^C \). Then \( \mathcal{D} \) is a connected subset of \( S^C \cap \mathcal{C}_\lambda^C \) and contains \( K \); hence \( \mathcal{D} = K \) (because \( K \) is a component). This contradicts \((γ)\), which proves \((ε)\).

Observe that \((δ)\) and \((ε)\) imply \( \mathcal{D} \subset \mathcal{C}_\lambda \) (because \( \mathcal{C}_\lambda \) is a component of \((\gamma^*)^C\)). Consequently by \((β)\)

\( K \subset \mathcal{C}_\lambda \),

which contradicts \((α)\). This disproves our assumption \((β)\). So \((f')\) holds, which proves \((f)\).

The assertions \((b), \ldots, (f)\) imply that \( A \) (resp. \( B \)) is a generalized open upper (resp. lower) half plane; in view of the assertion \((a)\) this completes the proof of the "existence" claim.

\(\Box\)