Random Walk in the Quadrant
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Chapter VI

Limit of the generating function $P(x, y, s)$ for $s \to 1$

VI.0 Summary of Chapter VI

In the preceding chapter the generating function $P(x, y, s)$ was determined for $x \in \mathcal{U}^2, \quad y \in \mathcal{U}^2$

(where $\mathcal{U}$ is the open unit disc) and $0 < s < 1$ (see Theorem V.7.5) in the case of skipfree, all-sided random walk. In this chapter we take, for fixed values of $(x, y)$, $x \notin \mathcal{K}_1(1)$, the limit for $s \to 1$ under the additional assumption that the random walk is driftless, and so prove Basic Theorem II.4.55. In the case of driftless random walk the kernel is degenerate for $s = 1$; for $s < 1$, $s \to 1$ the elliptic functions degenerate into simply periodic functions. Basic Theorem II.4.55 is proved separately in the non-even walk case (secs. 1–3) and in the even walk case (secs. 4–6). The basic theorem needs to be proved only with $y$ restricted to $\mathcal{U}^2$. In this chapter we prove the theorem, however, in a more general form, viz. for all values of $(x, y)$ which satisfy ($\ast$) and for which the power series $P(x, y, 1)$ (or its continuous continuation) exists; so we include all values of $y$ in the case $-1 < \rho < 0$, and in the case $0 \leq \rho < 1$ we exclude $y = (1, 1)$ (non-even walk) resp. $y = \pm(1, 1)$ (even walk), in short, we include: $y \in \mathcal{Y}(\hat{\sigma})$.

Remark In Chapter II we have defined the function $\hat{P}(x, y)$ (which is the limit function of $P(x, y, s)$ for $s \to 1$) for all $y \in \mathcal{U}^2$, and have, next, extended this limit function to all values of $y \in \mathcal{Y}(\hat{\sigma})$; see the Theorems II.4.58–60. However, once the Basic Theorem II.4.55 has been proved for all $y \in \mathcal{Y}(\hat{\sigma})$, this extension procedure is no longer necessary.

VI.0.1 Summary of Sec. VI.1

In Sec. VI.1 we compute for driftless, non-even random walk the limits for $s \to 1$ of the constants associated with the elliptic functions $f_1$ and $f_2$; also, we deter-
mine the limits of the functions themselves, and compute upper bounds for the remainder terms.
We start with showing that the $c$-values and the associated $d$-values converge to the corresponding values in the $s = 1$ case and that the imaginary period $\tau(s)$ converges to 0. Also, upper bounds for the remainder terms are computed. The tool for determining the limiting behaviour of the elliptic functions is Jacobi's Imaginary Transformation, viz. $\tilde{\tau} = -1/\tau$; the functions are transformed into elliptic functions with primitive periods 1, $\tilde{\tau}$ with $\tilde{\tau} \to \infty$. The transformed functions are shown to converge to the uniformizing functions in the $s = 1$ case (viz. $g_1$ and $g_2$, see Sec.II.2), which are simply periodic with period 1. This result appears in several forms (see Lemma 4 through Proposition 8) with different remainder terms. We use this result to show, in Proposition 9, that the remaining constants converge to the corresponding constants in the $s = 1$ case.

**VI.0.2 Summary of Sec. VI.2**

Here we prove, for driftless, non-even random walk Basic Theorem II.4.55, viz.

(i) $\tilde{P}(x, y, s) \to \tilde{P}(x, y)$ (s $\to$ 1)

(for $\tilde{P}(x, y, s)$ see V.7.(21) and for $\tilde{P}(x, y)$ see II.4.(150)). The nontrivial part of this limit is the convergence of the numerator on the right, i.e.

(ii) $\mathcal{L}P_0(x, y, s) \to LP_0(x, y)$ (s $\to$ 1)

(for $\mathcal{L}P_0(x, y, s)$ see V.7.(18) and for $LP_0(x, y)$ see II.4.(141)-(142)). On the left we substitute $x_k = f_k(z_k \tau + \frac{1}{2})$ (k=1,2) where $z_k = z_k(s)$, and on the right $x_k = g_k(z_k)$ where $z_k = z_k(1)$; here $z_k(s)$ is suitably chosen so $z_k(s)$ is continuous in $s = 1$. (Because $x_k$ is fixed and the function $f_k(\cdot \tau(s) + \frac{1}{2})$ changes with $s$, $z_k(s)$ too changes with $s$. The construction of $z_k(s)$ is given in Property 11.) As a result, the limit (ii) is transformed into

(iii) $(SF)(z_1(s)\tau(s) + \frac{1}{2}, z_2(s)\tau(s) + \frac{1}{2})$

$\to (\Lambda H_0)((z_1(1), z_2(1)), y)$ (s $\to$ 1)

with $SF$ given in (21) and $\Lambda H_0$ given in (51).

The function on the right in (iii) has been defined originally in II.4.(132)-(44)-(28) as the limit (for $M \to \infty$) of the sum of four integrals of the form II.4.(28) (indexed by $\lambda = \pm$ and $i = 1, 2$ with $j = 3 - i$). In these integrals we apply a shift (over $-\lambda \frac{1}{2} \tilde{\sigma}$) with the effect that the end points of the integration paths become the points $\pm M \hat{i}$ on the imaginary axis. In the limit these integrals are improper ones which, separately, do not necessarily converge (they do not converge, indeed, if $y_1 = 1$ or $y_2 = 1$, see Corollary II.4.30); however, their sum always converges (provided $y \in \mathcal{Y}(\tilde{\sigma})$).
In this section we first construct (for each one of the integrals) a convenient integration path. This path splits up into three parts, giving rise to three subintegrals.

The first part is a compact curve segment called $\Gamma_{\lambda}^j$, stretching from $-M$ to $+M$ (here $M$ is a fixed, sufficiently large positive number) and passing between the poles of the integrand in the right way. The construction takes place in (8), \ldots, (9 C); the subintegral belonging to it is called $I_{\lambda,M}^j$ (written in full in (41 A)). The two other parts of the integration path are the tails of the imaginary axis, viz. $\mu[M, \infty) (\mu = \pm)$. As the integrals over these half lines are improper and do not always converge, the next step is transforming these integrals into absolutely convergent integrals. To that end we modify the integrands by adding suitably chosen extra terms, in such a way that the eventual sum $\Phi_{\lambda,M}^j$ (written in full in (41 A)) remains intact. This is done in (46), \ldots, (51). The resulting absolutely convergent subintegrals over the half lines $\mu[M, \infty) (\mu = \pm)$ are called $\tilde{R}_{\lambda,M}^j$, cf. II.4.(122)-(115). The resulting expression for $\Phi_{\lambda,M}^j$ appears in (51).

The function on the left in (iii) (which occurs in the solution of the backward conditions V.7.(B 3′′)) appears in (3)–(4) likewise as the sum of four integrals called $I_{\lambda}^j$ (likewise indexed by $\lambda = \pm$ and $i = 1, 2$ with $j = 3 - i$), each one having a counterpart in $\Phi_{\lambda,M}^j((z_1(1), z_2(1)), y)$ of all four integrals remains intact. This is done in (46), \ldots, (51). The resulting absolutely convergent subintegrals over the half lines $\mu[M, \infty) (\mu = \pm)$ are called $\tilde{R}_{\lambda,M}^j$, cf. II.4.(122)-(115). The resulting expression for $\Phi_{\lambda,M}^j((z_1(1), z_2(1)), y)$ is given in (51).

The requirements on the integration paths depend on $s \in (0, 1)$. However, if $s$ is close to 1 we can find integration paths (almost) independent of $s$: we can use the integration path in the corresponding integral in $\Phi_{\lambda,M}^j$ (described above), but truncated at $\pm\tilde{\tau}(s)$. So the integration path in $I_{\lambda}^j$ depends on $s$ only through its end points. The construction of this path is given in (6), \ldots, (12). Proposition 3 asserts that this path satisfies all requirements if $s$ is close to 1; a formal proof is laborious, and therefore deferred to the next subsection. As above, each path consists of three parts, giving rise to three subintegrals. The compact, fixed part is $\Gamma_{\lambda}^j$ (which passes between the poles of the integrand in the right way if $s$ is close to 1), stretching from $-M$ to $+M$; the subintegral belonging to it is called $I_{\lambda,M}^j$, see (15).

The other two parts are the line segments $\mu[M, \infty) (\mu = \pm)$, of the imaginary axis; the subintegrals belonging to them are called $K_{\lambda,M}^j$, see (15 A). For technical reasons we modify the latter two subintegrals in a similar way as their counterparts in $\Phi_{\lambda,M}^j$, by adding extra terms to the integrands which leave the eventual sum $SF$ of all four integrals intact; the modified subintegrals are called $\tilde{K}_{\lambda,M}^j$, see (19). This modification is done in (16), \ldots, (20). The resulting expression for $SF$ appears in (21).
We prove (iii) by showing that each one of the four integrals on the left converges to its counterpart on the right; in fact, we show that their difference tends to zero. To that end we consider, for each integral on the left, the difference of each one of the three subintegrals and its counterpart on the right (on the right, however, we truncate the integration paths at $\pm \frac{1}{2} \tilde{\tau}(s)$ in order to have the same integration paths as on the left; the remaining integrals over the tails $\pm [\frac{1}{2} \tilde{\tau}(s), \infty)$ are in the limit negligible as $\tilde{\tau}(s) \to \infty$). Upper bounds for these differences for $s$ close to 1 are computed in Proposition 8. In addition, in Property 9 we compute an upper bound for the integrals on the right over the tails $\pm [\frac{1}{2} \tilde{\tau}(s), \infty)$ of the imaginary axis. For the estimation of the differences of integrals we need estimates of the differences of the integrands involved; each integrand being the product of two factors, we need estimates of these factors themselves, and of their differences as well, cf. (42). The estimates needed can be found in Proposition 4 through Property 7. The proofs (except of Property 7) are rather laborious, and therefore collected in the next subsection.

After these preparations, the proof of Basic Theorem II.4.55 consists of invoking the above mentioned results in the right order.

**VI.0.3 Summary of Sec. VI.3**

This section is a collection of proofs left over from the previous section.

**VI.0.4 Summary of Sec. VI.4**

This section is the analogue of Sec. VI.1 for driftless, even random walk.

**VI.0.5 Summary of Sec. VI.5**

This section is the analogue of Sec. VI.2 for driftless, even random walk. The main difference concerns the integration path. The integration path here begins at $w = -\frac{3}{4} \tilde{\tau}(s)$ and ends at $w = +\frac{1}{4} \tilde{\tau}(s)$, where $\tilde{\tau}(s) \to \infty$, and splits up into two parts:

- One part (indexed by $\nu = -$) from $-\frac{3}{4} \tilde{\tau}(s)$ to $-\frac{1}{4} \tilde{\tau}(s)$, which splits up into two parts:
  - a path (indexed by $\mu = -$) from $-\frac{3}{4} \tilde{\tau}(s)$ to $-\frac{1}{2} \tilde{\tau}(s)$
  - a path (indexed by $\mu = +$) from $-\frac{1}{2} \tilde{\tau}(s)$ to $-\frac{1}{4} \tilde{\tau}(s)$.

  The corresponding subintegrals turn out to be negligible in the limit.

- One part (indexed by $\nu = +$) from $-\frac{1}{4} \tilde{\tau}(s)$ to $+\frac{1}{4} \tilde{\tau}(s)$, which, as Sec. VI.2, splits up into three parts:
  - a path (indexed by $\mu = -$) from $-\frac{1}{4} \tilde{\tau}(s)$ to $-M i$
- a path (indexed by $\mu = 0$ and called $\Gamma^j_\lambda$) from $-M$ i to $+M$ i
- a path (indexed by $\mu = +$) from $+M$ i to $+\frac{1}{2}\tau(s)$.

The corresponding subintegrals are denoted $\tilde{K}_{\lambda, \mu, \nu}^j(\xi_i)$.

**VI.0.6 Summary of Sec. VI.6**

This section is a collection of proofs remaining from the previous section.
VI.1 Limits for \( s \to 1 \) in the case of skipfree, all-sided, driftless, non-even random walk

Our assumptions in this section are

\[
\begin{align*}
0 & \quad \text{the random walk is skipfree, all-sided, and driftless,} \\
0 < s < 1 & \quad \text{the random walk is non-even,}
\end{align*}
\]

Also it is assumed

\[ y \in \mathbb{H}^2. \]

In this section we consider some constants and functions which are defined in Chapter V, and depend on the parameter \( s \) with \( 0 < s < 1 \). We show that there is convergence for \( s \to 1 \) to the corresponding constants and functions defined in Section II.2 for \( s = 1 \); also, we determine the rate of convergence.

