Random Walk in the Quadrant
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Chapter V

Skipfree, all-sided random walk in the quadrant in the case $0 < s < 1$:
the uniformization of $Q(x, s) = 0$
for fixed $s$,
the unique solution of the backward conditions,
determination of the generating function $P(x, y, s)$

V.0 Summary of Chapter V

In this chapter we solve the backward conditions I.3.(B 1$''$). For that purpose we follow Groeneveld [27] in outline. In particular, we use Groeneveld’s approach to the uniformization of the zero locus of the kernel. Because our random walks (with absorbing boundary) have a simpler structure than the ones (with reflecting boundary) considered by Groeneveld, we need in the solution of the relevant functional equation only part of his machinery. However, we remove the technical condition that $s$ is close to 0, which is present in the available part of the manuscript.

In order to solve the backward conditions I.3.(B 1$''$) we first uniformize the zero locus of the kernel by means of elliptic functions.
Secondly, we use the uniformization to transform the conditions I.3.(B 2$''$) (which turn out to be equivalent to I.3.(B 1$''$)) into the conditions (B 3$''$).
Thirdly, we solve the conditions (B3′′) using the results of Chapter IV. The solution, finally, is transformed (using the inverse transformation) into the desired expression for the generating function $P(x, y, s)$. We now summarize the contents of the chapter’s seven sections.

V.0.1 Summary of Sec. V.1

The uniformizing functions are elliptic only if the zero locus of the kernel is equivalent (topologically) to a torus, and this is the case iff every point in the zero locus is “simple” (i.e. not a double point). In that case we call the kernel nondegenerate. In Sec. V.1 (see Theorem 1) we establish in the case $0 \leq s \leq 1$ necessary and sufficient conditions for nondegeneracy, using results from Chapter II and the subsequent section V.2. A sufficient condition is $0 < s < 1$ together with all-sidedness.

V.0.2 Summary of Sec. V.2

In Sec. V.2 we, first, investigate the location of the four zeros of each discriminant; these zeros are called the “c-values”. The results are collected in Theorem V.2.3. The zeros turn out to be real, two of them appearing inside the interval $(-1, +1)$, the remaining ones outside $[-1, +1]$ (in the case $0 < s < 1$). Also (Corollary 5) the values of the cross ratios of the four zeros are investigated.

Next, we investigate the so-called “d-values”. Each c-value is associated with a unique d-value: if for a point with coordinates $(x_1, x_2)$ belonging to the zero locus of the kernel one of the discriminants vanishes (i.e. the corresponding coordinate is a c-value), then the other coordinate is the associated d-value. The d-values are computed (Proposition 6) and their location determined (Proposition 7). We conclude this section with a corollary on the real part of the zero locus of the kernel. The real part consists of two distinct closed curves, with distinct projections on each axis. From Corollary 8 one sees that one of these curves is completely contained in the north-east quadrant.

V.0.3 Summary of Sec. V.3

The line of argument in this chapter is interrupted in Sec. V.3. The content of this section is principally taken from Groeneveld’s manuscript [27]. Here we study a special case of the biquadratic equation, viz. the one corresponding to all-sided, driftless, even random walk in the case $0 < s < 1$. The computations in this section are used in Secs. VI.4–6.

First (Subsec. V.3.1), we compute the c-values and the d-values.

Next (Subsec. V.3.2), we characterize this case by means of its c- and d-values.

We use this characterization in final Subsec. V.3.3 to construct a transformation from the general nondegenerate biquadratic equation into the special
case. The transformation consists of the cartesian product of two fractional linear transformations (i.e. each one of the two coordinates is transformed by a fractional linear transformation separately). This coordinate transformation can be used to construct a uniformization of the nondegenerate biquadratic equation in the general case. For that purpose it is necessary, first, to construct uniformizing functions in the special case, which are elliptic; next, in the general case the uniformizing functions are found immediately by using the above fractional linear transformations. An advantage of this procedure is that the elliptic functions needed in the special case are well-known, viz. modifications of Jacobi’s sn-functions (see V.5.(99)). However, we do not follow this route further; instead, in the next section we construct a uniformization in the general case directly, without first transforming the equation into a simpler one.

V.0.4 Summary of Sec. V.4

This section is a continuation of Sec. V.2 and is devoted to the uniformization of the biquadratic equation

$$Q(x, s) = 0$$

(where $$x = (x_1, x_2)$$, and $$s$$ is fixed, $$0 < s \leq 1$$) in the nondegenerate case, using results of Sec. V.2. The construction is as follows. Solving the equation for one of the two variables, say $$x_2$$, results in an expression which is a rational function of both $$x_1$$ and the square root of the discriminant $$D_1(x_1)$$ (omitting $$s$$). From this expression it can be seen that the problem of uniformizing the biquadratic equation is tantamount to the problem of uniformizing the equation

$$\tilde{x}_2^2 = D_1(x_1)$$

where $$\tilde{x}_2$$ is a linear function of $$x_2$$ (with coefficients quadratic in $$x_1$$).

The solution of the latter problem is well-known. Because of nondegeneracy, the discriminant is a polynomial of degree 4 (or 3, if one of the zeros is infinite). For $$x_1$$ substitute a second order elliptic function, say $$x_1 = f_1(v)$$, whose second order values are exactly the zeros of the discriminant; it turns out that $$\tilde{x}_2$$ equals $$df_1(v)/dv$$ (apart from a constant factor), and so $$x_2$$ is a second order elliptic function with the same period lattice, say $$x_2 = h_2(v)$$. The result is the uniformization

(i) \( (x_1, x_2) = (f_1(v), h_2(v)) \).

Following Groeneveld we treat $$x_1$$ and $$x_2$$ symmetrically. Reversing the role of $$x_1$$ and $$x_2$$ we obtain a second, similar uniformization

(ii) \( (x_1, x_2) = (h_1(v), f_2(v)) \).
The functions $h_1$ and $h_2$ still are in an awkward form. However, the two uniformizations are locally bijective and, therefore, essentially the same. In our case this leads to

$$ h_k(v) = f_k(v \pm \sigma \tau) \quad (k = 1, 2) $$

where $\tau$ is an imaginary period and $\sigma$ is a real constant.

The section can be summarized as follows.

After having stated the assumptions (which guarantee nondegeneracy) we first (in Subsec. V.4.1) construct the two elliptic functions $f_1$ and $f_2$, and specify the differential equations determining them (Theorem 1).

Next (in Subsec. V.4.2), we construct the two uniformizations and prove bijectivity (Theorem 2).

Finally (in Subsec. V.4.3), we express the $h$-functions in the $f$-functions; the tool here is Theorem 1.

We conclude this section with a different description of the uniformization Theorem 8: if $(x_1, x_2) = (f_1(z_1), f_2(z_2))$, and $(x_1, x_2)$ is a zero of the kernel, then the relationship between $z_1$ and $z_2$ is a linear one (of a special kind).

### V.0.5 Summary of Sec. V.5

Sec. V.5 is devoted to a description of the two elliptic functions $f_1$ and $f_2$ separately (which are even and second order elliptic with primitive periods 1 and $\tau$, with $\tau$ on the positive imaginary axis). In Subsec. V.5.1 we determine the inverse image under $f_i$ ($i = 1, 2$) of the real axis, and of the upper and lower half plane.

Next, in Subsec. V.5.2, we locate the zeros resp. poles (associated with the constants $\alpha_i$ and $\beta_i$, which are introduced here), and the inverse image under $f_i$ of the values +1 resp. −1 (associated with the constants $\gamma_i$ and $\delta_i$, which also are introduced here).

These constants also depend on $s$. Because we intend to vary $s$, we establish in the next Subsec. V.5.3 the regularity (resp. continuity) of these constants as functions of $s \in (0, 1]$ (in case of nondegeneracy).

We continue, in Subsec. V.5.4, with determining the inverse images under $f_1$ and $f_2$ of the interior of the unit circle. These are needed because in the backward conditions $x_1$ and $x_2$ are restricted to the open unit disc. It turns out that each of these inverse images consists of a series of disjoint “generalized open horizontal strips”, which differ only an imaginary period (a multiple of $\tau$), see Proposition 10, where the strip containing the real axis and related to $f_i$ is called $S_i$. According to Proposition IV.1.3, $S_i$ is the intersection of a unique “generalized open upper half plane” $A_i$ and a unique “generalized open lower half plane” $B_i$. We show that $A_i$ (resp. $B_i$) is the union of $S_i$ with the open upper (resp. lower) half plane. In order to complete the picture we also determine the inverse image of the exterior of the unit circle, with similar results; the generalized strips involved carry a prime; they separate the unprimed ones.
In the next Subsec. V.5.5 we identify the boundaries of the above “generalized strips” (see Corollary 14). Also, we describe the way the complex plane is partitioned by the associated “generalized half planes” (Property 15). The “generalized strips” and “generalized half planes” from the foregoing are pictured in Fig. 2.

We conclude this section (Subsec. V.5.6) with some special types of random walk. We characterize each type by means of conditions on the discriminants, and relate these conditions to conditions on the elliptic functions $f_1$ and $f_2$. It turns out that in the case of symmetric random walk all “generalized strips” and “generalized half planes” are ordinary ones (Property 21). Our last example concerns the driftless, even random walk, which also was dealt with in Sec. V.3. Here we show that in this case the elliptic functions $f_1$ and $f_2$ are a modification of Jacobi’s sn-function (see (97) and (99)), and collect some properties of $f_1$ and $f_2$ in this case.

**V.0.6 Summary of Sec. V.6**

After having investigated in Sec. V.5 the “generalized strips” associated with $f_1$ and with $f_2$ separately, we proceed in Sec. V.6 with investigating the relationship between the two sets, as the functions $f_1$ and $f_2$ are used to uniformize the zero locus of the kernel as described in Sec. V.4, see (i), (ii), (iii). The uniformization is used in Sec. V.7 to transform the backward conditions, and the transformed conditions will be shown to have the form (L) from Sec. IV.1.

First (Subsec. V.6.1 and Subsec. V.6.2), we show that the conditions (S0), (S1), (S2), and (S3) from Subsec. IV.1.2 are satisfied (0 < $s$ < 1), see Theorem 10.

Secondly (Subsec. V.6.3), we investigate the curve

$$K_1(s) = \{(x_1, x_2) : |x_1| < 1, |x_2| < 1, Q((x_1, x_2), s) = 0\}.$$ 

We construct the inverse image of $K_1(s)$ and related sets under the mapping (i) resp. (ii). The results can be found in Theorem 12 and in (27) and (30); they are pictured in Fig. 1.

We conclude this section (Subsec. V.6.4) with translating some technical conditions occurring in the foregoing into simple conditions on $s$ and the one step transition probabilities of the random walk. These technical conditions cover cases where the intersection of a “generalized open upper half plane” and an associated “generalized open lower half plane” fails to be a “generalized open horizontal strip” because their union doesn’t cover the whole plane (this doesn’t happen in the case 0 < $s$ < 1).
V.0.7 Summary of Sec. V.7

In the final Sec. V.7 we determine the generating function $P(x, y, s) \ (0 < s < 1)$ by solving the backward conditions I.3.(B 1′′).

In Subsec. V.7.1 we first check that the conditions I.3.(B 1′′) are equivalent to the conditions I.3.(B 2′′). Next, we transform the conditions I.3.(B 2′′) into a set of conditions called (B 3′′) (using the uniformization of the zero locus of the kernel constructed in Sec. V.4) and show equivalence (using results of the sections Sec. V.5 and V.6).

In the subsequent Subsec. V.7.2 we solve the conditions (B 3′′). The central theorem here is Theorem 3. This theorem is a straightforward application of Theorem IV.5.4, and all that needs to be done is checking the hypotheses of that theorem, which is easy.

The solution of (B 3′′) still needs to be transformed into the corresponding solution of I.3.(B 1′′). This is done in the final Subsec. V.7.3; the result is the desired expression for the generating function $P(x, y, s)$. 
V.1 Notations. Degeneracy of the biquadratic polynomial \( Q(x, s) \)

**Notations**

For convenience we repeat some notations introduced in Section I.3. Let the bivariate stochastic variable

\[ \xi = (\xi_1, \xi_2) \in \mathbb{Z}^2 \]

have the probability distribution of the step. We put (cf. I.1.(17))

\[ \xi^* = -\xi, \]

and define (cf. I.3.(3B))

\[ B(x) = IE x_1^{1+\xi_1} x_2^{1+\xi_2} = IE x_1^{1-\xi_1} x_2^{1-\xi_2}. \]

As in I.3.(11) we put

\[ Q(x, s) = sB(x) - x_1 x_2. \]

The random walk is supposed to be skipfree (cf. I.1.(4)), and, consequently, \( B(x) \) is a polynomial in \( x_1 \) and \( x_2 \) which is (at most) quadratic in each one of the variables separately. The coefficient polynomials \( a_i(x), b_i(x), \) and \( c_i(x), \) which are introduced in I.3.(21) ff., are quadratic in \( x_i, \) and one has for \( i = 1, 2 \) and \( j = 3 - i \)

\[ B(x) = a_i(x) x_j^2 + b_i(x) x_j + c_i(x), \]

\[ Q(x, s) = sa_i(x) x_j^2 + (sb_i(x) - x_i) x_j + sc_i(x); \]

where the polynomials \( B(x) \) and \( Q(x, s) \) are written explicitly as quadratic forms in \( x_j. \)

As in Chapter II, it will be convenient to allow \( x_1 \) or \( x_2 \) to be infinite; we refer to I.3.(26) and I.3.(27). The complex sphere is denoted by \( \mathbb{C} \) (in accordance with I.3.(29)), and we put

\[ K(s) = \{ x \in (\mathbb{C})^2 : Q(x, s) = 0 \} \]
(in accordance with I.3.(30)). As in I.3.(12) we put

\[ K_1(s) = \{ x \in \mathcal{U}^2 : Q(x,s) = 0 \}, \]

where \( \mathcal{U} \) is the open unit disc in \( \mathbb{C} \):

\[ \mathcal{U} = \{ w \in \mathbb{C}^2 : |w| < 1 \}. \]

The discriminant of \( Q(x,s) \) as a quadratic form in \( x_j \) is (cf. I.3.(25))

\[ D_1(x_j,s) = (s b_j(x_j) - x_j)^2 - 4 s^2 a_j(x_j) c_j(x_j), \]

which is a polynomial in \( x_j \) of degree 4 or less. If the degree is less than 4, say \( k \), then the polynomial \( x_j^4 + D_1(x_j,s) \) is said to have a zero at infinity of order \( 4 - k \) (see I.3.(28) for the value of \( D_1(\cdot,s) \) at infinity).

A point \( x \in \mathbb{C}^2 \) is called a double point (with respect to the function \( x + Q(x,s) \)) if (cf. I.3.(15) et seq.)

\[ (Q(x,s), \frac{\partial Q(x,s)}{\partial x_1}, \frac{\partial Q(x,s)}{\partial x_2}) = (0,0,0), \]

or, equivalently (cf. Corollary I.3.14)

\[ (Q(x,s), D_1(x_1,s), D_2(x_2,s)) = (0,0,0). \]

The point \( x \in \mathbb{C}^2 \) is called simple if it is not a double point.

**Degeneracy**

The characterization (12) of a finite double point \( x \) can be used to cover the case that \( x_1 \) or \( x_2 \) is infinite.

So \( x \in (\mathbb{C})^2 \) is called a double point (with respect to \( x + Q(x,s) \)) if (12) holds, and a simple point otherwise.

The polynomial \( x + Q(x,s) \) is called nondegenerate if every \( x \in (\mathbb{C})^2 \) is simple (with respect to \( x + Q(x,s) \)), and degenerate otherwise.

**Remark 1.** For \( \lambda = (\lambda_1, \lambda_2) \in \{-1, +1\}^2 \) write
\( Q_\lambda(x,s) = x_1^{1-\lambda_1} x_2^{1-\lambda_2} Q((x_1, x_2), s). \)

It is easily verified that the polynomial \( x + Q(x,s) \) has a (finite or infinite) double point iff for some \( \lambda \) the polynomial \( x + Q_\lambda(x,s) \) has a finite double point. Also it is easily verified that Corollary I.3.14 remains valid if \( x_1 \) or \( x_2 \) is allowed to be infinite.

**THEOREM 1.** (Degeneracy of \( x + Q(x,s) \) if \( s \in [0,1] \))

Let the random walk be skipfree (cf. I.1.(4)). Fix \( s \in [0,1] \).

I. The polynomial \( x + Q(x,s) \) is nondegenerate in each one of the following cases:
   (i) \( s \in (0,1) \) and the walk is all-sided (cf. I.1.(5)),
   (ii) \( s = 1 \) and the walk is all-sided and not driftless (cf. I.1.(5) and I.1.(6)).

II. The polynomial \( x + Q(x,s) \) is degenerate in each one of the following cases:
   (iii) \( s = 0 \),
   (iv) \( s = 1 \) and the walk is all-sided and driftless (cf. I.1.(5) and I.1.(6)),
   (v) the walk is one-sided (cf. I.1.(5)).

**PROOF.** Part I. Apply Theorem V.2.3 below to conclude that neither of the discriminants has a multiple zero (finite or infinite). By Corollary I.3.14 (Part III), cf. Remark 1, there is no double point.

Part II. Case (iii). If \( s = 0 \) then \( D(x,s) = x^2 \), so \( D(\cdot,s) \) has a double zero at the origin and at infinity; this implies degeneracy.

The case (iv) is dealt with in Chapter II; in this case \( x = (1,1) \) is a double point, see Property II.1.2 and Corollary II.2.4.

Consider case (v). Let the walk be one-sided. We consider two typical cases only.

Case 1: \( c_1(0) = c'_1(0) = b_1(0) \), or, equivalently,
\[ c_2(0) = c'_2(0) = b_2(0). \]

In this case both discriminants have a (multiple) zero at the origin, and because \( Q((0,0), s) = 0 \) the point \( (0,0) \) is a finite double point.
Case 2: \( a_2(0) = b_2(0) = c_2(0) = 0 \), or, equivalently,
\[ c_1(x_1) = 0. \]

In this case \( D_2(x_2, s) \) has the double zero \( x_2 = 0 \), and
\( D_1(x_1, s) = (sb_1(x_1) - x_1)^2 \) has a finite multiple zero, say \( x_1 = z_1 \) (cf. Remark 2 below). Because \( Q((x_1, 0), s) = 0 \) for all \( x_1 \), the point \((z_1, 0)\) is a double point (which is finite).

The remaining cases can be treated similarly.

\( \square \)

**Remark 2.** Assume \( s \in (0, 1) \) and \( s \notin \{ \xi = (0, 0) \} < 1 \). Then, in case 2 of the above proof, the zeros of \( D_1(\cdot, s) \) coincide with the zeros of the coefficient polynomial \( p(x_1) = sb_1(x_1) - x_1 \).

Let \( w, w' \) be these two zeros. We claim that each one is real or infinite, and that (if \( w \leq w' \)) they satisfy

\[ 0 \leq w \leq 1 \leq w' \leq \infty. \]

To see this, write \( b_1(x_1) = b_{-1} + b_0 x_1 + b_1 x_1^2. \)

If \( b_1 = 0 \) then

\[ w = \frac{s b_{-1}}{1 - s b_0} \in [0, 1] \quad \text{and} \quad w' = \infty, \]

and the claim is true.

If \( b_1 > 0 \) then the claim follows from:

\[ p(0) = s b_{-1} \geq 0, \quad p(1) = s b_1(1) - 1 \leq 0, \quad p_1(0) = +\infty. \]

Next assume \( s \in (0, 1) \) and \( s \notin \{ \xi = (0, 0) \} = 1 \), or \( s = 1 \notin \{ \xi = (0, 0) \} \).

Then \( Q(x, 1) = 0 \equiv D_2(x_2, 1) \) for \( i = 1, 2 \); so everything is trivial.
V.2  The zeros and the sign of the discriminants of $Q(x, s)$; the $d$-values

V.2.1  The zeros and the sign of the discriminants of $Q(x, s)$

Throughout this section we assume

(0) $\xi$ is skipfree and all-sided, and $s \in (0, 1]$.

As before we write

(1) $\nu \xi = -\xi$.

**PROPERTY 1.** Assume (0). Then for $i = 1, 2$ with $j = 3 - i$ the following holds.

I. $D_i(1, s) \geq 0$; equality holds if and only if $s = 1$ and $\nu \xi_j = 0$.

II. If $\nu \xi$ is even then $D_i(-\xi, s) = D_i(\xi, s)$;
if $\nu \xi$ is non-even then $D_i(-\xi, s) > D_i(\xi, s)$ for positive $\xi$.

III. If $D_i(0, s) \geq 0$ then the walk is non-even.

**PROOF.** Part I. Trivial from I.3.(45).

Part II. We use the notations I.3.(23), and drop the index $i$. We have

\[ (s^2b(-x) + x)^2 = (x - sb(x) + 2[sb(x) - \beta_0 x])^2 = (x - sb(x))^2 + 4sx(1 - \beta_0)(b(x) - \beta_0 x). \]

Also we have

\[ -4s^2\sigma(-x)c(-x) = -4s^2\sigma(x)(\sigma(x) - 2\alpha_0 x)(c(x) - 2\gamma_0 x) = -4s^2\sigma(x)c(x) + 8s^2\gamma_0 x(\sigma(x) - \alpha_0 x) + + 8s^2\alpha_0 x(c(x) - \gamma_0 x). \]

Adding (*) and (**) yields

\[ D(-x, s) = D(x, s) + 4sx(1 - \beta_0)(b(x) - \beta_0 x) + + 8s^2\gamma_0 x(\sigma(x) - \alpha_0 x) + 8s^2\alpha_0 x(c(x) - \gamma_0 x). \]
First assume that $\frac{y}{x}$ is even, i.e. $\beta_{-1} = \beta_1 = \alpha_0 = \gamma_0 = 0$. Then, obviously, $D(-x,s) = D(x,s)$, q.e.d.

Next assume that $\frac{y}{x}$ is non-even, and that $x > 0$.

Notice that $\sigma(x) - \alpha_0 x \geq 0$ (and that equality occurs for some positive $x$ iff $\alpha_{-1} = \alpha_1 = 0$); similarly for $b(x) - \beta_0 x$ and $c(x) - \gamma_0 x$. Consequently, $D(-x,s) \geq D(x,s)$. In order to show that the inequality is strict, assume, contrarily, that equality holds. By all-sidedness $1 - s\beta_0 > 0$, so by (***)

$$\beta_{-1} + \beta_1 = 0 \quad \text{and} \quad \gamma_0 (\alpha_{-1} + \alpha_1) = 0 = \alpha_0 (\gamma_{-1} + \gamma_1),$$

hence $\beta_{-1} = \beta_1 = \gamma_0 = \alpha_0 = 0$ by all-sidedness, which contradicts our assumption that $\frac{y}{x}$ is non-even.

Consequently, $D(-x,s) > D(x,s)$.

Part III. Assume, contrarily, that the walk is even. Then $b_\ell(0) = 0$, and $\sigma_\ell(0)c_\ell(0) \leq 0$ because of $D_\ell(0) \geq 0$, hence $\sigma_\ell(0)c_\ell(0) = 0$. Because the walk is even this implies that the walk is one-sided, which contradicts our assumption.

\[ \square \]

In imitation of Malyshev [1972] the discriminant $D_{\ell}(x_{\ell},s)$ is factorized in order to trace its zeros, as follows:

$$D_{\ell}(x_{\ell},s) = \prod_{\mu = \pm} D_{\ell}^\mu(x_{\ell},s)$$

where

$$D_{\ell}^\mu(x_{\ell},s) = sb_{\ell}(x_{\ell}) - x_{\ell} + \mu 2s \sqrt{\sigma_{\ell}(x_{\ell})c_{\ell}(x_{\ell})}$$

for $\ell = 1,2$ and $\mu = \pm$. The factors $D_{\ell}^\mu(x_{\ell},s)$ are needed only if $\sigma_{\ell}(x_{\ell})c_{\ell}(x_{\ell}) \geq 0$, and we use the nonnegative root of a nonnegative number. We put

$$D_{\ell} = \{x_{\ell} \in \mathbb{R} : \sigma_{\ell}(x_{\ell})c_{\ell}(x_{\ell}) > 0\}.$$

Notice:

$$D_{\ell}^+(x_{\ell},s) \quad \text{and} \quad D_{\ell}^-(x_{\ell},s) \quad \text{do not vanish simultaneously if} \quad x_{\ell} \in D_{\ell}.$$
The following property collects a number of statements to be applied in the proof of Theorem 3. We use the following notations: as before we write

\[ \text{sgn } 0 : = 0, \]

and analogous to I.3.(28) we put

\[ D^+_{t_i}(\omega, s) = s b_{t_i}(\omega) + \mu 2 \sqrt{\sigma_{t_i}(\omega)c_{t_i}(\omega)}, \]

where the notations I.3.(26) have been used.

**PROPERTY 2.** Under the assumptions (0) the following holds for \( t = 1, 2 \).

\[ (0, \infty) \in D_{t_i} \]

\[ \text{if } D_{t_i}(x_{t_i}, s) = 0 \text{ for a real nonvanishing } x_{t_i} \text{ then } x_{t_i} \in D_{t_i} \]

\[ D_{t_i}^+(\cdot, s) \text{ has no zero in } D_{t_i} \cap (-\infty, 0) \]

\[ \text{sgn } D_{t_i}^+(1, s) = - \text{sgn } D_{t_i}(1, s) \]

\[ \text{if } X_{t_i}^0 = 0 \text{ then } \frac{\partial D_{t_i}^+(x_{t_i}, s)}{\partial x_{t_i}} \bigg|_{x_{t_i}} = 1 = s - 1 + s X_{t_i}^0 \]

\[ \text{sgn } D_{t_i}^-(0, s) = \text{sgn } D_{t_i}(0, s) \]

\[ \text{sgn } D_{t_i}^-(\omega, s) = \text{sgn } D_{t_i}(\omega, s) \]

**PROOF of (8).** The all-sidedness implies \( a_{t_i}(x_{t_i}) > 0 \) and \( c_{t_i}(x_{t_i}) > 0 \) for positive \( x_{t_i} \).

Proof of (9). Observe that \( a_{t_i}(x_{t_i})c_{t_i}(x_{t_i}) \geq 0 \) if \( D_{t_i}(x_{t_i}, s) = 0 \) and \( x_{t_i} \in \mathbb{R} \). The all-sidedness implies that \( a_{t_i}(x_{t_i})c_{t_i}(x_{t_i}) < 0 \) if \( D_{t_i}(x_{t_i}, s) = 0 \) and \( 0 < |x_{t_i}| < \infty \) by Property I.3.12. This proves (9).

Proof of (10). Assume, contrarily, \( D_{t_i}^+(x_{t_i}, s) = 0 \) and \( x_{t_i} \in D_{t_i} \) and \( -\infty < x_{t_i} < 0 \). Then

\[ 0 = x_{t_i}^{-1} D_{t_i}^+(x_{t_i}, s) = sx_{t_i}^{-1} b_{t_i}(x_{t_i}) - 1 - 2s \sqrt{x_{t_i}^{-1} a_{t_i}(x_{t_i})x_{t_i}^{-1} c_{t_i}(x_{t_i})} < \]

\[ < sx_{t_i}^{-1} b_{t_i}(x_{t_i}) - 1 \leq s \mathbb{P}(\xi = (0, 0)) - 1 < 0, \]
which is a contradiction.

Proof of (11). Observe (use I.3.(31))

\[
D^u_\bar{\tau}(1,s) = s - 1 - s\{(\sqrt{\sigma_\bar{\tau}(1)) - \mu\sqrt{c_\bar{\tau}(1))}}\}^2.
\]

So

\[
D^-_\bar{\tau}(1,s) < 0
\]

by all-sidedness, and apply (2).

Proof of (12). From (3) one has

\[
\frac{\partial D^+_\bar{\tau}(x_\bar{\tau},s)}{\partial x_\bar{\tau}} = -1 + s \beta'(x_\bar{\tau}) + s \frac{\sigma'_\bar{\tau}(x_\bar{\tau})c'_\bar{\tau}(x_\bar{\tau})}{\sqrt{\sigma_\bar{\tau}(x_\bar{\tau})c_\bar{\tau}(x_\bar{\tau})}},
\]

and apply I.3.(35) and I.3.(37).

Proof of (13A). Observe that \(D^+_\bar{\tau}(0,s) \neq 0\) with equality holding only if \(D^-_\bar{\tau}(0,s) = 0\), and apply (2).

The equality (13B) is left to the reader.

From (9) and (5) it follows that any real nonvanishing zero of \(D^-_\bar{\tau}(\cdot, s)\) is a zero in \(D^-_\bar{\tau}\) of either \(D^+_\bar{\tau}(\cdot, s)\) or \(D^-_\bar{\tau}(\cdot, s)\) but not both. It is not excluded, however, that \(D^+_\bar{\tau}(\cdot, s)\) and \(D^-_\bar{\tau}(\cdot, s)\) vanish simultaneously in 0 or \(\infty\), see (39) through (42).

**THEOREM 3.** (The zeros and the sign of the discriminants)

Assume that the walk is skipfree and all-sided, and fix \(s \in (0, 1]\).

Also assume that \((s, E_{\bar{\tau}}) \neq 1, (0, 0))\).

Then for \(\bar{\tau} = 1, 2\) with \(j = 3 - \bar{\tau}\) the following holds.

I. \(D^+_\bar{\tau}(\cdot, s)\) has precisely two distinct positive zeros, say \(c_{\bar{\tau}1}\) and \(c_{\bar{\tau}3}\) with \(c_{\bar{\tau}1} < c_{\bar{\tau}3}\). These zeros satisfy

\[
\begin{align*}
(15A) & \quad c_{\bar{\tau}1} \in (0, 1], \\
(15B) & \quad c_{\bar{\tau}3}^{-1} \in (0, 1].
\end{align*}
\]

Moreover,
\begin{align}
(16A) & \quad c_{\xi_1} \neq 1 \neq c_{\xi_3} \iff (\sigma, \mathbb{E}_{\xi_j}^Y) \neq (1, 0), \\
(16B) & \quad c_{\xi_1} \neq 1 = c_{\xi_3} \iff (\sigma, \mathbb{E}_{\xi_j}^Y) = (1, 0) \text{ and } \mathbb{E}_{\xi_j}^Y > 0, \\
(16C) & \quad c_{\xi_1} = 1 \neq c_{\xi_3} \iff (\sigma, \mathbb{E}_{\xi_j}^Y) = (1, 0) \text{ and } \mathbb{E}_{\xi_j}^Y < 0.
\end{align}

II. \quad \tilde{D}_\xi(\cdot, s) \text{ has precisely two distinct zeros in } \mathbb{D}_\xi \cup \{0\} \cup \{\infty\},

say \quad c_{\xi_2} \quad \text{and} \quad c_{\xi_4} \quad \text{with } |c_{\xi_2}| \leq |c_{\xi_4}| \quad \text{(where } c_{\xi_2} \text{ possibly vanishes and } c_{\xi_4} \text{ possibly is infinite). These zeros satisfy}

\begin{align}
(17A) & \quad c_{\xi_2} \in [-c_{\xi_1}, c_{\xi_1}) \subset [-1, 1), \\
(17B) & \quad c_{\xi_4} \in [-c_{\xi_3}, c_{\xi_3}) \subset [-1, 1).
\end{align}

Moreover,

\begin{align}
(18A) & \quad c_{\xi_2} = -c_{\xi_1} \quad \text{iff the random walk is even}, \\
(18B) & \quad c_{\xi_4} = -c_{\xi_3} \quad \text{iff the random walk is even}.
\end{align}

The sign of these zeros is determined by

\begin{align}
(19A) & \quad \text{sgn } c_{\xi_2} = \text{sgn } \tilde{D}_\xi(0, s) \quad \text{(which doesn’t depend on } s), \\
(19B) & \quad \text{sgn } c_{\xi_4} = \text{sgn } \tilde{D}_\xi(\infty, s) \quad \text{(which doesn’t depend on } s)
\end{align}

(for the last expression see I.3.(28) and I.3.(26)).