VI.1.1 Limit behaviour of the zeros of the discriminants of \( Q(x, s) \)

The values \( c_{ik} \) (which depend on \( s \)) are the zeros of the discriminant \( x_{ik} = D_{ik}(x, s) \) (see V.1.(10)). The definition of these values is contained in Theorem V.2.3 in the case \( 0 < s < 1 \), and in II.2.(9) in the case \( s = 1 \).

**Proposition 1.** Under the assumptions \( (0) \) the following estimates for \( s \to 1 \) hold, for \( i = 1, 2 \):

\[
1 - c_{i1}(s), c_{i3}(s)-1 \in \lambda_i \sqrt{1-s} \left[ 1 + O(\sqrt{1-s}) \right]
\]

with

\[
\lambda_i := \left[ \frac{2}{(1-\rho^2)^2 \mathbb{E}(\xi_i^2)} \right]^{\frac{1}{2}} > 0,
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
c_{i2}(s) \in c_{i2}(1) + O(1-s), \\
c_{i4}(s) \in c_{i4}(1) + O(1-s), \end{array} \right.
\end{align*}
\]
(5) \((c_{\xi_2}(s), c_{\xi_2}(s), c_{\xi_3}(s), c_{\xi_4}(s)) \in \mathcal{T} \sqrt{1-s} \left[ 1 + O(\sqrt{1-s}) \right] \)

with (for \(Q_0\) see II.2.(64)-(70))

(6) \(\zeta := 8 Q_0^{-1} \left[ (1-\rho^2) \mathbb{E} (\xi_1^2) \mathbb{E} (\xi_2^2) \right]^{-1/2} \geq 0.\)

**Remark 1.** From (2) it follows, obviously,

(7) \(1 - c_{\xi_3}(s) \in \lambda_\zeta \mathcal{T} \sqrt{1-s} \left[ 1 + O(\sqrt{1-s}) \right].\)

**Proof.** We show (2). From V.2.(24) and II.2.(9) we know \(c_{\xi k}(s) + c_{\xi k}(1) = 1\) for \(s \to 1\) and \(k=1,3\). We develop \(D_{\xi}(x_\zeta, s)\) in a Taylor series around \((x_\zeta, s) = (1,1)\). Recall from Corollary II.2.2 that

\[ D_{\xi}(x_\zeta, s) = 0 = \frac{\partial D_{\xi}(x_\zeta, s)}{\partial x_\zeta} \quad \text{for } (x_\zeta, s) = (1,1). \]

A computation shows (for \(D_{\xi}\) see V.1.(10))

\[ \frac{\partial D_{\xi}(x_\zeta, s)}{\partial s} = \frac{2}{s} D_{\xi}(x_\zeta, s) + \frac{2}{s} x_\zeta \left[ s b_{\xi}(x_\zeta) - 1 \right], \]

which implies, cf. I.3.(45)-(31)-(36) and I.1.(17),

\[ \frac{\partial D_{\xi}(x_\zeta, s)}{\partial s} = -2 \mathbb{E} (\xi_j^2) \text{ for } (x_\zeta, s) = (1,1) \]

where \(j = 3-\xi\).

From Proposition II.2.1 (Part III) we know

\[ \frac{\partial^2 D_{\xi}(x_\zeta, s)}{\partial x_\zeta^2} = -2 \mathbb{E} (\xi_1^2) \mathbb{E} (\xi_2^2) (1-\rho^2) < 0 \text{ for } (x_\zeta, s) = (1,1). \]

So we find

\[(x)\]

\[ D_{\xi}(x_\zeta, s) = (s-1) \frac{\partial D_{\xi}(x_\zeta, s)}{\partial s} (1,1) + \frac{1}{2} (x_\zeta - 1)^2 \frac{\partial^2 D_{\xi}(x_\zeta, s)}{\partial x_\zeta^2} (1,1) + \]

\[+ O((x_\zeta - 1)(s-1)) + O((s-1)^2) + O((x_\zeta - 1)^3) \]

for \((x_\zeta, s) \to (1,1)\). Hence, if \(x_\zeta = c_{\xi k}(s)\) for \(k=1\) or 3, then
\[
\frac{(x_\varepsilon - 1)^2}{1 - s} + \frac{2}{(1 - \rho^2)^2 \Gamma (x_\varepsilon^2)} \quad \text{for } s < 1.
\]

Now use (*) again and V.2.(15A)-(15B) to conclude (2).

Proof of (4): By Proposition II.2.3 (Part I), \( c_{\varepsilon 2}^1(1) \) is a simple zero of \( D_{\varepsilon}(x_\varepsilon, 1) \) for \( k=2,4 \). Also, \( c_{\varepsilon 2}^1(1) \) and \( c_{\varepsilon 4}^{-1}(1) \) are finite, cf. I.2.(9).

Hence, by the implicit function theorem,

\[(8) \quad \text{the functions } s \sim c_{\varepsilon 2}^1(s), s \sim c_{\varepsilon 4}^{-1}(s) \text{ are regular near } s=1, \]

which implies (4).

Proof of (5): Use V.2.(29) to conclude from (2), (4), and (7), because of \( c_{\varepsilon 2}^1(1), c_{\varepsilon 4}^{-1}(1) \in (-1, +1) \) (cf. II.2.(9)), that

\[
(c_{\varepsilon 1}(s), c_{\varepsilon 2}(s); c_{\varepsilon 3}(s), c_{\varepsilon 4}(s)) \in \mu_{\varepsilon} \sqrt{1-s} + O(1-s)
\]

with

\[
\mu_{\varepsilon} := 2 \lambda_{\varepsilon} \frac{1 - c_{\varepsilon 2}^1(1) c_{\varepsilon 4}^{-1}(1)}{[1 - c_{\varepsilon 2}(1)][1 - c_{\varepsilon 4}^{-1}(1)]}
\]

The equality \( \mu_{\varepsilon} = \lambda \) remains to be shown. For this, use II.2.(23) to conclude

\[
\mu_{\varepsilon} = \frac{4 \lambda_{\varepsilon}}{a_{\varepsilon} + b_{\varepsilon}}
\]

which implies the claim by (3) and II.2.(67)-(71).

VI.1.2 Limit behaviour of the \( d \)-values

The values \( d_{\varepsilon k} \) (which depend on \( s \)) are defined by V.2.(32) in the case \( 0 < s < 1 \), and by II.2.(13) in the case \( s=1 \).

PROPOSITION 2. Under the assumptions (0) the following estimates for \( s \to 1 \) hold, for \( j=1,2 \) :

-1
\text{(9)} \quad \text{for } k \in \{1,3\} : \quad \frac{d_{\mathcal{J}k}(s) - 1}{c_{\mathcal{J}k}(s) - 1} \in -\rho + O(\sqrt{1-s}) \ ,

\text{(10)} \quad \text{for } k \in \{2,4\} : \quad d_{\mathcal{J}k}(s) \in d_{\mathcal{J}k}(1) + O(1-s) .

\text{PROOF.} \quad \text{Write } \xi := 3^{-\mathcal{J}} . \text{ Observe that } d_{\mathcal{J}k} \text{ depends on } s \text{ only through } c_{\xi k} , \text{ see V.2.(34)-(35).}

\text{Proof of (9): Take } k \in \{1,3\} . \text{ Recall } d_{\mathcal{J}k}(1) = 1 , \text{ see II.2.(14).}

\text{Differentiate } d_{\mathcal{J}k} \text{ with respect to } c_{\xi k} \text{ to find}

\frac{d}{dc_{\xi k}} d_{\mathcal{J}k} = \frac{1}{2} d_{\mathcal{J}k} \left[ \frac{c_{\xi}'}{c_{\xi}} (c_{\xi k}) - \frac{a_{\xi}'}{a_{\xi}} (c_{\xi k}) \right] .

\text{Hence, because of zero drift and I.3.(36)-(38),}

\frac{d_{\mathcal{J}k}'}{c_{\xi k}'} \bigg|_{c_{\xi k} = 1} = -\frac{\mathbb{E} \xi_{1,2}}{\mathbb{E} \xi_{2,2}} = -\rho \left[ \frac{\mathbb{E} (\xi_{2,2}^2)}{\mathbb{E} (\xi_{1,2}^2)} \right]^{1/2} .

\text{Now use the estimate}

\frac{d_{\mathcal{J}k}(s) - 1}{c_{\xi k}(s) - 1} \in d_{\mathcal{J}k}' + O(c_{\xi k}(s) - 1)

\text{and (2) and (3) to conclude (9).}

\text{Proof of (10): Take } k \in \{2,4\} . \text{ If } d_{\mathcal{J}k}(1) \in \{0,\infty\} \text{ then } d_{\mathcal{J}k}(s) = d_{\mathcal{J}k}(1)

\text{for all } s \in (0,1) , \text{ see V.2.(39),...,(42) and II.2.(17A),...,(17D), and}

\text{the claim is trivial.}

\text{It remains to consider the case } d_{\mathcal{J}k}(1) \in (-\infty,0) , \text{ or } d_{\mathcal{J}k}(s) \in (-\infty,0) \text{ for}

\text{all } s \in (0,1) , \text{ cf. V.2.(38) and II.2.(16). In this case } d_{\mathcal{J}k} \text{ depends}

\text{regularly on } c_{\xi k} , \text{ and the claim follows from (8).}

\text{VI.1.3 Limit behaviour of the constant } \tau

\text{The value } \tau \text{ is defined in V.5.(1) by means of the requirements}

\text{(11)} \quad \tau \in \mathbb{R}, \text{ Im } \tau > 0, \text{ m}(0|\tau) = (c_{\xi 1}, c_{\xi 2}, c_{\xi 3}, c_{\xi 4}).

\text{As before we put}
(12) \[ q := E \left( \frac{i}{2} \tau \right) \]

where

(13) \[ B(w) := e^{2\pi i w} \].

We define the value \( \tilde{\tau} \) by means of

(14) \[ \tilde{\tau} \tau = -1, \]

( imaginary transformation of Jacobi ) and put

(15) \[ \tilde{q} := E \left( \frac{i}{2} \tilde{\tau} \right). \]

**Remark 2.** Observe that \( q \in (0,1) \).

Also observe that \( \tilde{\tau} \in (0,\infty) \), so \( \tilde{q} \in (0,1) \), and \( m(0|\tilde{\tau}) = 1 - m(0|\tau) \), cf. III.II.(52B). Hence, \( \tilde{\tau} \) satisfies and is determined by the requirements

(16) \[ \tilde{\tau} \in i \mathbb{R}, \text{Im} \tilde{\tau} > 0, m(0|\tilde{\tau}) = (c_{\tilde{\tau}1}, c_{\tilde{\tau}3}, c_{\tilde{\tau}2}, c_{\tilde{\tau}4}), \]

cf. Property III.1.15 (Part IV, b)).

**Proposition 3.** Under the assumptions (0), if \( s \to 1 \) then

(17) \[ \tau(s) \to 0, \quad \tilde{\tau}(s) + i \infty, \]
(18) \[ q(s) \to 1, \quad \tilde{q}(s) \to 0, \]

and the following estimate holds:

(19) \[ \tilde{q}(s) \leq \frac{6}{15} \sqrt{1-s} [ 1 + o(\sqrt{1-s}) ] \]

with \( o \) defined in (6).

**Remark 3.** Observe that (15) and (19) imply
(20) \[ O\left( E(\frac{1}{2} \tau(s)) \right) = O(\bar{q}(s)) = O(\sqrt{1-s}). \]

**Proof.** We show (17) and (18). First observe from (11) and (5)

\[ m(0|\tau(s)) \rightarrow 0 \quad \text{for } s \rightarrow 1. \]

Next use Lemma III.1.21 to conclude \( \tau(s) \rightarrow 0 \). This implies the remaining claims by (12), (14), and (15). We show (19). On the one hand, use III.2.(69) to conclude

\[ m(0|\tau(s)) \in 16 \bar{q}(s) \left[ 1 + O(\bar{q}(s)) \right], \]

or

\[ 16 \bar{q}(s) \in m(0|\tau(s)) \left[ 1 + O(\bar{q}(s)) \right]. \]

On the other hand, use (11) and (5) to conclude

\[ m(0|\tau(s)) \in \epsilon \sqrt{1-s} \left[ 1 + O(\sqrt{1-s}) \right], \]

or

\[ \epsilon \sqrt{1-s} \in m(0|\tau(s)) \left[ 1 + O(\sqrt{1-s}) \right]. \]

Hence, because of \( m(0|\tau(s)) \neq 0, \infty, \)

\[ \frac{16 \bar{q}(s)}{\epsilon \sqrt{1-s}} \in \frac{1 + O(\sqrt{1-s})}{1 + O(\bar{q}(s))} \rightarrow 1. \]

This implies \( \bar{q}(s) \in O(\sqrt{1-s}) \), and the claim follows.

\[ \diamond \]

VI.1.4 Limit behaviour of the function \( f^\xi(.) \)

The function \( f^\xi(.) \) (which depends on \( s \)) is defined in V.5.(2), for \( i=1,2 \). In the following the constants \( a^\xi, b^\xi \), defined in II.2.(22), and the function \( g^\xi(.) \), defined in II.2.(20), occur.