III. \quad The four values \quad c_{\xi_1}, c_{\xi_2}, c_{\xi_3}, c_{\xi_4} \quad \text{are distinct, simple zeros of}

\quad \tilde{D}_\xi(\cdot, s), \quad \text{and } \tilde{D}_\xi(\cdot, s) \quad \text{does not have other zeros. Moreover,}

\begin{align}
(20) & \quad \tilde{D}_\xi(x_\xi, s) \geq 0 \quad \text{and } x_\xi \in \mathbb{R} \cup \{\infty\} \quad \text{iff } x_\xi \in [c_{\xi_1}, c_{\xi_3}] \cup [c_{\xi_4}, c_{\xi_2}].
\end{align}

where

\begin{align}
(21) & \quad [c_{\xi_4}, c_{\xi_2}] : = [c_{\xi_4}, \infty) \cup \{\infty\} \cup (-\infty, c_{\xi_2}) \quad \text{if } c_{\xi_4} > c_{\xi_2}.
\end{align}
PROOF. Part I. We claim:

(*) \[ D^+_L(\cdot,s) \text{ has at least two distinct positive zeros satisfying} \]
(15A) through (16C). (By (8) these zeros are in \( D_L \).)

To prove the claim, first observe (cf. (8))

(i) \( D^+_L(x_L,s) \) is real and continuous for positive \( x_L \).

Secondly one has for positive \( x_L \)

\[
x_L^{-1} D^+_L(x_L,s) =
-1 + s(x_L^{-1} b_L(x_L) + 2x_L^{-1} \sigma_L(x_L) \cdot x_L^{-1} c_L(x_L)),
\]

and one easily concludes applying all-sidedness

\[
x_L^{-1} D^+_L(x_L,s) \rightarrow +\infty \text{ for } x_L \rightarrow 0^+ \text{ and for } x_L \rightarrow +\infty,
\]

which implies

(ii) \( D^+_L(x_L,s) > 0 \) for small and for large positive \( x_L \).

Thirdly one has

(iii) \( D^+_L(1,s) \leq 0 \), with equality iff \((s, \xi L, 0, 0) = (1, 0)\),

because of (11) and Property 1 (Part I).

Finally one has

(iv) if \((s, \xi L, 0, 0) = (1, 0)\) then \( \left. \frac{\partial D^+_L(x_L,s)}{\partial x_L} \right|_{(x_L,s) = (1,1)} = \xi L \)

by (12).

The claim is an easy consequence of (i) through (iv).

For the proof that \( D^+_L(\cdot,s) \) has at most two distinct positive zeros, see the final step below.

Part II. We claim:

(**) \( D^-_L(\cdot,s) \) has at least one zero in \( D_L \cup \{0\} \) satisfying
(17A), (18A), and (19A).
To show the claim, we distinguish three cases.

Case 1: \( D_t^{-}(0,s) = 0 \), hence \( D_t^{-}(0,s) = 0 \) by (13A). The walk is non-even by Property 1 (Part III).

So the zero \( c_{t,2}^{-} = 0 \) satisfies the requirements.

Case 2: \( D_t^{-}(0,s) > 0 \), hence \( D_t^{-}(0,s) > 0 \) by (13A). The function \( D_t^{-}(\cdot,s) \) is real and continuous on \([0,\infty)\), and satisfies \( D_t^{-}(x_t,s) \leq D_t^{-}(x_t,s) \) for \( x_t \in [0,\infty) \); in particular \( D_t^{-}(c_{t,1},s) \leq D_t^{+}(c_{t,1},s) \) hence \( D_t^{-}(c_{t,1},s) \leq 0 \), where equality does not hold because of (5). So \( D_t^{-}(c_{t,1},s) < 0 \). Hence, \( D_t^{-}(\cdot,s) \) has a zero, say \( c_{t,2}^{-} \), in \((0,c_{t,1})\). The walk is non-even by Property 1 (Part III). Observe \( c_{t,2}^{-} \in D_t^{-} \) by (8).

So the zero \( c_{t,2}^{-} \) satisfies the requirements.

Case 3: \( D_t^{-}(0,s) < 0 \). Observe (cf. (2)) that \( c_{t,1}^{-} \) also is a zero of \( D_t(\cdot,s) \), hence, by Property 1 (Part II) \( D_t^{-}(c_{t,1}^{-},s) \geq 0 \) with equality iff the random walk is even.

Consequently, \( D_t^{-}(\cdot,s) \) has a zero, say \( c_{t,2}^{-} \), in \([-c_{t,1}^{-},0) \) with \( c_{t,2}^{-} = -c_{t,1}^{-} \) iff the walk is even. This zero also is a zero of \( D_t^{-}(\cdot,s) \) by (9) and (10), and so satisfies the requirements. The claim is proved.

Similarly one shows (e.g. by considering the polynomial \( x_t^4 D_t(\cdot,s) \) rather than \( D_t^{-}(x_t,s) \))

\[
(* *) \quad \text{\( D_t^{-}(\cdot,s) \) has at least one zero in \( D_t^{-} \cup \{0\} \)} \text{ satisfying (17B), (18B), and (19B).}
\]

For the proof that \( D_t^{-}(\cdot,s) \) has at most two distinct zeros in \( D_t^{-} \cup \{0\} \cup \{\infty\} \), see the final step below.

Part III. Observe that (17A) and (17B) imply that \( c_{t,2}^{-} \) and \( c_{t,4}^{-} \) are distinct because \( c_{t,1}^{-} \) and \( c_{t,3}^{-} \) are distinct, and also that \( \{c_{t,1}^{-}, c_{t,3}^{-}\} \) is disjoint from \( \{c_{t,2}^{-}, c_{t,4}^{-}\} \); hence, the four values \( c_{t,1}^{-}, c_{t,2}^{-}, c_{t,3}^{-}, c_{t,4}^{-} \) are distinct.

These values are zeros of \( D_t^{-}(\cdot,s) \) by (2) and the claims (*) , (**), and (***). Because the degree of \( D_t^{-}(\cdot,s) \) is (at most) four, these zeros are simple and there are no other zeros.

Final step. By (2) any zero of \( D_t^{+}(\cdot,s) \) or \( D_t^{-}(\cdot,s) \) in \( D_t^{-} \cup \{0\} \cup \{\infty\} \) is a zero of \( D_t(\cdot,s) \), so is element of the set \( \{c_{t,1}^{-}, c_{t,2}^{-}, c_{t,3}^{-}, c_{t,4}^{-}\} \).

This implies that the claims (*) , (**), and (***), remain true if the phrase
"at least" is replaced by "precisely". This completes the proofs of Part I and Part II.

We show (20). From the proof of Part I it follows

$$D_+(x, s) \leq 0 \text{ if } x \in [c_{11}, c_{13}].$$

Because of $D_-(x, s) < D_+(x, s)$ for positive $x$ (cf. (8)) one has

$$D_-(x, s) < D_+(x, s) \leq 0 \text{ if } x \in [c_{11}, c_{13}].$$

Consequently, by (2),

$$D_+(x, s) \geq 0 \text{ if } x \in [c_{11}, c_{13}].$$

This proves a part of (20), and easily implies the remaining parts, because the discriminant has a change of sign at each zero.

\[\square\]

**Remark 1. (Continuity and regularity of the zeros of the discriminants as functions of $s$)**

From the theory of algebraic functions it is known that the zeros of a polynomial in a complex variable are continuous functions of the coefficients. Moreover, the implicit function theorem (Gunning, Rossi [1965], Ch. 1, Sec. B, Theorem 4) implies that every simple zero is locally a regular function of the coefficients.

So the values $c_{i k}$ can be defined for arbitrary random walk and all $s \in [0,1]$ by means of continuity, and one has

\[(22)\]

$c_{i k}$ is a continuous function of $s$ on $[0,1]$;

moreover, by Theorem 3 (Part III),

\[(22')\]

$c_{i k}$ is a regular function of $s$

- on a neighbourhood of $(0,1)$,
- on a neighbourhood of $s = 1$ if the random walk is not driftless,

for $i = 1, 2$ and $k = 1, \ldots, 4$. 
REMARK 2. (Limits of the zeros for $s \to 0$ and for $s \to 1$)

If $s = 0$ then both discriminants have double zeros in the origin and at infinity (cf. the proof of Theorem V.1.1 (Part II); hence

$$c_{i1}, c_{i2}, c_{i3}^{-1}, c_{i4}^{-1} \to 0 \text{ if } s \to 0.$$  

Also, if the walk is driftless and $s = 1$, then the discriminants have a multiple zero at 1 (cf. II.1.5A) for even walk, and the remarks preceding II.2.(9) for even walk). Hence

$$c_{i1}, c_{i3} + 1 \text{ if } s = 1 \text{ and the walk is driftless}$$

(moreover, $c_{i2}, c_{i4} + 1$ if in addition the walk is even).

REMARK 3. (A characterization of one-sidedness in terms of the zeros of the discriminants)

Fix $s \in (0,1]$ and $i \in \{1,2\}$. Then one has:

$$c_{i1} = c_{i2} \text{ or } c_{i3} = c_{i4}.$$  

We show (25).

"Only if". See the proof of Theorem V.1.1 (Part II, case (v)) and Remark V.1.2.

"If". We leave the case $c_{i3} = c_{i4}$ to the reader, and assume $c_{i1} = c_{i2}$.

Then writing $x_{\dot{i}} := c_{i1} = c_{i2}$ one has $D_{\dot{i}}^+(x_{\dot{i}}, s) = 0 = D_{\dot{i}}^-(x_{\dot{i}}, s)$ and $|x_{\dot{i}}| \leq 1$ (by Theorem 3), hence

$$sb_{\dot{i}}(x_{\dot{i}}) - x_{\dot{i}} = 0 = a_{\dot{i}}(x_{\dot{i}})c_{\dot{i}}(x_{\dot{i}}).$$

First assume $x_{\dot{i}} \neq 0$. Then the walk is one-sided by Property I.3.12. Next assume $x_{\dot{i}} = 0$. Because $x_{\dot{i}} = 0$ is a multiple zero of $D_{\dot{i}}(x_{\dot{i}}, s)$ one has

$$3D_{\dot{i}}(x_{\dot{i}}, s)\frac{\partial D_{\dot{i}}(x_{\dot{i}}, s)}{\partial x_{\dot{i}}} = 0 \text{ for } x_{\dot{i}} = 0; \text{ differentiating the expression (10) and substituting (*) one finds}$$

$$a_{\dot{i}}'(x_{\dot{i}})c_{\dot{i}}(x_{\dot{i}}) + a_{\dot{i}}(x_{\dot{i}})c_{\dot{i}}'(x_{\dot{i}}) = 0.$$  

From (*) and (**) with $x_{\dot{i}} = 0$ it follows easily that the walk is one-sided, q.e.d.
COROLLARY 4. Under the hypotheses of Theorem 3 one has for all \( x \in \mathbb{C} \)

\[
D_{\bar{z}}(x, s) = D_{\bar{z}0}(c_{\bar{z}1} - x_{\bar{z}})(c_{\bar{z}2} - x_{\bar{z}})(1 - x_{\bar{z}}c_{\bar{z}3}^{-1})(1 - x_{\bar{z}}c_{\bar{z}4}^{-1}),
\]

where \( D_{\bar{z}0} \) is a constant (i.e., not depending on \( x_{\bar{z}} \)) satisfying

\[
D_{\bar{z}0} \in (0, \infty).
\]

PROOF. Theorem 3 (Part III) implies the equality (26) for some finite nonvanishing constant \( D_{\bar{z}0} \).

We prove (27). Observe that \( D_{\bar{z}}(-1, s), D_{\bar{z}}(0, s), \) and \( D_{\bar{z}}(\infty, s) \) do not vanish simultaneously (see (15A), (15B)).

First assume \( D_{\bar{z}}(-1, s) \neq 0 \). Then

\[
D_{\bar{z}0} = \frac{D_{\bar{z}}(-1, s)}{(1 + c_{\bar{z}1})(1 + c_{\bar{z}2})(1 + c_{\bar{z}3})(1 + c_{\bar{z}4})}.
\]

From Property 1 (Parts I and II) it follows \( D_{\bar{z}}(-1, s) > 0 \), hence \( D_{\bar{z}}(-1, s) > 0 \). Also one has \( c_{\bar{z}2} \neq -1 \) and \( c_{\bar{z}4} \neq -1 \); consequently the denominator in (28A) is positive by (17A), (17B).

Next assume \( D_{\bar{z}}(0, s) \neq 0 \). Then

\[
D_{\bar{z}0} = \frac{D_{\bar{z}}(0, s)}{c_{\bar{z}1} c_{\bar{z}2}}.
\]

which is positive by (15A) and (19A).

Finally assume \( D_{\bar{z}}(\infty, s) \neq 0 \). Then

\[
D_{\bar{z}0} = \frac{D_{\bar{z}}(\infty, s)}{c_{\bar{z}3} c_{\bar{z}4}}
\]

which is positive by (15B) and (19B).

\[ \square \]

The following corollary of Theorem 3 deals with the crossratio

\[(c_{\bar{z}1}, c_{\bar{z}2}; c_{\bar{z}3}, c_{\bar{z}4})\]

of the four zeros of the discriminant \( D_{\bar{z}}(\cdot, s) \). The crossratio is defined in III.1.(21), and can, more conveniently, be written as follows:

\[
(c_{\bar{z}1}, c_{\bar{z}2}; c_{\bar{z}3}, c_{\bar{z}4}) = \frac{1 - c_{\bar{z}1} c_{\bar{z}3}^{-1}}{1 - c_{\bar{z}1} c_{\bar{z}4}^{-1}} \cdot \frac{1 - c_{\bar{z}2} c_{\bar{z}4}^{-1}}{1 - c_{\bar{z}2} c_{\bar{z}3}^{-1}}.
\]
COROLLARY 5. Under the assumptions of Theorem 3

\[ 0 < (c_{\frac{1}{2}}; c_{\frac{3}{4}}; c_{\frac{1}{2}}; c_{\frac{1}{2}}) < 1. \]

PROOF. We show the first inequality. From Theorem 3 (Part I) it follows
\[ |c_{\frac{1}{2}}| < 1, \] and (17A) and (17B) imply
\[ |c_{\frac{3}{4}}| \leq |c_{\frac{1}{2}}| \] and
\[ |c_{\frac{1}{2}}| \leq |c_{\frac{1}{2}}|. \]
Consequently all four factors in the quotient on the right in (29) are
positive, which proves the first inequality.

We show the second inequality. One has, see Property III.1.15, Part IV, b),

\[ 1 - (c_{\frac{1}{2}}; c_{\frac{3}{4}}; c_{\frac{1}{2}}; c_{\frac{1}{2}}) = \]
\[ = (c_{\frac{1}{2}}; c_{\frac{3}{4}}; c_{\frac{1}{2}}; c_{\frac{1}{2}}) = \frac{c_{\frac{1}{2}} - c_{\frac{1}{2}}}{1 - c_{\frac{1}{2}} - c_{\frac{1}{2}}} \cdot \frac{c_{\frac{3}{4}} - c_{\frac{1}{2}}}{1 - c_{\frac{1}{2}} - c_{\frac{1}{2}}} \cdot \frac{c_{\frac{1}{2}} - c_{\frac{1}{2}}}{1 - c_{\frac{1}{2}} - c_{\frac{1}{2}}} . \]
The numerators on the right in (31) are positive because of (17A) and (17B),
which proves the second inequality.

\( \Box \)

V.2.2 The \( d \)-values

For \( k \in \{1,2,3,4\} \) the values \( d_{1k} \) and \( d_{2k} \) are defined by means of the
equations

\[ Q((d_{1k}, c_{2k}), s) = 0 \quad \text{and} \quad Q((c_{1k}, d_{2k}), s) = 0, \]
respectively. Because of the vanishing discriminants each equation
determines one \( d \)-value uniquely.

One has, using the notations V.1.(6), for \( i = 1,2 \) and \( j = 3 - i \)

\[ d_{jk} = \frac{x_i - s b_i(x_i)}{2 s \sigma_i(x_i)} = \frac{2 a_i(x_i) - s b_i(x_i)}{x_i - s b_i(x_i)} \quad \text{with} \quad x_i = c_{ik}, \]

(where at least one of the two quotients is well-defined, see below V.4.(20))
and \( d_{jk} = \frac{c_i(x_i)}{\sigma_i(x_i)} \) with \( x_i = c_{ik} \). The latter result can be improved.

PROPOSITION 6. Under the assumptions of Theorem 3 the following holds for
\( i = 1,2 \) with \( j = 3 - i \):

\[ \text{for } k = 1,3: \quad d_{jk} = \sqrt{\frac{c_i(x_i)}{\sigma_i(x_i)}} \quad \text{with} \quad x_i = c_{ik}, \]

\[ \text{for } k = 2,4: \quad d_{jk} = \sqrt{\frac{c_i(x_i)}{\sigma_i(x_i)}} \quad \text{with} \quad x_i = c_{ik}. \]
\( (35) \quad \text{for } k = 2, 4: d_{jk} = \sqrt{\frac{c_{ik}(x^+_i)}{a_{ik}'(x^+_i)}} \quad \text{with } x_i = x^+_i. \)

**Remark 4.** We show here that the square root expressions on the right are well-defined. Assume \( D_i(x^+_i, s) = 0 \) for some \( x^+_i \in \mathbb{R} \cup \{\infty\} \). Then \( a_i(x^+_i)c_i(x^+_i) \geq 0 \), so the quotients on the right are nonnegative. If, furthermore, \( a_i(x^+_i)c_i(x^+_i) = 0 \) then \( x^+_i \in \{0, \infty\} \) by (9), so \( a_i(x^+_i) = 0 \) because of \( D_i(x^+_i, s) = 0 \) (if \( x^+_i = \infty \) then use the notations I.3.26–(28)). Consequently, \( a_i(x^+_i) \) and \( c_i(x^+_i) \) do not vanish both (due to all-sidedness); so their quotient is well-defined.

**Proof of Proposition 6.**

Assume \( D_i(x^+_i, s) = 0 \). Then \( x_i^+ - sb_i(x_i^+) = \mu \cdot 2s \sqrt{\frac{a_i(x_i^+)c_i(x_i^+)}{a_i'(x_i^+)c_i'(x_i^+)}}, \) with \( \mu = +1 \) for \( x_i^+ \in \{c_{i1}, c_{i3}\} \) and \( \mu = -1 \) for \( x_i^+ \in \{c_{i2}, c_{i4}\} \) (cf. (2) and Theorem 3). Substituting this in (33) one finds (34) and (35), applying the following claim:

\[
\begin{align*}
\begin{cases}
  a_i(x_i^+) > 0 \quad \text{and} \quad c_i(x_i^+) > 0 \quad \text{if} \quad x_i^+ \in \{c_{i1}, c_{i3}\}, \\
  a_i(x_i^+) \geq 0 \quad \text{and} \quad c_i(x_i^+) \geq 0 \quad \text{if} \quad x_i^+ \in \{c_{i2}, c_{i4}\}.
\end{cases}
\end{align*}
\]

(36)

We prove (36). The claim is obvious for \( x_i^+ \in \{c_{i1}, c_{i3}\} \) because \( a_i(x_i^+) \) and \( c_i(x_i^+) \) are positive for positive \( x_i^+ \).

We show the claim for \( x_i^+ = c_{i2}^+ \). If \( w \in (c_{i2}, c_{i1}) \) then \( D_i(w, s) < 0 \), cf. (20), hence \( a_i(w)c_i(w) > 0 \).

This implies that \( a_i(\cdot) \) and \( c_i(\cdot) \) do not change their sign in \( (c_{i2}, c_{i1}) \).

Hence, \( a_i(c_{i2}) > 0 \) and \( c_i(c_{i2}) \geq 0 \) because of \( a_i(c_{i1}) > 0 \) and \( c_i(c_{i1}) > 0 \).

The claim for \( x_i^+ = c_{i4}^+ \) is proved similarly by considering \( x_i^2a_i(x_i^{-1}) \) and \( x_i^2c_i(x_i^{-1}) \) rather than \( a_i(x_i^+) \) and \( c_i(x_i^+) \), and \( x_i^2D_i(x_i^{-1}, s) \) rather than \( D_i(x_i^+, s) \).

\[\Box\]

**Remark 5.**

From Proposition 6 and (36) it follows, for \( j = 1, 2, \)

\[
(37) \quad 0 < d_{jk} < \infty \quad \text{for} \quad k = 1, 3,
\]

\[
(38) \quad -\infty < d_{jk} < 0 \quad \text{for} \quad k = 2, 4.
\]
We investigate the cases that \( d_{j2} \) or \( d_{j4} \) equals 0 or \( \infty \). Fix \( i = 1, 2 \) and \( j = 3 - i \). If \( d_{j2} = 0 \) then we have, writing temporarily \( x_i^2 = c_{i2} \) and \( x_i^4 = d_{j2}^2 \):

\[
Q(x, s) = 0, \quad x_j = 0, \quad D_i(x_i^2, s) = 0, \quad -1 \leq x_i < 1.
\]

Hence (cf. V.1.(6) and V.1.(10)), \( c_i(x_i) = 0 = sb_i(x_i) - x_i \), which implies \( x_i = 0 \) by Property I.3.12; consequently, \( c_i(0) = b_i(0) = 0 \).

Conversely, suppose \( c_i(0) = b_i(0) = 0 \).

Then \( D_i(0, s) = 0 \) and \( Q((0,0), s) = 0 \). Hence (cf. (32)) \( c_{i2} = 0 = d_{j2} \).

Also we have \( c_j(0) = 0 \), which implies \( D_j(0, s) = s^2b_j(0)^2 \geq 0 \); equality does not occur because of all-sidedness, so \( D_j(0, s) > 0 \); hence \( c_{j2} > 0 \) by (19A). In summation:

\[
\begin{cases}
d_{j2} = 0 \text{ iff } b_i(0) = c_i(0) = 0; \\
\text{this occurs only if } c_{i2} = 0 \text{ and } c_{j2} > 0.
\end{cases}
\]

In a similar way one shows:

\[
\begin{cases}
d_{j2} = \infty \text{ iff } b_i(0) = a_i(0) = 0; \\
\text{this occurs only if } c_{i2} = 0 \text{ and } c_{j4} > 0,
\end{cases}
\]

\[
\begin{cases}
d_{j4} = 0 \text{ iff } b_i(\infty) = c_i(\infty) = 0; \\
\text{this occurs only if } c_{i4} = 0 \text{ and } c_{j2} > 0,
\end{cases}
\]

\[
\begin{cases}
d_{j4} = \infty \text{ iff } b_i(\infty) = a_i(\infty) = 0; \\
\text{this occurs only if } c_{i4} = 0 \text{ and } c_{j4} > 0.
\end{cases}
\]

(Here we used the notational convention I.3.(26).)

**PROPOSITION 7.** Under the assumptions of Theorem 3 the following holds for \( i = 1, 2 \).

\[
D_i(d_{ik}, s) > 0 \text{ for } k = 1, \ldots, 4
\]

\[
d_{i1}, d_{i3} \in (c_{i1}, c_{i3})
\]
(45) \[ d_{\xi 2}, d_{\xi 4} \in (c_{\xi 1}, c_{\xi 2}) \cap [-\infty, 0] \]

(cf. (21)).

\textbf{Proof.} In the notations in this proof we suppress the dependence on \( s \).

Proof of (43). Take \( i = 1 \). Consider the equation in \( x_2 : Q(d_{1k}, x_2) = 0 \).

This is a quadratic equation with real coefficients (because \( d_{1k} \) is real) and a real solution (viz. \( x_2 = c_{2k} \)). Hence the discriminant is nonnegative: \( D_1(d_{1k}) \geq 0 \). Equality does not occur because \( D_2(c_{2k}) = 0 \), and the two discriminants do not vanish simultaneously because of nondegeneracy (see Theorem V.1.1 (Part I), and V.1.12 et seq.).

Proof of (44). Fix \( i \in \{1, 2\} \) and \( j \in 3 - i \).

For \( x_{\xi} \in [c_{\xi 1}, c_{\xi 3}] \) solve the equation \( Q(x) = 0 \) for \( x_j \) to find

\[ x_j = \frac{x_{\xi} - sb_{\xi}(x_{\xi}) + \lambda \sqrt{D_{\xi}(x_{\xi})}}{2s_{\xi}(x_{\xi})} \tag{46} \]

where \( \lambda \in \{-, +\} \), or (which amounts to the same thing)

\[ \frac{dQ(x)}{dx_j} = \lambda \sqrt{D_{\xi}(x_{\xi})}. \tag{47} \]

Notice that \( D_{\xi}(x_{\xi}) \geq 0 \) (see (20)), and that \( s_{\xi}(x_{\xi}) \neq 0 \) (because \( x_{\xi} \) is positive); so \( x_j \) is real and unambiguously determined by (46).

For \( x_{\xi} \in [c_{\xi 1}, c_{\xi 3}] \) and \( \lambda \in \{-, +\} \) define \( f_{\xi}^{\lambda}(x_{\xi}) \) by means of

\[ f_{\xi}^{\lambda}(x_{\xi}) = x_j \text{ with } x_j \text{ given by (46)}. \tag{48} \]

For later use notice

\[ \begin{cases} f_{\xi}^{-}(x_{\xi}) = f_{\xi}^{+}(x_{\xi}) & \text{if } x_{\xi} \in (c_{\xi 1}, c_{\xi 2}), \\ f_{\xi}^{-}(x_{\xi}) < f_{\xi}^{+}(x_{\xi}) & \text{if } x_{\xi} \in (c_{\xi 1}, c_{\xi 3}) \end{cases} \tag{48A} \]

because \( D_{\xi}(\cdot) \) is positive on \((c_{\xi 1}, c_{\xi 3})\), cf. Theorem 3 (Part III).

We claim

\[ \frac{d f_{\xi}^{\lambda}(x_{\xi})}{dx_{\xi}} \text{ has a unique zero, say } x_{\xi}^{\lambda}, \text{ in } (c_{\xi 1}, c_{\xi 3}). \tag{49} \]

To prove the claim, we first show the existence of a zero.

Differentiate (47) with respect to \( x_{\xi} \) to conclude from (48)
\[(*) \quad 2s \sigma_\zeta(x_\zeta) \frac{dx_\lambda}{dx_\zeta} + \frac{\partial^2 Q(x)}{\partial x_\lambda^2} \frac{\partial}{\partial x_\lambda} = \lambda \frac{D_\zeta(x_\zeta)}{2\nu D_\zeta(x_\zeta)} \cdot \]

For \( k = 1,3 \) one has \( \sigma_\zeta(c_{\zeta k}) > 0 \) (see (36)); also one has \( D_\zeta(c_{\zeta k}) = 0 \), and

\[(50) \quad D_\zeta^+(c_{\zeta 1}) > 0 \quad \text{and} \quad D_\zeta^+(c_{\zeta 3}) < 0, \]

cf. Theorem 3 (Part III). So one concludes from \((*)\)

\[\frac{df_\lambda^\zeta(x_\zeta)}{dx_\zeta} \begin{cases} \lambda \quad \text{if} \quad x_\zeta + c_{\zeta 1}, \quad x_\zeta > c_{\zeta 1}, \\ -\lambda \quad \text{if} \quad x_\zeta + c_{\zeta 3}, \quad x_\zeta < c_{\zeta 3}. \end{cases} \]

This proves the existence of a zero of \((f_\zeta^\lambda)\)  in \((c_{\zeta 1}, c_{\zeta 3})\).

To show the uniqueness, write \(H_\zeta^\lambda\) for the set of all zeros of \((f_\zeta^\lambda)\) in \((c_{\zeta 1}, c_{\zeta 3})\), and put

\[G_\zeta^\lambda \equiv \{(x_\zeta, f_\zeta^\lambda(x_\zeta)) : x_\zeta \in H_\zeta^\lambda\}.\]

It suffices to show that \(G_\zeta^\lambda\) contains at most one element.

From I.3.(52) we know

\[(51) \quad \frac{dx_\lambda}{dx_\zeta} \frac{D_\lambda^-(x_\zeta)}{D_\zeta^-(x_\zeta)} \quad \text{if} \quad x_\lambda = f_\zeta^\lambda(x_\zeta),\]

hence \(f_\zeta^\lambda(H_\zeta^\lambda) \subseteq \{c_{\zeta k} : k = 1, \ldots, 4\}\), or \(G_\zeta^\lambda \subseteq \{(d_{\zeta k}, c_{\zeta k}) : k = 1, \ldots, 4\}\). Because of \(H_\zeta^\lambda \subseteq (0, \infty)\) and \(d_{\zeta k} \in [-\infty, 0]\) for \( k = 2,4 \) (see (38)) one has \(H_\zeta^\lambda \subseteq \{d_{\zeta 1}, d_{\zeta 3}\}\), hence

\[(***) \quad G_\zeta^\lambda \subseteq \{(d_{\zeta k}, c_{\zeta k}) : k = 1,3\}.\]

So \(\#(\bigcup_{\lambda=\pm} G_\zeta^\lambda) \leq 2\). We know \(\# G_\zeta^\lambda \geq 1\).

Observe \( \cap G_\zeta^\lambda = \emptyset \) (if \( (x_\zeta, x_\lambda) \in \cap G_\zeta^\lambda \) then \( x_\lambda = f_\zeta^\lambda(x_\zeta) \) for \( \lambda = \pm \)), hence \(D_\zeta(x_\zeta) = 0\) by (46), which contradicts \(D_\zeta(\cdot) > 0\) on \((c_{\zeta 1}, c_{\zeta 3})\).

Consequently, \(\# G_\zeta^\lambda = 1\), which proves (49).

Therefore (**) is tantamount to

\[(52) \quad \{(x_\zeta^\lambda, f_\zeta^\lambda(x_\zeta^\lambda)) : \lambda = \pm \} = \{(d_{\zeta k}, c_{\zeta k}) : k = 1,3\}.\]
This proves (44) because of $x^\lambda_i \in (c_i^1, c_i^3)$.

Proof of (45). From (43) and Theorem 3 (Part III) one has

\[ \{ d_{i^2}, d_{i^4} \} \subset \{ x_i^\lambda : D_i(x_i) > 0 \} = (c_i^1, c_i^3) \cup (c_i^4, c_i^2). \]

From (15A) and (15B) one has $(c_i^1, c_i^3) \subset (0, \infty)$, which implies the claim because of (38).

\[ \square \]

REMARK 6. In addition to (52) we show

\[ \begin{align*}
(x_i^-, f_i^-(x_i^-)) &= (d_{i^1}, c_{f^1}), \\
(x_i^+, f_i^+(x_i^+)) &= (d_{i^3}, c_{f^3}).
\end{align*} \tag{53} \]

It suffices to show for $\lambda = \pm$ (we still suppress the dependence on $s$ in our notations)

\[ \lambda D_j^i(f_i^\lambda(x_i^\lambda)) < 0, \tag{*} \]

because of the inequalities (50).

We show (*). Put $x_i^\pm = f_i^\lambda(x_i^\pm)$. Differentiate the identity $Q(x) = 0$ two times with respect to $x_i^\lambda$, substitute $x_i^\pm = x_i^\lambda$, and apply $\frac{dz}{dx_i} = 0$ to get

\[ \frac{3}{2} Q(x) + \frac{3}{2} Q(x) \frac{d^2 x}{dx_i^2} = 0 \text{ for } x_i = x_i^\lambda. \tag{+} \]

From (51) it follows similarly (apply: $D_j^i(x_i^\lambda) = 0$ if $x_i = x_i^\lambda$, cf. (52))

\[ \frac{d^2 x_i}{dx_i^2} - \frac{1}{2} \left( \frac{D_j^i(x_i^\lambda)}{D_j^i(x_i^\lambda)} \right) \text{ for } x_i = x_i^\lambda. \]

Substitute this in (+), and apply the equalities $\frac{3}{2} Q(x) = 2 s g_j^i(x_j)$ and (47) to find

\[ 2 s g_j^i(f_i^\lambda(x_i^\lambda)) + \lambda \frac{1}{2} D_j^i(x_i^\lambda) \frac{1}{2} D_j^i(f_i^\lambda(x_i^\lambda)) = 0. \]

Now apply $g_j^i(f_i^\lambda(x_i^\lambda)) > 0$ (because of $f_i^\lambda(x_i^\lambda) > 0$, cf. (52), (15A) and (15B)) and $D_j^i(x_i^\lambda) > 0$ (because of (52) and (43)) to conclude (*).
COROLLARY 8. Under the assumptions of Theorem 3 the following holds. If 
$Q(x,s) = 0$ then 

\[(54) \quad x_1 \in [c_{11}, c_{13}] \iff x_2 \in [c_{21}, c_{23}].\]

PROOF. Consider the set 

\[K_j(s) := \{ x_j \in \mathbb{R} \cup \{\infty\} : Q(x,s) = 0 \text{ and } x_j \in [c_{j1}, c_{j3}] \}. \]

It must be shown $K_j(s) = [c_{j1}, c_{j3}]$. First observe 

(i) \[K_j(s) \supseteq \{ x_j \in \mathbb{R} \cup \{\infty\} : D_j(x_j, s) \geq 0 \}. \]

This follows from the fact that $Q(x,s) = 0$ implies: 

\[ x_j \in \mathbb{R} \cup \{\infty\} \& D_j(x_j, s) \geq 0 \Rightarrow x_j \in \mathbb{R} \cup \{\infty\} \& D_j(x_j, s) \geq 0. \]

Secondly we show 

(ii) \[K_j(s) \text{ is a connected subset of } \mathbb{R} \text{ (hence a real interval).} \]

Notice $K_j(s) = \bigcup_{\lambda = \pm} K_j^\lambda(s)$ where \[K_j^\lambda(s) := \{-\lambda / \lambda \in \mathbb{R} \cup \{\infty\} \text{ and } K_j^\lambda(s) = \{ x_j \in \mathbb{R} \cup \{\infty\} : D_j(x_j, s) \geq 0 \}. \]

The sets $K_j^\lambda(s)$ are connected subsets of $\mathbb{R}$ with nonempty intersection, for \( \{d_{j1}, d_{j3}\} \cap \bigcap_{\lambda = \pm} K_j^\lambda(s) \text{ because of } \frac{f_j^\lambda(c_{z_k})}{d_{jk}} = d_{jk} \text{ for } k = 1, 3. \)

This implies (ii).

Finally observe $c_{j1} \in K_j^-(s)$ and $c_{j3} \in K_j^+(s)$, cf. (53) and (44), hence 

(iii) \[\{c_{j1}, c_{j3}\} \subseteq K_j(s).\]

From (20) it follows that $[c_{j1}, c_{j3}]$ is the unique real interval that contains $\{c_{j1}, c_{j3}\}$ and is contained in the set on the right in (i); so (i), (ii), and (iii) imply $K_j(s) = [c_{j1}, c_{j3}]$, q.e.d. \[\square\]
V.3 Skipfree, all-sided, driftless, even random walk. A special form for the nondegenerate biquadratic equation $Q(x,s) = 0$

V.3.1 Skipfree, all-sided, driftless, even random walk

In this section we consider the random walk which is studied in Section II.1; so we assume that the walk is skipfree, all-sided, driftless, and even (cf. Sec. I.1, (4) through (7)). We also assume that the probability that the step has the size 0 vanishes. From Section II.1 we know

\begin{equation}
Q(x,s) = s \frac{1+\rho}{4} (x_1^2 x_2^2 + 1) + s \frac{1-\rho}{4} (x_1^2 + x_2^2) - x_1 x_2,
\end{equation}

where $\rho$ is the correlation coefficient of the two coordinates of the step, and (because of all-sidedness)

\begin{equation}
\rho \in (-1,1).
\end{equation}

From Theorem V.1.1 we know that the polynomial $x \mapsto Q(x,s)$ is degenerate if $s \in \{0,1\}$, and nondegenerate if $s \in (0,1)$. The case $s = 1$ is considered in Section II.1.

In the sequel we assume

\begin{equation}
s \in (0,1)
\end{equation}

The polynomial $Q(x,s)$ has the following symmetries:

\begin{equation}
\begin{cases}
Q(x,s) = Q(-x,s), \\
Q(x,s) = x_1^2 x_2^2 Q((x_1^{-1},x_2^{-1}),s), \\
Q(x,s) = Q((x_2,x_1),s).
\end{cases}
\end{equation}

As always $i$ and $j$ are indices with $i \in \{1,2\}$ and $j = 3 - i$.

We put

\begin{equation}
p(x_i) = s \frac{1+\rho}{2} x_i + s \frac{1-\rho}{2} x_i^{-1},
\end{equation}

and have

\begin{equation}
Q(x,s) = \frac{1}{2} x_i x_j (x_j p(x_i) + x_i^{-1} p(x_i^{-1}) - 2).
\end{equation}
The discriminant of $Q(x,s)$ as a quadratic polynomial in $x_j$ is

\begin{equation}
D_j(x_j,s) = x_j^2 \left[ 1 - p(x_j) p(x_j^{-1}) \right],
\end{equation}

which can be rewritten as follows

\begin{equation}
D_j(x_j,s) = x_j^2 \left[ 1 - s^2 \rho^2 - \frac{1}{4} s^2 (1 - \rho^2) (x_j^2 + x_j^{-2})^2 \right].
\end{equation}

Observe $D_1(\cdot) \equiv D_2(\cdot)$.

We consider the zeros of the discriminants. If $c$ is a zero of $D_j(\cdot, s)$, then $c \neq 0$ and the complete set of zeros is $\{c, -c, c^{-1}, -c^{-1}\}$. Because the zeros are real and distinct (Theorem V.2.3, Part III), precisely one of these zeros belongs to $(0,1)$. Let $c$ be this zero, so

\begin{equation}
D_j(c,s) = 0 \quad \text{and} \quad 0 < c < 1
\end{equation}

for $j = 1,2$. Using the notations from Theorem V.2.3 one has

\begin{equation}
c_{i1} = c, \quad c_{i2} = -c, \quad c_{i3} = c^{-1}, \quad c_{i4} = -c^{-1}.
\end{equation}

Let the constant $d$ be determined by

\begin{equation}
Q((c,d),s) = 0 \quad \text{or} \quad Q((d,c),s) = 0).
\end{equation}

Hence, with the notations V.2.(32), because of (4),

\begin{equation}
d_{j1} = d, \quad d_{j2} = -d, \quad d_{j3} = d^{-1}, \quad d_{j4} = -d^{-1}
\end{equation}

for $j = 1,2$. The cross ratio of the $c$-values is (see V.2.(29))

\begin{equation}
(c_{i1}; c_{i2}; c_{i3}; c_{i4}) = \left(\frac{1-c^2}{1+c^2}\right)^2,
\end{equation}

and that of the $d$-values

\begin{equation}
(d_{j2}; d_{j2}; d_{j3}; d_{j4}) = \left(\frac{1-d^2}{1+d^2}\right)^2.
\end{equation}
PROPERTY 1. If \( 0 < s < 1 \) and \( -1 < p < 1 \) then

\[
(15) \quad c = \frac{\sqrt{1-s^2}p^2 - \sqrt{1-s^2}}{s\sqrt{1-p^2}} = \frac{s\sqrt{1-p^2}}{\sqrt{1-s^2}p^2 + \sqrt{1-s^2}}.
\]

\[
(16) \quad d = \frac{\sqrt{1-s^2}p^2 + p\sqrt{1-s^2}}{\sqrt{1-p^2}} = \frac{\sqrt{1-p^2}}{\sqrt{1-s^2}p^2 - p\sqrt{1-s^2}}.
\]

**PROOF.** Assume

\[ (*) \quad 0 < s < 1 \text{ and } -1 < p < 1. \]

The value \( c \) is determined by (9), and the value \( d \) by (11). Apply (8) to conclude from (9)

\[
\left( \frac{c^{-1}+c}{2} \right)^2 = 1 - \frac{s^2p^2}{1-s^2}. \]

from which (15) follows. Next apply (9), (7) and (6) to conclude from (11) (cf. V.2.(33))

\[ d^{-1}p(c^{-1}) = 1 = d^{-1}p(c^{-1}). \]

Now substitute (15) to get (16).

\[ \square \]

**REMARK 1.** From (15) it follows easily

\[
(17) \quad \left( \frac{1-c^2}{1+c^2} \right)^2 = 1 - \frac{s^2}{1-s^2},
\]

and from (16)

\[
(18) \quad \left( \frac{1-d^2}{1+d^2} \right)^2 = \frac{p}{1-s^2} \frac{1-s^2}{1-s^2}. \]

Hence,

\[
(19) \quad s^2 = \frac{1 - \left( \frac{1-c^2}{1+c^2} \right)^2}{1 - \left( \frac{1-d^2}{1+d^2} \right)^2} = \frac{(d^{-1}+d)^2}{(c^{-1}+c)^2} - \frac{(c^{-1}-c)^2}{(d^{-1}+d)^2} - \frac{(d^{-1}-d)^2}{(c^{-1}+c)^2} = \frac{(d^{-1}+d)^2}{(c^{-1}+c)^2}. \]
(20) \[ \rho^2 = \left( \frac{1-d^2}{1+d^2} \right)^2 \left( \frac{1-c^2}{1+c^2} \right)^2. \]

**PROPERTY 2.** The mapping \((s, \rho) + (c, d)\) given by (15) and (16) is a bijection

\[
\{(s, \rho) : -1 < \rho < 0 < s < 1\} + \\
\{(c, d) : 0 < c < d < 1\}.
\]

The inverse of this mapping is given by

(21) \[ s = \frac{d^{-1} + d}{c^{-1} + c}, \quad \rho = \frac{d^{-1} - d}{c^{-1} + d} \left/ \frac{c^{-1} - c}{c^{-1} + c} \right. \]

**PROOF.** Consider the restrictions:

(*) \[-1 < \rho < 0 < s < 1\]

(**) \[0 < c < d < 1\]

We must show

\[ (*), (15) \& (16) \iff (**), (21). \]

To prove this we show:

(i) \((15) \& (16) \& (*) \Rightarrow (**),\)

(ii) \((21) \& (**) \Rightarrow (*),\)

(iii) \((15) \& (16) \iff (21), \text{under the assumptions} \,(*) \text{and} \,(**).\)

We show (i). From (15) and (*) it follows

\[ c > 0 \quad \text{and} \quad c^2 = \frac{\sqrt{1-s^2} \rho^2}{\sqrt{1-s^2}} - \frac{\sqrt{1-s^2}}{\sqrt{1-s^2}} < 1, \]

and from (16) and (*)

\[ d > 0 \quad \text{and} \quad d^2 = \frac{\sqrt{1-s^2} \rho^2}{\sqrt{1-s^2}} + \frac{\sqrt{1-s^2} \rho^2}{\sqrt{1-s^2}} \left/ \frac{\sqrt{1-s^2} \rho^2 - \rho \sqrt{1-s^2}} \right. \in (c^2, 1), \]
which implies (i).

We leave (ii) to the reader, and show (iii). One easily checks that under the assumptions (*) and (**) the following equivalences hold:

(15) $\iff$ (17), (16) $\iff$ (18), and (21) $\iff$ (19) & (20).

The claim now follows from the equivalence (17) & (18) $\iff$ (19) & (20).

\(\square\)

We notice that (21) implies

\[
\frac{s}{2} \frac{1+\rho}{1-\rho} = \frac{c-1}{c^2} - \frac{c^{-1}-1}{c^2} = \frac{c^{-1}d^{-1}}{c^2} - \frac{cd}{c^2}.
\]

Also one has from (21)

\[
\text{sgn } (d-1) = \text{sgn } \rho,
\]

and, writing \(d = d(s, \rho)\), one has from (16)

\[
d(s, -\rho) = d(s, \rho)^{-1}.
\]

\[\text{V.3.2 A special form for the nondegenerate biquadratic equation } Q(x, s) = 0\]

Let \(Q(x)\) be a biquadratic polynomial. We will say that it has standard form (with parameters \(s\) and \(\rho\)) if there exist a nonvanishing constant \(Q_0\) and constants \(s\) and \(\rho\) such that

\[
Q_0 \cdot Q(x) = s \frac{1+\rho}{4} (x_1^2 + x_2^2 + 1) + s \frac{1-\rho}{4} (x_1^2 + x_2^2) - x_1 x_2.
\]

So, apart from a constant factor, \(Q(x)\) coincides with the polynomial \(Q(x, s)\) in (1).

**Proposition 3.** Let \(c\) and \(d\) be finite constants satisfying \(c \neq 0 \neq d\) and \(c^4 \neq 1 \neq d^4\).

Let \(Q(x)\), where \(x = (x_1, x_2)\), be a biquadratic polynomial (not vanishing identically). For \(i = 1, 2\) let \(D_i(x)\) be the discriminant of \(Q(x)\) as a quadratic polynomial in \(x_i\).

Assume
\[ Q(x) = 0 = D_1(x_1) \text{ if } x \in \{(\lambda c^\mu, \lambda d^\mu) : \lambda, \mu = \pm 1\}. \]

Then \(Q(x)\) has standard form, with parameters \(s\) and \(p\) satisfying (21).

**Proof.** For the coefficients in \(Q(x)\) we use the following notation:

\[ x_1^{-1} x_2^{-1} Q(x) = \gamma(x_1) x_2^{-1} + \beta(x_1) + \alpha(x_1) x_2 \]

with

\[
\begin{align*}
\alpha(x_1) &= \alpha_{-1} x_1^{-1} + \alpha_0 + \alpha_1 x_1, \\
\beta(x_1) &= \beta_{-1} x_1^{-1} + \beta_0 + \beta_1 x_1, \\
\gamma(x_1) &= \gamma_{-1} x_1^{-1} + \gamma_0 + \gamma_1 x_1.
\end{align*}
\]

If \(x_1 \neq 0 \neq x_2\) then the condition \(Q(x) = 0 = D_1(x_1)\) is equivalent to

\[ x_2^{-1} \gamma(x_1) = -\frac{1}{2} \beta(x_1) = x_2 \alpha(x_1). \]

So substitution of \(x = (\lambda c^\mu, \lambda d^\mu)\), \(\lambda = \pm 1\), \(\mu = \pm 1\), yields the following eight linear equations in the unknown coefficients:

\[
\begin{align*}
(i) \quad d^{-1} \gamma(c) &= -\frac{1}{2} \beta(c) = d \alpha(c), \\
(ii) \quad -d^{-1} \gamma(-c) &= -\frac{1}{2} \beta(-c) = -d \alpha(-c), \\
(iii) \quad d \gamma(c^{-1}) &= -\frac{1}{2} \beta(c^{-1}) = d^{-1} \alpha(c^{-1}), \\
(iv) \quad -d \gamma(-c^{-1}) &= -\frac{1}{2} \beta(-c^{-1}) = -d^{-1} \alpha(-c^{-1}).
\end{align*}
\]

These equations will be solved for the coefficients \(\alpha_k, \beta_k, \gamma_k\).

Subtract (ii) from (i) to find

\[ 2d^{-1} \gamma_0 = -\beta_{-1} c^{-1} - \beta_1 c = 2d \alpha_0, \]

and subtract (iv) from (iii) to find

\[ 2d \gamma_0 = -\beta_{-1} c - \beta_1 c^{-1} = 2d^{-1} \alpha_0. \]

It follows \(\gamma_0 = 0 = \alpha_0\) because of \(d^4 \neq 1\), and \(\beta_{-1} = 0 = \beta_1\) because
of $c^4 \neq 1$.

Add (i) and (ii) to find

$$2d^{-1}(\gamma_{-1}c^{-1} + \gamma_1c) = -\beta_0 = 2d(\alpha_{-1}c^{-1} + \alpha_1c),$$

and add (iii) and (iv) to find

$$2d(\gamma_{-1}c + \gamma_1c^{-1}) = -\beta_0 = 2d^{-1}(\alpha_{-1}c + \alpha_1c^{-1}).$$

It follows

$$\gamma_{-1} = \alpha_1 = -\frac{1}{2} \beta_0 \frac{c^{-1}d - cd^{-1}}{c^{-2} - c^2},$$

$$\gamma_1 = \alpha_{-1} = -\frac{1}{2} \beta_0 \frac{c^{-1}d^{-1} - cd}{c^{-2} - c^2},$$

hence $\beta_0 \neq 0$. It follows that $Q(\cdot, s)$ satisfies (25) with $Q_0 = -\beta_0^{-1}$, and $s$ and $p$ satisfying (22), hence satisfying (21).

V.3.3 Transformation of the general nondegenerate biquadratic equation

$Q(x,s) = 0$ into the special form

Let $T_i$ for $i = 1, 2$ be a nonconstant fractional linear transformation, say

$$\dot{x}_i = T_i(x_i) := \frac{a_i x_i + b_i}{c_i x_i + d_i}, \quad \alpha_i = c_i, \quad \beta_i = d_i, \quad \gamma_i, \delta_i \neq 0.$$ (26)

Observe that $T_i$ is a bijection of the complex sphere onto itself, with inverse

$$x_i = T_i^{-1}(\dot{x}_i) = \frac{\delta_i \dot{x}_i - \beta_i}{\gamma_i \dot{x}_i + \alpha_i}. \quad (26')$$

We put $x := (x_1, x_2)$ (as usual), and $\dot{x} := (\dot{x}_1, \dot{x}_2)$ and

$$T := (T_1, T_2), \quad \text{so} \quad \dot{x} = T(x) = (T_1(x_1), T_2(x_2)).$$ (27)

If $Q(x)$ is a biquadratic polynomial (in $x$) then the polynomial $\dot{Q}(\dot{x})$, defined by
\[ \dot{Q}(\dot{x}) := (-\gamma_1 \dot{x}_1 + \alpha_1)^2 (-\gamma_2 \dot{x}_2 + \alpha_2)^2 Q(T^{-1}(\dot{x})), \]

also is biquadratic (in \( \dot{x} \)).

**PROPERTY 4.** For \( i = 1, 2 \) let \( T_i \) be a nonconstant fractional linear transformation, say the one in (26).

Let \( Q(x) \) be an arbitrary biquadratic polynomial (not vanishing identically). Let \( \dot{Q}(\dot{x}) \) be the biquadratic polynomial given in (28).

I. Then

\[ Q(x) = 0 \iff \dot{Q}(\dot{x}) = 0. \]

II. Let \( D_1(x_1), D_2(x_2) \) be the two discriminants of \( Q(x) \), and \( \dot{D}_1(\dot{x}_1), \dot{D}_2(\dot{x}_2) \) the two discriminants of \( \dot{Q}(\dot{x}) \). Then for \( i = 1, 2 \)

\[ D_i(x_i) = 0 \iff \dot{D}_i(\dot{x}_i) = 0. \]

III. \( Q(x) \) is degenerate iff \( \dot{Q}(\dot{x}) \) is degenerate.

**PROOF.** Part I. This is trivial.

Part II. We take \( i = 1 \), and show the "only if"-claim.

Assume \( D_1(c) = 0 \). Then the discriminant of the quadratic form \( x_2 + Q(c, x_2) \) vanishes, so \( Q(c_1, x_2) \) has a unique zero, say \( x_2 = d \).

Put \((\dot{c}, \dot{d}) := T(c, d)\), cf. (27).

Because of (29) and the bijectivity of \( T(\cdot) \), also the quadratic function \( \dot{x}_2 + \dot{Q}(\dot{c}, \dot{x}_2) \) has a unique zero, viz. \( \dot{x}_2 = \dot{d} \). Hence the discriminant \( \dot{D}_1(\dot{c}) \) vanishes.

Part III. This is a trivial consequence of (29) and (30), see V.1.(12) ff.

**THEOREM 5.** (Transformation of the biquadratic form to the standard form)

Consider the biquadratic form \( Q(x, s) \) for fixed \( s \), see V.1.(4) and V.1.(6). Under the assumptions of Theorem V.2.3 the following holds.

I. The cross ratio \((c_{i1}, c_{i2}, c_{i3}, c_{i4})\) does not depend on \( i \in \{1, 2\} \).

Define the value \( c \) by means of
(31) \[ 0 < c < 1 \quad \text{and} \quad \left( \frac{1-c^2}{1+c^2} \right)^2 = \left( c_{i1}, c_{i2}, c_{i3}, c_{i4} \right) \]

for \( i = 1 \) or \( 2 \) (which is possible by Corollary V.2.5).

Define the fractional linear transformation \( T_i \) for \( i = 1, 2 \) by

(32) \[ (T_i(x_i), -c; c^{-1}, -c^{-1}) = (x_i, c_{i2}; c_{i3}, c_{i4}). \]

II. Then the value \( T_i(d_i) \) does not depend on \( i \).

Define the constant \( d \) by means of

(33) \[ d = T_i(d_{i1}) \]

for \( i = 1, 2 \). Define the biquadratic polynomial \( \hat{q}(x) \) by means of (28), cf. (26) and (27).

III. Then

(34) \[ c < d < c^{-1}, \]

and the polynomial \( \hat{q}(x) \) has standard form with parameters \( \hat{s} \in (0, 1) \) and \( \hat{p} \in (-1, 1) \) given by

(35) \[ \hat{s} = \frac{d^{-1}+d}{c^{-1}+c}, \quad \hat{p} = \frac{d_{i3} - d_{i4}}{c_{i3} - c_{i4}}. \]

Furthermore,

(36) \[ T_i \text{ maps } d_{i2}, d_{i3}, d_{i4} \text{ on } -d, d^{-1}, -d^{-1}, \text{ respectively}, \]

for \( i = 1, 2 \).

PROOF. Here we prove the theorem only under the additional assumption:

(37) \[ \text{the four numbers } d_{2k}, k = 1, \ldots, 4, \text{ are distinct.} \]

Once the theorem is proved under the restriction (37) one can get rid of this restriction by means of a continuity argument; this is left to the reader.

We prove the three parts simultaneously.
First define the constant \( c \) by means of (31) for \( i = 1 \) (which is possible by Corollary V.2.5), and the fractional linear transformation \( T_1 \) by means of (32) for \( i = 1 \).
Next let \( d \) be a constant satisfying

\[
(i) \quad \left(\frac{1-d^2}{1+d^2}\right)^2 = (d_{21}, d_{22}; d_{23}, d_{24}).
\]

Because of the assumption (37), the cross ratio on the right has a value distinct from 0, 1, \( \infty \); so \( d \notin (0, \infty) \) and \( d^4 \neq 1 \). Hence there are four distinct possible values for \( d \in \mathbb{C} \) (with \( d \) also the values \( \pm d^{-1} \) satisfy \( (i) \)). A choice will be made later on.
Let the fractional linear transformation \( T_2 \) be defined by

\[
(ii) \quad (T_2(x_2), -d; d^{-1}, -d^{-1}) = (x_2, d_{22}; d_{23}, d_{24}).
\]

We put \( T := (T_1, T_2) \), cf. (27), and write for convenience

\[
c_1 := c, c_2 := -c, c_3 := c^{-1}, c_4 := -c^{-1},
\]
\[
d_1 := d, d_2 := -d, d_3 := d^{-1}, d_4 := -d^{-1}.
\]

Hence (cf. Property III.1.15 (Part II), and apply (31) and (i))

\[
(iii) \quad T \text{ maps } (c_{1k}, d_{2k}) \text{ on } (c_k, d_k) \text{ (for } k = 1, \ldots, 4). \]

So (33) and (36) are satisfied for \( i = 2 \).
Through the relationship \( \dot{x} := T(x) \) the equation \( Q(x, s) = 0 \) is equivalent to a biquadratic equation \( \dot{Q}(\dot{x}) = 0 \), see (26) through (28), with discriminants \( \dot{D}_1(\dot{x}_1) \) and \( \dot{D}_2(\dot{x}_2) \). Apply Property 4 to conclude from (iii)

\[
(+) \quad \dot{Q}(\dot{x}) = 0 = \dot{D}_1(\dot{x}_1) \iff \dot{x} = (c_k, d_k) \text{ for some } k.
\]

Due to Proposition 3 this implies that \( \dot{Q}(\dot{x}) \) has standard form with parameters \( \dot{s}, \dot{p} \) satisfying (35).
Recall \( \dot{Q}(\dot{x}_1, \dot{x}_2) = \dot{Q}(\dot{x}_2, \dot{x}_1) \) and \( \dot{D}_1(\cdot) = \dot{D}_2(\cdot) \).
So (\(+) \text{ is identical with } (++)\):

\[
(++) \quad \dot{Q}(\dot{x}) = 0 = \dot{D}_2(\dot{x}_2) \iff \dot{x} = (\dot{a}_k, \dot{c}_k) \text{ for some } k.
\]
Recall \( Q(x) = 0 = D_2(x_2) \) iff \( x = (d_{1k}, c_{2k}) \) for some \( k \), cf. Theorem V.2.3 (Part III) and V.2.(32). Now apply Property 4 again to conclude

\[ (*) \quad T \text{ maps the set } \{(d_{1k}, c_{2k}) : k = 1, \ldots, 4\} \]

onto the set \( \{(d_k, c_k) : k = 1, \ldots, 4\} \).

In particular conclude that \( T_1(d_{11}) \in \{d_k : k = 1, \ldots, 4\} \).

Hence we may and will choose

\[ (iv) \quad d := T_1(d_{11}) \]

So (33) is satisfied for \( \iota = 1 \).

Next we show (34). From (32) one derives by brute force (putting \( \dot{x}_1 := T_1(x_1) \))

\[
\frac{2c}{(1+c^2)} \frac{1-c^2}{(1+\dot{x}_1c)} \frac{d\dot{x}_1}{dx} = \frac{c_{11}^{-1} - c_{14}^{-1}}{(1-c_{14}^{-1})^2} \frac{1-c_{12}c_{14}^{-1}}{(1+c^2)^2},
\]

hence \( T_1'(x_1) > 0 \) (cf. the proof of Corollary V.2.5 and V.2.(17B)), so

\[ (38) \quad T_1(\cdot) \text{ is increasing.} \]

Because \( T_1 \) maps \( c_{11} \) on \( c \) and \( c_{13} \) on \( c^{-1} \), it follows that

\[ (**) \quad T_1 \text{ maps the interval } (c_{11}, c_{13}) \text{ onto the interval } (c, c^{-1}). \]

This implies (34) because of V.2.(44). Observe that \( 0 < c < 1 \) and (34) imply \( 0 < s < 1 \) and \( -1 < \hat{p} < 1 \), cf. Property 2 and (24).

We now prove (31) and (32) for \( \iota = 2 \), and (36) for \( \iota = 1 \), by proving the following extension of (*)

\[ (v) \quad T \text{ maps } (d_{1k}, c_{2k}) \text{ on } (d_k, c_k) \text{ for } k = 1, \ldots, 4. \]

Because of (iv) one has from (*) \( T_2(c_{21}) = c \). From V.2.(44) and (**) one has \( c < T_1(d_{13}) < c^{-1} \), so \( T_1(d_{13}) = d_{13}^{-1} \), hence from (*) \( T_2(c_{23}) = c^{-1} \).

So we have shown \( T(d_{1k}, c_{2k}) = (d_k, c_k) \) for \( k = 1, 3 \).

We show that the assumption

\[ (a) \quad T_2(c_{22}) = -c^{-1} \]
leads to a contradiction. Assume (a). Then necessarily $T_2(c_{24}) = -c$ because of (*). From V.2.(30), and because $T_2$ preserves cross ratios, and in view of Property III.1.15 (Part IV), one has

\[
0 < (c_{21}, c_{22}; c_{23}, c_{24}) = (c_1, c_4; c_3, c_2) = [1 - (c_1, c_2; c_3, c_4)^{-1}]^{-1} < 0.
\]

(30)

which is a contradiction. So the assumption (a) is false, and one has from (*), necessarily,

\[
T(d_{1k}, c_{2k}) = (d_k, c_k) \text{ for } k = 2, 4.
\]

This proves (v).

Finally Part I is a consequence of (31) for $i = 1, 2$.

\[ \square \]

Theorem 5 has the following corollary.

**Corollary 6.** Under the assumptions of Theorem V.2.3 the following holds.

Let $c$ be given by (31), and $d$ by (33) and (32).

I. If $c < d < 1$ then $d_{i1} < d_{i3}$ and $d_{i2} > d_{i4}$ for $i = 1, 2$.
   If $d = 1$ then $d_{i1} = d_{i3}$ and $d_{i2} = d_{i4}$ for $i = 1, 2$.
   If $1 < d < c^{-1}$ then $d_{i1} > d_{i3}$ and $d_{i2} < d_{i4}$ for $i = 1, 2$.

II. If $d = 1$ then $(d_{i1}, d_{i2}; d_{i3}, d_{i4}) = 0$ else
    \[ 0 < (d_{i1}, d_{i2}; d_{i3}, d_{i4}) < (c_{i1}, c_{i2}; c_{i3}, c_{i4}). \]

III. The cross ratio $(d_{i1}, d_{i2}; d_{i3}, d_{i4})$ does not depend on $i \in \{1, 2\}$.

**Proof.** Part I follows from (33) and (36), because for $i = 1, 2$ $T_\varepsilon(\cdot')$ is increasing, cf. (38).

Part II. By V.2.(37-38) the cross ratio is well-defined. From (33) and (36) it follows for $i = 1, 2$

\[
\left(\frac{1-d^2}{1+d^2}\right)^2 = (d_{i1}, d_{i2}; d_{i3}, d_{i4}),
\]

(39)

because $T_\varepsilon$ preserves cross ratios. The claim follows from (31), (34) and (39).

Part III. From (39) for $i = 1, 2$.

\[ \square \]
V.4 Uniformization of the nondegenerate biquadratic equation

\[ Q(x, s) = 0 \]

Throughout this section our assumptions are those of Theorem V.2.3, viz.

\[
\begin{cases}
\xi \text{ (see V.1.(1)) is skipfree and all-sided,} \\
s \in (0,1) , \\
(s, \text{E}\xi) \neq (1, (0,0)).
\end{cases}
\]

Under these assumptions the polynomial \( x \sim Q(x, s) \) (cf. V.1.(6)) is nondegenerate, see Theorem V.1.1 (Part I). In this section we will for fixed \( s \) uniformize the curve \( K(s) \) (cf. V.1.7), i.e. parametrize the curve by means of meromorphic functions. In fact two parametrizations will be constructed, viz. for \( i = 1, 2 \)

\[(\dag) \quad x_i^\prime = f_i^\prime(v) , \quad x_j^\prime = h_j^\prime(v)\]

where \( j = 3-i \). These two parametrizations turn out to be essentially the same (cf. Theorem 7). The functions involved turn out to be second order elliptic. Furthermore, the mapping \( v \in \mathbb{C} \sim x \in (\mathbb{C})^2 \) given by (\dag) for fixed \( i \) and \( j \), where

\[(1) \quad \mathbb{C} \text{ denotes the complex sphere,}\]

turns out to be a surjective and locally injective mapping from \( \mathbb{C} \) onto \( K(s) \) (see (23) and Theorem 2).

V.4.1 Introduction of the functions \( f_1 \) and \( f_2 \) and some constants

We start with defining two second order elliptic, even functions \( f_1 \) and \( f_2 \). In Section V.2 for \( i = 1 \) and 2 the constants \( c_{i1}, c_{i2}, c_{i3}, c_{i4} \) are defined as the zeros of \( D_i(x_i, s) \), which is (cf. V.1.(10)) the discriminant of \( Q(x, s) \) considered as a quadratic polynomial in \( x_i \) (where \( j := 3-i \)). The numbering and properties of these zeros are described in Theorem V.2.3. For their cross ratio \( (c_{i1}, c_{i2}; c_{i3}, c_{i4}) \) see V.2.(29).
We define the constants $\tau_1$ and $\tau_2$ by

$$(2) \quad \tau_i \in i\mathbb{R}, \quad \text{Im} \, \tau_i > 0, \quad m(\tau_i) = (c_{i1}, c_{i2}, c_{i3}, c_{i4})$$

for $i = 1$ and 2. The function $m(\tau)$ is given in III.1.(18).
Such constants exist, and are uniquely determined by (2), because of the Corollaries III.1.22 and V.2.5. Below (see Corollary 5) we will prove $\tau_1 = \tau_2$, and this results (see Remark 5) in another proof of Theorem V.3.5 (Part I).

We define the functions $f_1(\cdot)$ and $f_2(\cdot)$ by

$$(3) \quad m(\nu|\tau_i) = (\xi_i(\nu), c_{i2}, c_{i3}, c_{i4})$$

for $i = 1$ and 2, where $m(\cdot|\tau)$ is the standard second order elliptic, even function defined in Sec. III.1.(9)...(12). This defines $\xi_i(\cdot)$ as the second order elliptic, even function with

$$(4) \quad \Omega(\tau_i) := \mathbb{Z} + i\tau_i \mathbb{Z}$$

as its complete period lattice, and $c_{i1}, c_{i2}, c_{i3}, c_{i4}$ as its second order values, in such a way that

$$(5) \quad f_1(0) = c_{i1}, \quad f_2(\frac{1}{2}) = c_{i2}, \quad f_2(\frac{1}{2} \tau_i) = c_{i3}, \quad f_2(\frac{1}{2} + \frac{1}{2} \tau_i) = c_{i4},$$

cf. the discussion following Corollary III.1.18.