**Lemma 4.** Under the assumptions (0) the following estimate holds for \( s \rightarrow 1 \). For \( i \in \{1,2\} \) and \( \mu = \pm \)

(21) \[ f^\xi((\mu \nu+\rho)\tau + \frac{1}{2}) = \frac{\cos 2\pi (\mu \nu+\rho) - a^\xi + R}{\cos 2\pi (\mu \nu+\rho) + b^\xi + S} \]
where \( R, S \) are remainder terms satisfying

\[
(22) \quad R, S \in O(E \left( \frac{1}{2} \tau(s) - \omega \right))
\]

uniformly in \((w, v)\) provided

\[
(23) \quad 0 \leq \text{Im} \ w \leq \frac{1}{2} \text{Im} \ \tau(s) \quad \text{and} \quad \text{Im} \ v \ \text{is bounded}.
\]

**Proof.** The function \( f(\cdot) \) is defined by

\[
(24) \quad m(\cdot | \tau(s)) = (f(\cdot), c(\cdot) \tau(\cdot); c(\cdot) \tau(\cdot), c(\cdot) \tau(\cdot)),
\]

which can be rewritten as follows (cf. V.2.(29))

\[
f(\cdot) = \frac{(1 - c(\cdot) \tau(\cdot)) - m(\cdot | \tau) (1 - c(\cdot) \tau(\cdot))}{\frac{1}{c(\cdot) \tau(\cdot) - m(\cdot | \tau) (1 - c(\cdot) \tau(\cdot))}}
\]

From (2), (4), (20), and II.2.(9)-(21) we have the following estimates (we suppress the dependence of \( \tilde{q} \) on \( s \)):

\[
1 - c(\cdot) \tau(\cdot) \frac{1 - 0(1-s)}{1 - c(\cdot) \tau(\cdot)} = 1 - c(\cdot) \tau(\cdot) (1 + 0(\tilde{q})) = 1 - c(\cdot) \tau(\cdot) + 0(\tilde{q}) =
\]

\[
= \frac{a(\cdot) + b(\cdot)}{1 + b(\cdot)} + 0(\tilde{q}),
\]

\[
c(\cdot) \tau(\cdot) \in 1 + 0(\tilde{q}),
\]

\[
c(\cdot) \tau(\cdot) \in c(\cdot) \tau(\cdot) (1 + 0(\tilde{q}^2)) = \frac{1 - b(\cdot)}{1 + a(\cdot)} + 0(\tilde{q})
\]

Hence, with II.2.(23)-(25),

\[
f(\cdot) \in \frac{2 + 0(\tilde{q}^2) - m(\cdot | \tau) [1 + a(\cdot) + 0(\tilde{q})]}{2 + 0(\tilde{q}) - m(\cdot | \tau) [1 - b(\cdot) + 0(\tilde{q})]}.
\]

From III.2.(70) we know
(25) \[ m((\mu w + v)\tau + \frac{1}{2}) = \frac{1 + R}{\cos^2(\pi(w + v)) + S} \]

where \( R \) and \( S \) are remainder terms satisfying (22) for \( s + 1 \), uniformly in \((w,v)\) provided (23) holds.

This easily implies the claim in view of the estimate

(26) \[ \cos \pi(w + v) \in O(E(-\frac{1}{2}w)) \]

provided \( \text{Im} \ w \geq 0 \) and \( \text{Im} \ v \) is bounded.

**Theorem 5. I.** Under the assumptions (0) the following estimates for \( s + 1 \) hold. For \( i \in \{1,2\} \) and \( \lambda \in \{-1,+1\} \):

(27) \[ f_i((w+v)\tau + \frac{1}{2})^\lambda - g_i(w+v)^\lambda \in O(E(\frac{1}{2} \tau(s))) \]

uniformly in \((w,v)\) provided

(28) \[ |\text{Im} \ w| \leq \frac{1}{2} \text{Im} \ \tau(s) \text{ and } \text{Im} \ v \text{ is bounded} \]

and

(29A) \[ g_i(w+v)^\lambda \text{ is bounded.} \]

**II.** In Part I the restriction (29A) can be replaced by

(29B) \[ f_i((w+v)\tau + \frac{1}{2})^\lambda \text{ is bounded.} \]

**Proof.** Because \( f_i(\cdot) \) and \( g_i(\cdot) \) are even with period 1, we may and do assume \( \text{Im} \ w \geq 0 \). We take \( \lambda = +1 \) (the case \( \lambda = -1 \) goes similar), and use the abbreviation

\[ x_i := f_i((w+v)\tau + \frac{1}{2}). \]

**Part I.** By II.2.(20)-(23) the condition (29A) is equivalent to

(29C) \[ [\cos 2\pi(w+v) + b_i]\ ^{-1} \text{ is bounded}, \]

which in turn, by Aux. Prop. 6 below, is equivalent to
\[ [\cos 2\pi (w+v) + b_{\zeta}]^{-1} \in O(E(w)). \]

Now apply Lemma 4 to conclude
\[ x_{\zeta} \in \frac{g_{\zeta}(w+v) + O(E(\frac{1}{2}))}{1 + O(E(\frac{1}{2}))} \]

which implies the claim because of (29A).

Part II. Assume (28) and (29B). First suppose \( 0 \leq \Im w \leq \frac{1}{4} \Im \bar{\tau}. \) We use Lemma 4. By (22) the remainder terms now satisfy \( R, S \in O(E(\frac{1}{4})) \rightarrow 0. \)

Let \( s \) be sufficiently close to 1, viz. so that \( |R - S| < a_{\zeta} + b_{\zeta}. \) Then by (21) \( x_{\zeta} \neq 1, \) and, consequently, it follows from (21)
\[ g_{\zeta}(w+v) = \frac{x_{\zeta} + \tilde{R}}{1 + \tilde{R}} \]

where \( \tilde{R} \in O(E(\frac{1}{4})). \) Hence,
\[ g_{\zeta}(w+v) \in x_{\zeta} + O(E(\frac{1}{4})). \]

This implies (29A) with \( \lambda = +1. \)

Next suppose \( \frac{1}{4} \Im \bar{\tau} \leq \Im w \leq \frac{1}{2} \Im \bar{\tau}. \) Also assume that \( s \) is sufficiently close to 1, viz. such that \( E(\frac{1}{2}) |E(2v)| \leq \frac{1}{2}, \) hence
\[ (*) \quad |1 + E(2w + 2v)| \geq \frac{1}{2}. \]

Then one has
\[ |\cos 2\pi (w+v) + b_{\zeta}| \geq |\cos 2\pi (w+v)| - b_{\zeta} \geq \frac{1}{4} |E(-w-v)| - b_{\zeta} \geq \frac{1}{4} |E(-\frac{1}{4} - v)| - b_{\zeta} \to \infty \]

for \( s \to 1. \) This implies (29C), which is equivalent to (29A).

\[ \text{AUXILIARY PROPERTY 6. Consider} \]

\[ f(w,z) := \frac{1}{\cos 2\pi w + z}. \]
If \( z \) and \( f(w,z) \) are bounded, then \( E(-w) f(w,z) \) is bounded.

**Proof.** Assume

\[
|z| \leq m \quad \text{and} \quad |f(w,z)| \leq M.
\]

Let \( p \) be a sufficiently large positive number, viz.

\[
|E(w)| |E(w) + 2z| \leq \frac{1}{2} \quad \text{if \( \text{Im} w \geq p \).}
\]

First assume \( \text{Im} w \leq p \). Then

\[
|E(-w) f(w,z)| \leq E(-ip) M.
\]

Next assume \( \text{Im} w \geq p \). Then

\[
|E(-w) f(w,z)| = \frac{2}{1 + E(2w) + 2zE(w)} \leq 2 \leq \frac{2}{1 - |E(w)| |E(w) + 2z|} \leq 4,
\]

which proves the claim.

\[ \Box \]

**Lemma 7.** Under the assumptions \((0)\) the following estimates for \( s + 1 \) hold.

For \( \zeta \in \{1,2\} \) and \( \mu = \zeta \)

\[
(30) \quad \frac{1 - f_\zeta((uw+v)\tau(s))}{1 - c_{\zeta,1}(s)} = \frac{\cos 2\pi(uw+v) + R}{1 + S}
\]

where \( R \) and \( S \) are remainder terms satisfying \((22)\) uniformly in \((w,v)\) provided \((23)\) holds.

**Proof.** From \((24)\) and \((11)\) it follows, cf. V.2.\((29)\),

\[
\frac{m((uw+v) \tau(s)|\tau(s))}{m(0|\tau(s))} =
\]
which can be rewritten as follows:

\[
\frac{c_{\xi 3} - c_{\xi 1}}{1 - c_{\xi 1}} \cdot \frac{1 - c_{\xi 1}^{-1}}{1 - c_{\xi 1}^{-1} c_{\xi 4}^{-1}} \cdot \frac{m((\mu \nu + v)\tau | \tau)}{m(0|\tau)} \cdot \frac{c_{\xi 3} - 1}{1 - c_{\xi 1}^{-1} c_{\xi 4}^{-1}} \cdot \frac{m((\mu \nu + v)\xi | \tau)}{m(0|\tau)}
\]

where the variable \( s \) has been omitted. We proceed with estimating the factors in the quotient on the right.

Use (2), (4) and (20) to conclude:

\[
c_{\xi 3} - c_{\xi 1} \in O(\tilde{q}),
\]

\[
c_{\xi 3} - 1 \in 1 + O(\tilde{q}),
\]

\[
\frac{c_{\xi 3} - c_{\xi 1}}{1 - c_{\xi 1}} \in 2 + O(\tilde{q})
\]

and in view of II.2.(9)

\[
\frac{1 - c_{\xi 1}^{-1}}{1 - c_{\xi 1}^{-1} c_{\xi 4}^{-1}} \in 1 + O(\tilde{q})
\]

\[
\frac{c_{\xi 4}^{-1}}{1 - c_{\xi 1}^{-1} c_{\xi 4}^{-1}} \in O(1).
\]

The above estimates, together with the estimates (25) and (26), lead to (30).

\[\square\]

**PROPOSITION 8.** Assume (0) and fix 6 \( \in \{0, \frac{1}{2}\} \).
Then for \( i=1,2 \) and \( \mu=\xi \) the following estimate for \( s \to 1 \) holds:
(31) \[ f_{\tau}(\mu w + v) \in \]
\[ \in 1 - [1 - c_{\tau 1}(s)] \cos 2\pi(\mu w + v) + O(E(\tau - 2w)) \]

uniformly in \((w, v)\) provided

(32) \[ 0 \leq \text{Im } w \leq \delta \text{ Im } \tau \text{ and } \text{Im } v \text{ is bounded.} \]

**Proof.** From Lemma 7 conclude

\[ \frac{1 - f_{\tau}(\mu w + v)\tau(s))}{1 - c_{\tau 1}(s)} = [\cos 2\pi(\mu w + v) + R] [1 + S] \]

where \(R\) and \(S\) are remainder terms satisfying (22) for \(s > 1\), uniformly in \((w, v)\) provided (32) holds. The product on the right can be estimated as follows (cf. (26)):

\[ \cos 2\pi(\mu w + v) + O(E(\frac{1}{2} - \tau - w)) + O(E(\frac{1}{2} - 2w)) \in \]
\[ \in \cos 2\pi(\mu w + v) + O(E(\frac{1}{2} - 2w)), \]

and with (2) and (20) the claim follows.

---

VI.1.5 Limit behaviour of the other constants

For \(\sigma = \sigma(s)\) see V.5.(5)-(6),
for \(\alpha_1 = \alpha_1(s)\) see V.5.(19), ..., (20C),
for \(\beta_1 = \beta_1(s)\) see V.5.(22), ..., (23C),
for \(\gamma_1 = \gamma_1(s)\) see V.5.(27),
for \(\delta_1 = \delta_1(s)\) see V.5.(31).

For \(\hat{\sigma}\) see II.2.(42)-(48) or II.2.(50),
for \(\hat{\alpha}_1\) see II.2.(149a) or II.2.(150a), and II.2.(152),
for \(\hat{\beta}_1\) see II.2.(149b) or II.2.(150b), and II.2.(152),
for \(\hat{\delta}_1\) see II.2.(83).

**Proposition 9.** Under the assumptions (0) the following estimates for \(s > 1\) hold, for \(i \in \{1, 2\}\) (for \(O(\hat{\eta})\) see (20)).
\begin{align*}
(33) \quad \sigma(s) & \in \bar{\sigma} + O(\bar{q}), \\
(34) \quad \alpha_\tau(s) & \in \bar{\alpha}_\tau + O(\bar{q}), \\
(35) \quad \beta_\tau(s) & \in \bar{\beta}_\tau + O(\bar{q}), \\
(36) \quad \gamma_\tau(s) & \in \frac{1}{4} + O(\bar{q}), \\
(37) \quad \delta_\tau(s) & \in \bar{\delta}_\tau + O(\bar{q}).
\end{align*}

**Proof.** Proof of (33): Apply Theorem 5 (Part II) with \( \lambda = +1, \ w := 0, \ v := \sigma(s), \) and (10) with \( k = 2 \) to conclude (cf. (20))

\begin{align*}
(38) \quad g_\tau(s) - g_\tau(\bar{\sigma}) & = \\
& = g_\tau(s) - f_\tau(s)\tau(s) + \frac{1}{2} + d_\tau(\bar{\tau}) - d_\tau(1) \in O(\bar{q}),
\end{align*}

which implies (33) because \( g_\tau(.) \) is regular and injective on \((0, \frac{1}{2})\) with nonvanishing derivative. Alternatively, apply Lemma 7 with \( w := 0, \ v := \sigma(s), \) and (9) with \( k = 1 \) to conclude

\begin{align*}
(39) \quad \cos 2\pi \sigma(s) & \in \rho + O(\bar{q}),
\end{align*}

which implies (33).

Proof of (34): We distinguish two cases, according to the sign of \( \alpha_\tau_2(s) \), which does not depend on \( s \in (0, 1] \) (see V.2.(19A)-(19B) and Proposition II.2.3 (Part III)).

The case \( \alpha_\tau_2(s) = 0 \). Recall \( \alpha_\tau(s) = 0 = \bar{\alpha}_\tau \), cf. Property II.2.32 (Part I), and (34) is trivial.

The case \( \alpha_\tau_2(s) \neq 0 \). Noticing \( \alpha_\tau(s) \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \) (cf. (14)) apply Theorem 5 (Part II) with \( w := \alpha_\tau(s), \ v := 0 \) and \( \lambda := +1 \) to conclude

\begin{align*}
(40) \quad g_\tau(\alpha_\tau(s)) - g_\tau(\bar{\alpha}_\tau) & = \\
& = g_\tau(\alpha_\tau(s)) - f_\tau(\alpha_\tau(s)\tau(s) + \frac{1}{2}) \in O(\bar{q}).
\end{align*}

This implies (34) because both \( \alpha_\tau(s) \) and \( \bar{\alpha}_\tau \) are in the set \((-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \) (cf. II.2.(152)) and \( g_\tau(.) \) is regular and injective on this set with nonvanishing derivative.
VI.1

Proof of (35): Similar to the proof of (34) (this time use (27) with \( w := \xi(s), v := \frac{1}{2} \) and \( \lambda := -1 \)).

Proof of (36): Apply Lemma 7 with \( w := 0 \) and \( v := \gamma(s) \) to conclude

\[
\cos 2\pi \gamma(s) \in \mathcal{O}(\bar{q}),
\]

which implies (36).

Proof of (37): Notice (cf. V.5.(33)) that \( \delta_\xi(s), \delta_\xi \in (0, \frac{1}{2}) \) and conclude from Theorem 5 (Part II) with \( w := 0, v := \delta_\xi(s) \) and \( \lambda := +1 \)

\[
g_\xi(\delta_\xi(s)) - g_\xi(\delta_\xi) =
g_\xi(\delta_\xi(s)) - f_\xi(\delta_\xi(s) \tau(s) + \frac{1}{2}) \in \mathcal{O}(\bar{q}),
\]

which implies (37).
VI.2 Continuation of Sec. VI.1 — Proof of Basic Theorem II.4.55 for non-even walk

This section continues the previous one, and so our assumptions remain the same, viz.

\[
\begin{align*}
(0) & \quad \text{the random walk is skipfree, all-sided, and driftless,} \\
& \quad \text{the random walk is non-even,} \\
& \quad 0 < s < 1,
\end{align*}
\]

and

\[
(1) \quad y \in (\mathbb{U})^2.
\]

Using the results of the previous section we will now prove the Basic Theorem II.4.55 in the case of non-even walk. Some proofs are deferred to the next section (which serves as an appendix to the present one).

VI.2.1 The function \((\text{SF}) (z_1 \tau + \frac{1}{2}, z_2 \tau + \frac{1}{2})\) for \(s\) close to 1

The function \((\text{SF})(z)\) is defined in IV.5.(37) et seq. Replace \((z_1, z_2)\) by \((z_1 \tau + \frac{1}{2}, z_2 \tau + \frac{1}{2})\) to find the following. If

\[
(2) \quad z_\kappa \tau + \frac{1}{2} \in S_\kappa \quad \text{for} \ \kappa = 1, 2
\]

(where \(\tau\) (see V.5.(1)) and \(S_\kappa\) (see V.5.(35)) depend on \(s\), then

\[
(3) \quad (\text{SF}) (z_1 \tau + \frac{1}{2}, z_2 \tau + \frac{1}{2}) = \sum_{\zeta_1, 2} \sum_{\lambda = \pm} \lambda \cdot I^\lambda_{\zeta}(z_\zeta)
\]

(where \(\zeta := 3 - \zeta\)) with

\[
(4) \quad I^\lambda_{\zeta}(z_\zeta) = \int \frac{du}{2\pi i} \tilde{\eta}(z_\zeta \tau + \frac{1}{2} - u|2\sigma\tau) \cdot F^\lambda_{\zeta}(u) \quad \text{with the function} \ F^\lambda_{\zeta} \text{given in V.7.(7)-(8), and}
\]

where \(A^\lambda_{\zeta}(\cdot)\) is a path to be specified presently.

Let \(f^\lambda_{\zeta}, \zeta = 1, 2, \lambda = \pm\), be four numbers satisfying IV.5.(38A)-(38B). The path \(A^\lambda_{\zeta}(\cdot)\) is required to satisfy:
\[
\Lambda_j^\lambda(\cdot) \text{ is a path in } F_j^\lambda (S^\lambda \cap (S^\lambda + \lambda \sigma \tau), \text{ cf. IV.1.}(2)) \text{ from } f_j^\lambda \text{ to } f_j^\lambda + 1,
\]

\[
\Lambda_j^\lambda(\cdot) \text{ satisfies the conditions for } \Lambda \text{ in IV.4.}(25A)-(25B) \text{ with } v := z_\tau^\lambda + \frac{1}{2}.
\]

**Choice of the integration paths for } s \text{ close to } 1**

In the following we assume

\[
\tilde{z}_k \in \tilde{S}_k \text{ for } k = 1, 2
\]

(for the sets \( \tilde{S}_k \) (which do not depend on \( s \)) see II.2.(87)).

**PROPERTY 1.** Assume (0) and (6). Then there exist a neighbourhood of 1 and a neighbourhood of \( \hat{z}_k \) so that (2) holds if \( s \) is in the neighbourhood of 1 and \( z \) is in the neighbourhood of \( \hat{z}_k \).

**PROOF.** Let \( \zeta_k \subset \tilde{S}_k \) be a compact neighbourhood of \( \hat{z}_k \) for \( k = 1, 2 \). Then, by definition, for all \( z_k \in \zeta_k \)

(a) \[|\text{Re } z_k| < \frac{1}{2},\]

(b) \[|g_k(z_k)| < 1.\]

From (a) it follows (because \( \tau \) is purely imaginary)

(a') \[|\text{Im } (z_k \tau + \frac{1}{2})| = |\text{Re } z_k| \cdot |\text{Im } \tau| < \frac{1}{2} |\text{Im } \tau|.\]

From (b) it follows (by compactness) that for some positive \( \epsilon \) and all \( z_k \in \zeta_k \)

\[|g_k(z_k)| \leq 1 - \epsilon < 1.\]

Use Theorem VI.1.5 (Part I with \( v := z_k, w := 0 \) and \( \lambda = +1 \)) to conclude that for all \( z_k \in \zeta_k \)

\[|f_k(z_k \tau + \frac{1}{2}) - g_k(z_k)| < \frac{1}{2} \epsilon.\]
if \( s \) is sufficiently close to 1, hence

(b') \[ |f_k(z_k + \frac{1}{2})| < 1 - \frac{1}{2} \varepsilon < 1. \]

From (a') and (b') it follows that (2) holds for all \( z_k \in C_k \) if \( s \) is near 1, which proves the claim.

\[ \Box \]

**PROPERTY 2.** Assume \((0)\). Then the conditions IV.5.(38A)-(38B) are satisfied if

\[ f^\lambda_j = \lambda \frac{1}{2} \sigma(s) \tau(s) \]

(for \( j=1,2 \) and \( \lambda = \pm \)) and \( s \) is sufficiently close to 1.

**PROOF.** Obviously, IV.5.(38B) is satisfied. For IV.5.(38A) use VI.1.(33)-(36) and \( 0 < \delta < \frac{1}{2} \) to conclude

\[ \lambda \frac{1}{2} \sigma \in (-\gamma_j, \gamma_j) \cap (-\gamma_{\hat{z}^*} + \lambda \sigma, \gamma_{\hat{z}^*} + \lambda \sigma) \]

if \( s \) is close to 1. This implies the claim because of VI.5.(47).

\[ \Box \]

For given \( \hat{z} = (\hat{z}_1, \hat{z}_2) \) satisfying (6) we next construct a path \( \lambda^\lambda_j(\cdot) \) (depending on \( s \)) which will be shown to meet the requirements (5A) (with \( f^\lambda_j \) given in (7)) and (5B) for all \( z_{\hat{z}} \) in some neighbourhood of \( \hat{z}_{\hat{z}} \), if \( s \) is sufficiently close to 1.

First, choose a fixed sufficiently large positive number \( M \), viz. an \( M \) satisfying:

\[ M > |\text{Im} \hat{z}_k| \text{ for } k=1,2, \text{ and } M > M_0, \]

where \( M_0 \) is the positive constant from Property II.4.2.

Secondly, let \( \Gamma^\lambda_j(\cdot) \) be a path (not depending on \( s \)) satisfying (9A) and (9B):

(9A) \[ \Gamma^\lambda_j(\cdot) \text{ is a path from } -M \text{ to } +M, \]
(9B) \( (\Gamma_j^\lambda)^* + \lambda \frac{1}{2} \mathcal{O} \subset \mathcal{D}(z) \cap H_j^\lambda \)

where the star indicates the range; for the constant \( \mathcal{O} \) see II.2.(50), for the set \( H_j^\lambda \) see II.4.(3), and for the set \( \mathcal{D}(z) \) see II.4.(18)-(17). In other words, the path \( \Gamma_j^\lambda(\cdot) + \lambda \frac{1}{2} \mathcal{O} \) is assumed to meet the conditions for the integration path in the integral II.4.(28) with \( \nu := z \). Such a path does exist, cf. Proposition II.4.5 and Remark II.4.1 (see II.4.(14) and (6), and (1)). Observe that (9B) implies

(9C) \( (\Gamma_j^\lambda)^* + \lambda \frac{1}{2} \mathcal{O} \subset \mathcal{D}(z) \cap H_j^\lambda \)

for all \( z \) in some neighbourhood of \( z \), because \( (\Gamma_j^\lambda)^* \) is compact and \( \mathcal{D}(z) \) is open (use II.4.(19)).

Thirdly, assume that \( s \) is sufficiently close to 1, viz. so that (for \( \tau \) see VI.1.(14))

(10) \( M \leq \frac{1}{2} \text{Im} \tau(s) \).

Then extend the path \( \Gamma_j^\lambda(\cdot) \) to a path, to be called \( \gamma_j^\lambda(\cdot) \), from \( -\frac{1}{2} \tau \) to \( \frac{1}{2} \tau \) by adding the straight line segments \( \pm[M_i, \frac{1}{2} \tau] \). More formally, let \( L^+(\cdot) \) be a straight path from \( M_i \) to \( \frac{1}{2} \tau \), and \( L^-(\cdot) \) a straight path from \( -\frac{1}{2} \tau \) to \( -M_i \); then the chain \( \gamma_j^\lambda(\cdot) \) consists of the collection \( \{L^-(\cdot), \Gamma_j^\lambda(\cdot), L^+(\cdot)\} \).

**REMARK 1.** Observe that for all \( z \) near \( z \)

(11) \( (\gamma_j^\lambda)^* + \lambda \frac{1}{2} \mathcal{O} \subset \mathcal{D}(z) \cap H_j^\lambda \)

if \( s \) is sufficiently close to 1; this follows from (9C), (8) and Property II.4.2.

So the path \( \gamma_j^\lambda(\cdot) + \lambda \frac{1}{2} \mathcal{O} \) satisfies the conditions for the integration path in the integral II.4.(28) with \( M_i \) replaced by \( \frac{1}{2} \tau \), for all \( \nu \) near \( z \).