For future use we define the constants $\sigma_1$ and $\sigma_2$. The constants $d_{i1}, d_{i2}, d_{i3},$ and $d_{i4}$ are defined in V.2.(32). From V.2.(44) we know $d_{i1} \in (c_{i1}, c_{i3})$, and it will be shown in the next section (see V.5.(16))

$$(*) \quad t \sim f_2(t \tau_i) \text{ is an increasing bijection } (0, \frac{1}{2}) \sim (c_{i1}, c_{i3}).$$

Therefore the following defines the constants $\sigma_1$ and $\sigma_2$ uniquely:

$$(6) \quad \sigma_i \in (0, \frac{1}{2}) \quad \text{and} \quad f_2(\frac{\sigma_i}{2} \tau_i) = d_{i1}$$

for $i = 1, 2$. Below (see Proposition 6) it will be shown that $\sigma_1 = \sigma_2$. 
For future use we also note that (*) implies

\[(***) \quad \frac{d}{dt} f_i(t \tau_i) > 0 \quad \text{for} \quad 0 < t < \frac{1}{2}, \]

hence,

\[(7) \quad \tau_i f_i'(\sigma_i \tau_i) > 0 \]

for \(i = 1, 2\).

The function \(f_i(\cdot)\) has the following property.

**Theorem 1.** Fix \(i \in \{1, 2\}\). Let \(c\) be a complex constant (possibly infinity), and let \(\gamma\) satisfy \(f_i(\gamma) = c\). Then the two functions of \(v\)

\[(8) \quad x_i = f_i(\lambda v + \gamma), \quad \lambda \in \{+, -\} \]

are the only non-constant solutions of the differential equation:

\[(9) \quad \begin{cases} \begin{array}{l} x_i \text{ is a meromorphic function of } v \text{ on some neighbourhood } U(0) \text{ of the origin,} \\ x_i = c \text{ for } v = 0, \\ \left(\frac{dx_i}{dv}\right)^2 = \omega_i^2 D_i(x_i, s) \text{ for } v \in U(0) \setminus \{0\} \text{ (i.e. in a reduced neighbourhood of the origin),} \end{array} \end{cases} \]

where \(\omega_i\) is a constant satisfying

\[(10) \quad \omega_i^2 = \frac{-\omega_0(\tau_i)^2}{D_{i0} \left(1 - c_i 1 c_i^{-1}\right) \left(1 - c_i 2 c_i^{-3}\right)}. \]

For the function \(\omega_0(\cdot)\) see III.1.(19), for the constant \(D_{i0}\) see V.2.(26)-(27).

**Proof.** See Theorem III.1.19.

**Remark 1.** Apply the substitutions
\[ \tilde{x}_\zeta := \frac{1}{x_\zeta} \quad \text{and} \quad \tilde{D}_\zeta(\tilde{x}_\zeta, s) := \tilde{x}_\zeta^{-4} D_\zeta(x_\zeta^{-1}, s) \]

to see that the set of conditions (9) is equivalent to the set (12):

\[\begin{align*}
\tilde{x}_\zeta & \quad \text{is a meromorphic function of } v \text{ on some neighbourhood} \\
\tilde{x}_\zeta & = c^{-1} \quad \text{for } v = 0 , \\
\left( \frac{d \tilde{x}_\zeta}{dv} \right)^2 & = \omega_\zeta^2 \tilde{D}_\zeta(\tilde{x}_\zeta, s) \quad \text{for } v \text{ in a reduced neighbourhood} \\
& \quad \text{of the origin.}
\end{align*}\]

**Remark 2.** The relationship (10) determines \( \omega_\zeta \) up to the sign.

Notice \( \omega_0(\zeta)^2 > 0 \) (from (2) and III.1.(19)). Also notice that all factors in the denominator on the right in (10) are positive (see V.2.(27) and the proof of Corollary V.2.5). Hence,

\[ -\infty < \omega_\zeta^2 < 0 . \]

We fix the sign of \( \omega_\zeta \) by making a rather arbitrary choice, viz.

\[ \text{Im } \omega_\zeta > 0 . \]

Below we will show \( \omega_1 = \omega_2 \) (see Corollary 5).

**Remark 3.** In the sequel we will need the inequality

\[ f_\zeta(0) < 0 . \]

To see this, in (9) take \( c := c_1 \zeta \) and \( \gamma := 0 \) (cf. (5)). Differentiate the two sides of the differential equation in (9) with respect to \( v \), divide both sides by \( \frac{dx_\zeta}{dv} \), and substitute \( v := 0 \). The result is

\[ 2 f_\zeta(0) = \omega_\zeta^2 D_\zeta(c_1, s) \]

(where the prime denotes differentiation of \( D_\zeta(x_\zeta, s) \) with respect to \( x_\zeta \)), which implies (15) because of (13) and V.2.(50).
V.4.2 Construction of two uniformizations of the biquadratic equation

\[ Q(x, s) = 0 \]

In the sequel fix \( s \), and fix \( \iota \in \{1, 2\} \) and \( j = 3 - \iota \).

In the equation \( Q(x, s) = 0 \) substitute

(17) \[ x_\iota = f_\iota(v) \]

and solve the equation \( Q(x, s) = 0 \) for \( x_\iota \). We define \( h_\iota(v) \) to be one of the two possible values of \( x_\iota \) as follows, using the notations \( V.1.(6)-(10) \). Theorem 1 implies that the two square roots of the discriminant \( D_\iota(x_\iota, s) \) are \( \pm \omega^{-1}_\iota f_\iota'(v) \), and we define:

if \( x_\iota \) is finite then

(18) \[ h_\iota(v) := \frac{x_\iota - s b_\iota(x_\iota) + \omega^{-1}_\iota f_\iota'(v)}{2 s a_\iota(x_\iota)} = \frac{2 s c_\iota(x_\iota)}{x_\iota - s b_\iota(x_\iota) - \omega^{-1}_\iota f_\iota'(v)} \]

and if \( x_\iota^{-1} \) is finite then (cf. Remark 1)

(19) \[ h_\iota(v) := \frac{x_\iota^{-1} - s x_\iota^{-2} b_\iota(x_\iota) - \omega^{-1}_\iota f_\iota'(v)}{2 s x_\iota^{-2} a_\iota(x_\iota)} = \frac{2 s x_\iota^{-2} c_\iota(x_\iota)}{x_\iota^{-1} - s x_\iota^{-2} b_\iota(x_\iota) + \omega^{-1}_\iota f_\iota'(v)} \]

where

(20) \[ \tilde{f}_\iota(v) := \frac{1}{f_\iota(v)} \]

Obviously, \( (18) \) and \( (19) \) coincide for \( 0 < |x_\iota| < \infty \). We note that at least one of the two quotients on the right in \( (18) \) is well-defined (i.e. the numerator and the denominator do not vanish simultaneously) because of all-sidedness, cf. Property I.3.12. A similar remark applies to the quotients in \( (19) \). It follows that

(21) \[ h_\iota(\cdot) \text{ is a meromorphic, non-constant function} \]

with periods \( \Omega(\tau_\iota) \) (hence elliptic)

(in order to see that \( h_\iota(\cdot) \) is non-constant, e.g. observe that \( h_\iota(0) = d_{\iota 1} > 0 \) and \( h_\iota(\frac{1}{2}) = d_{\iota 2} \leq 0 \); cf. \( (5) \), \( V.2.(32)-(37)-(38) \)).

\textbf{REMARK 4.} Because \( f_\iota'(\cdot) \) is odd, the two square roots of the discriminant \( D_s(f_\iota(v), s) \) are \( \omega^{-1}_\iota f_\iota'(\pm v) \). Hence, from the definition of \( h_\iota(\cdot) \) it follows that for \( \iota = 1, 2 \) and \( j = 3 - \iota \):
(22) \[ \text{if } x_i = f_i(v) \quad \text{(hence, } x_i = f_i(-v)) \]
then: \[ Q(x,s) = 0 \iff x_j \in \{ h_j(v), h_j(-v) \} \]

We define the functions \( \Phi_1 \) and \( \Phi_2 : \mathbb{C} \sim (\mathbb{C}^2) \) (cf. (1)) by means of

(23) \[ \Phi_i(v) = (x_1, x_2) \quad \text{with} \quad x_i = f_i(v), \quad x_j = h_j(v) \]
for \( i = 1,2 \) and \( j = 3-i \), or

(24) \[ \Phi_1(v) = (f_1(v), h_2(v)) \quad \text{and} \quad \Phi_2(v) = (h_1(v), f_2(v)) \]

Notice that

(25) \[ \Omega(\tau) \] is the complete period set of \( \Phi_i(\cdot) \), because it is the complete period set of \( f_i(\cdot) \). We put (in agreement with V.1.(7))

(26) \[ K(s) := \{ x \in (\mathbb{C}^2) : Q(x,s) = 0 \} \]

and have

**Theorem 2.** For \( i = 1,2 \) the mapping \( \Phi_i \) is a bijection \( \mathbb{C}/\Omega(\tau) \sim K(s) \).

**Proof.**

By construction, \( \Phi_i(v) \in K(s) \) for all \( v \).

To prove surjectivity, take an \( x \in K(s) \). Then \( x_i = f_i(v) \) for some \( v \).

Hence from (22) \( x \in \{ \Phi_i(v), \Phi_i(-v) \} \). To prove injectivity, suppose \( \Phi_i(v) = \Phi_i(w) \). It follows from (17) and (18):

\[ \begin{align*}
\text{(*) } & \quad \text{if } f_i(v) \text{ is finite then } (f_i(v), f_i'(v)) = (f_i(w), f_i'(w)), \\
\text{(**) } & \quad \text{if } f_i(v)^{-1} \text{ is finite then } (f_i(v), f_i'(v)) = (f_i(w), f_i'(w)).
\end{align*} \]

Because of Proposition III.1.8 it follows from (*) or (**) that \( v-w \in \Omega(\tau) \).

\( \Box \)
Theorem 2 implies that both \( \Phi_1 \) and \( \Phi_2 \) are surjective and locally injective uniformizations of the curve \( Q(x,s) = 0 \) for fixed \( s \).

V.4.3 The relationship between the two uniformizations

The function \( h_j(\cdot) \) has been expressed (see (18)-(19)) in the function \( f_j(\cdot) \). Our next goal is to express \( h_j(\cdot) \) in the function \( f_j(\cdot) \).

The first step is the following corollary to Theorem 2.

**Corollary 3.** The complete period set of \( h_j(\cdot) \) is \( \Omega(\tau_j) \).

**Proof.** Because of (21) we only need to prove completeness. Because of the bijectivity of \( \Phi_2(\cdot) \) it follows from the fact that \( K(s) \) contains only one point \( x \) satisfying \( x_j = c_{j_1} \), that \( h_j(\cdot) \) assumes the value \( c_{j_1} \) only once in \( \mathbb{C}/\Omega(\tau_j) \). Hence, if \( h_j(v) = c_{j_1} \) then \( \{ w \in \mathbb{C} : h_j(w) = c_{j_1} \} = v + \Omega(\tau_j) \). Now assume that \( \omega \) is a period of \( h_j \). Then \( h_j(v + \omega) = c_{j_1} \), hence \( v + \omega \in v + \Omega(\tau_j) \), or \( \omega \in \Omega(\tau_j) \), q.e.d.

**Proposition 4.** For \( i = 1,2 \) with \( j = 3-i \) the identity

\[
(27) \quad h_j(\frac{\omega_j}{\omega_i} v) = f_j(v - \nu_j \sigma_j \tau_j)
\]

holds, where \( \nu_j \) is a constant satisfying

\[
(28) \quad \nu_j \in \{-1, +1\}.
\]

**Proof.** Fix \( i \in \{1,2\} \), and \( j = 3-i \). We put \( x := \Phi_i(v) \), or

\[
(*) \quad x_i = \xi_i(v) \quad \text{and} \quad x_j = h_j(v).
\]

Observe that \( x \) is finite if \( v \) is in some reduced neighbourhood of the origin. From Property I.3.15 it follows

\[
\left( \frac{dx_i}{dv} \right)^2 D_i(x_i, s) = \left( \frac{dx_j}{dv} \right)^2 D_j(x_j, s)
\]

for all \( v \) in the above reduced neighbourhood of the origin. From Theorem 1 we know
\[
\left( \frac{dx_i}{dv} \right)^2 = \omega_i^2 D_i(x_i,s)
\]

for \( v \) in some reduced neighbourhood of the origin. Hence,

\[(**)
\left( \frac{dx_j}{dv} \right)^2 = \omega_j^2 D_j(x_j,s)
\]

for all \( v \) in some reduced neighbourhood of the origin. From \( f_\xi(0) = c_\xi \) (see (5)) it follows (cf. (22) and V.2.(32))

\[(***) x_j = d_j1 \quad \text{if} \quad v = 0.
\]

Putting \( H_j(v) := h_j(\frac{v}{\omega_j},\frac{\omega_j}{\omega_\xi} v) \) one obtains from \((*)\), \((**)\), and \((***)\)

\[
\begin{cases}
H_j(0) = d_j1, \\
H_j'(v)^2 = \omega_j^2 D_j(H_j(v),s) \quad \text{for} \ v \ \text{in a reduced neighbourhood of the origin.}
\end{cases}
\]

Now apply Theorem 1 (with \( \xi \) replaced by \( j \), and with \( c := d_j1 \) and \( \gamma := \sigma_j \gamma_j \)), cf. (6) and (21).

\[
\begin{align}
\omega_1 &= \omega_2 \\
\tau_1 &= \tau_2
\end{align}
\]

**COROLLARY 5.** The following equalities hold.

(29) \[\omega_1 = \omega_2\]

(30) \[\tau_1 = \tau_2\]

**PROOF.** Because \( \Omega(\tau_\xi) \) is the complete period lattice of \( h_\xi \) (cf. Corollary 3), and \( \Omega(\tau_j) \) that of \( f_j \) (cf. (3) et seq.), the identity (27) implies \( \frac{\omega_\xi}{\omega_j} \Omega(\tau_\xi) = \Omega(\tau_j) \), or \( \omega_1 \Omega(\tau_1) = \omega_2 \Omega(\tau_2) \). This equality implies (29) and (30), because of (2), (13) and (14).

\[
\begin{align}
\begin{cases}
c_{11}, c_{12}, c_{13}, c_{14} \\
c_{21}, c_{22}, c_{23}, c_{24}
\end{cases}
\end{align}
\]

**REMARK 5.** In view of (2) the equality (30) implies

\[(31)\]

Thus we have given a general proof of Theorem V.3.5 (Part I).
Because of (30) the constant $\tau$ can be defined as follows:

\[(32) \quad \tau := \tau_1 = \tau_2 \]

or (by (2))

\[(33) \quad \tau \in i\mathbb{R}, \quad \text{Im } \tau > 0, \quad m(0|\tau) = (c_{\tau_1}, c_{\tau_2}; c_{\tau_3}, c_{\tau_4}) \]

for $\tau = 1$, or (equivalently) for $\tau = 2$.

By (29) and (32) the identity (27) goes over in

\[(33A) \quad h_j(v) \equiv f_j(v - \mu_j \sigma_j \tau) \]

**PROPOSITION 6.** The following equalities hold.

\[(34) \quad \sigma_1 = \sigma_2 \]
\[(35) \quad \mu_1 = \mu_2 = +1 \]

**PROOF.** Proof of (34). Find a $v$ so that $\Phi_1(0) = \Phi_2(v)$, or (due to (24) and (33A))

\[f_1(0) = h_1(v) = f_1(v - \mu_1 \sigma_1 \tau), \]
\[f_2(v) = h_2(0) = f_2(-\mu_2 \sigma_2 \tau). \]

Consequently (cf. Proposition III.1.7),
\[v - \mu_1 \sigma_1 \tau \in \Omega(\tau), \]
\[v + \lambda \mu_2 \sigma_2 \tau \in \Omega(\tau) \]

for some $\lambda \in \{+,-\}$; hence
\[\mu_1 \sigma_1 + \lambda \mu_2 \sigma_2 \in \mathbb{Z} \]

This implies (34) because of (6).

Proof of (35). We compute $h_j'(0)$ in two ways. On the one hand it follows from (18) (differentiate with respect to $v$, take $v=0$, and use $f_\xi(0) = c_{\tau_1}$ and $f_\xi'(0) = 0$ because $f_\xi$ is even, cf. (3)-(5))

\[2s_\xi'(c_{\tau_1}) h_j'(0) = + \omega_\xi^{-1} f_\xi'(0). \]

On the other hand it follows from (33A)

\[h_j'(0) = f_j'(-\mu_j \sigma_j \tau) = -\mu_j f_j'(\sigma_j \tau). \]
Hence,

\[ 2 \sigma \vartheta \left( c_{j1} \right) \left( -u_j \right) \vartheta f_j' \left( \frac{c_{j1}}{\vartheta} \right) = \vartheta \omega_j^{-1} f_j''(0). \]

Observe that \( \sigma \vartheta \left( c_{j1} \right) \) is positive because of \( c_{j1} > 0 \) and all-sidedness; hence, because of (7) and (15),

\[ \text{sgn}(u_j) = \text{sgn}(\vartheta \omega_j^{-1}). \]

This implies the claim because of (13), (14) and (33).

\[ \Box \]

**Definition of the constant \( \sigma \)**: we put

\[ \sigma := \sigma_1 = \sigma_2, \]

or (from (6))

\[ \sigma \in \left( 0, \frac{1}{2} \right) \quad \text{and} \quad f_{j1}(\sigma \vartheta) = d_{j1} \]

for \( i = 1 \), or (equivalently) for \( i = 2 \).

**THEOREM 7.** Let \( h_1(*) \) and \( h_2(*) \) be defined by (17) through (20), and \( \Phi_1(*) \) and \( \Phi_2(*) \) by (24). Then the following identities hold for \( i = 1, 2 \) with \( j = 3 - i \).

\[ \begin{align*}
\text{(39)} & & h_{ij}(v) & = f_{ij}(v - \sigma \vartheta) \\
\text{(40)} & & \Phi_{ij}(v) & = \Phi_{ij}(-v + \sigma \vartheta)
\end{align*} \]

**PROOF.** The equality (39) follows from (33A) because of (35) and (37). The equality (40) follows from (39) (see the definition (23)), because \( f_{i1} \) and \( f_{i2} \) are even.

\[ \Box \]

**REMARK 6.** We show

\[ f_{j1}\left( \frac{1}{2} + \sigma \vartheta \right) = d_{j1} \]

as follows. Put \( x := \Phi_{j1}(\frac{1}{2}) \). Then \( x_j = f_{j1}(\frac{1}{2}) = c_{j2} \) (cf. (5)), hence \( x_j = d_{j2} \) (cf. V.2.(32)), so \( d_{j2} = h_{j1}(\frac{1}{2}) = f_{j1}(\frac{1}{2} - \sigma \vartheta) = f_{j1}(\frac{1}{2} + \sigma \vartheta) \)

because \( f_{j1}(\cdot) \) is even and has period 1. Similarly one shows
(42) \[ f_{\tau}^{i} \left( \frac{1}{2} \tau - \sigma \tau \right) = d_{\tau}^{i3} \]

(43) \[ f_{\tau}^{i} \left( \frac{1}{2} + \frac{1}{2} \tau - \sigma \tau \right) = d_{\tau}^{i4} \]

Observe that \( \sigma \) also is determined by the requirements \( 0 < \sigma < \frac{1}{2} \) and any one of the equalities (41), \ldots, (43), because \( f_{\tau}^{i}(\ast) \) is injective on the straight line segments \( (0, \frac{1}{2} \tau) \) and \( (\frac{1}{2}, \frac{1}{2} + \frac{1}{2} \tau) \), cf. V.5.(16)–(18).

**Theorem 8.** (Uniformization of the curve \( Q(x,s) = 0 \) for fixed \( s \in (0,1) \) in the nondegenerate case)

**Assume** (0). Let the constant \( \tau \) be defined by (33).

Let for \( i = 1, 2 \) the function \( f_{\tau}^{i} \) be defined by

(44) \[ m(v|\tau) = (f_{\tau}^{i}(v), c_{\tau}^{i2}; c_{\tau}^{i3}, c_{\tau}^{i4}) \]

Let the constant \( \sigma \) be defined by (38).

Let the set \( K(s) \) be defined by (26) and (1).

**Assume** \( \omega \in \Omega(\tau) \) (cf. (4)), \( \lambda_{1}, \lambda_{2} \in \{+,-\} \), and \( i \in \{1,2\} \) and \( j = 3 - i \).

**I.** Let \( z_{1} \) and \( z_{2} \) be related through

(45) \[ \lambda_{1}z_{1} + \lambda_{2}z_{2} = \sigma \tau + \omega \]

Let \( x := (x_{1}, x_{2}) \) and \( z := (z_{1}, z_{2}) \) be related through

(46) \[ (x_{1}, x_{2}) = (f_{\tau}^{i}(z_{1}), f_{\tau}^{i}(z_{2})) \]

Then the mapping \( z_{i} \sim x \) is a bijection \( \mathbb{C}/\Omega(\tau) \sim K(s) \).

**II.**

(47) \[ K(s) = \{ x \in (\mathbb{C})^{2} : \exists z \in \mathbb{C}^{2} \text{ such that (45) and (46) hold} \} \]

**III.** If \( x \in K(s) \) and \( x_{i} = f_{\tau}^{i}(v) \) then \( x_{j} \in \{ f_{j}^{i}(v - \sigma \tau), f_{j}^{i}(v + \sigma \tau) \} \).

**Proof.** Part I: Put \( v := \lambda_{1}z_{1}^{i} \), so \( v - \sigma \tau = -\lambda_{2}z_{2}^{j} + \omega \). Then \( x_{i} = f_{\tau}^{i}(z_{i}) = f_{\tau}^{j}(v) \) and \( x_{j} = f_{\tau}^{j}(z_{j}) = f_{\tau}^{j}(v - \sigma \tau) \) (because the \( f \)-functions are even and \( \omega \) is a period), so \( x_{j} = h_{\tau}^{j}(v) \) by (39), hence \( x = \Phi_{\tau}(v) \) by (23).

The claim now follows from Theorem 2.

**Part II:** Trivial from Part I.

**Part III:** From (22) and (39), because \( f_{j}(\ast) \) is even.

\( \Box \)
V.5 Descriptive properties of the elliptic functions $f_1$ and $f_2$

Throughout this section we make the same assumptions as in the preceding section, viz. the assumptions V.4.(0), which coincide with those made in Theorem V.2.3, and are the following:

\[
\begin{align*}
0 < s \leq 1, \\
(s, E_\alpha^s) \neq (1, (0,0)),
\end{align*}
\]

where $E_\alpha^s$ is the drift vector of the random walk.

V.5.1 Mapping properties of the functions $f_\tau (\tau = 1, 2)$

We have at our disposal for $\tau = 1, 2$ the set of constants $\{c_{\tau 1}, \ldots, c_{\tau 4}\}$ with properties described in Theorem V.2.3 and Corollary V.2.5. The two sets are related through the equality V.4.(31). We repeat the definition of the function $f_\tau (\cdot)$ for $\tau = 1, 2$ from the preceding section. For more details see Subsection V.4.1.

First define the constant $\tau$ by means of the requirements

\[
\tau \in i\mathbb{R}, \quad \text{Im} \tau > 0, \quad m(0|\tau) = (c_{\tau 1}, c_{\tau 2}; c_{\tau 3}, c_{\tau 4})
\]

for $\tau = 1$ or $2$, where the cross ratio on the right does not depend on $\tau$, see V.4.(31); the function $m(0|\tau)$ is given in III.1.(18).

Next define the function $f_\tau (\cdot)$ for $\tau = 1, 2$ by

\[
m(v|\tau) = (f_\tau (v), c_{\tau 2}; c_{\tau 3}, c_{\tau 4}),
\]

cf. V.4.(3). This defines $f_\tau (\cdot)$ as the second order elliptic, even function with complete period lattice

\[
\Omega(\tau) = \mathbb{Z} + \tau \mathbb{Z},
\]

and second order values $c_{\tau 1}, \ldots, c_{\tau 4}$, in such a way that

\[
f_\tau (0) = c_{\tau 1}, \quad f_\tau (\frac{1}{2}) = c_{\tau 2}, \quad f_\tau (\frac{1}{2} \tau) = c_{\tau 3}, \quad f_\tau (\frac{1}{2} + \frac{1}{2} \tau) = c_{\tau 4}.
\]

In this section we also need the constant $\sigma$. This constant is defined
in V.4.(38), cf. Remark V.4.6, by

\begin{equation}
0 < \sigma < \frac{1}{2}
\end{equation}

and any one of the following four equivalent requirements:

\begin{equation}
\begin{aligned}
f_{\zeta}(\sigma \tau) &= d_{\zeta 1}, \quad f_{\zeta}(\frac{1}{2} \tau + \sigma \tau) = d_{\zeta 2}, \\
f_{\zeta}(\frac{1}{2} \tau - \sigma \tau) &= d_{\zeta 3}, \quad f_{\zeta}(\frac{1}{2} \tau + \frac{1}{2} \tau - \sigma \tau) = d_{\zeta 4},
\end{aligned}
\end{equation}

where \( d_{\zeta 1}, \ldots, d_{\zeta 4} \) are the constants defined in V.2.(32). Notice that, if \( \omega \) is a period then

\begin{equation}
f_{\zeta}(v + \frac{1}{2} \omega) = f_{\zeta}(-v + \frac{1}{2} \omega),
\end{equation}

see Remark III.1.1.

**REMARK 1.** Due to III.1.(28), the identity (2) can be replaced by any one of the following identities:

\begin{equation}
m(v|\tau) \equiv (c_{\zeta 1}, f_{\zeta}(v + \frac{1}{2}); c_{\zeta 3}, c_{\zeta 4}),
\end{equation}

\begin{equation}
m(v|\tau) \equiv (c_{\zeta 1}, c_{\zeta 2}; f_{\zeta}(v + \frac{1}{2} \tau), c_{\zeta 4}),
\end{equation}

\begin{equation}
m(v|\tau) \equiv (c_{\zeta 1}, c_{\zeta 2}; c_{\zeta 3}, f_{\zeta}(v + \frac{1}{2} + \frac{1}{2} \tau)).
\end{equation}

In combination with (2) these identities can be used to express \( f_{\zeta}(v + \frac{1}{2} \omega) \) in \( f_{\zeta}(v) \) where \( \omega \) is a period.

We next collect a number of mapping properties of the function \( f_{\zeta} \). The following notations (already introduced in Section III.1, see III.1.(64) and III.1.(65)) will be used. We put for \( \lambda = \pm \)

\begin{equation}
R_{\lambda} := \{ v \in \mathbb{C} : 0 < \lambda \Re v < \frac{1}{2} \text{ and } 0 < \Im v < \frac{1}{2} \Im \tau \},
\end{equation}

\begin{equation}
\mathbb{C}_{\lambda} := \{ w \in \mathbb{C} : \lambda \Im w > 0 \}.
\end{equation}
So $\mathbb{R}_\lambda$ is the open rectangle with sides the half periods $\frac{1}{2} \tau$, $\frac{1}{2} \tau$, and $\mathbb{C}_+ (\mathbb{C}_-)$ is the open upper (lower) half plane. For a picture of $\mathbb{R}_\pm$ see Figure 1 (which also contains the second order values of $f_z$). In this figure the arrows indicate the direction in which $f_z(\cdot)$ increases, see Property 1 (Part III).

**FIGURE 1**

**PROPERTY 1.** (Mapping properties of $f_z$.) Assume (0).

I. The mapping $v \sim f_z(v)$ is
- a bijective bijection $\mathbb{R}_\lambda \sim \mathbb{C}_\lambda$,
- a bicontinuous bijection $\overline{\mathbb{R}_\lambda} \sim \mathbb{C}_{\lambda} \cup \mathbb{R} \cup \{ \infty \}$ (where the bar denotes closure).

II. One has

$$f_z^*(v) = f_z^{-1}(v^*)$$

(where the star denotes complex conjugation) and

$$f_z(v) \in \mathbb{R} \cup \{ \infty \} \text{ iff } \frac{\operatorname{Re} v}{\operatorname{Im} v} \in \frac{1}{2} \mathbb{Z} \text{ or } \frac{\operatorname{Im} v}{\operatorname{Im} \tau} \in \frac{1}{2} \mathbb{Z}.$$  

III. If $\tau$ increases from 0 to $\frac{1}{2}$ then:

$$f_z(\tau) \text{ decreases strictly from } c_{i4} \text{ to } c_{i2},$$

$$f_z(\tau \tau) \text{ increases strictly from } c_{i4} \text{ to } c_{i3},$$

$$f_z(\tau + \frac{1}{2} \tau) \text{ increases strictly from } c_{i3} \text{ to } c_{i4} \star,$$

$$f_z(\frac{1}{2} + \tau \tau) \text{ decreases strictly from } c_{i2} \text{ to } c_{i4} \star.$$
*) If \( c_{\xi 4} < c_{\xi 3} \), then (17) is to be read as follows: \( f_\xi(t + \frac{1}{2}\tau) \) increases from \( c_{\xi 3} \) to \( +\infty \) and from \( -\infty \) to \( c_{\xi 4} \). A similar remark applies to (18).

**Proof.** Part I. Consider the fractional linear transformation \( T_\xi(\cdot) \) defined by

\[
(18a) \quad x_\xi = T_\xi(m) \text{ if } m = (x_\xi, c_{\xi 2}, c_{\xi 3}, c_{\xi 4}),
\]

so \( f_\xi(v) = T_\xi(m(v|\tau)) \). Observe, cf. (1) and Property III.1.15 (Part I),

\[
(*) \quad T_\xi \text{ maps } m(0|\tau), 1, 0, \infty \text{ onto } c_{\xi 1}, c_{\xi 2}, c_{\xi 3}, c_{\xi 4},
\]

respectively.

So \( T_\xi \) is nonconstant, so a bicontinuous bijection from the complex sphere onto itself, and a bijection from \( \mathbb{R} \cup \{\infty\} \) onto itself (because the \( c \) values are (extended) real). It also is a bijection from \( \mathbb{C}_\lambda \) onto \( \mathbb{C}_{-\lambda} \) because of

\[
\frac{\text{Im } m}{\text{Im } x_\xi} = \frac{c_{\xi 3}^{-1} - c_{\xi 4}^{-1}}{|1 - x_\xi c_{\xi 4}^{-1}|^2} = \frac{1 - c_{\xi 2} c_{\xi 4}^{-1}}{1 - c_{\xi 2} c_{\xi 3}^{-1}} < 0
\]

(cf. V.2. (17b) and the proof of Corollary V.2.5). Now apply Proposition III.1.30 for the first claim, and Proposition III.1.31 for the second claim.

Part II. For (13) observe \( T_\xi(m^*) = T_\xi(m^*) \), and apply Property III.1.27.

For (14) apply Property III.1.28.

Part III. Observe that \( T_\xi(m) \) is a strictly decreasing function of \( m \in \mathbb{R} \), because of

\[
\frac{dm}{dx_\xi} = -\frac{c_{\xi 3}^{-1} - c_{\xi 4}^{-1}}{(1 - x_\xi c_{\xi 4}^{-1})^2} = \frac{1 - c_{\xi 2} c_{\xi 4}^{-1}}{1 - c_{\xi 2} c_{\xi 3}^{-1}} < 0
\]

as above. Now apply (*) and Property III.1.29 to conclude (15) through (18).

\( \Box \)
V.5.2 Definition and properties of some constants

Next we locate a zero and a pole of the function \( f_{\xi}(\cdot) \). Also we locate a point at which the value 1, and a point at which the value -1 is assumed.

V.5.2.1 The constants \( a_{\xi} \) and \( b_{\xi} \) (\( \xi=1,2 \))

Notation. For straight line segments in the complex plane we use the interval notation. So \([z_1, z_2]\) denotes the straight line segment connecting \( z_1 \) and \( z_2 \) including the end points, \((z_1, z_2)\) denotes the same segment excluding the end points, and so on.

First we locate a zero of the function \( f_{\xi}(\cdot) \) for \( \xi=1,2 \).

If \( c_{\xi 2} < 0 \) then \( f_{\xi}(\cdot) \) has a unique zero in \((0, \frac{1}{2})\) because of \( c_{\xi 1} > 0 \) and (15) (hence, \( f_{\xi}(\cdot) \) has a unique zero in \((\frac{1}{2}, 1)\) by (7)), if \( c_{\xi 2} > 0 \) then \( f_{\xi}(\cdot) \) has a unique zero in \((\frac{1}{2}, \frac{1}{2} + \frac{1}{2} \tau)\) because of (18) and \( c_{\xi 4}^{-1} \in [-1, 1] \), if \( c_{\xi 2} = 0 \) then \( f_{\xi}(\frac{1}{2}) = 0 \) (see (4)). Hence, the constant \( a_{\xi} \) can be defined by the requirements (19) through (20C):

\[
\begin{align*}
(19) \quad & f_{\xi}(a_{\xi} \tau + \frac{1}{2}) = 0, \\
(20A) \quad & 0 < a_{\xi} \tau < \frac{1}{2} \quad \text{if} \quad c_{\xi 2} < 0, \\
(20B) \quad & 0 < a_{\xi} < \frac{1}{2} \quad \text{if} \quad c_{\xi 2} > 0, \\
(20C) \quad & 0 = a_{\xi} \quad \text{if} \quad c_{\xi 2} = 0.
\end{align*}
\]

For the sign of \( c_{\xi 2} \) see V.2.(19A). Due to (8) the equality (19) is equivalent to

\[
(21) \quad m(a_{\xi} | \tau) = (c_{\xi 1}, 0; c_{\xi 3}, c_{\xi 4}) = \frac{1 - c_{\xi 1} c_{\xi 3}^{-1}}{1 - c_{\xi 1} c_{\xi 4}^{-1}}.
\]

Next we locate a pole of \( f_{\xi}(\cdot) \) for \( \xi=1,2 \). For that purpose consider the function \( f_{\xi}(\cdot)^{-1} \), which is strictly increasing (decreasing) whenever the function \( f_{\xi}(\cdot) \) is strictly decreasing (increasing). If \( c_{\xi 4}^{-1} < 0 \) then \( f_{\xi}(\cdot)^{-1} \) has a unique zero in \((\frac{1}{2} \tau, \frac{1}{2} + \frac{1}{2} \tau)\) because of \( c_{\xi 3}^{-1} > 0 \) and (17) (hence, \( f_{\xi}(\cdot)^{-1} \) has a unique zero in \((\frac{1}{2} + \frac{1}{2} \tau, 1 + \frac{1}{2} \tau)\) by (7)),
if \( c_{\xi 4}^{-1} > 0 \) then \( f_{\xi}(\cdot)^{-1} \) has a unique zero in \((\frac{1}{2} + \frac{1}{2} \tau, \frac{1}{2} + \tau)\) (hence, in \((\frac{1}{2} + \frac{1}{2} \tau, \frac{1}{2} + \tau)\) by (7)) because of \( c_{\xi 2} \in [-1, 1) \) and (18); if \( c_{\xi 4}^{-1} = 0 \) then \( f_{\xi}(\cdot)^{-1} \) has a zero in \( \frac{1}{2} + \frac{1}{2} \tau \). So the constant \( \beta_{\xi} \) is well-defined by the requirements (22) through (23C):

(22) \[ f_{\xi}(\beta_{\xi} \tau + \frac{1}{2} + \frac{1}{2} \tau) = \infty, \]

(23A) \[ 0 < \beta_{\xi} \tau < \frac{1}{2} \text{ if } c_{\xi 4}^{-1} < 0, \]

(23B) \[ 0 < \beta_{\xi} < \frac{1}{2} \text{ if } c_{\xi 4}^{-1} > 0, \]

(23C) \[ 0 = \beta_{\xi} \text{ if } c_{\xi 4}^{-1} = 0. \]

For the sign of \( c_{\xi 4}^{-1} \) see V.2.(19B). Due to (10) the equality (22) is equivalent to

(24) \[ m(\beta_{\xi} \tau | \tau) = (c_{\xi 1}, c_{\xi 2}, c_{\xi 3}, \infty) = \frac{1 - c_{\xi 1} c_{\xi 3}^{-1}}{1 - c_{\xi 2} c_{\xi 3}^{-1}}. \]

Some of the above inequalities can be sharpened.

**Proposition 2.** Assume (0).

I. The inequalities (20A) and (20B) can be sharpened as follows:

(25A) \[ \text{if } c_{\xi 2} < 0 \text{ then } 0 < \alpha_{\xi} \tau \leq \frac{1}{4}, \]

where equality holds iff the random walk is even;

(25B) \[ \text{if } c_{\xi 2} > 0 \text{ then } 0 < \alpha_{\xi} \leq [\sigma \wedge (\frac{1}{2} - \sigma)] \leq \frac{1}{4}, \]

where \( \alpha_{\xi} = \sigma \) iff \( d_{\xi 2} = 0 \), and \( \alpha_{\xi} = \frac{1}{2} - \sigma \) iff \( d_{\xi 4} = 0 \). (For the case \( d_{\xi 2} = 0 \) see V.2.(39), and for the case \( d_{\xi 4} = 0 \) see V.2.(41).)

II. The inequalities (23A) and (23B) can be sharpened as follows:

(26A) \[ \text{if } c_{\xi 4}^{-1} < 0 \text{ then } 0 < \beta_{\xi} \tau \leq \frac{1}{4}, \]

where equality holds iff the random walk is even;

(26B) \[ \text{if } c_{\xi 4}^{-1} > 0 \text{ then } 0 < \beta_{\xi} \leq [\sigma \wedge (\frac{1}{2} - \sigma)] \leq \frac{1}{4}, \]
where $\beta_2 = \sigma$ and $d_2^{-1} = 0$ and $\beta_2 = \frac{1}{2} - \sigma$ and $d_2^{-1} = 0$. (For the case $d_4 = 0$ see V.2.(42), and for the case $d_2^{-1} = 0$ see V.2.(40).)

**Proof.** Part I. Proof of (25A): Because of (20A) and the monotonicity III.1.(58) it suffices to show

$(\dagger)$ \[m(a_\tau | \tau) \leq m\left(\frac{1}{4} | \tau\right)\]
with equality iff the random walk is even.

We show $(\dagger)$. By III.1.(63) with $\nu = -\frac{1}{4}$ the quantity $m\left(\frac{1}{4} | \tau\right)$ is one of the square roots of $m(0 | \tau)$. From $m(0 | \tau) > 0$ (by (1) and V.2.(30)) and $m\left(\frac{1}{4} | \tau\right) > 0$ (by III.1.(58)) it follows that $m\left(\frac{1}{4} | \tau\right)$ is the positive square root of $m(0 | \tau)$. By (21) and (1) the inequality in $(\dagger)$ is equivalent to

$(\ast)$ \[\frac{1 - c_{\ell_1} c_{\ell_3}^{-1}}{1 - c_{\ell_1} c_{\ell_4}^{-1}} \leq \sqrt{\frac{1 - c_{\ell_1} c_{\ell_3}^{-1}}{1 - c_{\ell_1} c_{\ell_4}^{-1}} \cdot \frac{1 - c_{\ell_2} c_{\ell_4}^{-1}}{1 - c_{\ell_2} c_{\ell_3}^{-1}}},\]

where all factors are positive (cf. the proof of Corollary V.2.5). Hence, the inequality $(\ast)$ is equivalent to

\[(1 - c_{\ell_1} c_{\ell_3}^{-1})(1 - c_{\ell_2} c_{\ell_4}^{-1}) \leq (1 - c_{\ell_1} c_{\ell_4}^{-1})(1 - c_{\ell_2} c_{\ell_3}^{-1})\]

hence to

\[0 \leq (c_{\ell_3}^{-1} - c_{\ell_4}^{-1})(c_{\ell_1} + c_{\ell_2} - c_{\ell_1} c_{\ell_2} [c_{\ell_3}^{-1} + c_{\ell_4}^{-1}]).\]

By V.2.(17B) the first factor on the right is positive, so $(\ast)$ is equivalent to

$(\ast\ast)$ \[c_{\ell_1} c_{\ell_2} (c_{\ell_3}^{-1} + c_{\ell_4}^{-1}) \leq c_{\ell_1} + c_{\ell_2}.\]

If the random walk is even, then both sides of $(\ast\ast)$ vanish (by V.2.(18A)-(18B)), which proves equality. Now assume that the walk is non-even. Because of

\[0 < c_{\ell_1} + c_{\ell_2} \quad \text{and} \quad 0 < c_{\ell_3}^{-1} + c_{\ell_4}^{-1}\]
by V.2.(17A), ..., (18B), and \( c_2 < 0 \), the product on the left in (**), is negative and the sum on the right is positive, which proves (**), with strict inequality. This proves (†).

Proof of (25B): From (6), V.2.(38), and (19) one has

\[
\begin{align*}
\xi_c a \left( s \tau + \frac{1}{2} \right) & = d_2 \leq 0 = f_c \left( \alpha_2 \tau + \frac{1}{2} \right), \\
f_c \left( \frac{1}{2} - \sigma \right) \tau + \frac{1}{2} & = d_4 \leq 0 = f_c \left( \alpha_4 \tau + \frac{1}{2} \right).
\end{align*}
\]

This implies (25B) because of monotonicity, see (18), (5) and (20B).

Part II. The proof runs similar to the proof of Part I, and we leave the details to the reader. For (26A) it suffices to show

\[
m(\xi_2 \tau \mid \tau) \leq m(\frac{1}{4} \mid \tau)
\]

with equality iff the random walk is even,

where the inequality turns out to be equivalent to

\[
c_3^{-1} c_4^{-1} (c_1 + c_2) \leq c_3^{-1} + c_4^{-1}.
\]

For the proof of (26B) apply:

\[
\begin{align*}
\xi_c \left( \frac{1}{2} - \sigma \right) \tau + \frac{1}{2} & = d_4^{-1} \leq 0 = \xi_c \left( \frac{1}{2} - \beta \right) \tau + \frac{1}{2}^{-1}, \\
\xi_c \left( \alpha \tau + \frac{1}{2} \right) & = d_2^{-1} \leq 0 = \xi_c \left( \frac{1}{2} - \beta \right) \tau + \frac{1}{2}^{-1}.
\end{align*}
\]

\[\blacklozenge\]

V.5.2.2 The constants \( \xi_c \) and \( \beta_c \) (\( \xi = 1, 2 \))

From (16) and V.2.(15A)-(15B) it follows that \( \xi_c (\cdot) \) assumes the value \( +1 \) precisely once in \([0, \frac{1}{2} \tau]\). We define for \( \xi = 1, 2 \) the constant \( \xi_c \) by

\[
0 \leq \xi_c \leq \frac{1}{2} \quad \text{and} \quad \xi_c \left( \xi \tau \right) = +1,
\]

and have
\[
\begin{align*}
0 < \gamma_z < \frac{1}{2} & \quad \text{iff} \quad c_z' < 1 < c_z^3, \\
\gamma_z = \frac{1}{2} & \quad \text{iff} \quad c_z' = 1 = c_z^3, \\
\gamma_z = 0 & \quad \text{iff} \quad c_z' = 1 < c_z^3.
\end{align*}
\]

(28)

See V.2.(16A),...,16C) in order to translate the conditions on the right into conditions on our random walk. In particular notice (from V.2.(16A))

(29) \quad \text{if } 0 < s < 1 \text{ then } 0 < \gamma_z < \frac{1}{2}. \]

Note that by (2)

(30) \quad m(\gamma_z^+ | \tau) = \frac{1 - c_z^3}{1 - c_z^2 c_z^4} \frac{1 - c_z^2}{1 - c_z^2 c_z^3}.

From (18) and V.2.(17A)-(17B) it follows that \( f_z(\cdot) \) assumes the value -1 precisely once in \([\frac{1}{2}, \frac{1}{2} + \frac{1}{2} \tau] \). So we define the constant \( \delta_z \) by

(31) \quad 0 \leq \delta_z \leq \frac{1}{2} \quad \text{and} \quad f_z(\delta_z^+ + \frac{1}{2}) = -1,

and have

\[
\begin{align*}
0 < \delta_z < \frac{1}{2} & \quad \text{iff} \quad -1 < c_z^2 \quad \text{and} \quad -1 < c_z^4^{-1}, \\
\delta_z = \frac{1}{2} & \quad \text{iff} \quad -1 = c_z^2 \quad \text{and} \quad -1 = c_z^4^{-1}, \\
\delta_z = 0 & \quad \text{iff} \quad -1 = c_z^2 \quad \text{and} \quad -1 < c_z^4^{-1}.
\end{align*}
\]

(32)

See Theorem V.2.3 (Part II) in order to relate the conditions on the right to conditions on our random walk. In particular conclude from V.2.(17A),...,18B)

(33) \quad \text{if the random walk is non-even then } 0 < \delta_z < \frac{1}{2}. \]

Combine the propositions 17 and 18 below to conclude

(33A) \quad \text{if the random walk is even then } \gamma_z = \delta_z. \]
Note that (by (8))

\[
m(\delta \tau | \tau) = \frac{1 - c_{i1} c_{i3}^{-1}}{1 - c_{i1} c_{i4}^{-1}} \cdot \frac{1 + c_{i4}^{-1}}{1 + c_{i3}^{-1}}.
\]

V.5.3 Continuity and regularity of the constants as functions of \( s \)

Notation. We put

\[
J := \{(0, 1) \text{ if the random walk is driftless,} \}
\]

\[
(0, 1]\] otherwise.
\]

So the conditions (0) imply \( s \in J \).

Definition. Let \( \mathcal{V} \) be an arbitrary subset of \( \mathbb{C}^d \) (with \( d \in \mathbb{N} \)) and \( f(\cdot) \) a complex function on \( \mathcal{V} \). Then \( f(\cdot) \) is said to be regular on \( \mathcal{V} \) if \( f(\cdot) \) is the restriction to \( \mathcal{V} \) of a function which is regular on some open set containing \( \mathcal{V} \).

**Property 3.** Let the random walk be skipfree and all-sided. Then the function \( s \sim r \) is a regular function \( J \sim i \mathbb{R}_+ \).

**Proof.** By V.2.(22') and Corollary V.2.5 the function \( s \sim (c_{i1}, c_{i2}; c_{i3}, c_{i4}) \) is a regular function from some neighbourhood of \( J \) into some neighbourhood of \( (0, 1) \). Now apply (1) and Proposition III.1.33.

**Meromorphic functions.** Let \( \mathcal{O} \) be an open subset of \( \mathbb{C}^d \), and let \( f(\cdot) \) be a function \( 0 \sim \mathcal{O} \cup \{\infty\} \). Let us call \( f(\cdot) \) meromorphic on \( \mathcal{O} \) if:

- \( f(\cdot) \) is continuous,
- if \( v \in \mathcal{O} \), then \( f(\cdot) \) (or \( \frac{1}{f(\cdot)} \)) is regular in some neighbourhood of \( v \) if \( f(v) \) (resp. \( \frac{1}{f(v)} \)) is finite.

If \( \mathcal{V} \) is an arbitrary subset of \( \mathbb{C}^d \), and \( f(\cdot) \) a function \( \mathcal{V} \sim \mathcal{O} \cup \{\infty\} \), then \( f(\cdot) \) is said to be meromorphic on \( \mathcal{V} \), if \( f(\cdot) \) is the restriction to \( \mathcal{V} \) of a function which is meromorphic on some open set \( \mathcal{O} \) containing \( \mathcal{V} \).
PROPOSITION 4. Let the random walk be skipfree and all-sided.

I. The function \( (v, s) \sim f_\hat{\nu}(v) \) is meromorphic (in the above sense) on \( \mathbb{C} \times J \).

II. Fix \( \lambda \in \{-, +\} \). For fixed \( s \) let \( \tilde{\nu}_\lambda(\cdot) \) be the restriction of \( \nu_\lambda(\cdot) \) to \( \overline{R_\lambda^+} \), and \( \tilde{\nu}_\lambda(\cdot) \) the inverse of \( \nu_\lambda(\cdot) \). Then \( (x_\hat{\nu}, s) \sim \tilde{\nu}_\lambda(x_\hat{\nu}) \) is a continuous function \( \overline{C_{-\lambda}} \times J \sim \overline{R_\lambda} \). (For \( R_\lambda \) see (11).)

PROOF. Part I: One has \( f_\hat{\nu}(v) = T_\hat{\nu}(m(v \mid \gamma(s))) \), where \( T_\hat{\nu} \) is the fractional linear mapping (18A), the function \( s \sim \gamma(s) \) is regular \( J \sim \mathbb{R}^+ \) (Property 3), and the function \( (v, \gamma(s)) \sim m(v \mid \gamma(s)) \) is meromorphic on \( \mathbb{C} \times \mathbb{C}_+ \) (see Proposition III.1.13 at sec.). This implies the claim.

Part II: By Property 1 (Part I) the mapping \( v \sim \tilde{\nu}_\lambda(v) \) is for fixed \( s \in J \) a homeomorphism \( \overline{R_\lambda^+} \sim \mathbb{C}_{-\lambda} \). Now use Lemma II.4.79, cf. the proof of Proposition III.1.32.

PROPOSITION 5. Let the random walk be skipfree and all-sided.

I. The mappings

\[ s \sim \sigma, \quad s \sim \alpha_\hat{\nu}, \quad s \sim \beta_\hat{\nu} \]

are regular on \( J \).

II. The mappings

\[ s \sim \gamma_\hat{\nu}, \quad s \sim \delta_\hat{\nu} \]

are continuous on \( J \), and regular on \( (0, 1) \).

If the random walk is not driftless, and \( 0 < \gamma_\hat{\nu} < \frac{1}{2} \) (resp. \( 0 < \delta_\hat{\nu} < \frac{1}{2} \)) for \( s = 1 \), then \( \gamma_\hat{\nu} \) (resp. \( \delta_\hat{\nu} \)) is regular on \( J = (0, 1) \).

PROOF. Part I: We prove the claim concerning \( \alpha_\hat{\nu} \) and leave the remaining claims to the reader. We distinguish three cases, according to the sign of \( c_{\hat{\nu}} \); if \( s \in J \) then (by V.2.(19A)) this sign equals the sign of \( D_\hat{\nu}(0, s) \) and so does not depend on \( s \), cf. V.1.(10). First assume \( c_{\hat{\nu}} = 0 \), or (by (20C)) \( \alpha_\hat{\nu} = 0 \) for all \( s \in J \). Then the claim is trivial.

Next assume \( c_{\hat{\nu}} > 0 \), or (by (20B)) \( \alpha_\hat{\nu} \in (0, \frac{1}{2}) \) for all \( s \in J \). Then \( f_\hat{\nu}(\alpha_\hat{\nu} + \frac{1}{2}) = 0 \neq f'_\hat{\nu}(\alpha_\hat{\nu} + \frac{1}{2}) \) (cf. (19), (3) and Proposition III.1.7) and \( \tau \neq 0 \) (cf. (1)). Now apply the implicit function theorem (GUNNING, ROSSI...
(1965, Ch.1, Sec. B, Th.4)] using Property 3 and Proposition 4 (Part I).
In the case $c_{\xi^2} < 0$ the proof is similar.

Part II: We prove the continuity. For fixed $s$ let $f_{\xi}(\cdot)$ be the function from Proposition 4 (Part II) with $\lambda = +$. Then $\varphi_{\xi} = \hat{g}_{\xi}(1)$ and $\beta_{\xi} + \frac{1}{2} = \tilde{g}_{\xi}(-1)$, and the claim follows from Proposition 4 (Part II) and Property 3.

The regularity claims can be proved with the implicit function theorem, cf. the proof of Part I.

$\Box$

V.5.4 Some generalized strips and some generalized half planes

Notations. In the sequel a bar denotes closure, and a star denotes complex conjugation.

For $\xi = 1, 2$ we define the open (see Property 6) sets

\begin{align*}
S_{\xi} := \{ v \in \mathbb{C} : \frac{1}{2} \leq \frac{\text{Im} v}{\text{Im } \tau} \leq \frac{1}{2} \text{ and } |f_{\xi}(v)| < 1 \}, \\
S'_{\xi} := \{ v \in \mathbb{C} : \frac{1}{2} \leq \frac{\text{Im} v}{\text{Im } \tau} \leq \frac{1}{2} \text{ and } |f_{\xi}(v + \frac{1}{2} \tau)| > 1 \}.
\end{align*}

In (35) and (36) the equality signs can be dropped, see Property 6.

Notice (for $a_{\xi}$ see (19) et seq., for $b_{\xi}$ see (22) et seq.)

\begin{equation}
\alpha_{\xi} \tau + \frac{1}{2} \in S_{\xi}, \quad \beta_{\xi} \tau + \frac{1}{2} \in S'_{\xi}.
\end{equation}

Due to the period $\tau$ one has

\begin{align*}
f_{\xi}(v) \in U \iff v \in S_{\xi} + \mathbb{Z} \tau, \\
f_{\xi}(v) \in U' \iff v \in S'_{\xi} + \frac{1}{2} \tau + \mathbb{Z} \tau,
\end{align*}

where

\begin{align*}
U & \text{ is the open unit disc,} \\
U' & \text{ is the complement (including } \infty \text{) of the closed unit disc.}
\end{align*}

Because $f_{\xi}$ (as an elliptic function) assumes every value (including $\infty$) in a period parallelogram, one has
\( f_{\varphi}(S_{\varphi}) = U \),
\( f_{\varphi}(S_{\varphi}^1 + \frac{1}{2} \tau) = U^1 \).

**Property 6. Assume (0).**

I. The set \( S_{\varphi} \) is open, and has the properties:

\( S_{\varphi} = S_{\varphi} + 1 = -S_{\varphi} = S_{\varphi}^* \),

\( \mathbb{R} \cap S_{\varphi} = \begin{cases} 
\mathbb{R} & \text{iff } \gamma_{\varphi} > 0, \\
\mathbb{R} \setminus \mathbb{Z} & \text{iff } \gamma_{\varphi} = 0 < \delta_{\varphi}, \\
\mathbb{R} \setminus \left( \frac{1}{2} \mathbb{Z} \right) & \text{iff } \gamma_{\varphi} = 0 = \delta_{\varphi},
\end{cases} \)

\( \mathbb{R} + \frac{1}{2} \tau \subset S_{\varphi}^c \),

\( (\tau \mathbb{R}) \cap S_{\varphi} = (-\gamma_{\varphi} \tau, \gamma_{\varphi} \tau) \),

\( \left( \frac{1}{2} + \tau \mathbb{R} \right) \cap S_{\varphi} = \frac{1}{2} + (\delta_{\varphi} \tau, \delta_{\varphi} \tau) \).

II. The set \( S_{\varphi}^1 \) is open, and has the properties:

\( S_{\varphi}^1 = S_{\varphi}^1 + 1 = -S_{\varphi}^1 = (S_{\varphi}^1)^* \),

\( \mathbb{R} \cap S_{\varphi}^1 = \begin{cases} 
\mathbb{R} & \text{iff } \gamma_{\varphi} < \frac{1}{2}, \\
\mathbb{R} \setminus \mathbb{Z} & \text{iff } \delta_{\varphi} < \frac{1}{2} = \gamma_{\varphi}, \\
\mathbb{R} \setminus \left( \frac{1}{2} \mathbb{Z} \right) & \text{iff } \delta_{\varphi} = \frac{1}{2} = \gamma_{\varphi},
\end{cases} \)

\( \mathbb{R} + \frac{1}{2} \tau \subset (S_{\varphi}^1)^c \),

\( (\tau \mathbb{R}) \cap S_{\varphi}^1 = ((\gamma_{\varphi} - \frac{1}{2}) \tau, (\frac{1}{2} - \gamma_{\varphi}) \tau) \),

\( \left( \frac{1}{2} + \tau \mathbb{R} \right) \cap S_{\varphi}^1 = \frac{1}{2} + ((\delta_{\varphi} - \frac{1}{2}) \tau, (\frac{1}{2} - \delta_{\varphi}) \tau) \).

**Proof.** We show Part I and leave Part II to the reader.

Proof of (44): Use the fact that \( f_{\varphi} \) has period 1 and is even, and apply (13).

Proof of (45): If \( \tau \in (0, \frac{1}{2}) \) then \( f_{\varphi}(\tau) \in (c_{\varphi}, c_{\varphi}^1) \subset (-1, 1) \) by
(15) and V.2.(15A)-(17A), hence always \( (0, \frac{1}{2}) \subset S_{\varepsilon}' \). Also observe
\[ 0 \in S_{\varepsilon}' \iff c_{\varepsilon}^{-1} < 1 \iff \gamma_{\varepsilon} > 0, \quad \text{and} \quad \frac{1}{2} \in S_{\varepsilon}' \iff c_{\varepsilon} > -1 \iff \delta_{\varepsilon} > 0, \]
cf. (4), (28) and (32). This implies the claim because of
\[ \gamma_{\varepsilon} > 0 \iff \delta_{\varepsilon} > 0, \]
cf. (33)-(33A).

Proof of (46): If \( t \in \frac{1}{2} \mathbb{T} + \mathbb{R} \) then \( f_{\varepsilon}'(t)^{-1} \in [c_{\varepsilon}^{-1}, c_{\varepsilon}^{-1} c_{\varepsilon}^{-1}] \subset [-1, 1] \) by (17) and V.2.(15B)-(17B).

Proof of (47): Use (27), (16), and V.2.(15A)-(15B).

Proof of (48): Use (31), (18), and V.2.(17A)-(17B).

Proof that \( S_{\varepsilon}' \) is open. By (46) the equality signs in (35) can be dropped, so \( S_{\varepsilon}' \) is open by the continuity of \( f_{\varepsilon}' \).
\[ \square \]

In the following property (as before) \( U \) is the open unit disc, \( U^c \) its complement (including \( \infty \)), \( U' \) is the complement (including \( \infty \)) of the closed unit disk; the set \( \mathbb{E}_+ (\mathbb{E}_-) \) is the open upper (lower) half plane (see (12)), and their closures include \( \infty \). For the open rectangles \( R_{\varepsilon} \) see (11).

**PROPERTY 7.** Assume (0). Then the following holds for \( i = 1, 2 \) and \( \lambda = \varepsilon, z \):

(54) \[ f_{\varepsilon}' \text{ is a biregular bijection } S_{\varepsilon}' \cap R_{\lambda} \sim U \cap \mathbb{E}_{-\lambda}, \]

(55) \[ f_{\varepsilon}' \text{ is a bicontinuous bijection } \begin{cases} S_{\varepsilon}' \cap \overline{R_{\lambda}} \sim U \cap \mathbb{E}_{-\lambda}, \\ S_{\varepsilon}' \cap \overline{R_{\lambda}} \sim U \cap \overline{\mathbb{E}_{-\lambda}} \end{cases}, \]

(56) \[ f_{\varepsilon}' \text{ is a biregular bijection } (S_{\varepsilon}' + \frac{1}{2} \mathbb{T}) \cap R_{\lambda} \sim U' \cap \mathbb{E}_{-\lambda}, \]

(57) \[ f_{\varepsilon}' \text{ is a bicontinuous bijection } \begin{cases} (S_{\varepsilon}' + \frac{1}{2} \mathbb{T}) \cap \overline{R_{\lambda}} \sim U' \cap \mathbb{E}_{-\lambda}, \\ (S_{\varepsilon}' + \frac{1}{2} \mathbb{T}) \cap \overline{R_{\lambda}} \sim U \cap \mathbb{E}_{-\lambda} \end{cases}. \]

**PROOF.** Apply Property 1 (Part I), and use the definitions (35) and (36).
\[ \square \]
For \( i = 1,2 \) the open sets \( A_i \) and \( B_i \) are defined by

\[
A_i := S_i \cup \mathcal{C}_+ \quad \text{and} \quad B_i := S_i \cup \mathcal{C}_-
\]

(for \( S_i \) see (35), and for the half planes \( \mathcal{C}_\pm \) see (12)), and similarly the open sets \( A'_i \) and \( B'_i \) by

\[
A'_i := S'_i \cup \mathcal{C}_+ \quad \text{and} \quad B'_i := S'_i \cup \mathcal{C}_-
\]

(for \( S'_i \) see (36)). By (44) and (49) one has

\[
A_i = A_i + 1 = -B_i = B_i^* \quad \text{and} \quad B_i = B_i + 1 = -A_i = A_i^*,
\]

\[
A'_i = A'_i + 1 = -B'_i = (B'_i)^* \quad \text{and} \quad B'_i = B'_i + 1 = -A'_i = (A'_i)^*.
\]

Obviously one has

\[
S_i = A_i \cap B_i \quad \text{and} \quad S'_i = A'_i \cap B'_i.
\]

Our aim is to show that the \( A \)-sets and the \( B \)-sets are generalized open half planes, and that (in general) the \( S \)-sets are generalized open horizontal strips (see the definitions at the beginning of Section IV.1).

**PROPOSITION 8.** Assume (0). Then for \( i = 1, 2 \) the following holds.

I. The sets \( A_i, B_i, A'_i, B'_i \) and their complements are connected.

II. The sets \( A_i, B_i, A'_i, B'_i \) are simply connected.

**PROOF.** Part I: We show that \( A_i \) and its complement are connected and leave the remaining claims to the reader.

First we show that \( A_i \) is connected. Observe that the set \( S_i \cap R_{\lambda\lambda}^- \) (which is by (55) the continuous image of the connected set \( \mathcal{U} \cap \mathcal{R}_{\lambda\lambda}^- \)) is connected. Let \( \mathcal{R} \) be the closed rectangle \( \mathcal{U} \cup \mathcal{R}_{\lambda\lambda}^- \). The set \( S_i \cap \mathcal{R} \) is connected because the intersection \( \cap \{ S_i \cap \mathcal{R}_{\lambda\lambda}^- \} \) contains \( (0, \frac{1}{2}) \), so is nonempty. It follows that the set \( [S_i \cap (\mathcal{L}\mathcal{R})] \cup \mathcal{C}_+ \) is connected because of \( S_i \cap (\mathcal{L}\mathcal{R}) \cap \mathcal{C}_+ \neq \emptyset \), see (54). Finally, one has
\[ \bigcup_{\lambda=\pm} \bigcup_{k \in \mathbb{Z}} \{ [(S_{\lambda} \cap \lambda R) \cup E_{+}] + k \} = \]
\[ = \bigcup_{\lambda} \bigcup_{k \in \mathbb{Z}} [S_{\lambda} \cap (\lambda R + k)] \cup E_{+} = A_{t}^{c}, \]

where the individual sets in the union on the left are connected, and have the common intersection \( E_{+} \). Hence, their union is connected.

Next we show that \((A_{t}^{c})^{c}\) is connected. Observe

\[(*) \quad A_{t}^{c} \cap S_{t}^{c} \cap E_{-} = (35), (46) \]
\[ = \bigcup_{\lambda=\pm} \bigcup_{k \in \mathbb{Z}} \{ ([S_{t}^{c} \cap (-R_{\lambda})] \cup [-\frac{1}{2}t + E_{-}]) + k \}. \]

Observe that the set \(S_{t}^{c} \cap (-R_{\lambda})\) is nonempty and connected (by (55)), and has a nonempty intersection (viz. the interval \([-\frac{1}{2}\tau, -\frac{1}{2}\tau - \lambda \frac{1}{2}\]) with the half plane \(-\frac{1}{2}\tau + E_{-}\), cf. (46). Hence the individual sets in the union on the right in (*) are connected; they have the common intersection \(-\frac{1}{2}\tau - E_{-}\), which implies that their union is connected.

Part II. We show that \(A_{t}^{c}\) is simply connected, and leave the remaining claims to the reader. Observe that \(A_{t}^{c}\) is open and connected, and that its complement is connected and unbounded. Now the claim is a consequence of Theorem IV.6.7.

\[ \diamond \]

**COROLLARY 9.** Assume (0).

Then the sets \(A_{t}, A_{t}^{c}\) are generalized open upper half planes, and the sets \(B_{t}, B_{t}^{c}\) are generalized open lower half planes.

**PROOF.** From (58) through (61), and Proposition 8 (Part II).

\[ \diamond \]

**PROPOSITION 10.** Assume (0).

I. If \(\gamma_{t} > 0\) then \(S_{t}\) is a generalized open horizontal strip.

If \(\gamma_{t} = 0\) then \(S_{t}\) is disconnected.

II. If \(\gamma_{t} < \frac{1}{2}\) then \(S_{t}^{c}\) is a generalized open horizontal strip.

If \(\gamma_{t} = \frac{1}{2}\) then \(S_{t}^{c}\) is disconnected.
PROOF. We show Part I and leave Part II to the reader.