Finally,

(12) \( \Lambda_j^\lambda(\cdot) \) is the opposite path to \( [\gamma_j^\lambda(\cdot) + \lambda \frac{1}{2} \mathcal{O}(s)] \tau(s) + \frac{1}{2} \).
PROPOSITION 3. Assume (0) and (6). Then there exist a neighbourhood of 1 and a neighbourhood of \( \hat{z}_i \) with the property that the path \( \Lambda^\lambda_j(\cdot) \) defined in (12) satisfies the requirements (5A) (with \( f^\lambda_j \) given in (7)) and (5B) if \( s \) is in the neighbourhood of 1 and \( z_i \) is in the neighbourhood of \( \hat{z}_i \).

PROOF. Postponed to subsection VI.3.1.

Transformation of the integral (4)

In the sequel we assume that \( \hat{z} = (\hat{z}_1, \hat{z}_2) \) satisfies (6), that \( s \) is sufficiently close to 1, and that \( z = (z_1, z_2) \) is sufficiently close to \( \hat{z} \), so Proposition 3 is in force. In the integral (4) we choose the integration path \( \Lambda^\lambda_j(\cdot) \) given in (12), and next transform the integral into an integral over \( \gamma^\lambda_j(\cdot) \). To that end we introduce the new integration variable \( w \) given by

\[
w := \frac{1}{\tau} (u - \frac{1}{2}) - \frac{1}{2} \eta, \quad \text{or} \quad u = (w + \frac{1}{2} \eta) \tau + \frac{1}{2},\]

and use IV.5.(36F) to conclude

\[
I^\lambda_j(z_i) = \int_{\gamma^\lambda_j} \frac{dw}{2\pi i} \tau \sim \left( (w + \frac{1}{2} \eta - z_i) \tau \right) 2\sigma \tau). \quad \gamma^\lambda_j \quad \eta^\lambda_j \quad f^\lambda_j ((w + \frac{1}{2} \eta) \tau + \frac{1}{2}).
\]

The integral (13) splits up in three parts, because the integration path is so divided, giving

\[
I^\lambda_j(z_i) = I^\lambda_j M(z_i) + \sum_{\mu=\pm} K^\lambda_j \mu(z_i)
\]

where

\[
I^\lambda_j M(z_i) := \int_{\gamma^\lambda_j} \frac{dw}{2\pi i} \tau \sim \left( (w + \frac{1}{2} \eta - z_i) \tau \right) 2\sigma \tau). \quad \gamma^\lambda_j \quad \eta^\lambda_j \quad f^\lambda_j ((w + \frac{1}{2} \eta) \tau + \frac{1}{2}).
\]

and for \( \mu=\pm \) (with the same integrand)
\[ K^\lambda, \mu(z) = \mu \left( \frac{1}{\mu i} \int \frac{dw}{2\pi i} \right) \cdots. \]

We rewrite this last integral. Introduce the new integration variable \( w' := \mu w \) to conclude

\[ K^\lambda, \mu(z) = \frac{1}{2i} \int \frac{dw'}{\mu i} \cdot \frac{\tau \eta}{2\pi i} \left( (\mu w + \frac{1}{2} \sigma - z) \nu \right| 2\sigma \tau) \cdot F^\lambda \left( (\mu w + \frac{1}{2} \sigma) \nu + \frac{1}{2} \right). \]

We change the above integral by adding an extra term which involves the constant

\[ \varphi' (\omega i) := - \frac{\pi i}{2\sigma(s)}, \]

as follows. We put (using (5A))

\[ L^\lambda, \mu := \frac{1}{2i} \int \frac{dw'}{2\pi i} \cdot \mu \frac{\varphi'}{\varphi} (\omega i) \cdot F^\lambda \left( (\mu w + \frac{1}{2} \sigma) \nu + \frac{1}{2} \right), \]

and define

\[ K^\lambda, \mu(z) := K^\lambda, \mu(z) - I^\lambda, \mu, \]

or, in full,

\[ K^\lambda, \mu(z) = \frac{1}{2i} \int \frac{dw'}{2\pi i} \cdot \mu \frac{\varphi'}{\varphi} (\omega i) \cdot \left[ \tau \eta \left( (\mu w + \frac{1}{2} \sigma - z) \nu \right| 2\sigma \tau \right) - \mu \frac{\varphi'}{\varphi} (\omega i) \right]. \]

In (14) we replace each \( K^\lambda, \mu \) by \( K^\lambda, \mu \).

The addition of the extra terms does not change the sum on the right in (3), as will be shown presently. From V.7.9 it follows...
\[ P_j^+ (\cdot + \frac{1}{2} \sigma \tau) - P_j^- (\cdot - \frac{1}{2} \sigma \tau) \equiv 0 \]

(where \( j = 3 - \bar{j} \)), which implies \( L_j^{+, \mu} - L_j^{-, \mu} = 0 \), hence

\[
\sum_{j=1,2} \sum_{\lambda=\pm} \lambda L_j^{\lambda, \mu} = 0.
\]

Summarizing we have the following. Let \( \tilde{z} = (\tilde{z}_1, \tilde{z}_2) \) satisfy (6), and assume that \( s \in (0,1) \) is in some fixed (sufficiently small) neighbourhood of \( 1 \), and that \( z = (z_1, z_2) \) is in some fixed (sufficiently small) neighbourhood of \( \tilde{z} \), so that Property 1, Property 2, (10), and Proposition 3 are in force. Then, using (14), (18) and (20), we can rewrite (3) as follows:

\[
(SF) (z_1^\tau + \frac{1}{2}, z_2^\tau + \frac{1}{2}) = \\
= \sum_{\lambda=\pm} \sum_{j=1,2} \lambda \left[ L_j^{\lambda, \mu}(z_j) + \sum_{\mu=\pm} \tilde{K}_j^{\lambda, \mu}(z_j) \right]
\]

with \( L_j^{\lambda, \mu}(\cdot) \) defined in (15), and \( \tilde{K}_j^{\lambda, \mu}(\cdot) \) given in (19).

VI.2.2 Limit of the function \((SF)(z_1^\tau + \frac{1}{2}, z_2^\tau + \frac{1}{2})\) for \( s \to 1 \)

Our aim is Theorem 10 below. We need some notations. We put

\[
m_{\bar{j}} = m_{\bar{j}}(y) := \begin{cases} 
1 & \text{if } y_{\bar{j}} = 1 \\
0 & \text{otherwise}
\end{cases}
\]

for \( \bar{j} = 1,2 \), and with \( y = (y_1, y_2) \)

\[
n(y) := \max_{\bar{j}=1,2} \{ 2m_{\bar{j}} + m_{\bar{j}} \}
\]

where \( j := 3 - \bar{j} \), or
(23') \[ n(y) = \begin{cases} 0 & \text{if } m_1 + m_2 = 0 \\ 2 & \text{if } m_1 + m_2 = 1 \\ 3 & \text{if } m_1 + m_2 = 2. \end{cases} \]

Also we recall from II.4.(101)-(96)-(83)

(24) \[ \mathcal{V}(\delta) := \begin{cases} (\overline{U})^2 & \text{if } 0 < \delta < \frac{1}{4} \\ (\overline{U})^2 \setminus \{(1,1)\} & \text{if } \frac{1}{4} \leq \delta < \frac{1}{2} \end{cases} \]

where $\overline{U}$ is the closed unit disc. Observe

(25) \[ \frac{1}{2\delta} - (m_1 + m_2) > 0 \text{ if } y \in \mathcal{V}(\delta). \]

In the following the function $H^\lambda_j(\cdot,y)$ occurs, which is defined in II.2.(135)-(128).

**PROPOSITION 4.** Assume (0) and (1). Consider for $j=1,2$ and $\lambda, \mu = \pm$ the difference

(26) \[ F^\lambda_j((\mu w + \lambda \frac{1}{2} \sigma) + \frac{1}{2}) - H^\lambda_j(\mu w + \lambda \frac{1}{2} \sigma, y). \]

I. Assume $\mu = +$. Then this difference belongs to $
0 \left( E(\frac{1}{2}) \right) \]
for $s \to 1$, uniformly in $w$ provided $w \in (\Gamma^\lambda_j)^*$.

II. For $\mu = \pm$ this difference belongs to \n$0 \left( E(\frac{1}{2} - n(y)w) \right)$ \nfor $s \to 1$, uniformly in $w$ provided

(27) \[ w \in i\mathbb{R} \text{ and } M \leq \text{Im } w \leq \frac{1}{2} \text{Im } \tau. \]

**PROOF.** Postponed to subsection VI.3.2.

**PROPERTY 5.** Assume (1). Consider for $j=1,2$ and $\lambda, \mu = \pm$ the quantity

(28) \[ H^\lambda_j(\mu w + \lambda \frac{1}{2} \sigma, y). \]
I. Assume $\mu = +$. Then this quantity, as a function of $w$ on the set $(\Gamma_j^*)^*$, is bounded.

II. For $\mu = \pm$ this quantity, as a function of $w$ on the set

$$\{w \in \mathbb{R} : M \leq \text{Im } w\},$$

is bounded by a constant times $E(-{(m_1 + m_2)}w)$.

**PROOF.** Part I: Because of (9B) the function is finite on the compact set $(\Gamma_j^*)^*$.

Part II: Postponed to subsection VI.3.2.

For the function $\varphi/\varphi$ see II.4.(23), and for the constant $\varphi'/\varphi(\omega i)$ see II.4.(56). For $\bar{\varphi}/\varphi$ resp. $\bar{\varphi}'/\varphi(\omega i)$ see VI.3.(34) resp. (16).

**PROPOSITION 6.** Assume (0) and (6), and (8) (first inequality).

I. For $\lambda = \pm$ the difference

$$\tau \vec{h}_\mu((w + \frac{1}{2}\sigma - z_i)\tau|2\sigma \tau) - \frac{\varphi'}{\varphi}(w + \frac{1}{2}\sigma - z_i)$$

belongs to $O(E(1/\tau))$ uniformly in $w \in (\Gamma_j^*)^*$ and uniformly in $z_i$ in some (sufficiently small) neighbourhood of $\hat{z}_i$.

II. For $\lambda = \pm$ and $\mu = \pm$ the difference

$$\left[\tau \vec{h}_\mu((\mu w + \frac{1}{2}\sigma - z_i)\tau|2\sigma \tau) - \mu \frac{\varphi'}{\varphi}(\omega i)\right] +$$

$$-\left[\frac{\varphi'}{\varphi}(\mu w + \frac{1}{2}\sigma - z_i) - \mu \frac{\varphi'}{\varphi}(\omega i)\right]$$

belongs to

$$O(E(\frac{1-w}{2\sigma})) + O(\frac{\tau}{\tau E(\frac{1}{2}\tau + \frac{w}{2\sigma})})$$

for $s + 1$, uniformly in $w$ provided (27) holds and uniformly in $z_i$ in some (sufficiently small) neighbourhood of $\hat{z}_i$.

**PROOF.** Postponed to Subsec. VI.3.3.
PROPERTY 7. Assume (6), and (8).

I. Then for \( \lambda = \pm \) the function

\[
\frac{\varphi'}{\varphi}(w + \frac{1}{2} \zeta - z_i)
\]

is bounded if \( w \in (R^*)_i \) and \( z_i \) is in some (sufficiently small) neighbourhood of \( \hat{z}_i \).

II. For \( \lambda, \mu \in \{-,+\} \) the function

\[
\frac{\varphi'}{\varphi}(\mu w + \frac{1}{2} \zeta - z_i) - \mu \frac{\varphi'}{\varphi}(\mu \varphi)
\]

is bounded by a constant times \( \mathcal{E}(\frac{w}{\zeta}) \) if \( w \) is in the set (29) and \( z_i \) is in some (sufficiently small) neighbourhood of \( \hat{z}_i \).

PROOF. Part I. Left to the reader.

Part II. Use (8), and Property II.4. 22 with \( t = \text{Im} \ w \).

The function \( I_{\phi}^{\lambda,M} H_{j}^{\lambda}(\cdot,y) \) is defined in II.4.(28). For the function \( R_{\phi,\mu}^{\lambda,M,M'} H_{j}^{\lambda}(\cdot,y) \) see II.4.(104) or, equivalently, II.4.(115). In the following proposition we use the abbreviation

\[
M' := \frac{1}{2} \text{Im} \ \hat{\gamma}(s).
\]

PROPOSITION 8. Assume (0), (6), (8), and

\[
y \in \gamma(\mathcal{G}).
\]

Then for \( i=1,2 \) (with \( j=3-i \)) and \( \lambda = \pm \) there exists a neighbourhood \( N \) of \( \hat{z}_i \) such that the following estimates for \( s \to 1 \) hold uniformly in \( z_i \) in \( N \).

I. The difference

\[
I_{j}^{\lambda,M}(z_i) - I_{\phi}^{\lambda,M}(z_i,y)
\]

can be estimated by
(37) \[ O(E(\frac{1}{T_n})). \]

II. For \( \mu = \pm \) consider the difference

\[
\sum_{j}^\sim \Phi_j^\lambda, \mu(z_{\tilde{z}}, y) = \sum_{j}^\sim \Phi_j^\lambda, \mu H_j^\lambda(y),
\]

with \( \mu \) given in (34). This difference can be estimated by (39) + (40) + (41):

(39) \[ O(\tau E(\frac{1}{T_n})) , \]

(40) \[ O(E(\frac{1}{T_n}) [1 + \frac{1}{2\sigma} - n(y)]) , \]

(41) \[ O(E(\frac{1}{T_n}) [1 + \frac{1}{2\sigma} - n(y)]) . \]

**PROOF.** Part I. Use II.4.(28) to conclude from (8) and (9A)-(9B)

(41 A) \[ I_{\Phi}^\lambda, H_j^\lambda(z_{\tilde{z}}, y) = \]

\[ = \int \frac{dw}{2\pi} \frac{\phi}{\phi^*}(w + \frac{1}{2\sigma} - z_{\tilde{z}}) H_j^\lambda(w + \frac{1}{2\sigma}, y), \]

because both (9C) and II.4.(34) (cf. II.4.(14)) are satisfied if \( z_{\tilde{z}} \) is close to \( \tilde{z}_{\tilde{z}}. \) Now use (15) and the equality

(42) \[ a'b' - ab = (a' - a)b + a(b' - b) + (a' - a)(b' - b) \]

to conclude that the difference (36) equals

\[ \int \frac{dw}{2\pi} \{ (30).[(28) with \ \mu = +] + (32).[(26) with \ \mu = +] + (30).[(26) with \ \mu = +] \} . \]

Because \( I_{\Phi}^\lambda \) is compact, the claim easily follows with the Propositions 4 and 6 (Parts I) and the Properties 5 and 7 (Parts I).

Part II. The difference (38) can be rewritten as follows (cf. (19) and II.4.(115) with \( v := z_{\tilde{z}} \) and \( w := ti, \) and use the equality (42))
\[
\frac{1}{\sqrt{t}} \int \frac{dw}{2w \text{I}} \{ (31).(28) + (33).(26) + (31).(26) \}.
\]

The contribution of each one of the three terms in the integrand now will be estimated separately, with the use of the Propositions 4 and 6 (Parts II) and the Properties 5 and 7 (Parts II).

First observe

\[
(31).(28) \in O\left(E\left(\frac{1}{\sqrt{t}} - w[\frac{1}{2\sigma} + \frac{1}{4m_1 + m_2}]\right) + O\left(\frac{1}{\sqrt{t}} E\left(\frac{1}{\sqrt{t}} - w[\frac{1}{2\sigma} + \frac{1}{4m_1 + m_2}]\right)\right).
\]

The contribution of the first order-term to the integral can be estimated as follows (because we can do the integration; observe that the coefficient of \(w\) in the exponent is positive)

\[
(*) \quad O\left(E\left(\frac{1}{2} \left[ \frac{1}{2\sigma} - (m_1 + m_2) \right] \right)\right).
\]

Similarly, an estimate of the contribution of the second one is (observe that the coefficient of \(w\) in the exponent is positive because of (35) and (25))

\[
O\left(\frac{1}{\sqrt{t}} E\left(\frac{1}{2} \left[ \frac{1}{2\sigma} - (m_1 + m_2) \right] \right)\right),
\]

which is negligible in comparison with (*) because of VI.1.(33).

Next observe

\[
(33).(26) \in O\left(E\left(\frac{1}{\sqrt{t}} + w[\frac{1}{2\sigma} - n(y)]\right)\right).
\]

An estimate of its contribution to the integral is given by (39) if the coefficient of \(w\) in the exponent vanishes, and by (40) + \(O\left(E\left(\frac{1}{\sqrt{t}}\right)\right)\) otherwise. So the contribution can be estimated by (39) + (40).

Finally observe

\[
(31).(26) \in O\left(E\left(\frac{1}{\sqrt{t}} + \frac{1}{2\sigma} - w[\frac{1}{2\sigma} + n(y)]\right)\right) + O\left(\frac{1}{\sqrt{t}} E\left(\frac{1}{\sqrt{t}} + w[\frac{1}{2\sigma} - n(y)]\right)\right).
\]
An estimate of the contribution of the first order-term to the integral is given by (41).
An estimate of the contribution of the second order-term to the integral is (see (4) et seq.)

\[ O(\tau E(\frac{1}{\tilde{z}})) \cdot [(39) + (40)], \]

which is negligible in comparison with (39) + (40).

To complete the proof, observe that the estimate (*) can be omitted because it is negligible in comparison with (39) if \( m_1 + m_2 = 0 \), and coincides with (41) if \( m_1 + m_2 \in \{1, 2\} \).

From II.4.(122) we recall (cf. also II.4.(197))

\[ \tilde{R}_{\lambda, M}^{\phi, \mu} \tilde{H}_{\lambda}^{\phi}(z, y) \cdot \lim_{M' \to \infty} \tilde{R}_{\lambda, M}^{\phi, \mu} \tilde{H}_{\lambda, M'}^{\phi}(z, y) \]

provided the limit exists. By Proposition II.4.26 the limit exists if 
\( y \in Y(\delta) \) and \( |\text{Im } z_\delta| < M \).

**PROPERTY 9.** Assume (6), (8), and (35). Consider for \( i = 1, 2 \) (with \( j := 3-i \)) and \( \lambda, \mu = \pm \) the difference

\[ (\tilde{R}_{\lambda, M}^{\phi, \mu} - \tilde{R}_{\lambda, M'}^{\phi, \mu}) \tilde{H}_{\lambda}^{\phi}(z, y) \]

with \( M' \) given in (34). This difference belongs to

\[ O \left( E(\frac{1}{\tilde{z}} - (m_1 + m_2) \right)) \]

uniformly in \( z_\delta \) in some neighbourhood of \( \tilde{z} \).

**PROOF.** Observe that the difference (44) depends only through \( M' \), and use Property 5 (Part II) and Property 7 (Part II).

In order to prove Theorem 10 below we need some additional notations. We introduce the following integrals
These integrals are well-defined if

\[
\begin{align*}
\left\{ \begin{array}{l}
y \in (\bar{\omega})^2, \\
|\text{Im} \ z_{\bar{\iota}}| < M \text{ for } \iota = 1, 2, \\
M_0 \leq M \leq M',
\end{array} \right.
\end{align*}
\]

where \( M_0 \) is the positive constant from Property II.4.2 (cf. (8)), and have from II.4.(104)

\[
(R_{\phi,\mu}^{\lambda, M, M'} - \tilde{R}_{\phi,\mu}^{\lambda, M, M'}) H_j^\lambda(z_{\bar{\iota}}, y) = S_{\mu}^{\lambda, M, M'} H_j^\lambda(y).
\]

Observe (use II.4.(1)) \( S_{\mu}^{\lambda, M, M'} H_j^\lambda(y) = S_{\mu}^{-\lambda, M, M'} H_{j-\iota}^\lambda(y) \) where \( j = 3-\iota, \) hence

\[
(49) \quad \sum_{\iota=1, 2} \sum_{\lambda = \pm} \lambda S_{\mu}^{\lambda, M, M'} H_j^\lambda(y) \equiv 0.
\]

So, if in addition \( z_{\bar{\iota}} \in \tilde{S}_{\bar{\iota}} \) for \( \iota = 1, 2, \) we can write

\[
(50) \quad \sum_{\iota=1, 2} (R_{\phi,\mu}^{M', H_j^\lambda}(z_{\bar{\iota}}, y) = \text{II.4.}(44)
\]

\[
= \sum_{\iota=1, 2} \sum_{\lambda = \pm} \lambda (I_{\phi}^{\lambda, M'} + \sum_{\mu = \pm} R_{\phi,\mu}^{\lambda, M, M'}) H_j^\lambda(z_{\bar{\iota}}, y) = \text{II.4.}(107)-(108)
\]

\[
= \sum_{\iota=1, 2} \sum_{\lambda = \pm} \lambda (I_{\phi}^{\lambda, M'} + \sum_{\mu = \pm} R_{\phi,\mu}^{\lambda, M, M'}) H_j^\lambda(z_{\bar{\iota}}, y).
\]

\[
= \sum_{\iota=1, 2} \sum_{\lambda = \pm} \lambda (I_{\phi}^{\lambda, M'} + \sum_{\mu = \pm} R_{\phi,\mu}^{\lambda, M, M'}) H_j^\lambda(z_{\bar{\iota}}, y).
\]
Because of (43) we can take the limit for \( M' \to \infty \) provided
\[
y \in Y(\mathcal{G})
\]
and we find, with II.4.(132),
\[
(\Lambda H_0) (z,y) =
\sum_{i=1,2} \sum_{\lambda = \pm} -\lambda (1 - 1^\lambda M + \sum_{\mu = \pm} R_{\phi,\mu}^\lambda M) H_i^\lambda (z^i, y).
\]
This expression will be used in the proof of the following theorem.

**THEOREM 10.** Assume (6) and (35). Then the difference
\[
(SF) (z_1, z_2) - (\Lambda H_0) (z,y)
\]
(where \( z = (z_1, z_2) \)) can be estimated for \( s + 1 \) by (39) + (41) + (45) uniformly in \( z \) on some neighbourhood of \( \tilde{z} \).

(For the function \((\Lambda H_0) (z,y)\) see II.4.(130)-(132).)

**REMARK 2.** It follows that the above estimate of (52) holds uniformly in \( z_k \) on compact subsets of \( \tilde{z}_k \) (for \( k = 1,2 \)) (cf. also Property 1).

**PROOF** of Theorem 10. Choose an \( M \) satisfying (8) and (for \( \sigma = 1,2 \) and \( \lambda = \pm \)) an integration path \( \Gamma_{\sigma}^\lambda (\cdot) \) satisfying (9A)-(9B). In (52) substitute the expressions (21) and (51), and use Proposition 8 and Property 9 to conclude that the difference (52) can be estimated by (37) + (39) + (40) + (41) + (45). Observe that (37) is negligible in comparison with (39), and that (40) is negligible in comparison with (45) because of \( n(y) - 1 \leq m_1 + m_2 \) , cf. (23').

VI.2.3 Proof of Basic Theorem II.4.55 for non-even walk

**PROPERTY 11.** Assume \( k \in \{1,2\} \) and \( x_k \in \mathcal{U} \). Then there exists a continuous function
\[
s \in (0,1] \sim x_k (s \mid x_k) \in \mathcal{G}
\]
satisfying:
(i) \[ z_k \in \tilde{S}_k \quad \text{and} \quad g_k(z_k) = x_k \]

where

\[ z_k = \zeta_k(1|x_k), \]

(ii) if \( 0 < s < 1 \) then \[ w_k \in S_k \quad \text{and} \quad f_k(w_k) = x_k \]

where

\[ w_k = \zeta_k(s|x_k) \tau(s) + \frac{1}{2}. \]

**Proof.** First we define a function \( \zeta_k(s|x_k) \) satisfying (i) and (ii). In the following a bar denotes closure. Assume \( x_k \in \tilde{U} \). Choose a fixed \( \lambda \in \{-, +\} \) so that \( x_k \in \overline{\theta} \). (cf. V.5.(12)). By Property II.2.21 there exists a unique \( z_k \) satisfying

\[ g_k(z_k) = x_k \quad \text{and} \quad z_k \in \overline{R}_\lambda \]

(cf. II.2.(85)), and by Property V.5.1 (Part I) there exists, for each \( s \in (0,1) \), a unique \( w_k \) satisfying

\[ f_k(w_k) = x_k \quad \text{and} \quad w_k \in \begin{cases} \overline{R}_\lambda & \text{if} \; \lambda = +, \\ \overline{R}_\lambda + 1 & \text{if} \; \lambda = - \end{cases} \]

(cf. V.5.(11)). Because of \( x_k \in \tilde{U} \) we have \( z_k \in \tilde{S}_k \) (use II.2.(87)-(88)-(89)) and \( w_k \in S_k \) (use V.5.(35)). Now define the function \( \zeta_k(s|x_k) \) by means of (54) and (55).