Assume $\gamma > 0$. Because of (62) (first equality) and Corollary 9 we can apply Proposition IV.1.2.; so it suffices to show $A_\zeta \cup B_\zeta = E$. Observe

$$A_\zeta \cup B_\zeta = S_\zeta \cup E_+ \cup E_- \supset E$$

because of $R \subset S_\zeta$, see (45). This proves the claim. Assume $\gamma = 0$. Then $(\tau R) \cap S_\zeta = \emptyset$ by (47). Let $E_+ (E_-)$ be the open right (left) half plane. Then $S_\zeta$ is the disjoint union of the open nonempty sets $S_\zeta \cap E_\lambda (\lambda = \pm)$, which proves disconnectedness.

The sets $S_\zeta, S_\zeta', A_\zeta, B_\zeta, A_\zeta', B_\zeta'$ are pictured in Fig. 2 below.

V.5.5 The boundaries of the generalized strips and the generalized half planes

We put

$$\Gamma_\zeta := \{ \nu \in E: 0 \leq \frac{\text{Im} \nu}{\text{Im} \tau} \leq \frac{1}{2} \text{ and } |f_\zeta(\nu)| = 1 \}$$  \hspace{1cm} (63)

$$\Delta_\zeta := \{ \nu \in E: -\frac{1}{2} \leq \frac{\text{Im} \nu}{\text{Im} \tau} \leq 0 \text{ and } |f_\zeta(\nu)| = 1 \}$$  \hspace{1cm} (64)

These sets are pictured in Fig. 1 below. Also we put (anticipating (80) through (83))

$$\Gamma_\zeta' := \Delta_\zeta + \frac{1}{2} \tau$$  \hspace{1cm} (65)

$$\Delta_\zeta' := \Gamma_\zeta - \frac{1}{2} \tau$$  \hspace{1cm} (66)

Obviously,

$$|f_\zeta(\nu)| = 1 \text{ iff } \nu \in (\Gamma_\zeta \cup \Delta_\zeta) + Z \tau.$$  \hspace{1cm} (66A)

The analogue of Property 6 is
PROPERTY 11. Assume (0).

I. The sets $\Gamma_i$ and $\Lambda_i$ are closed, and have the properties:

\begin{equation}
\Gamma_i = \Gamma_i + 1 = -\Lambda_i = (\Lambda_i)^* \quad \text{(star denotes complex conjugation)}
\end{equation}

where $\Gamma_i$ and $\Lambda_i$ may be interchanged,

\begin{align}
(\tau R) \cap \Gamma_i &= \{ r_i \tau \} , \\
(\frac{1}{2} + \tau R) \cap \Gamma_i &= \{ \frac{1}{2} + \delta_i \tau \} , \\
\Gamma_i \cap \Lambda_i &= R \setminus S_i .
\end{align}

II. The sets $\Gamma'_i$ and $\Lambda'_i$ are closed, and have the properties:

\begin{equation}
\Gamma'_i = \Gamma'_i + 1 = -\Lambda'_i = (\Lambda'_i)^* \quad \text{(star denotes complex conjugation)}
\end{equation}

where $\Gamma'_i$ and $\Lambda'_i$ may be interchanged,

\begin{align}
(\tau R) \cap \Gamma'_i &= \{ (\frac{1}{2} - r_i \tau) \} , \\
(\frac{1}{2} + \tau R) \cap \Gamma'_i &= \{ \frac{1}{2} + (\frac{1}{2} - \delta_i \tau) \} , \\
\Gamma'_i \cap \Lambda'_i &= R \setminus S'_i .
\end{align}

PROOF. Left to the reader (for (70) use (15), and for (74) use (17)).

In the following we put

\begin{equation}
C \quad \text{is the unit circle.}
\end{equation}

PROPERTY 12. Assume (0).

I. For $\lambda = \pm$ the following holds (with the notations (11) and (12)):

\begin{equation}
f_i(\cdot) \text{ is a bicontinuous bijection } \Gamma_i \cap \overline{R_\lambda} \rightarrow C \cap \overline{E_{-\lambda}} .
\end{equation}

II. $\Gamma_i$ is connected.
PROOF. Part I. Apply Property 1 (Part I).

Part II. For \( \lambda = \pm \) the set \( \Gamma \cap \overline{R_{\lambda}} \) is connected because it is by (76) the continuous image of a connected set. Observe:

\[
\Gamma_\lambda = \bigcup_{k \in \mathbb{Z}} \Gamma_\lambda(k) \text{ with } \Gamma_\lambda(2k-1) := (\Gamma_\lambda \cap \overline{R_-}) + k, \\
\Gamma_\lambda(2k) := (\Gamma_\lambda \cap \overline{R_+}) + k,
\]

where each individual set in the union is connected, and (by (68) and (69)) two consecutive sets have a nonempty intersection. This proves the connectedness.

In the following the boundary of a set \( A \) is denoted by \( \partial A \),

\[
(77) \quad \partial A := \overline{A} \cap (A^C).
\]

**PROPOSITION 13.** Assume (0). Then:

\[
(78) \quad (\partial S_\lambda') \cap \overline{E_+} = \Gamma' \ 	ext{,} \quad (\partial S_\lambda') \cap \overline{E_-} = \Lambda',
\]

\[
(79) \quad (\partial S_\lambda') \cap \overline{E_+} = \Gamma' \ 	ext{,} \quad (\partial S_\lambda') \cap \overline{E_-} = \Lambda',
\]

where \( \overline{E_+} (\overline{E_-}) \) is the closed upper (lower) half plane.

**PROOF.** We show the first claim in (78), and leave the remaining ones to the reader. Because of the period 1 it suffices to prove for \( \lambda = \pm \) (for \( R_{\lambda} \) see (11))

\[
(\ast) \quad (\partial S_\lambda') \cap \overline{R_{\lambda}} = \Gamma' \cap \overline{R_{\lambda}}.
\]

Fix \( \lambda \) and write \( \hat{f}_\lambda \) for the restriction of \( f_\lambda \) to \( \overline{R_{\lambda}} \). From (35) and Property 1 (Part I) we know

\[
S_\lambda \cap \overline{R_{\lambda}} = \hat{f}_\lambda^{-1} (U \cap \overline{E_{-\lambda}}),
\]

which implies, because \( \hat{f}_\lambda \) is a homeomorphism (cf. Property 1 (Part I))
\[
\overline{S}_i \cap \overline{R}_\lambda = \overline{F}_i^{-1} (\overline{U} \cap \overline{E}^{-\lambda}) = \overline{F}_i^{-1} (\overline{U}).
\]

Obviously we have from (35)

\[
S_i^C \cap \overline{R}_\lambda = \overline{F}_i^{-1} (u^C).
\]

Hence

\[
(\partial S_i) \cap \overline{R}_\lambda = \overline{F}_i^{-1} (au),
\]

which by (63) implies (*)

\[\Box\]

**COROLLARY 14.**

Assume (0). Then the following equalities hold:

\[(80) \quad \partial S_i = \Gamma_i \cup \Lambda_i,\]

\[(81) \quad \partial \Lambda_i = \Lambda_i, \quad \partial B_i = \Gamma_i,\]

\[(82) \quad \partial S_i' = \Gamma_i' \cup \Lambda_i',\]

\[(83) \quad \partial \Lambda_i' = \Lambda_i', \quad \partial B_i' = \Gamma_i'.\]

**PROOF.** We show (80) and the second equality in (81), and leave the remaining claims to the reader.

Proof of (80): Trivial from (78).

Proof of the second equality in (81): Apply Property IV.6.2 to conclude from the second equality in (58) (because $S_i$ and $E_+$ are open)

\[
\partial B_i = [ (\partial S_i) \cap (\overline{E}_+) ] \cup [ R \cap (S_i^C) ].
\]

Observe that the first intersection equals $\Gamma_i$ (by (78)) and that the second intersection is contained in $\Gamma_i$ by (70).

\[\Box\]
PROPOSITION 15.
Assume (0). Then the following are partitions of $E$:

\[(84)\] $E = B_\varepsilon^i \cup \Gamma_\varepsilon^i \cup (A_\varepsilon^i + \frac{1}{2} \tau)$,

\[(85)\] $E = (B_\varepsilon^i - \frac{1}{2} \tau) \cup \Delta_\varepsilon \cup A_\varepsilon^i$.

**Proof.** We show (84). Let $R$ be the strip $\{v: 0 \leq \frac{\text{Im} \ v}{\text{Im} \ \tau} \leq \frac{1}{2}\}$. Observe that the following is a partition of $E$:

$$E = E_- \cup (S_\varepsilon^i \cap R) \cup \Gamma_\varepsilon \cup ([S_\varepsilon^i + \frac{1}{2} \tau] \cap R) \cup (E_+ + \frac{1}{2} \tau).$$

This easily implies (84).
Multiply (84) by $-1$ to find (85).

\[
\square
\]

**Corollary 16.**
Assume (0). Then the following holds for $i = 1, 2$:

\[(86)\] $(A_\varepsilon^i)^c = B_\varepsilon^i - \frac{1}{2} \tau$, \quad $(B_\varepsilon^i)^c = A_\varepsilon^i + \frac{1}{2} \tau$,

\[(87)\] $(A_\varepsilon^i)^c = B_\varepsilon^i - \frac{1}{2} \tau$, \quad $(B_\varepsilon^i)^c = A_\varepsilon^i + \frac{1}{2} \tau$

(where the bar denotes closure).

**Proof.** For the first equality in (86) substitute the first equality from (81) in (85).

The remaining claims are left to the reader.

\[
\square
\]

The location of the strips $S_\varepsilon, S_\varepsilon^i$ (see (35), (36)), the generalized half planes $A_\varepsilon, B_\varepsilon$ (see (58)) and $A_\varepsilon^i, B_\varepsilon^i$ (see (59)), and the curves $\Gamma_\varepsilon, \Delta_\varepsilon$ (see (63), (64)) is illustrated in Figure 2. This figure is a mere sketch and is not based on a computation; it is assumed $0 < \varepsilon < \gamma < \frac{1}{2}$.
V.5.6 Some special random walks
We conclude this section with some examples, viz. successively
- even random walk (see I.1.(7))
- symmetric random walk
- coordinate-symmetric random walk
- even and driftless random walk.
V.5.6.1 Even random walk

PROPOSITION 17. Assume that the random walk is skipfree and all-sided.
Then the following are equivalent:

(88A) the random walk is even (cf. I.1.(7))

(88B) \( \xi (-x, s) = \xi (-x, s) \)
   for all \( x \in \mathbb{E} \) and all \( s \in \mathbb{E} \)

(88C) \( \xi (-x, s) = \xi (-x, s) \)
   for some \( i \in \{1, 2\} \), some \( x > 0 \), and some \( s > 0 \).

PROOF.
(88A) \( \Rightarrow \) (88B): From the assumption (88A) it follows that \( \alpha \xi \) and \( c \xi \) are even polynomials and that \( b \xi \) is odd, which implies (88B).

(88B) \( \Rightarrow \) (88C): Obvious.

(88A) \( \Leftrightarrow \) (88C): See Property V.2.1 (Part II).

PROPOSITION 18.
Assume (0). Fix \( s \in J \) (see (34A)) and \( i \in \{1, 2\} \). Then the following are equivalent:

(89A) \( \xi (-x, s) = \xi (-x, s) \) for all \( x \in \mathbb{E} \),

(89B) \( c_1 + c_2 = 0 = c_3 + c_4 \),

(89C) \( f \xi (v) + f \xi (v+1/2) = 0 \) for all \( v \in \mathbb{E} \),

(89D) \( \alpha \xi \tau = \frac{1}{4} = \beta \xi \tau \) and \( \gamma \xi = \delta \xi \).

PROOF.
(89A) \( \Leftrightarrow \) (89B): From Proposition 17 it follows that (89A) is equivalent to (88A), which in turn (by V.2.(18A) - (18B)) is equivalent to (89B).

(89B) \( \Rightarrow \) (89C): One has from (8), first applying Property III.1.15 (Part IV, a)), next observing that the cross ratio is invariant under the transformation \( x \sim -x \) (idem, Part III), and finally assuming (89B),
\[ m(v|\tau) = (f_\ell(v+\frac{1}{2}), c_\ell_1, c_\ell_4, c_\ell_3) =
= (- f_\ell(v+\frac{1}{2}), - c_\ell_1, - c_\ell_4, - c_\ell_3) = (89B)
= (- f_\ell(v+\frac{1}{2}), c_\ell_2, c_\ell_3, c_\ell_4). \]

Because of (2) this implies (89C).

(89B) \iff (89C): Substitute \( v = 0, \frac{1}{2} \) and apply (4).

(89C) \iff (89D): In (89C) substitute \( v = \ell \cdot \tau \) and use (19) to conclude
\[ f_\ell(\ell \cdot \tau + \frac{1}{2}) = 0 = f_\ell(\ell \cdot \tau), \]

hence \( \ell \cdot \tau + \frac{1}{2} + \lambda \ell \cdot \tau \in \Omega(\tau) \) (see (3)) for some
\( \lambda \in \{-, +\} \), cf. III.1.5. Clearly \( \lambda = + \), so \( 2\ell \cdot \tau + \frac{1}{2} \in \Omega(\tau) \). By (20A)\ldots, (20C) this implies \( \ell \cdot \tau = \frac{1}{4} \). Similarly one shows \( \ell \cdot \tau = \frac{1}{4} \). Finally in
(89C) substitute \( v = \gamma_\ell \cdot \tau \) to find \( \gamma_\ell = \delta_\ell \), cf. (27) and (31).

(89C) \iff (89D): Assume (89D) and consider the two functions \( f_\ell(.) \) and
\( - f_\ell(. + \frac{1}{2}) \). They have the same simple zeros (viz. \( \pm \frac{1}{4} \)) and the same simple
poles (viz. \( \pm \frac{1}{4} + \frac{1}{2} \)), and the common value \( +1 \) in \( \gamma_\ell \cdot \tau \). This implies
(89C), cf. Property III.1.5.

\[ \square \]

V.5.6.2 Symmetric random walk

The random walk is called symmetric if the distribution of the step is symmetric with respect to the origin, or (cf. V.1.(1))

\[ \mathbb{P}\{ \xi = h \} = \mathbb{P}\{ \xi = -h \} \]

for all \( h \).

PROPOSITION 19. Assume that the random walk is skipfree and all-sided. Then the following are equivalent:

(90A) the random walk is symmetric,

(90B) \[ D_\ell(x, s) = x_\ell^4 D_\ell(x_\ell^{-1}, s) \]

for \( \ell \in \{1, 2\} \), all \( x_\ell \in \mathbb{E} \), and all \( s \in \mathbb{E} \).

PROOF.

(90A) \implies (90B): Obviously, the random walk is symmetric iff
(90C) \[ Q(x_1, x_2, s) \equiv x_1^2 x_2^2 Q(x_1^{-1}, x_2^{-1}, s) , \]
cf. V.1.(2) - (4), or

(90D) \[ \beta_\ell(x_\ell) \equiv x_\ell^2 \beta_\ell(x_\ell^{-1}) , \quad \alpha_\ell(x_\ell) \equiv x_\ell^2 \alpha_\ell(x_\ell^{-1}) \]

for \( \ell = 1 \) or (equivalently) for \( \ell = 2 \), cf. V.1.(6). This implies the claim, cf. V.1.(10).

(90A) \( \iff \) (90B) : The assumption (90B) implies

(90E) \[ \begin{cases} \beta_1 = \beta_{-1} , \\ \alpha_1 \gamma_1 = \alpha_{-1} \gamma_{-1} , \\ \alpha_0 = \gamma_0 , \\ \alpha_1 \alpha_{-1} = \gamma_1 \gamma_{-1} , \\ \beta_0 \alpha_1 + \alpha_1 \gamma_0 = \alpha_0 \gamma_{-1} + \alpha_{-1} \gamma_0 , \\ \beta_1 \alpha_{-1} + \alpha_1 \beta_0 = \beta_0 \gamma_{-1} + \gamma_1 \beta_{-1} , \end{cases} \]

We now use the notation I.3.(23) for \( \ell = 1 \). With this notation the condition (90E) can be rewritten as follows:

(i) \[ \begin{cases} \beta_1 = \beta_{-1} , \\ \alpha_0 = \gamma_0 , \end{cases} \]

(ii) \[ \begin{cases} (\alpha_1 + \gamma_{-1})(\alpha_{-1} - \gamma_1) = 0 , \\ (\alpha_1 - \gamma_{-1})(\alpha_{-1} + \gamma_1) = 0 , \end{cases} \]

(iii) \[ \begin{cases} \alpha_0 [ (\alpha_1 - \gamma_{-1}) - (\alpha_{-1} - \gamma_1) ] = 0 , \\ \beta_1 [ (\alpha_1 - \gamma_{-1}) + (\alpha_{-1} - \gamma_1) ] = 0 . \end{cases} \]

We distinguish three cases.

a) \( \alpha_1 + \gamma_{-1} = 0 \), or \( \alpha_1 = 0 = \gamma_{-1} \). Then by (iii) \( \alpha_0 (\alpha_{-1} - \gamma_1) = 0 = \beta_1 (\alpha_{-1} - \gamma_1) \). Observe that \( \alpha_0 = 0 = \beta_1 \) implies \( \xi_1 + \xi_2 = 0 \) a.s. (because of \( \alpha_0 = \alpha_1 = \beta_1 = \beta_{-1} = \gamma_{-1} = \gamma_0 = 0 \), which contradicts all-sidedness. Consequently \( \alpha_0, \beta_1 \) \( \not\equiv (0,0) \), hence \( \alpha_{-1} = \gamma_1 \), so the walk is symmetric.
b) \( a_{-1} + \gamma_1 = 0 \), or \( a_{-1} = 0 = \gamma_1 \). Then by (iii) \( \alpha_0 (a_{1} - \gamma_{-1}) = 0 = \beta_1 \cdot (a_{1} - \gamma_{-1}) \). Observe that \( \alpha_0 = 0 = \beta_1 \) implies \( \xi_1 - \xi_2 = 0 \) a.s. (because of \( \alpha_0 = \alpha_{-1} = \beta_1 = \gamma_1 = \gamma_0 = 0 \)), which contradicts all-sidedness. Consequently \( \alpha_0, \beta_1 \neq (0,0) \), hence \( a_1 = \gamma_1 \), which implies symmetry.

c) \( a_{1} + \gamma_{-1} > 0 \) and \( a_{-1} + \gamma_1 > 0 \). Then by (ii) \( a_{-1} - \gamma_1 = 0 = a_1 - \gamma_{-1} \), which implies symmetry.

Example. The following is an example of a all-sided random walk which is not symmetric, but nevertheless satisfies \( D_{\xi}(w,s) = w^4 D_{\xi}(w^{-1},s) \) for \( \xi = 1 \) (hence, not for \( \xi = 2 \)). Put

\[
\begin{align*}
\sigma_1 (w) &= m (p + q w^2), \\
\sigma_1 (w) &= r (1 + w^2), \\
c_1 (w) &= m \left( \frac{1}{p} + \frac{1}{q} w^2 \right),
\end{align*}
\]

where \( m, p, q, r \) are positive numbers satisfying

\[
m (p + \frac{1}{p} + q + \frac{1}{q}) + 2r = 1.
\]

This walk satisfies (90E), hence (90B), with \( \xi = 1 \), but is not symmetric unless \( pq = 1 \).

PROPOSITION 20.
Assume (0). Fix \( s \in J \) (see (34A)) and \( \xi \in \{1,2\} \). Then the following are equivalent:

\[
\begin{align*}
(91A) & \quad D_{\xi}(x_{\xi},s) = x_{\xi}^4 D_{\xi}(x_{\xi}^{-1},s) \text{ for all } x_{\xi} \in \mathbb{E}, \\
(91B) & \quad c_{\xi 1} c_{\xi 3} = 1 = c_{\xi 2} c_{\xi 4}, \\
(91C) & \quad f_{\xi}(v) f_{\xi}(v + \frac{1}{2}r) = 1 \text{ for all } v \in \mathbb{E}, \\
(91D) & \quad \alpha_{\xi} = \beta_{\xi} \text{ and } \gamma_{\xi} = \frac{1}{4} = \delta_{\xi}.
\end{align*}
\]

PROOF.

(91A) \( \Rightarrow \) (91B): The assumption (91A) implies
\begin{align*}
\{c_{i1}^{-1}, c_{i2}^{-1}, c_{i3}^{-1}, c_{i4}^{-1}\} &= \{c_{i1}, c_{i2}, c_{i3}, c_{i4}\}
\end{align*}

where the four \( c \)-values are distinct, cf. Theorem V.2.3 (Part III). With Theorem V.2.3 (Parts I and II) we show that \((*)\) implies \((91B)\). First we show

(i) \( c_{i1} \in (0,1) \).

Assume contrarily \( c_{i1} = 1 \), or \( c_{i1}^{-1} = c_{i1} \). Then by V.2.(15B) \( c_{i3}^{-1} < 1 \), or \( c_{i3}^{-1} \notin \{c_{i1}, c_{i3}\} \). Due to V.2.(17B) it follows \( c_{i3}^{-1} \notin c_{i4} \). Consequently by \((*)\) \( c_{i3}^{-1} = c_{i2} \) and so \( c_{i4}^{-1} = c_{i4} \). This implies \( c_{i4}^{-1} = -1 < c_{i3}^{-1} \), which contradicts V.2.(17B). This proves (i). Similarly one shows

(ii) \( c_{i3}^{-1} \in (0,1) \).

Consequently, by V.2.(17A) - (17B),

(iii) \( c_{i2}, c_{i4}^{-1} \in (-1,1) \).

Next use \((*)\) to conclude

\[ \{c_{i3}^{-1}, c_{i4}^{-1}\} = \{c_{i1}, c_{i2}\} \]

The assumption \( c_{i3}^{-1} = c_{i2} \) (hence \( c_{i4}^{-1} = c_{i1}^{-1} \)) leads to a contradiction, as follows. From V.2.(17A) we know \( c_{i2} < c_{i3}^{-1} \), hence by our assumption \( c_{i3}^{-1} < c_{i4}^{-1} \) which contradicts V.2.(17B). This proves \((91B)\).

\((91B) \Rightarrow (91C)\): In the same way as in the proof of \((89B) \Rightarrow (89C)\) one derives from (9), assuming \((91B)\),

\[
\begin{align*}
\mathbb{P}(v|\tau) &= (f_{i}(v + \frac{1}{2})^{-1}, c_{i4}^{-1}, c_{i1}^{-1}, c_{i2}^{-1}) = \\
&= (f_{i}(v + \frac{1}{2})^{-1}, c_{i4}^{-1}, c_{i1}^{-1}, c_{i2}^{-1}) \quad (91B) \\
&= (f_{i}(v + \frac{1}{2})^{-1}, c_{i2}^{-1}, c_{i3}^{-1}, c_{i4}^{-1}).
\end{align*}
\]

Because of (2) this implies \((91C)\).

\((91B) \Leftarrow (91C)\): In \((91C)\) substitute \( v = 0, \frac{1}{2} \), and apply (4).
(91A) = (91C): From Theorem V.4.1. we know

\[ f'(z)(v)^2 = \omega_\tau(z) D_\tau(z)(f(z), s). \]

Put \( D_\tau(x, s) := x^{\tau} D_\tau(x^{-1}, s) \) and \( \tau^2(z) := \frac{1}{f(z)} \) for \( w \in E \). From Remark V.4.1 we know

\[ \tau^2(z) f'(z)(w)^2 = \omega_\tau(z) D_\tau(\tau(z)(w), s). \]

Assume (91C), or \( \tau(z)(w) \equiv f(z)(w + \frac{1}{2} \tau) \), and put \( v := w + \frac{1}{2} \tau \) to find

\[ f'(z)(v)^2 = \omega_\tau(z) D_\tau(f(z)(v), s). \]

From (†) and (‡‡) conclude (91A).

(91C) ⇒ (91D): In (91C) substitute \( v = \alpha \tau + \frac{1}{2} \) to conclude (cf. (19))

\[ f(z)(\alpha \tau + \frac{1}{2}) = 0 = f(z)(\alpha \tau + \frac{1}{2} + \frac{1}{2} \tau)^{-1}, \]

hence (cf. (22))

\[ \alpha \tau + \frac{1}{2} + \frac{1}{2} \tau = \lambda (\beta \tau + \frac{1}{2} + \frac{1}{2} \tau) \in \Omega(\tau) \text{ for some } \lambda = \pm, \text{ hence } \]

\( \alpha + \lambda \beta = \frac{1}{2} \tau \in \Omega(\tau). \) By (20A), ..., (20C) and (23A), ..., (23C) this implies \( \alpha - \beta = 0. \)

In (91C) substitute \( v = \gamma \tau \) to conclude (by (27)) \( f(z)(\gamma \tau + \frac{1}{2} \tau) = 1 \),

hence \( \gamma \tau + \frac{1}{2} \tau + \lambda \gamma \tau \in \Omega(\tau) \text{ for some } \lambda \in \{-, +\}. \) Obviously, this implies \( \lambda = + \) and (by (27)) \( \gamma \tau = \frac{1}{4}. \) Similarly one shows \( \gamma = \frac{1}{4}. \)

(91C) ⇒ (91D): Assume (91D). Consider the two functions \( f(z)(v) \) and

\[ f(z)(v + \frac{1}{2} \tau)^{-1}. \]

They have the common zeros \( \pm \alpha \tau + \frac{1}{2} \) and the common poles \( \pm \alpha \tau + \frac{1}{2} + \frac{1}{2} \tau \) (which are double if \( \alpha = 0 \) and simple otherwise). In addition both have the value 1 at \( v = \frac{1}{4} \tau. \) This implies (91C).

**PROPERTY 21.** Assume \( (0). \) Fix \( s \in J \) and \( i \in \{1, 2\}. \)

If any of the conditions (91A), ..., (91D) is satisfied then

\[ \Gamma[z] = \{ z \in E : \frac{\text{Im } v}{\text{Im } \tau} = \frac{1}{4} \}, \]

\[ S[z] = S[z] = \{ z \in E : |\frac{\text{Im } v}{\text{Im } \tau}| < \frac{1}{4} \}. \]
PROOF. Call $\Gamma$ the line in (92) on the right, and $S$ the strip in (93) on the right.

First we show

$$\Gamma \subseteq \Gamma_{s}, \quad S \subseteq S_{r}, \quad S' \subseteq S'_{r}. \quad (*)$$

We show $\Gamma \subseteq \Gamma_{s}$. Take $v \in \Gamma$, or $v = v^* + \frac{1}{2} \tau$ (where the star denotes complex conjugation). Then $|f_{s}(v)|^{2} = f_{s}(v) f_{s}(v)^* \neq 0$ by (13) and $f_{s}(v^* + \frac{1}{2} \tau) f_{s}(v^*) = 1$ by (91C), hence $v \in \Gamma_{s}$ by (63).

We show $S \subseteq S_{r}$ and $S' \subseteq S'_{r}$. First observe that $v \in S_{r}$ if $v \in S$ and $\Re v \in \frac{1}{2} Z$ (because of (47), (48), and (91D)). Let $R$ be the closed rectangle with vertices $\pm \frac{1}{2} \pm \frac{1}{4} \tau$. Then $|f_{s}(v)| \leq 1$ on the boundary of $R$ and $f_{s}$ is regular on $R$ (the poles are outside $R$, cf. Proposition 2 (Part II)). Consequently, if $v$ is in the interior of $R$ then $|f_{s}(v)| < 1$ by the maximum modulus principle (RUDIN [1966, Chapter 12]), and $|f_{s}(v + \frac{1}{2} \tau)| > 1$ by (91C). This implies $S \subseteq S_{r}$ by (35), and $S' \subseteq S'_{r}$ by (36), which proves $(*)$.

Next we show that equality holds in $(*)$. Put

$$R := \{ w \in \mathbb{C} : 0 \leq \frac{\Im w}{\Im \tau} \leq \frac{1}{2} \}.$$

Then

$$S \cap R \Gamma, \quad S \cap (-R) + \frac{1}{2} \tau$$

have union $R$

and (cf. (35), (36))

$$S_{r} \cap R, \quad \Gamma_{s}, \quad S'_{r} \cap (-R) + \frac{1}{2} \tau \quad \text{are disjoint subsets of } R.$$

Because of the inclusions $(*)$ this implies

$$S \cap R = S_{r} \cap R, \quad \Gamma = \Gamma_{s}, \quad S \cap (-R) = S'_{r} \cap (-R),$$

and these equalities remain valid if $R$ is replaced by $-R$ (multiply by $-1$). This proves equality in $(*)$. 

\[ \]
V.5.6.3 Coordinate-symmetric random walk

Let us call the random walk coordinate-symmetric if the distribution of the step ~ξ (see V.1.1) is symmetric in its two coordinates, i.e. the vectors (ξ₁, ξ₂) and (ξ₂, ξ₁) have the same distribution.

**PROPOSITION 22.** Let the random walk be skipfree and all-sided. Then the following are equivalent:

(94A) the random walk is coordinate-symmetric, 
(94B) \( D₁(\cdot, \cdot) = D₂(\cdot, \cdot) \).

**PROOF.**

(94A) ⇒ (94B): If (94A) holds, then the biquadratic polynomial \( Q((x₁, x₂), s) \) (see V.1.4) is symmetric in \( x₁ \) and \( x₂ \), which implies (94B).

(94A) ⇔ (94B): Assuming (94B) one has

(94C) \( b₁(\cdot) = b₂(\cdot) \) and \( a₁(\cdot) c₁(\cdot) = a₂(\cdot) c₂(\cdot) \).

We show that (94E) implies (94A) (hence, (94D) and (94E) are equivalent). With the notations I.3.23 for \( i=1 \) the condition (94C) can be rewritten in the form (*) and (**):

\[
(*) \quad \begin{cases} 
β₁ = α₀, \\
β₋₁ = γ₀,
\end{cases}
\]

\[
(**) \quad \begin{cases} 
α₇₁Y₁ = α₁α₋₁, \\
α₀Y₁ + α₁γ₀ = β₁α₋₁ + α₁β₋₁, \\
α₋₁Y₁ + α₀Y₀ + α₁γ₋₁ = γ₁α₋₁ + β₁β₋₁ + α₁γ₋₁, \\
α₀Y₋₁ + α₋₁γ₀ = β₁γ₋₁ + γ₁β₋₁, \\
α₋₁γ₋₁ = γ₁γ₋₁.
\end{cases}
\]

Because of (*) the conditions (***) imply

\[
(***) \quad \begin{cases} 
α₁(α₋₁ - γ₁) = 0, \\
α₀(α₋₁ - γ₁) = 0, \\
γ₀(α₋₁ - γ₁) = 0, \\
γ₋₁(α₋₁ - γ₁) = 0.
\end{cases}
\]
Observe that \((\alpha_1, \alpha_0, \gamma_0, \gamma_1) \neq (0,0,0,0)\), because otherwise \(L_1 + L_2 = 0\) a.s., which is excluded by all-sidedness. Hence, \((***)\) implies \(\alpha_{-1} = \gamma_1\), which together with \((*)\) implies \((94A)\).

\[\]

**PROPOSITION 23.**

Assume \((0)\) and fix \(s \in J\) (see \((34A)\)). Then the following are equivalent:

\[
\begin{align*}
(95A) & \quad D_1(w,s) = D_2(w,s) \text{ for all } w \in \mathcal{L}, \\
(95B) & \quad c_{1k} = c_{2k} \text{ for } k = 1, \ldots, 4, \\
(95C) & \quad f_1(v) = f_2(v) \text{ for all } v \in \mathcal{L}, \\
(95D) & \quad \alpha_{\dot{i}}, \beta_{\dot{i}}, \gamma_{\dot{i}}, \delta_{\dot{i}} \text{ do not depend on } \dot{i} \in \{1,2\}.
\end{align*}
\]

**PROOF.**

\((95A) \Rightarrow (95B)\) : Trivial from Theorem V.2.3 (Part III).

\((95B) \Rightarrow (95C)\) : Trivial from \((2)\).

\((95A) \Rightarrow (95C)\) : From Theorem V.4.1 we know

\[
\{f_1'(v)\}^2 = \omega_{\dot{i}}^2 D_{\dot{i}}(f_1'(v),s)
\]

for \(\dot{i} = 1,2\), where \(\omega_{\dot{i}}\) does not depend on \(\dot{i}\), see V.4.(29). This implies the claim.

\((95C) \Rightarrow (95D)\) : Obvious.