Next we show that the above defined function is continuous. For \( s \in (0,1) \) this follows from Proposition V.5.4 (Part II; cf. V.5.(34A)). The continuity in \( s=1 \) remains to be shown. We use the abbreviation \( z_k(s) := \zeta_k(s|x_k) \). One easily checks (using VI.1.11-(14)) that

\[ z_k(s) \in \overline{R}_\lambda \quad \text{for all} \; s \in (0,1), \]

\[ |\text{Im} \; z_k(s)| \leq \frac{1}{2} \text{Im} \; \tau(s) \quad \text{if} \; 0 < s < 1. \]

Use Theorem VI.1.5 (Part I, with \( \lambda = +, \tau := k, \; v := z_k(s) \) and \( v := 0 \)) and the relationship \( f_k(z_k(s) \tau + \frac{1}{2}) = g_k(z_k(1)) \) to conclude
\[ g_k(z_k(1)) - g_k(z_k(s)) \to 0 \]

for \( s \to 1 \). Now use (*) and Property II.2.21 to conclude \( z_k(s) \to z_k(1) \).

\[ \]

\[ \]

**PROOF of Basic Theorem II.4.55 in the case of non-even walk**

Choose a fixed \( x = (x_1, x_2) \in u^2 \) so that \( Q(x, 1) \neq 0 \) and a fixed \( y = (y_1, y_2) \in y(0) \) (notice that this includes the case \( y \in u^2 \), see (24)). We have by continuity that \( Q(x, s) \neq 0 \) if \( s \) is close to 1; hence, by Theorem V.7.5 it suffices to show

\[ \]

(a) \[ \lim_{s \to 1, \ 0 < s < 1} \widetilde{P}(x, y, s) = \tilde{P}(x, y) \]

with \( \widetilde{P}(x, y, s) \) defined in V.7.(21) and \( \tilde{P}(x, y) \) in II.4.(150). Obviously, (a) is equivalent to (b):

\[ \]

(b) \[ \lim_{s \to 1, \ 0 < s < 1} (LP_0)(x, y, s) = (LP_0)(x, y) \]

where the function \( LP_0 \) is defined in V.7.(19) (with \( AF_0(\cdot) \equiv (SF)(\cdot) \), cf. V.7.(18)), and the function \( LP_0 \) is defined in II.4.(141) (with the function \( AH_0 \) given in (51)). Put for \( s \in (0, 1] \)

\[ z(s) := (z_1(s), z_2(s)), \quad z_k(s) := \zeta_k(s|z_k) \text{ for } k = 1, 2 \]

with \( \zeta_k(s|z_k) \) given in Property 11, and for \( s \in (0, 1) \)

\[ w(s) := (w_1(s), w_2(s)), \quad w_k(s) := z_k(s)\tau(s) + \frac{1}{2} \text{ for } k = 1, 2. \]

Then

\[ (LP_0)(x, y, s) = (SF)(w(s)) \]

for \( s \in (0, 1) \), and

\[ (LP_0)(x, y) = (AH_0)(z(1), y). \]
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Hence, (b) is equivalent to (c):

(c) \[ \lim_{s \to 1, \, 0 < s < 1} (SF)(w(s)) = (\Lambda H_0)(z(1), y). \]

We prove (c). First, apply Theorem 10 with \( \hat{z} := z(1) \) and \( z := z(s) \), using \( z(s) \to z(1) \), to conclude that the difference

\[ (SF)(w(s)) - (\Lambda H_0)(z(s), y) \]

can be estimated by (39) + (41) + (45) for \( s \to 1 \), hence, has limit 0. Next, use the continuity of the function \( z \to (\Lambda H_0)(z, y) \) (see Property II.4.40, Part III) to conclude that the difference

\[ (\Lambda H_0)(z(s), y) - (\Lambda H_0)(z(1), y) \]

has limit 0 for \( s \to 1 \). This proves (c). □
VI.3 Deferred proofs

This section is devoted to proofs which are omitted in the previous section. It contains:
- subsec. VI.3.1: proof of Proposition VI.2.3
- subsec. VI.3.2: proof of Proposition VI.2.4 and Property VI.2.5
- subsec. VI.3.3: proof of Proposition VI.2.6.

Notations. For positive $\delta$ we introduce two rectangles:

\[(1A) \quad S_M(\delta) := i [M, \infty) + [-\frac{1}{2} \delta - \delta, +\frac{1}{2} \delta + \delta],\]

where $M$ satisfies VI.2.(8), and

\[(1B) \quad R_M(\delta) := i [M, \infty) + [-\delta, +\delta] = \bigcap_{\mu = \pm} (S_M(\delta) + \mu \frac{1}{2} \delta).\]

In this section $\delta$ is a fixed sufficiently small positive number, i.e. $\delta$ satisfies (cf. II.2.(87) et seq.)

\[(2A) \quad S_M(\delta) \subseteq \hat{S}_1 \cap \hat{S}_2,\]
\[(2B) \quad 0 < \delta < \frac{1}{4} - \frac{1}{2} \delta.\]

Such a $\delta$ does exist, cf. the proof of Property II.4.2. We remark that (by (2A) and II.4.(3))

\[(3) \quad R_M(\delta) \subseteq (\hat{S}_{j} - \lambda \frac{1}{2} \delta) \cap (\hat{S}_{j} + \lambda \frac{1}{2} \delta) = \mu^\lambda - \lambda \frac{1}{2} \delta\]

for $j = 1, 2$, $i = 3 - j$ and $\lambda = \pm$. Assuming that $s$ is sufficiently close to $1$ (viz. so that VI.2.(10) is satisfied) we have (cf. VI.2.(9B))

\[(4) \quad (\gamma_j^\lambda)^* = (\gamma_j^\lambda)^* U (U \mu [M, \frac{1}{2} \tau(s)]) \subseteq\]
\[\subseteq (\gamma_j^\lambda)^* U (U \mu R_M(\delta)) \subseteq\]
\[\subseteq \mu^\lambda - \lambda \frac{1}{2} \delta.\]
VI.3.1 Proof of Proposition VI.2.3

First, we verify VI.2.(5A). By substitution verify that

\[ \text{the initial point of } \lambda_j^\lambda(\cdot) \text{ is } \lambda_j^\frac{1}{2}\sigma \tau, \text{ and} \]
\[ \text{the end point is } \lambda_j^\frac{1}{2}\sigma \tau + 1. \]

It remains to be proved \((\lambda_j^\lambda)^* \subset F_j^\lambda\) (cf. VI.2.(5A)) if \(s\) is sufficiently close to 1. For fixed \(\lambda\) and \(j\) (with \(\epsilon = 3-j\)) we put

\[ \mu_j := \lambda, \quad \mu_j := -\lambda, \]

and for \(k=1,2\) and complex \(w\) we use the abbreviations

\[ \hat{w}_k := w + \mu_k \frac{1}{k}\sigma \]
\[ w_k := w + \mu_k \frac{1}{k}\sigma(s) \]
\[ u_k := w_k \tau(s) + \frac{1}{2}, \]

and

\[ \Delta_k(s) := w_k - \hat{w}_k = \mu_k \frac{1}{k}\sigma(s) - \sigma \]

which has limit 0 if \(s \to 1\), cf. VI.1.(33). It must be shown: if \(s\) is sufficiently close to 1 then \(u_k \in S_k\) for \(k=1,2\) and all \(w \in (\lambda_j^\lambda)^*\), or (equivalently)

\[ | \frac{\text{Im } u_k}{\text{Im } \tau} | < \frac{1}{2} \text{ for } k=1,2 \text{ and all } w \in (\lambda_j^\lambda)^* \]

(II) \[ | f_k(u_k) | < 1 \text{ for } k=1,2 \text{ and all } w \in (\lambda_j^\lambda)^*. \]

Observe (cf. the equality in (3))

\[ H_j^\lambda - \lambda_j^\frac{1}{2}\sigma = \bigcap_{k=1,2} (\hat{S}_k - \mu_k \frac{1}{k}\sigma). \]

So, if \(w \in (\lambda_j^\lambda)^*\) then we know from (4) that \(\hat{w}_k \in S_k\) for \(k=1,2\), or (equivalently)
(I') \quad |\Re \hat{w}_k| < \frac{1}{2} \quad \text{for } k=1,2 \text{ and all } w \in (\gamma_j^\lambda)^*.

(II') \quad |g_k(\hat{w}_k)| < 1 \quad \text{for } k=1,2 \text{ and all } w \in (\gamma_j^\lambda)^*.

We prove (I) from (I'). Observe \( \frac{\Im u_k}{\Im \tau} = \Re \hat{w}_k \). First restrict \( w \) to the compact subset \((\gamma_j^\lambda)^*\) of \((\gamma_j^\lambda)^*\). Then (I') can be sharpened to

\[
\sup_{w \in (\gamma_j^\lambda)^*} |\Re \hat{w}_k| < \frac{1}{2}
\]

giving

\[
\sup_{w \in (\gamma_j^\lambda)^*} |\Re \hat{w}_k| \leq \eta + |\Delta_k(s)| < \frac{1}{2}
\]

if \( s \) is sufficiently close to 1, which proves (I) with \( w \) restricted to \((\gamma_j^\lambda)^*\).

Next restrict \( w \) to \((\gamma_j^\lambda)^* \setminus (\gamma_j^\lambda)^*\). Then \( \Re w = 0 \), hence

\[
|\Re \hat{w}_k| = \frac{1}{2^\sigma} \Im(s) \rightarrow \frac{1}{2} \epsilon \in (0, \frac{1}{4}).
\]

This proves (I).

We show (II). First restrict \( w \) to \((\gamma_j^\lambda)^*\). We have (all suprema are over \( w \in (\gamma_j^\lambda)^*\))

\[
\sup |f_k(u_k)| \leq \\
\leq \sup |f_k(u_k) - g_k(w_k)| + \\
+ \sup |g_k(w_k) - g_k(\hat{w}_k)| + \\
+ \sup |g_k(\hat{w}_k)|.
\]

The last term is smaller than 1 by (II') and compactness. The term in the middle has limit 0 for \( s \to 1 \) by VI.1.(33) and again compactness, cf. Lemma II.4.78. This also implies that \( \sup g_k(w_k) \) is finite if \( s \) is close to 1. Now apply Theorem VI.1.5 (Part I, with \( v := \mu \frac{1}{k^2} \sigma(s) \) and \( \lambda = + \) to conclude that the first term has limit 0 for \( s \to 1 \). This proves (II) with \( w \) restricted to \((\gamma_j^\lambda)^*\).

Finally we prove (II) with \( w \) restricted to \((\gamma_j^\lambda)^* \setminus (\gamma_j^\lambda)^*\), i.e. \( w \in i\mathbb{R} \) and \( M \leq |\Im w| \leq \frac{1}{2} \Im \tau(s) \). The above compactness argument fails, and we prove the claim by means of a detailed computation. We restrict ourselves to \( \Im w > 0 \). This is allowed because of \( u_k(-w) = -u_k(w)^* - 1 \) (where the star denotes complex conjugation), and because \( f_k(\cdot) \) has period 1, is even, and satisfies V.5.(13).

We assume that \( s \) is so close to 1 that \( |\Delta_k(s)| < \delta \) where \( \delta \) satisfies (2A) - (2B). Then
VI.3

(a) \( w_k \in S_M(\delta) \)

for \( k = 1, 2 \), so by (2A)

(b) \( w_k \in \tilde{S}_\delta \).

We distinguish two cases.

Case 1. We suppose

(*) \( w \in i\mathbb{R} \) and \( M \leq \text{Im } w \leq \frac{1}{4} \text{Im } \tilde{\tau} \).

Observe:

(c) \( 1 - |f_k(w_k)|^2 = 1 - |g_k(w_k)|^2 - \varepsilon_k(w) \)

with

\[ \varepsilon_k(w) := |f_k(w_k)|^2 - |g_k(w_k)|^2 \]

and (from the definition II.2.(20))

(d) \( 1 - |g_k(w_k)|^2 = G_k(w_k) H_k(w_k) \),

where

\[ G_k(w_k) := \frac{2(a_k + b_k) \text{Re} (\cos 2\pi w_k)}{|\cos 2\pi w_k + b_k|^2} \]

\[ H_k(w_k) := 1 - \frac{a_k - b_k}{2 \text{Re} (\cos 2\pi w_k)} \]

Write \( w_0 := \frac{1}{2} \tilde{\tau} + \delta + \text{Mi} \in S_M(\delta) \). Then

\( \text{Re} (\cos 2\pi w_k) = \cos 2\pi w \cdot \cos \pi \sigma(s) \geq (\text{by (a)}) \)

\( \cos 2\pi M \cdot \cos 2\pi (\frac{1}{2} \tilde{\tau} + \delta) = \text{Re} (\cos 2\pi w_0), \)

where

\( \text{Re} (\cos 2\pi w_0) > 0 \)

because of (2B), and
\[ \text{Re} \left( \cos 2\pi w_0 \right) > \frac{a_k - b_k}{2} \]

because of \( w_0 \in \tilde{S}_k \) by (2A), cf. II.2.(86)-(87). Consequently,

(e) \[ G_k(w_K) > 0 \]

(cf. II.2.(23)), and

(f) \[ H_k(w) \geq H_k(w_0) > 0. \]

It follows from (c) through (f)

(g) \[ 1 - |f_k(u_K)|^2 \geq \]

\[ \geq G_k(w_K) \left[ H_k(w_0) - \frac{|\varepsilon_k(w)|}{G_k(w_K)} \right]. \]

We proceed with estimating the quotient in the right-hand member. Observe that \( g_k(w_K) \) is bounded (in fact, bounded by 1) because of (b). Now apply VI.1.27 with \( \lambda = \pm \) first to conclude that \( f_k(u_K) \) is bounded if \( s \) is sufficiently close to 1, and again to conclude

\[ \varepsilon_k(w) \in O(\varepsilon(\frac{1}{2})) \]

for \( s \to 1 \), uniformly in \( w \) provided (\( \ast \)).