\((95C) \Rightarrow (95D)\) : The assumption \((95D)\) implies that \(f_1(.)\) and \(f_2(.)\) have the same zeros and the same poles (with the same multiplicities), and that both have the value 1 in \(\gamma_1 = \gamma_2\). This implies \((95C)\).

\[\]

**V.5.6.4 Even driftless random walk**

For even random walk (cf. I.1.(7)) one easily checks the implications

\[
\text{driftless } \Rightarrow \text{ symmetric } \Rightarrow \text{ coordinate-symmetric}.
\]

So all the foregoing (viz. Proposition 17 through Proposition 23) applies to even driftless random walk. In this case, we will express the function \(f_{\dot{i}}(.)\) in terms of \(\theta\)-functions and summarize some of its properties.
Let us introduce the elliptic function

\[(96A) \quad f(v|\tau) := \frac{\theta_2}{\theta_3} (2v|2\tau) = \frac{\theta_4}{\theta_3} (v^\tau | \frac{1}{2}\tau) \quad (\tau \tau = -1)\]

where \(\tau\) is a parameter from the open upper half plane (for the \(\theta\)-functions see III.2.(41)...(44)). Equivalently,

\[(96B) \quad f(v|\tau) := \frac{\theta_1}{\theta_3 \theta_4} (v + \frac{1}{4}|\tau).\]

To see the equivalence, check the periods 1, \(\tau\), the zeros \(\pm \frac{1}{4}\), the poles \(\frac{1}{4} + \frac{1}{2}\tau\), and the value 1 in \(v = \frac{1}{4}\tau\) of the right-hand members (e.g. use ERDÉLYI, MAGNUS, OBERHETTINGER, TRICOMI [1953; Sec. 13.19, Tables 8 and 9]. In the same way the following proposition can be proved (e.g. use (3)-(9D)-(91D)).

**Property 24.** If the random walk is skipfree, all-sided, driftless and even, then

\[(97) \quad f_z(v) = f_0(v|\tau) \quad (z=1,2)\]

with \(\tau\) given in (1) (cf. V.3.(13)-(17)).

Next we summarize some properties of the above function:

\[(98A) \quad f.(|\tau) \text{ has primitive periods } 1, \tau\]

\[(98B) \quad f.(|\tau) \text{ is even}\]

\[(98C) \quad f(v|\tau) + f(v + \frac{1}{2}|\tau) = 0\]

\[(98D) \quad f(v|\tau) \quad f(v + \frac{1}{2}|\tau) = 1\]

\[(98E) \quad f(\pm \frac{1}{4}|\tau) = 0, \quad f(\pm \frac{1}{4} + \frac{1}{2}\tau|\tau) = \infty\]

\[(98F) \quad f(\pm \frac{1}{4}|\tau) = 1\]

\[(98G) \quad f(\frac{1}{4} + \frac{1}{4}\tau|\tau) = -i\]

\[(98H) \quad f(v^\tau | \tau) = \frac{1-f(v|\tau)}{1+f(v|\tau)} \quad (\tau \tau = -1)\]

\[(98I) \quad f(v^*| -\tau^*) = f(v|\tau)^* \text{ (where the star denotes complex conjugation).}\]
The properties (98A) through (98F) have been proved, and we leave the remaining ones to the reader. The function $f(v|\tau)$ is closely related to Jacobi's $sn$-function, viz. as follows:

\begin{equation}
(99) \quad f(v + \frac{1}{4}|\tau) = -c \, sn(4Kv, k)
\end{equation}

with the constant $c = f(0|\tau)$ given in V.3.(15), and the modulus $k$ and the real quarter period $K$ given by

\begin{align}
(100) \quad k &= c^2 \\
(101) \quad 4K &= \frac{\omega_0(\tau)}{1 + c^2} > 0
\end{align}

where the constant $\omega_0(\tau) > 0$ is given in III.1.(19); this constant can be rewritten as follows:

\begin{equation}
(102) \quad \omega_0(\tau) = 4 \frac{\varphi_4}{\varphi_4} \frac{\varphi_4^2}{\varphi_3^2 - \varphi_2^2} (0|\tau).
\end{equation}

The relationship (99) can be proved with the help of the differential equations determining the $sn$-function and the function $f(.)|\tau)$, respectively (cf. Theorem V.4.1, and use V.2.(26), V.3.(10) and V.4.(10)).

\[\dagger\] The equality (98H) can also be obtained by a straightforward calculation, as follows. From III.2.(43)-(44) compute

\begin{align}
(103) \quad (\theta_3 - \theta_2)(2v|2\tau) &= \theta_4(v|\frac{1}{2}) \\
(\theta_3 + \theta_2)(2v|2\tau) &= \theta_3(v|\frac{1}{2})
\end{align}

and with (96A) and (98B) this leads to (98H).
V.6 The curve \( \{ x \in U^2 : Q(x, s) = 0 \} \)

In this section our assumptions are the same as in the sections 5 and 4, viz. V.4.(0); these assumptions are those made in Theorem V.2.3, and are the following:

\[
\begin{cases}
\text{the random walk is skipfree and all-sided} \\
0 < s \leq 1 \\
(s, E\xi) \neq (1, (0,0))
\end{cases}
\]

(0)

where \( E\xi \) is the drift of the random walk.

Our aim in this section is twofold. Firstly, we will check whether the generalized open half planes \( A_{\ddot{z}}, B_{\ddot{z}} \) \((\ddot{z}=1,2)\), which are defined in V.5.(58) and (by Corollary V.5.9) satisfy the condition (S1) from Subsection IV.1.2, also satisfy the conditions (S2) and (S3) from Subsection IV.1.2. It turns out (see Theorem 10) that under the assumptions (0) the condition (S2) is satisfied, and that the condition (S3) holds under the additional assumption \( 0 < s < 1 \) (i.e. in the case (i) from Theorem V.1.1), and partially fails to hold under the complementary assumption \( s=1 \) (i.e. in the case (ii) from Theorem V.1.1). Secondly, we will investigate the curve

\[ K_1(s) := \{ x \in U^2 : Q(x, s) = 0 \} \]

(1)

where \( U \) is the open unit disc. The results can be found in Theorem 12.

V.6.1 Relations between the generalized strips

Consider the generalized strips \( S_{k}, S'_{k} \) \((k=1,2)\), defined in V.5.(35)-(36). We investigate the relations between the strips \( S_{1}, S'_{1} \) (and their boundaries) on the one hand, and the strips \( S_{2}, S'_{2} \) (and their boundaries) on the other hand.

Notation. As before \( \ddot{z} \in \{1,2\} \) and \( j:=3-\ddot{z} \).

The following proposition extends Proposition II.2.25 (which holds if \( s=1 \)) to the case \( 0 < s < 1 \).
PROPOSITION 1. Let the random walk be skipfree and all-sided. Assume 
0 < s ≤ 1. Fix i ∈ {1,2}.
Fix an arbitrary x₁, satisfying |x₁| = 1.
Call the two values of xₙ satisfying Q((x₁,x₂), s) = 0 xₙ' and xₙ''
in such a way that |xₙ'| ≤ |xₙ''|. Then the inequalities 

\[ 0 ≤ |xₙ'| < 1 < |xₙ''| ≤ ∞ \]

hold in each one of the following three cases:
(i) 0 < s < 1,
(ii) s = 1, x₁ ≠ 1, and the random walk is non-even,
(iii) s = 1, x₁ ≠ ± 1, and the random walk is even.

PROOF. The cases (ii) and (iii) are considered in Proposition II.2.25. So we only consider case (i), which is the simplest one.
Assume 0 < s < 1. We use the notations introduced in V.1.(4)-(6) for the coefficient polynomials in the biquadratic form x ~ Q(x,s). We apply:

\[ a₁(1) + b₁(1) + c₁(1) = 1, \ b₁(1) ≥ 0, \text{ and (because of all-sidedness) } a₁(1) > 0 \text{ and } c₁(1) > 0. \]

Also we apply the following inequalities (from the triangle inequality and |x₁| = 1):

\[ |a₁(x₁)| ≤ a₁(1), \ |b₁(x₁)| ≤ b₁(1), \ |c₁(x₁)| ≤ c₁(1), \]
\[ 1 - s, b₁(1) ≤ |x₁ - s b₁(x₁)|. \]

We show the claim. First assume \( a₁(x₁) = 0 \). Then we have

\[ xₙ'' = ∞ \text{ and } xₙ' = \frac{s c₁(x₁)}{x₁ - s b₁(x₁)}. \]

Hence,

\[ |xₙ'| ≤ \frac{s c₁(1)}{1 - s b₁(1)} < \frac{s c₁(1)}{1 - s + s c₁(1)} < 1. \]

Next suppose \( a₁(x₁) ≠ 0 \), or both \( xₙ' \) and \( xₙ'' \) are finite. Now we have
\[(1 - |x_j^2|) (1 - |x^0_j|) \leq 1 - |x^0_j + x_j| + |x_jx^0_j| = s |\sigma_i(x_i) / \sigma_i(x_i)| - \{ s |\sigma_i(x_i) / \sigma_i(x_i)| - |x_i^0| - s \delta_i(x_i) | + s |c_i(x_i)| \} \leq \frac{s - 1}{s |\sigma_i(x_i)|} < 0, \text{ which proves the claim.}\]

\[\Box\]

The following proposition relates the constants \( \gamma_1 \) and \( \gamma_2 \) (defined in V.5.(27)) and also the constants \( \delta_1 \) and \( \delta_2 \) (defined in V.5.(31)).

**Proposition 2.** Assume (0).

I. The following inequalities hold:

\[(3) \quad - \sigma \leq \gamma_1 - \gamma_2 \leq \sigma \leq \gamma_1 + \gamma_2 \leq 1 - \sigma, \]

\[(4) \quad - \sigma \leq \delta_1 - \delta_2 \leq \sigma \leq \delta_1 + \delta_2 \leq 1 - \sigma. \]

II. If \( 0 < s < 1 \) then all inequalities in (3) and (4) are strict.

III. If \( s = 1 \) then in (3) at least one equality holds.

IV. If \( s = 1 \) and the random walk is non-even, then all inequalities in (4) are strict.

V. If the random walk is even then \( \gamma_i = \delta_i \) for \( i = 1, 2 \) (so in (4) at least one equality holds if \( s = 1 \)).

**Remark 1.** For fixed \( i \in \{1, 2\} \) and \( j = 3 - \hat{i} \) the inequalities (3) can be rewritten in the form

\[(5) \quad - \gamma_j \leq \gamma_i - \sigma \leq \gamma_j \leq \sigma \leq 1 - \gamma_j, \]

and similarly the inequalities (4) in the form

\[(6) \quad - \delta_j \leq \delta_i - \sigma \leq \delta_j \leq \sigma \leq 1 - \delta_j. \]

**Proof** of Proposition 2. Fix \( i \in \{1, 2\} \) and \( j = 3 - \hat{i} \). Parts II and IV: Assuming \( 0 < s < 1 \) we prove (5) with strict inequalities. Apply Theorem V.4.8 (Part III) with \( v := \gamma_i \tau \) to conclude from Proposition 1 that either for \( \mu = + \) or for \( \mu = - \).
\[ |f_j(\gamma \cdot \tau - \mu \sigma \tau)| < 1 < |f_j(\gamma \cdot \tau + \mu \sigma \tau)|, \]

or, equivalently (by V.5.(38)-(39)), there exist integers \( k \) and \( k' \) such that

\[(\ast) \quad \gamma \cdot \tau - \mu \sigma \tau + k \tau \in S_j \quad \text{and} \quad \gamma \cdot \tau + \mu \sigma \tau + k' \tau \in S_j' + \frac{1}{2} \tau \]

or, equivalently, (by V.5.(47))

(a) \[ - \gamma_j < \gamma - \mu \sigma + k < \gamma_j \]

and (by V.5.(52))

(b) \[ \gamma_j < \gamma + \mu \sigma + k' < 1 - \gamma_j. \]

Add (a) and (b), and use V.5.(29), to conclude \( k + k' = 0 \). Hence (subtract (a) from (b)) \( 0 < 2\mu \sigma - 2k < 1 \), or \(-1/2 + \mu \sigma < k < \mu \sigma.\)

Because of \( |\mu \sigma| < 1/2 \) (see V.5.(5)) this implies \( k = 0 \) (so \( k' = 0 \)), hence \( \mu = + \) because of \( \sigma > 0 \). Substitute this in (a) and (b) to find (5) with strict inequalities.

Assuming either \( s = 1 \) and non-even random walk or \( 0 < s < 1 \) one shows (6) with strict inequalities in the same way from Proposition 1 by applying Theorem V.4.8 (Part III) with \( v = \delta \tau + 1/2.\)

Part I: By Part II the inequalities (5) and (6) hold if \( 0 < s < 1.\)

Now use continuity (see Proposition V.5.5 (Part II)) to conclude that (5) and (6) hold for \( s = 1.\)

Part III: Assume \( s = 1.\) From I.3.(40) recall \( Q((1,1), s) = 0 \) for \( s = 1.\)

Hence (Theorem V.4.8, Part III) \( f_j(\gamma \cdot \tau) = 1 = f_j(\gamma \cdot \tau + \lambda \sigma \tau) \) for some \( \lambda \in \{ -, +\}, \) or \( \gamma \cdot \tau + \lambda \sigma \tau = \mu \gamma \cdot \tau + h + k \tau \) for some \( \mu \in \{ -, +\} \) and some \( h, k \in \mathbb{Z}. \) Obviously \( h = 0, \) so \( \gamma + \lambda \sigma = \mu \gamma + k. \)

This implies, because of \( \gamma_1, \gamma_2 \in [0,1/2] \) and \( \sigma \in (0,1/2), \) that in (5) at least one equality holds.

Part V: See V.5.(33A).

\( \Box \)
The following proposition deals with the curves $\Gamma_k, \delta_k$ \((k=1,2)\) defined in V.5.(63)-(64). Recall $\delta_k = -\Gamma_k$, cf. V.5.(67).

**PROPOSITION 3.** Assume (0).

Fix $i \in \{1,2\}$ and $j = 3-i$. Then the following equalities hold, in which the abbreviation

\[
G_i := \begin{cases} 
\gamma_i^\tau + \mathbb{Z} & \text{if the random walk is non-even} \\
\gamma_i^\tau + \frac{1}{2} \mathbb{Z} & \text{if the random walk is even}
\end{cases}
\]

is used.

(7A) 
\[
(\delta_j + \sigma \tau) \cap \Gamma_i = \begin{cases} 
-G_j^* + \sigma \tau = G_i^* & \text{if } \gamma_i^\tau + \gamma_j = \sigma \\
\emptyset & \text{otherwise},
\end{cases}
\]

(7B) 
\[
\Gamma_i \cap (\Gamma_j^* + \sigma \tau) = \begin{cases} 
G_i = G_j^* + \sigma \tau & \text{if } \gamma_i - \gamma_j = \sigma \\
\emptyset & \text{otherwise},
\end{cases}
\]

(7C) 
\[
(\Gamma_j^* + \sigma \tau) \cap (\delta_j + \tau) = \begin{cases} 
G_j^* + \sigma \tau = -G_i^* + \tau & \text{if } \gamma_i^\tau + \gamma_j = 1-\sigma \\
\emptyset & \text{otherwise},
\end{cases}
\]

(7D) 
\[
\delta_i \cap (\delta_j + \sigma \tau) = \begin{cases} 
-G_j^* = -G_i^* + \sigma \tau & \text{if } \gamma_i^\tau - \gamma_j = -\sigma \\
\emptyset & \text{otherwise}.
\end{cases}
\]

**REMARK 2.** The above proposition implies that all four intersections are empty if \(0 < s < 1\), because of Proposition 2 (Part II).

**PROOF** of Proposition 3. We prove (7A) and leave the remaining claims to the reader. First, assuming that there exists an element \(v\) of the intersection on the left such that \(\text{Re } v \notin \frac{1}{2} \mathbb{Z}\), we derive a contradiction.

So assume \(v = z_i^\tau + \sigma \tau = z_j^\tau\) with \(z_i \in \Delta_j, z_j \in \Gamma_i\), and \(\text{Re } v \notin \frac{1}{2} \mathbb{Z}\).

Put \(x_k := f_k(z_k^\tau)\) for \(k=1,2\). Then \(|x_k^\tau| = 1\), and \(x_k \neq \pm 1\). By Theorem V.4.8 (Part I) the pair \(x := (x_1, x_2)\) satisfies \(Q(x,s) = 0\), which contradicts Proposition 1. So such a \(v\) does not exist.

Secondly, one has from V.5.(67)-(68)

\[
(\delta_j + \sigma \tau) \cap \Gamma_i \cap (i \mathbb{R}) = \{ -\gamma_j^\tau + \sigma \tau \} \cap \{ \gamma_i^\tau \} = \begin{cases} 
\{ -\gamma_j^\tau + \sigma \tau \} = \{ \gamma_i^\tau \} & \text{if } \gamma_i^\tau + \gamma_j = \sigma \\
\emptyset & \text{otherwise}.
\end{cases}
\]
Finally, one has from V.5.(67)-(69)

\[
(\delta_i + \sigma \tau) \cap \Gamma_i \cap (\tau \mathbf{R} + 1/2) = \{ -\delta_i \tau + 1/2 + \sigma \tau \} \cap \{ \delta_i \tau + 1/2 \} = \\
= \begin{cases} 
\{ -\delta_i \tau + 1/2 + \sigma \tau \} = \{ \delta_i \tau + 1/2 \} & \text{if } \delta_i + \delta_j = \sigma, \\
\emptyset & \text{otherwise};
\end{cases}
\]

from Proposition 2 conclude that for non-even walk $\delta_i + \delta_j \neq \sigma$, and that for even walk $\delta_k = \gamma_k$ for $k=1,2$, hence $\delta_i + \delta_j = \gamma_i + \gamma_j$.

The above easily implies the claim.

\[\square\]

**PROPOSITION 4.** Assume \((0)\). Then the following holds for $i=1,2$:

\[
(8) \quad \Gamma_i - \sigma \tau \subset S_j \quad \text{iff} \quad \gamma_i - \gamma_j < \sigma < \gamma_i + \gamma_j
\]

\[
(8') \quad \Gamma_i + \sigma \tau \subset S_j^\prime + \frac{1}{2} \tau \quad \text{iff} \quad \gamma_i - \gamma_j < \sigma < 1 - (\gamma_i + \gamma_j)
\]

**REMARK 3.** The above proposition implies that the inclusions in \((8)\) and \((8')\) on the left hold if \(0 < \sigma < 1\), because of Proposition 2 (Part II).

**PROOF of Proposition 4.** We show \((8)\) and leave \((8')\) to the reader.

"If": Assume $\gamma_i - \gamma_j < \sigma < \gamma_i + \gamma_j$. Because of (7A) and (7B) this implies (cf. V.5.(80))

\[
(\ast) \quad \Gamma_i - \sigma \tau \text{ does not intersect the boundary of } S_j.
\]

Because of V.5.(47) our assumption also implies $\gamma_i \tau - \sigma \tau \in S_j$, hence by V.5.(68)

\[
(\ast\ast) \quad (\Gamma_i - \sigma \tau) \cap S_j \neq \emptyset.
\]

Because $\Gamma_i - \sigma \tau$ is connected (Property V.5.12 (Part II)) it follows from \((\ast)\) and \((\ast\ast)\) that $\Gamma_i - \sigma \tau \subset \text{Int } S_j = S_j$.

"Only if": Assume $\Gamma_i - \sigma \tau \subset S_j$. Then in particular $\gamma_i \tau - \sigma \tau \in S_j$, which (by V.5.(47)) implies $\gamma_i - \gamma_j < \sigma < \gamma_i + \gamma_j$.

\[\square\]
PROPERTY 5. Assume (0). Fix $i \in \{1, 2\}$.
Assume $v \in \Gamma_i^\nu$. Then:

\begin{align}
(9) & \quad v - \sigma t \subseteq S_j^i \quad \text{iff} \quad |f_j^i(v - \sigma t)| < 1,
& \quad v - \sigma t \subseteq aS_j^i \quad \text{iff} \quad |f_j^i(v - \sigma t)| = 1,
(9') & \quad v + \sigma t \subseteq S_j^i + \frac{1}{2} t \quad \text{iff} \quad |f_j^i(v + \sigma t)| > 1,
& \quad v + \sigma t \subseteq (aS_j^i) + \frac{1}{2} t \quad \text{iff} \quad |f_j^i(v + \sigma t)| = 1.
\end{align}

PROOF. We show the first claim in (9).
"If": Assume $|f_j^i(v - \sigma t)| < 1$ and $v \in \Gamma_i^\nu$. Then $v - \sigma t \in S_j^i + k\tau$ for some integer $k$ (by V.5.(38)) and $v - \sigma t \in \{w: \frac{\text{Im} w}{\text{Im} t} \in (-1/2, 1/2)\}$ (by V.5.(5)-(63)), where $S_j^i \subseteq \{w: \frac{\text{Im} w}{\text{Im} t} \in [-1/2, +1/2]\}$ (by V.5.(35)). This implies $k=0$, q.e.d.
"Only if": Left to the reader.
The proof of the second claim in (9) runs similarly (use V.5.(80), V.5. (66A), and V.5.(63)-(64)).
Proof of (9'): Left to the reader.
\[ \Box \]

PROPOSITION 6. Assume (0). Fix $i \in \{1, 2\}$.
Assume $v \in \Gamma_i^\nu$. Then

\begin{align}
(10) & \quad |f_j^i(v - \sigma t)| \leq 1 \leq |f_j^i(v + \sigma t)|.
\end{align}

If $0 < s < 1$ then both inequalities are strict.

PROOF. If $0 < s < 1$ then we have strict inequalities because of Property 5 and Remark 3. In order to prove (10) for $s=1$ we use a continuity argument, which runs as follows.
Write $f_k^i(.,s)$ for $f_k^i(\cdot)$ and $\Gamma_k^i(s)$ for $\Gamma_k^i$(where $k=1,2$) in order to indicate the dependence on $s$. Because of the period 1 we may assume $v \in \bar{\Gamma}_\lambda^\nu$ for $\lambda = +$ or $-$, cf. V.5.(11).
Assume $v \in \Gamma_i^\nu(1) \cap \bar{\Gamma}_\lambda^\nu$. We construct for $0 < s < 1$ an element $v(s)$ of $\Gamma_i^\nu(s)$ satisfying

\begin{align}
(\xi) & \quad v(s) + v \text{ if } s = 1.
\end{align}
Let \( w \sim \tilde{f}_j(w, s) \) be the restriction of \( w \sim f_j(w, s) \) to \( \mathbb{R}_+ \), and let 
\[ x^*_j \sim g_j(x^*_j, s) \] 
be the inverse of \( w \sim \tilde{f}_j(w, s) \), cf. Property V.5.1 (Part I). 
Put \( t = \tilde{f}_j(v, 1) \) and \( v(s) = g_j(t, s) \), so \( v = v(1) \). Because of \( |t| = 1 \) 
one has \( v(s) \in \Gamma_j(t, s) \), and (*) holds because of the continuity of 
\( s \sim g_j(t, s) \), see Proposition V.5.4 (Part II). 
Writing \( \sigma(s) \) for \( \sigma \), and \( \tau(s) \) for \( \tau \) we have 
\[ |f_j(v(s) - \sigma(s)\tau(s), s)| < 1 < |f_j(v(s) + \sigma(s)\tau(s), s)| \]
for \( s \in (0,1) \). This implies (10) for \( s=1 \) due to (*) and the continuity of 
\( \sigma(s) \), \( \tau(s) \), and \( f_j(w, s) \), see Proposition V.5.5 (Part I), Property 
V.5.3, and Proposition V.5.4 (Part I).

\[ \square \]

**Remark 4.** By V.4.(39) (and because \( f_j(\cdot) \) is even) the inequalities (10) can be rewritten as follows:

\[ |h_j(v)| \leq 1 \leq |h_j(-v)|. \]  

For the definition of \( h_j(\cdot) \) see V.4.(17),..., (20).

**Remark 5.** If \( x^*_j \) is an arbitrary value satisfying \( |x^*_j| = 1 \) and \( x^*_j \neq \pm 1 \), 
then the values \( x'_j, x''_j \) from Proposition 1 satisfy the inequalities (2). 
Therefore, using V.4.(22) one concludes from Remark 4

\[ \begin{cases} 
\text{if } v \in \Gamma_j^1, \text{ Re } v \in \frac{1}{2} \mathbb{Z}, \text{ and } x^*_j = f_j(v), \text{ then} \\
\quad x'_j = h_j(v) = f_j(v - \sigma \tau), \quad x''_j = h_j(-v) = f_j(v + \sigma \tau). 
\end{cases} \]  

**Corollary 7.** Assume (0). Then for \( i=1,2 \)

\[ \Gamma_i - \sigma \tau \subset \tilde{\Gamma}_j, \]

\[ \Gamma_i + \sigma \tau \subset \tilde{\Gamma}_j + \frac{1}{2} \tau. \]

**Proof.** From Proposition 6 and Property 5.

\[ \square \]
V.6.2 Relations between the generalized half planes

We next consider the generalized open half planes defined in V.5.(58) - (59). We consider four cases:

\[
\begin{align*}
(A, B) : &= \left( A_j + \sigma \tau, \ B_j \right), \\
(3A, 3B) &= \left( A_j + \sigma \tau, \ B_j \right), \\
(14A) \\
(A, B) : &= \left( A_j + \frac{1}{2} \tau, \ B_j + \sigma \tau \right), \\
(3A, 3B) &= \left( A_j, \ B_j + \sigma \tau \right), \\
(14B) \\
(A, B) : &= \left( A_j + \frac{1}{2} \tau + \sigma \tau, \ B_j + \frac{1}{2} \tau \right), \\
(3A, 3B) &= \left( A_j + \sigma \tau, \ B_j + \tau \right), \\
(14C) \\
(A, B) : &= \left( A_j, \ B_j - \frac{1}{2} \tau + \sigma \tau \right), \\
(3A, 3B) &= \left( A_j, \ B_j + \sigma \tau \right). \\
(14D)
\end{align*}
\]

The statements about boundaries follow from V.5.(81)-(83)-(65)-(66).

**Proposition 8.** Assume \((0)\) and fix \(i \in \{1, 2\}\).

I. Then one has in each one of the cases \((14A), \ldots, (14D)\)

\[
A \cup \overline{B} = C = A \cup B.
\]

II. Furthermore, the stronger equality

\[
A \cup B = \mathcal{C}
\]

holds iff

\[
\begin{align*}
\gamma_i + \gamma_j &\not\in \sigma \quad \text{in case } (14A), \\
\gamma_i - \gamma_j &\not\in \sigma \quad \text{in case } (14B), \\
\gamma_i + \gamma_j &\not\in 1-\sigma \quad \text{in case } (14C), \\
\gamma_i - \gamma_j &\not\in -\sigma \quad \text{in case } (14D).
\end{align*}
\]
PROOF. Observe that the pair of equalities (15) is equivalent to

\[ (A \cup B) \cup [(3A) \cap (3B)] = \emptyset, \]

which is a disjoint union because \( A \) and \( B \) are open. By Property IV.6.5, in order to prove (17) it suffices to show

(i) \( C \setminus [(3A) \cap (3B)] \) is connected,

(ii) \( 3A \subset \overline{C} \) and \( 3B \subset \overline{C} \).

Obviously, if (17) holds then (16) is equivalent to

(iii) \( (3A) \cap (3B) = \emptyset \).

We check (i), (ii), (iii) in case (14A). Claim (i) holds because of (7A). Claim (ii) follows from (13) because of \( \overline{S}_j \subset \overline{A}_j \cap \overline{B}_j \) (cf. V.5.(62)). Hence, (15) holds. Finally, (7A) implies that (iii) holds iff \( \gamma_i + \gamma_j = \sigma \), q.e.d.

Similarly one checks the remaining claims using (7B), (7C), (7D), and (13), (13').

\[ \Box \]

COROLLARY 9. The following inclusions hold for \( i = 1, 2 \) and \( j = 3-i \):

\[ A_i \supset A_j + \sigma \tau \supset A_i' + \frac{1}{2} \tau \supset A_i + \frac{1}{2} \tau + \sigma \tau \supset A_i + \tau, \]

\[ B_i' - \frac{1}{2} \tau + \sigma \tau \subset B_i \subset B_j + \sigma \tau \subset B_i' + \frac{1}{2} \tau + \sigma \tau \subset B_j' + \frac{1}{2} \tau + \sigma \tau. \]

If \( 0 < s < 1 \) then each inclusion can be read as follows: the closure of the smaller set is contained in the larger set.

PROOF. We show (18A). The inclusions follow from (15) (first equality) in the four cases (14A),..., (14D) as follows. For the first inclusion apply (15) (case (14D)), and use V.5.(87) to find

\[ A_i \supset (B_j')^c - \frac{1}{2} \tau + \sigma \tau = A_j + \sigma \tau. \]
For the second, third, and last inclusion apply in the same way (15)
cases (14A), (14B), and (14C), respectively, and use V.5.86)-87).
Proof of (18B): Multiply (18A) by $-1$, add $\sigma \tau$, and interchange $i$ and $j$.

Proof of the last claim. This claim follows by using in the above proof
(16) rather than (15); observe that (16) holds in all four cases
(14A), ..., (14D) if $0 < s < 1$, cf. Proposition 2 (Part II).

We notice that (18A) and (18B) can be rewritten as follows:

(19A) \[ A'_i - \frac{1}{2} \tau \supset A'_j - \frac{1}{2} \tau + \sigma \tau \supset A'_i \supset A'_j + \sigma \tau \supset A'_i + \frac{1}{2} \tau, \]

(19B) \[ B'_i - \frac{1}{2} \tau \subset B'_j - \frac{1}{2} \tau + \sigma \tau \subset B'_i \subset B'_j + \sigma \tau \subset B'_i + \frac{1}{2} \tau. \]

**THEOREM 10.** Assume (0).

I. Let the constant $\tau$ be defined by V.5.(1), and the constant $\sigma$ by
V.5.(5)-(6). Then the constant $\sigma \tau$ satisfies the condition (S0) from
Subsection IV.1.2.

II. The two pairs of sets $(A'_1, A'_2)$ and $(B'_1, B'_2)$ defined in V.5.58)
satisfy the conditions (S1) and (S2) from Subsection IV.1.2.

The condition (S3) from Subsection IV.1.2 is satisfied iff

(20) \[ \gamma_1 + \gamma_2 \neq \sigma. \]

III. The condition (20) is satisfied if $0 < s < 1$.

The condition (20) can be translated into a simple condition on $s$ and
the drift vector of our random walk, see (31A).

**PROOF.** Part I. Obvious.

Part II. For (S1) see Corollary V.5.9.

For (S2) see (19A) and (19B).

For (S3) see (16) in case (14A).

Part III. See Proposition 2 (Part II).

\[ \Box \]
V.6.3 The set \( \{ x \in U^2 : Q(x, a) = 0 \} \) and related sets

**Proposition 11.** Assume (0).

Then the following holds for \( i=1,2 \) and \( j=3-i \):

\[
(21A) \quad (S_i^x + \mathbb{Z} \tau) \cap (S_j^x + \sigma \tau) = S_i^x \cap (S_j^x + \sigma \tau) \quad (= A \cap B \text{ in case } (14A)) ,
\]

which is a generalized open horizontal strip iff \( \gamma_i^x + \gamma_j^x \neq \sigma \);

\[
(21B) \quad (S_i^y + \frac{1}{2} \tau + \mathbb{Z} \tau) \cap (S_j^y + \sigma \tau) = (S_i^y + \frac{1}{2} \tau) \cap (S_j^y + \sigma \tau) \quad (= A \cap B \text{ in case } (14B)) ,
\]

which is a generalized open horizontal strip iff \( \gamma_i^y - \gamma_j^y \neq \sigma \);

\[
(21C) \quad (S_i^x + \frac{1}{2} \tau + \mathbb{Z} \tau) \cap (S_j^x + \frac{1}{2} \tau + \sigma \tau) = (S_i^x + \frac{1}{2} \tau) \cap (S_j^x + \frac{1}{2} \tau + \sigma \tau) \quad (= A \cap B \text{ in case } (14C)) ,
\]

which is a generalized open horizontal strip iff \( \gamma_i^x + \gamma_j^y \neq 1-\sigma \);

\[
(21D) \quad (S_i^x + \mathbb{Z} \tau) \cap (S_j^y - \frac{1}{2} \tau + \sigma \tau) = S_i^x \cap (S_j^y - \frac{1}{2} \tau + \sigma \tau) \quad (= A \cap B \text{ in case } (14D)) ,
\]

which is a generalized open horizontal strip iff \( \gamma_i^x - \gamma_j^y \neq -\sigma \).

All above intersections are generalized open horizontal strips if \( 0 < s < 1 \).

**Proof.** Proof of (21A): The set on the left equals the union \( \bigcup_{k \in \mathbb{Z}} D^x_i(k) \) where

\[
D^x_i(k) := (S_i^x + k\tau) \cap (S_j^x + \sigma \tau) .
\]

Use V.5.(5) and the definition V.5.(35) to conclude that \( D^x_i(k) = \emptyset \) if \( k \notin \{0,1\} \). Next we show \( D^x_i(1) = \emptyset \). From V.5.(62) it follows

\[
(a) \quad D^x_i(k) = (A_i^x + k\tau) \cap (B_i^x + k\tau) \cap (A_j^x + \sigma \tau) \cap (B_j^x + \sigma \tau) .
\]

Hence, by the inclusions (19A) - (19B).
\[ \mathcal{D}_i(1) = (A_i + \tau) \cap (B_j + \sigma \tau) \]

or, by V.5.(87),

\[ \mathcal{D}_i(1)^c = (B_i - \frac{1}{2} \tau) \cup (A_j - \frac{1}{2} \tau + \sigma \tau) = \mathcal{E} \]

by (15) (case (14D)). So \( \mathcal{D}_i(1) = \emptyset \), which proves the first equality.

The second equality follows from (a) with \( k=0 \), because of the inclusions (19A) - (19B).

Proof of (21B): We put

\[ \mathcal{D}_i(k) := (S_i + \frac{1}{2} \tau + k \tau) \cap (S_j + \sigma \tau), \]

and have (as above) \( \mathcal{D}_i(k) = \emptyset \) if \( k \notin \{-1,0\} \).

Also we have

\[ \mathcal{D}_i(k) = (A_i + \frac{1}{2} \tau + k \tau) \cap (B_i - \frac{1}{2} \tau + \sigma \tau) \cap (A_j + \sigma \tau) \cap (B_j + \sigma \tau). \]

Hence, by the inclusions (19A) - (19B),

\[ \mathcal{D}_i(-1) = (A_j + \sigma \tau) \cap (B_i - \frac{1}{2} \tau), \]

or, by V.5.(87)-(86),

\[ \mathcal{D}_i(-1)^c = (B_j - \frac{1}{2} \tau + \sigma \tau) \cup A_i = \mathcal{E} \]

by (15) (case (14D)). So \( \mathcal{D}_i(-1) = \emptyset \), which proves the first equality.

The second equality follows from (b) with \( k=0 \), because of the inclusions (19A) - (19B).

Proof of (21C): Similar to the proof of (21A).

Proof of (21D): We put

\[ \tilde{\mathcal{D}}_i(k) := (S_i + k \tau) \cap (S_j - \frac{1}{2} \tau + \sigma \tau) \]

and have, with \( \mathcal{D}_i(k) \) given by (*),

\[ \tilde{\mathcal{D}}_i(k) = - \mathcal{D}_j(k) + \sigma \tau + k \tau. \]
So (21D) easily follows from (21B).
For the remaining claims apply Proposition IV.1.2 (check (iii) to prove (i)), Proposition 8 (Part II), and Proposition 2 (Part II).

\[ \square \]

**THEOREM 12.** Assume (0). Fix \( \lambda_1, \lambda_2 \in \{-, +\} \).

Let \( z_1 \) and \( z_2 \) be related through

\[ \lambda_1 z_1 + \lambda_2 z_2 = \tau, \]

and let \( x = (x_1, x_2) \) and \( z = (z_1, z_2) \) be related through

\[ x_k = f_k(z_k) \text{ for } k = 1, 2 \]

(for the functions \( f_k(.) \) see V.5.(1)-(2)).

I. Then for \( i \in \{1, 2\} \) the mapping \( z_i \mapsto x \) is a bijection

\[ [S_i \cap (S_j + \lambda_i \tau)]/\mathbb{Z} \sim K_i(s) \]

(for the set \( K_i(s) \) see (1)).

II. One has

\[ K_1(s) = \{ x \in \mathbb{Z}^2 : \exists z \in S_1 \times S_2 \text{ such that (22) and (23) hold} \}. \]

**PROOF.** Part I. Fix \( i \) and \( j \). "Into": Assume (22) and (23), and \( z_i \in S_i \cap (S_j + \lambda_i \tau) \). Then \( z_j \in S_j \cap (S_i + \lambda_j \tau) \) (because the sets \( S_k \) are even, cf. V.5.(44)), so \( z \in S_1 \times S_2 \), hence \( x \in \mathbb{Z}^2 \) by V.5.(38).

Also \( x \in K(s) \) by Theorem V.4.8 (Part I), which implies \( x \in K_1(s) \).

"Onto": Assume \( x \in K_1(s) \). By Theorem V.4.8 (Part II) there exists a \( z \) satisfying (22) and (23). Because of the period \( \tau \) we may assume \( |\text{Im } z_j| \leq 1/2 \text{ Im } \tau \). From V.5.(38) it follows \( z_k \in S_k + \mathbb{Z} \tau \) for \( k = 1, 2 \), hence (cf. V.5.(35)) \( z_i \in S_i \), or \( z_i \in S_j + \lambda_i \tau \). So

\[ z_i \in (S_i + \mathbb{Z} \tau) \cap (S_j + \lambda_i \tau) = S_i \cap (S_j + \lambda_i \tau), \]

where the last equality is (21A) multiplied by \( \lambda_i \). This implies surjectivity.

"One-to-one": By Theorem V.4.8 (Part I) this mapping is injective on the larger set \( S_i/\mathbb{Z} \).

Part II. This is a trivial consequence of Part I.

\[ \square \]
Consider the following four open subsets of $K(s)$ (see V.1.7), viz. (for $K_1(s)$ also see (1))

\[
\begin{align*}
K_1(s) &= \{ x \in K(s) : |x_1| < 1 \text{ and } |x_2| < 1 \}, \\
K_2(s) &= \{ x \in K(s) : |x_1| > 1 \text{ and } |x_2| < 1 \}, \\
K_3(s) &= \{ x \in K(s) : |x_1| > 1 \text{ and } |x_2| > 1 \}, \\
K_4(s) &= \{ x \in K(s) : |x_1| < 1 \text{ and } |x_2| > 1 \}.
\end{align*}
\]

(24A)

Also consider the following subsets of $K(s)$:

\[
\begin{align*}
C_0(s) &= \{ x \in K(s) : |x_1| = 1 = |x_2| \}, \\
C_1(s) &= \{ x \in K(s) : |x_1| < 1 \text{ and } |x_2| = 1 \}, \\
C_2(s) &= \{ x \in K(s) : |x_1| = 1 \text{ and } |x_2| < 1 \}, \\
C_3(s) &= \{ x \in K(s) : |x_1| > 1 \text{ and } |x_2| = 1 \}, \\
C_4(s) &= \{ x \in K(s) : |x_1| = 1 \text{ and } |x_2| > 1 \}.
\end{align*}
\]

(24B)

The above collection partitions $K(s)$. One has

\[
C_0(s) = \emptyset \text{ iff } 0 < s < 1
\]

(25)

(if $0 < s < 1$ then $C_0(s) = \emptyset$ by Proposition 1; if $s=1$ then $(1,1) \in C_0(s)$ because of $Q((1,1),1) = 0$, see I.3.40). Uniformize the curve $K(s)$ by means of the substitution (cf. V.4.23-24 and V.4.39)

\[
x = \Phi_1(v) = (f_1(v), f_2(v-\sigma t)).
\]

(26)

We determine the inverse images under this mapping of the above sets. Apply V.5.38-39 and (21A),...,(21D) to find

\[
\begin{align*}
x \in K_1(s) \iff & \quad v \in (S_1 + \tau) \cap (S_2 + \sigma + \tau) = \\
& = [(S_1 + \tau) \cap (S_2 + \sigma)] + \tau,
\end{align*}
\]

(27)

\[
\begin{align*}
x \in K_2(s) \iff & \quad v \in [(S_1' + 1/2 \tau) \cap (S_2 + \sigma)] + \tau,
\end{align*}
\]

\[
\begin{align*}
x \in K_3(s) \iff & \quad v \in [(S_1' + 1/2 \tau) \cap (S_2' + 1/2 \tau + \sigma)] + \tau,
\end{align*}
\]

\[
\begin{align*}
x \in K_4(s) \iff & \quad v \in [S_1 \cap (S_2' - 1/2 \tau + \sigma)] + \tau.
\end{align*}
\]
Under the assumption $0 < s < 1$ we next determine the inverse images under the mapping (26) of the sets (24C), using the following property.

**PROPERTY 13.** Assume (0) and $0 < s < 1$. Then the following holds for $i = 1, 2$ and $j = 3 - i$:

\[
\text{(28A)} \quad [S_{\frac{i}{j}} + \mathbb{Z} \tau] \cap [\sigma + \Delta_j] = \Delta_j + \sigma_j
\]

\[
\text{(28B)} \quad [(\sigma_j + S_{\frac{i}{j}}) \cap [\sigma + \Delta_j + \mathbb{Z} \tau]. = \Gamma_i
\]

\[
\text{(28C)} \quad [S_{\frac{i}{j}} + \frac{1}{2} \tau + \mathbb{Z} \tau] \cap [\sigma + \Delta_j] = \Gamma_j + \sigma_j
\]

\[
\text{(28D)} \quad [\sigma_i + S_{\frac{i}{j}} + \frac{1}{2} \tau + \mathbb{Z} \tau] = \Delta_i
\]

**PROOF.** Proof of (28B) and (28D): The inclusions "$\supseteq"$ follow (because of V.5.(80)) from (8) and (8') (the latter one multiplied by $-1$), respectively.

This implies the claims because the set on the left in (28B) is disjoint from the corresponding one in (28D), and the sets on the right unite to $\Delta_j$.

Proof of (28A): Multiply (28B) by $-1$, add $\sigma_j$, and interchange $i$ and $j$.

Proof of (28C): Similarly from (28D).

Let $x$ and $v$ be related by (26), and assume $0 < s < 1$. From V.5.(66A)-(80) one has for $k = 1, 2$

\[
|E_k(v)| = 1 \text{ iff } v \in (\sigma_j + \mathbb{Z} \tau),
\]

and one deduces using (29), V.5.(38)-(39), and (28A), ..., (28D) the following:

\[
\begin{cases}
x \in C_1(s) \text{ iff } v \in \Delta_2 + \sigma + \mathbb{Z} \tau, \\
x \in C_2(s) \text{ iff } v \in \Gamma_1 + \mathbb{Z} \tau, \\
x \in C_3(s) \text{ iff } v \in \Gamma_2 + \sigma + \mathbb{Z} \tau, \\
x \in C_4(s) \text{ iff } v \in \Delta_1 + \mathbb{Z} \tau.
\end{cases}
\]
The location of the four curves occurring in (30) (viz. the curves (28A), ..., (28D) with $i=1$ and $j=2$) has been illustrated, under the assumption $0 < s < 1$, in Figure 1. This figure, which extends Figure V.5.1, is a mere sketch and is not based on a computation. It also illustrates the location of the four strips occurring in (27) (viz. the strips (21A), ..., (21D) with $i=1$ and $j=2$). The generalized open half planes occurring in (14A), ..., (14D) with $i=1$ and $j=2$, whose boundaries appear in the sketch, are not indicated themselves; the reader is referred to Figure V.5.1.
V.6.4 On some conditions on the constants $\gamma_1$ and $\gamma_2$

Several propositions from the foregoing (e.g. the propositions 3, 8, and 11) contain conditions on the constants $\gamma_1$ and $\gamma_2$. We conclude this section with a translation of these conditions into simple conditions on $s$ and the drift vector of our random walk.

**Proposition 14.** Assume (0). Then the following holds:

\[\gamma_1 + \gamma_2 = \sigma \quad \text{iff} \quad (s=1, \mathbb{E} \frac{Y}{\xi_1} \leq 0, \mathbb{E} \frac{Y}{\xi_2} \leq 0),\]

\[\gamma_1 - \gamma_2 = \sigma \quad \text{iff} \quad (s=1, \mathbb{E} \frac{Y}{\xi_1} \geq 0, \mathbb{E} \frac{Y}{\xi_2} \leq 0),\]

\[\gamma_1 + \gamma_2 = 1-\sigma \quad \text{iff} \quad (s=1, \mathbb{E} \frac{Y}{\xi_1} \geq 0, \mathbb{E} \frac{Y}{\xi_2} \geq 0),\]

\[\gamma_1 - \gamma_2 = -\sigma \quad \text{iff} \quad (s=1, \mathbb{E} \frac{Y}{\xi_1} \leq 0, \mathbb{E} \frac{Y}{\xi_2} \geq 0).\]

For the stochastic vector $(\frac{Y}{\xi_1}, \frac{Y}{\xi_2})$ see V.1.(1)-(2).

**Proof.** The proof consists of four steps.

**First step.** We claim:

\[\gamma_1 + \gamma_2 = \sigma \quad \text{iff} \quad f_1(\gamma_2 \tau - \sigma \tau) = f_2(\gamma_1 \tau - \sigma \tau),\]

\[\gamma_1 - \gamma_2 = \sigma \quad \text{iff} \quad f_1(\gamma_2 \tau + \sigma \tau) = f_2(\gamma_1 \tau + \sigma \tau),\]

\[\gamma_1 + \gamma_2 = 1-\sigma \quad \text{iff} \quad f_1(\gamma_2 \tau + \sigma \tau) = f_2(\gamma_1 \tau + \sigma \tau),\]

\[\gamma_1 - \gamma_2 = -\sigma \quad \text{iff} \quad f_1(\gamma_2 \tau - \sigma \tau) = f_2(\gamma_1 \tau + \sigma \tau).\]

We show (32A) and leave the remaining claims to the reader.

If $\gamma_1 + \gamma_2 = \sigma$ then trivially $f_1(\gamma_2 \tau - \sigma \tau) = f_2(\gamma_1 \tau - \sigma \tau) = 1$ (cf. V.5.(27) for $i=1,2$ and $j=3-\iota$).

Conversely, assume $f_1(\gamma_2 \tau - \sigma \tau) = 1$ for $\iota=1,2$.

Then $\gamma_2 \tau - \sigma \tau \in \{\gamma_2 \tau, -\gamma_2 \tau\} + \mathbb{Z} \tau$ for $\iota=1,2$, hence (because of $\gamma_1, \gamma_2 \in [0, \frac{1}{2}]$ and $\sigma \in (0, \frac{1}{2})$) $\gamma_2 \tau - \sigma \in \{\gamma_2 \tau, -\gamma_2 \tau\}$ for $\iota=1,2$.

This implies (because of $\sigma \neq 0$) $\gamma_1 + \gamma_2 = \sigma$, which proves (32A).

**Second step.** Consider the four real functions $f_\iota^\lambda(\cdot)$, $i=1,2$, $\lambda=\iota$, defined in V.2.(48).
We claim: for \( i=1,2 \) and \( \lambda = \pm \)

\[(33)\quad f_j^\lambda(\gamma_j^i \tau + \lambda \sigma \tau) = f_j^\lambda(1).\]

Proof of (33): In the equation \( Q((x_1, x_2), s) = 0 \) substitute \( x_1^i = 1 \); the resulting equation in \( x_2^i \) has two solutions (which are positive and finite by \( V.2.34 \) and \( V.2.15A-15B \)), say \( x_2^i \) and \( x_2^\prime \), such that \( x_2^i \leq x_2^\prime \). On the one hand one has \( \{ x_2^i, x_2^\prime \} = \{ f_j^\lambda(1) \}, \) hence by \( V.2.48A \)

\[(\ast)\quad x_2^i = f_j^\lambda(1), \quad x_2^\prime = f_j^\lambda(1).\]

On the other hand one has
\[\{ x_2^i, x_2^\prime \} = \{ f_j^\lambda(\gamma_j^i \tau \pm \sigma \tau) \}\]
by Theorem \( V.4.8 \) (Part III). So \( 0 < f_j^\lambda(\gamma_j^i \tau \pm \sigma \tau) < \infty \), and hence \( f_j^\lambda(\gamma_j^i \tau - \sigma \tau) \leq f_j^\lambda(\gamma_j^i \tau + \sigma \tau) \) by (10). Consequently

\[x_2^i = f_j^\lambda(\gamma_j^i \tau - \sigma \tau) , \quad x_2^\prime = f_j^\lambda(\gamma_j^i \tau + \sigma \tau) ,\]

which proves (33).

Third step. We claim: for \( i=1,2 \) and \( \lambda = \pm \)

\[(34)\quad f_j^\lambda(1) = 1 \quad \text{iff} \quad (s=1, \lambda \equiv X, \gamma \geq 0).\]

Proof of (34): If \( 0 < s < 1 \) then \( f_j^\lambda(1) \neq 1 \) because of (\( \ast \)) and Proposition 1. So assume \( s=1 \). From the definition \( V.2.48 \) one has

\[f_j^\lambda(1) = \frac{1 - b_j^\lambda(1) + \lambda \sqrt{D_j}(1)}{2c_j^\lambda(1)}\]

Use I.3.31-(35)-(36)-(45) to conclude

\[f_j^\lambda(1) = \frac{\gamma_j^i + \lambda \gamma_j^\prime}{\gamma_j^i + \lambda \gamma_j^\prime}\]

Consequently, \( f_j^\lambda(1) = 1 \) iff \( \gamma_j^i = \lambda \gamma_j^\prime \), or \( \lambda \gamma_j^\prime \leq 0 \), which proves (34).

Fourth step. Combine (32A), ..., (32D), (33), and (34).

\[\Box\]
V.7 The solution of the backward conditions. Determination of the generating function 
\( P(x, y, s) \)

The assumptions are the same as in the preceding sections, viz.

\[
(0) \quad \begin{cases} 
\text{the random walk is skipfree and all-sided,} \\
\ s \in (0, 1) , \\
\ (s, \ \mathbb{E}_x^s) \neq (1, (0,0)) , 
\end{cases}
\]

where \( \mathbb{E}_x^s \) is the drift of our random walk. We assume that \( s \) is fixed. Also it is assumed that \( y = (y_1, y_2) \) is a fixed pair of numbers satisfying

\[
(1) \quad y \in (\overline{U})^2 ,
\]

where \( \overline{U} \) is the open unit disc.

Our aim in this section is the determination of the generating function \( P(x, y, s) \) of the transition probabilities of the random walk in the first quadrant with absorbing boundary, see I.1.(28). This generating function converges if \( (x, y, s) \) satisfies I.1.(29). In particular the function \( x \sim P(x, y, s) \) is well-defined for \( x \in \overline{U}^2 \) if

\[
(2) \quad y \in (\overline{U})^2 \text{ and } 0 < s < 1 .
\]

Furthermore, under the assumption (2) the function \( x \sim P(x, y, s) \) satisfies the backward conditions I.3.(B1'') (following I.3.(17)). It will be shown presently that, conversely, the conditions I.3.(B1'') determine the unknown function \( x \sim P(x, y, s) \) completely; consequently, this unknown function coincides with \( x \sim P(x, y, s) \). Also, this unknown function will be determined.

V.7.1 Transformation of the backward conditions.

In Sec. I.3 the conditions I.3.(B1'') were transformed into the conditions I.3.(B2'') (preceding Proposition I.3.10) by means of the substitution I.3.(18). This was possible by Proposition I.3.10 provided a certain condition is satisfied. We can now check this condition.
PROPERTY 1. Under the assumptions (0) and (1) the conditions I.3.\((B1')\) and I.3.\((B2')\) are equivalent through I.3.\((18)\), and also through I.3.\((20)\).

PROOF. By Proposition I.3.10 it suffices to check that every \(x \in \Omega^2\) is simple with respect to \(Q(.,s)\). Use Corollary I.3.14 (Part II) to conclude that this is true because neither of the discriminants has a multiple zero, see Theorem V.2.3 (Part III).

Next, the conditions I.3.\((B2')\) are transformed into the conditions \((B3'')\) by means of a parametrization of the curve \(K_i(s)\) (see I.3.\((12)\)). A parametrization is described in Theorem V.6.12. So we apply the substitution

\[
x_k := f_k(z_k) \quad \text{for} \quad k = 1, 2,
\]

and transform the triplet \((x_0, x_1, x_2)\) (see I.3.\((18)\)) into a triplet of functions \((z_0, z_1, z_2)\) defined by

\[
\begin{align*}
  z_0 &= z_0(y, s) = x_0(y, s), \\
  z_k(z_k, y, s) &= z_k(x_k, y, s) = x_k(x_k, y, s) \quad \text{for} \quad k = 1, 2, \quad \text{with} \quad x_k \quad \text{and} \quad z_k
\end{align*}
\]

Also we put (for \(P_0(x, y)\) see I.3.\((10)\))

\[
F_0(z) := P_0(x, y) = x_1(1-x_1y_1)^{-1} \cdot x_2(1-x_2y_2)^{-1}
\]

with \(z = (z_1, z_2)\) and \(x = (x_1, x_2)\) related by (3).

In the following we need for \(k = 1, 2\) the set \(S_k\) (see V.5.\((35)\)) and the constant \(a_k\) (see V.5.\((19), \ldots, (20)\)). It will be shown presently (see the next Property) that through (4) the set of conditions I.3.\((B2')\) is equivalent to the following set of conditions \((B3'')\):

\[
(B3'')
\]

\[
\begin{align*}
\text{Regularity:} & \quad \quad Z_k(.) \text{ is regular on } S_k, \text{ and has period } 1 \quad \text{(for } k = 1, 2) \\
\text{Symmetry:} & \quad \quad Z_k(.) \text{ is even function (for } k = 1, 2) \\
\text{Initial Value:} & \quad \quad Z_k(x_k + \frac{1}{2}) = 0 \quad \text{(for } k = 1, 2) \\
\text{Equations:} & \quad \quad \text{for fixed } \lambda_1, \lambda_2 \in \{+, -\} \text{ the equation } (6) \text{ is satisfied:}
\end{align*}
\]
\[
\begin{cases}
\text{if } (z_1, z_2) \in S_1 \times S_2 \text{ and } \lambda_1 z_1 + \lambda_2 z_2 = \sigma t \\
\text{then } Z_0 + Z_1(z_1) + Z_2(z_2) = F_0(z_1, z_2).
\end{cases}
\]

Observe that the function \( F_0(z_1, z_2) \) is even in \( z_1 \) and \( z_2 \) (because the functions \( f_k(\cdot) \) are even), and that the sets \( S_k \) are even (cf. V.5.(44)). Hence, the four (viz. for different \( \lambda_1, \lambda_2 \)) statements (6) are equivalent.

**PROPERTY 2.** Assume (0) and (1). Then the conditions I.3.(B2') and (B3') are equivalent through (4).

**PROOF.** The proof is similar to the proof of Proposition II.2.33. Assume that I.3.(B2') is satisfied, and consider (B3'). The regularity and the symmetry condition are satisfied because of V.5.(42) and because \( f_k(\cdot) \) is regular on \( S_k \), is even and has period 1. The initial value condition is satisfied because of the definition of \( \alpha_k \). Finally, the equations (6) are satisfied because of Theorem V.6.12 (Part II) and (5).

Conversely, assume that (B3') is satisfied. First observe that the relations (4) do define \( X_k(\cdot) \) as a function on \( U \) because of V.5.(42) and the implication (cf. III.1.(5) and V.5.(35)-(46))

\[
v, v' \in S_k \text{ and } f_k(v) = f_k(v') \Rightarrow v \neq v' \in Z \Rightarrow Z_k(v) = Z_k(v').
\]

Secondly observe that \( X_k(\cdot) \) is regular on \( U \) because of Lemma II.1.6.

The remaining claims are left to the reader.

We rewrite the equations (6) in a slightly different form. We put

\[
\varphi_k(v) := \frac{x_k}{1 - x_k y_k}, \quad \text{with } x_k := f_k(v)
\]

for \( k = 1, 2 \), and define

\[
F^1_\varepsilon(v) := \varphi_\varepsilon(v) \varphi_j(v - \lambda \sigma t)
\]

for \( \varepsilon = 1, 2 \) (with \( j = 3 - \varepsilon \)) and \( \lambda = \pm \). Observe (cf. (5)) that

\[
F_0(z_1, z_2) = \varphi_\varepsilon(z_2) \varphi_j(z_j), \text{ hence}
\]

\[
F^1_\varepsilon(v) = F_0(z_1, z_2) \text{ with } z_\varepsilon = v \text{ and } z_j = v - \lambda \sigma t.
\]
Also observe

\[(9) \quad F^\lambda_z(v) = F^{-\lambda}_{j}(v - \lambda\sigma \tau) \]

and (because \(F_0(z)\) is even in \(z\))

\[(10) \quad F^\lambda_z(v) = F^{-\lambda}_{z}(v) \]

Now fix \(i\) and \(j\); in (6) substitute \(v := -\lambda_j^j z^j_j + \lambda\sigma \tau\) and \(\lambda := -\lambda_j\) (so \(z^j_j = v - \lambda\sigma \tau\)) to conclude that the equation (6) is equivalent to the equation (11) (because the functions \(Z^j_k(\cdot)\) and the sets \(S^j_k\) are even, and \(F_0(z)\) is even in both \(z_1\) and \(z_2\)):

\[(11) \quad \begin{cases} 
\text{if } v \in S^j_1 \cap (S^j_j + \lambda\sigma \tau) \\
\text{then } Z_0^j + Z^j_z(v) + Z^j_j(v - \lambda\sigma \tau) = F^\lambda_j(v). 
\end{cases} \]

Obviously, given \((B3')\) the four equations (11) (viz. for \(i = 1, 2\) and \(j = 1\)) are equivalent (by (9) and (10)).

V.7.2 The solution of the backward conditions

Consider the conditions \((\widetilde{\Gamma})\) from subsection IV.1.2. The set of conditions \((B3')\) (with (6) replaced by (11)) coincides with the set of conditions \((\widetilde{\Gamma})\) (with the substitution \(v_0^j := \alpha^\tau_j + \frac{1}{2}\), cf. IV.1.2. In order to determine from these conditions the unknown triplet \((Z_0, Z_1, Z_2)\) we are to apply Theorem IV.5.4. Therefore we now check the hypotheses from that theorem, assuming (0) and (1).

First, consider the conditions \((S0)\) through \((S3)\) from subsection IV.1.2. The generalized open half planes \(A^j_k, B^j_k \quad (k = 1, 2)\) are defined in V.5.(58) and satisfy \(A^j_k \cap B^j_k = S^j_k\) (see V.5.(62)). By Theorem V.6.10 the conditions \((S0), (S1), (S2)\) are satisfied, and \((S3)\) is satisfied provided \(\gamma_1 + \gamma_2 \neq 0\) or (by V.6.(31A)) equivalently (cf. V.1.(2))

\[(12) \quad (s, \mathbb{E}_z) \notin \{1\} \times ([0, \infty)^2) \]

Next, observe that the conditions \((F0)\) (by (9)), \((F1)\) (by (1), cf. IV.1.(2)), and \((F2)\) from subsection IV.1.3 are satisfied; also, the condition IV.1.(8) is satisfied (by V.5.(19)-(37))
So the assumptions of Theorem IV.5.4 are satisfied. Also observe that (because of (10)) the condition IV.1.(F7) (following IV.1.(22)) is satisfied; hence (cf. Remark IV.1.7) the set of conditions (L) coincides with the set of conditions (L) from Subsec. IV.1.2. In particular the Fourier coefficient $F^\lambda_1(0)$ (see IV.1.(18)) does not depend on $\hat{\iota}$ or $\hat{\lambda}$ (i.e. the condition IV.1.(F5) (following IV.1.(22)) is satisfied); also, the constant $Z_0^\lambda$ does not depend on $\hat{\lambda}$ and $\lambda$ (i.e. the condition IV.1.(25) (following IV.1.(23)) is satisfied), and is identified with $Z_0$.

Hence, Theorem IV.5.4 can be applied in order to determine the triplet $(Z_0, Z_1, Z_2)$, see Theorem 3. The representations (14) and (15) involve the functions $(\Omega F)_{\iota}(\cdot)$ (defined by IV.5.(17) and IV.5.(10)-(11)), and the Fourier coefficients $F^\lambda_\iota(0)$. Also, Corollary IV.5.11 can be applied in order to find a different expression for the sum $Z_0 + Z_1(z_1) + Z_2(z_2)$, see (16).

**Remark 1.** Assuming (0) and (1) we now show that the definitions of the functions $(\Omega F)_{\iota}(\cdot)$ and the constants $F^\lambda_\iota(0)$ fail if the condition (12) fails to hold, in other words (by V.6.(31A)) if

\[(*) \quad \gamma_1 + \gamma_2 = \sigma \]

Observe that

\[
\begin{align*}
S_j \cap (S_\iota + \lambda \sigma \tau) \cap (\tau \mathbb{R}) &= \text{ (see V.5.(47))} \\
&= (-\gamma_j \tau, \gamma_j \tau) \cap (-\gamma_{\iota} \tau + \lambda \sigma \tau, \gamma_{\iota} \tau + \lambda \sigma \tau) \quad \text{ (see V.6.(5))} \\
&= \lambda (-\gamma_{\iota} \tau + \sigma \tau, \gamma_{\iota} \tau),
\end{align*}
\]

which is empty because of (*). Therefore, a path $\Lambda(\cdot)$ in $S_j \cap (S_\iota + \lambda \sigma \tau)$ satisfying $\Lambda(1) - \Lambda(0) = 1$ does not exist. This implies the claim.

We now give the result.

**Theorem 3.** Assume (0), (1), and (12), and fix $y$ and $s$. Then the following holds.

1. The conditions (B3') are equivalent to (14) and (15):

\[
\begin{align*}
(14) \quad &Z_{\iota}(v) = (\Omega F)_{\iota}(v) - (\Omega F)_{\iota}(a_{\iota} \tau + \frac{1}{2}) \quad \text{for} \quad v \in S_\iota \quad \text{and} \quad \iota = 1, 2, \\
(15) \quad &Z_0 = F^\lambda_k(0) + \sum_{\iota = 1, 2} (\Omega F)_{\iota}(a_{\iota} \tau + \frac{1}{2}) \quad \text{for} \quad \lambda = \pm \quad \text{and} \quad k = 1, 2,
\end{align*}
\]

where $F^\lambda_k(0)$ does not depend on $k$ or $\lambda$.
II. The conditions (B3')" imply

\[ z_0 + \sum_{i=1,2} z_i(z_i) = F_k^\lambda(Q) + \sum_{i=1,2} (DF)_i(z_i) = (SF)(z) \]

for \( z \in S_1 \times S_2 \)

with \( (SF)(z) \) defined in IV.5.(37) et seq.

**PROOF.** Part I. Application of Theorem IV.5.4 (see above).

Part II. From Part I and Corollary IV.5.11.

\[ \square \]

V.7.3 Determination of the generating function \( P(x,y,s) \)

**Notations.** Assuming (0), (1), and (12) we put for \( x \in U^2 \)

\[ (LP_0)(x) = (LP_0)(x,y,s) := (SF)(z) \]

with \( x \) and \( z \) related by (3), and \( z \in S_1 \times S_2 \).

This is possible by (17) and (4) (cf. Property 2), and one has

\[ (LP_0)(x,y,s) = X_0(y,s) + \sum_{i=1,2} X_i(x_i,y,s) \]

Finally we put for \( x \in U^2 \cap \{x: Q(x,s) \neq 0\} \)

\[ \tilde{F}(x,y,s) := \frac{(LP_0)(x,y,s) - P_0(x,y)}{Q(x,s)} \]

**COROLLARY 4.** Assume (0), (1), and (12), and fix \( y \) and \( s \). Then the following holds.

I. The function \( x \mapsto \tilde{F}(x,y,s) \) can be continued analytically as a regular function in \( U^2 \cap \{x: Q(x,s) = 0\} \).

II. The function \( x \mapsto \tilde{F}(x,y,s) \) satisfies and is uniquely determined by the conditions I.3.(B1')

**PROOF.** This follows immediately from Theorem 3 because of the Properties 2 and 1.

\[ \square \]

**THEOREM 5.** Assume (0), (1), and \( 0 < s < 1 \).

Then \( P(x,y,s) = \tilde{F}(x,y,s) \) for \( x \in U^2 \).

**PROOF.** Recall that under the above assumptions the power series \( P(x,y,s) \) converges, and that the function \( x \mapsto P(x,y,s) \) satisfies the conditions I.3.(B1'). Now apply Corollary 4 (Part II).

\[ \square \]