Also observe that the function \( w \sim E(w_K)/G_k(w_K) \) is bounded on \( S_M(\delta) \) (cf. Aux. Property VI.1.6), hence by (a)

\[ w \sim \frac{E(w)}{G_k(w_K)} \text{ is bounded} \]

under the assumption (\( \ast \)). Consequently,

(h) \[ \frac{\varepsilon_k(w)}{G_k(w_K)} \in O(\varepsilon(\frac{1}{2} - w)) \in O(\varepsilon(\frac{1}{4})) \]

for \( s \to 1 \), uniformly in \( w \) provided (\( \ast \)) holds, where \( E(\frac{1}{4}) = 0 \). From (e), (g), and (h) it follows that \( 1 - |f_k(u_K)|^2 > 0 \) for all \( w \) satisfying (\( \ast \)) if \( s \) is sufficiently close to 1. This proves (II) with
w restricted by (\#).

\textit{Case 2.} Next we suppose

\[(\#\#) \quad w \in i \mathbb{R} \text{ and } \frac{1}{4} \text{ Im } \tau \leq \text{ Im } w \leq \frac{1}{2} \text{ Im } \tau.\]

Putting \(w' := \frac{1}{2}i - w\) one has (because \(f(\cdot)\) is even and VI.1.(14))

\[(i) \quad 1 - |f_{\kappa}(w')|^2 = 1 - |f_{\kappa}(w' - \mu_{k/2}i)|^2 = \]
\[= 1 - |1 - F_{\kappa}(w',s) + R_{\kappa}(w',s)|^2 = \]
\[= 2\text{Re}[F_{\kappa}(w',s) - R_{\kappa}(w',s)] - |F_{\kappa}(w',s) - R_{\kappa}(w',s)|^2 \]

where

\[F_{\kappa}(w',s) := [1 - c_{k1}(s)] \cos 2\pi(w' - \mu_{k/2}i) \]

and \(R_{\kappa}(w',s)\) is the remainder term from Proposition VI.1.8 (with \(\mu := +,\nu := -\mu_{k/2}i\) and \(\delta = \frac{1}{4}\)), satisfying

\[R_{\kappa}(w',s) \in O(E(\tau - 2w')) = O(E(2w)) \text{ for } s \to 1,\]

uniformly in \(w\) provided (\#\#) holds.

Observe \(\text{Re} F_{\kappa}(w',s) = [1-c_{k1}(s)] \cos 2\pi w'.\cos \pi\sigma(s)\) where \(\cos \pi\sigma(s) \geq \cos \pi(\delta + 2\delta) > 0\), hence

\[(j) \quad \text{Re} F_{\kappa}(w',s) > 0,\]

and for \(\mu = \pm 1\) (use VI.1.(2)-(3)-(20))

\[(k) \quad [\text{Re} F_{\kappa}(w',s)]^\mu \in O(E(\mu_{k/2} - \mu w')) = O(E(\mu w)).\]

Also observe \(F_{\kappa}(w',s) \in O(E(w)),\) hence

\[(l) \quad |F_{\kappa}(w',s) - R_{\kappa}(w',s)|^2 \in [O(E(w)) + O(E(2w))]^2 \subset O(E(2w)).\]

Hence from (i),
\[ 1 - |\xi_k(w', s)|^2 \in 2 \text{Re} \, F_k(w', s) + O(E(2w)) \in \epsilon 2 \text{Re} \, F_k(w', s) [1 + O(E(w))] \]

for \( s \to 1 \), uniformly in \( w \) provided (**),

where \( E(1/2) \leq E(w) \leq E(1/4) \to 0 \). This proves (II) with \( w \) restricted by (**). So VI.2.(5A) is satisfied if \( s \) is sufficiently close to 1.

Next, we verify VI.2.(5B). It must be shown that, if \( s \) is in some fixed (sufficiently small) neighbourhood of 1 and \( z \) in some fixed (sufficiently small) neighbourhood of \( \hat{z} \), then

\[(III) \quad (\Lambda_j^\lambda)^* \cap P(z, \tau + 1/2) = \emptyset \]

\[(IV) \quad \text{Ind}^* (w | \Lambda_j^\lambda) = \begin{cases} 1 & \text{if } w \in P^+(z, \tau + 1/2), \\ 0 & \text{if } w \in P^-(z, \tau + 1/2), \end{cases} \]

where the sets \( P(\cdot), P^+ (\cdot) \) are given in IV.4.(15A),..., (16), and the function \( \text{Ind}^* \) in IV.2.(19). Consider the half lines

\[ L^\mu(z) := z + \frac{1}{2} + \mu \tau [\sigma(s), \infty) \]

for \( \mu = \pm \). It suffices to show:

\[(III') \quad (\Lambda_j^\lambda)^* \cap (k + L^\mu(z)) = \emptyset \quad \text{for } k \in \mathbb{Z} \text{ and } \mu = \pm \]

\[(IV') \quad \text{Ind}^* (w | \Lambda_j^\lambda) = \begin{cases} 1 & \text{if } w \in L^+(z) \\ 0 & \text{if } w \in L^-(z) \end{cases} \]

if \( s \) is sufficiently close to 1. \( s \)

First we show \( (III') \). We put

\[ \hat{L}^\mu(\hat{z}) := \hat{z} + \mu [\hat{\sigma}, \infty) \]

and have

\[ L^\mu(z) = \frac{1}{2} + \tau [\hat{L}^\mu(\hat{z}) + (z - \hat{z}) + \mu(\sigma(s) - \hat{\sigma})] \]

Then \( (III') \) is equivalent to \( (III'') \) (cf. VI.2.(12)).
(III'') \quad I(s) \cap [\hat{L}^H(\hat{z}_\xi) + \varepsilon(z_\xi, s, \mu)] = \emptyset \quad \text{for} \quad \mu = \pm

where

\begin{align*}
I(s) &:= \bigcup_{k \in \mathbb{Z}} [(\gamma_j^\lambda)^* + \frac{1}{2} \lambda_j^\lambda + k \hat{\tau}(s)], \\
\varepsilon(z_\xi, s, \mu) &:= (z_\xi - \hat{z}_\xi) + [\mu 1 - \frac{1}{2} \lambda_j^\lambda] \sigma(s) - \hat{\sigma}.
\end{align*}

Write (for $\mathfrak{M}^M$ see II.4.(32))

\begin{align*}
J(s) &:= I(s) \setminus \mathfrak{M}^M, \\
K(s) &:= I(s) \cap \mathfrak{M}^M.
\end{align*}

Observe that the distance between $(\mathfrak{M}^M)^c$ and $\hat{L}^H(\hat{z}_\xi)$ is $M - |\Im \hat{z}_\xi| > 0$; hence, if $\varepsilon(z_\xi, s, \mu)$ is sufficiently small (and this is the case if $s$ is sufficiently close to 1 (cf. VI.1.(33)) and $z_\xi$ is sufficiently close to $\hat{z}_\xi$) then

\[ J(s) \cap [\hat{L}^H(\hat{z}_\xi) + \varepsilon(z_\xi, s, \mu)] = \emptyset. \]

Assuming VI.1.(10) we have $((\gamma_j^\lambda)^*)^* \cap \mathfrak{M}^M \subset (\gamma_j^\lambda)^*$; hence, if $s$ is sufficiently close to 1 then (by VI.1.(17))

\[ K(s) \subset (\gamma_j^\lambda)^* + \frac{1}{2} \lambda_j^\lambda. \]

From VI.2.(9B) we know $[(\gamma_j^\lambda)^* + \frac{1}{2} \lambda_j^\lambda] \cap \hat{L}^H(\hat{z}_\xi) = \emptyset$; this implies that the two sets (the first one being compact and the second one closed) have a positive distance. Hence, if $\varepsilon(z_\xi, s, \mu)$ is sufficiently close to 0 then

\[ K(s) \cap [\hat{L}^H(\hat{z}_\xi) + \varepsilon(z_\xi, s, \mu)] = \emptyset. \]

This proves (III'').

Finally we show (IV'). Observe that for $\nu = \pm$ the set $E(\nu L^H(z_\xi))$ is connected and (by (III'')) disjoint from $E(\nu (\Lambda_j^\lambda)^*)$. Because the set $E(\nu L^{-\nu}(z_\xi))$ is unbounded, one has

\[ \text{Ind}^*(w|\nu \Lambda_j^\lambda(\cdot)) = 0 \quad \text{for} \quad w \in \nu L^{-\nu}(z_\xi). \]

With $\nu = +$ this proves the second part of (IV'). To see the first part, observe from (5) (cf. IV.2.(17)-(18))
(n) \[ \text{Ind}^* (\omega_1 | \Lambda_j^\lambda) = 1. \]

Now apply IV.2.(22), and use (m) with \( v^= - \) and (n) to conclude that for \( w \in L^+(z_{\lambda_j}) \)

\[ \text{Ind}^* (w | \Lambda_j^\lambda) = \]
\[ = \text{Ind}^* (\omega_1 | \Lambda_j^\lambda) + \text{Ind}^* (-w | \Lambda_j^\lambda) = \]
\[ = 1 + 0 = 1. \]

This proves (IV').

VI.3.2 Proof of Proposition VI.2.4 and Property VI.2.5

Notations. We use the notations (1A)-(1B), and in addition we write for positive \( \delta \)

\[ (12) \quad C_j^\lambda (\delta) := (\Gamma_j^\lambda)^* + [-\delta, \delta]. \]

It will be assumed that \( \delta \) is a fixed, sufficiently small positive number satisfying (2A)-(2B) and, furthermore,

\[ (13) \quad G_j^\lambda (\delta) \subset H_j^\lambda - \frac{1}{2} \pi \]

(in view of VI.2.(9B) this is possible because \( (\Gamma_j^\lambda)^* \) is compact and \( H_j^\lambda \) is open). Also we put

\[ (14A) \quad G_j^\lambda (\delta) := C_j^\lambda (\delta) \cup R_M (\delta), \]
\[ (14B) \quad G_j^\lambda := G_j^\lambda (0) = (\Gamma_j^\lambda)^* \cup i[M, \infty), \]

which is the upper part of the integration path \((\gamma_j^\lambda)^*\) (cf. (4)), and have, due to (2A) and (13) (cf. (3)),

\[ (15) \quad G_j^\lambda \subset G_j^\lambda + [-\delta, +\delta] = G_j^\lambda (\delta) \subset H_j^\lambda - \frac{1}{2} \pi. \]

For the function \( f_k (\cdot) \) see V.5.(2) and environs, for the function \( \varphi_k (\cdot) \) (which depends also on \( y_k \)) see V.7.(7). For the function \( g_k (\cdot) \) see II.2.(20), and for the function \( \psi_k (\cdot, y_k) \) see II.2.(128).
**Lemma 1.** Assume VI.2.0-(1). Then for $k=1,2$ the following estimates for $s+1$ hold.

I.

\begin{equation}
\varphi_k(w + \frac{1}{2}) \in O(1)
\end{equation}

uniformly in $w$ provided

\begin{equation}
\begin{cases}
|\text{Im } w| \leq \frac{1}{2} \text{ Im } \tilde{\tau}, \\
\delta_k(w) \text{ is bounded}, \\
\psi_k(w, y_k) \text{ is bounded}.
\end{cases}
\end{equation}

II. If, in addition, $y_k = 1$ then

\begin{equation}
\varphi_k(w + \frac{1}{2}) \in O(E(-w))
\end{equation}

uniformly in $w$ provided

\begin{equation}
0 \leq \text{Im } w \leq \frac{1}{2} \text{ Im } \tilde{\tau} \text{ and } \frac{1}{\cos 2\pi \left(\frac{1}{2} - w\right)} \text{ is bounded.}
\end{equation}

**Proof.** We use the abbreviations $x_k := f_k(w + \frac{1}{2})$ and $\hat{x}_k := g_k(w)$.

Part I. Assume (17). Use Theorem VI.1.5 (Part I) to conclude $x_k \in O(1)$ (uniformly in $w$), so it suffices to prove

\begin{equation}
\frac{1}{1 - x_k y_k} \in O(1) \text{ (uniformly in } w\text{).}
\end{equation}

From (17) it follows that $(1 - \hat{x}_k y_k)^{-1} = 1 + y_k \Psi_k(w, y_k)$ is bounded, so assume $|1 - \hat{x}_k y_k| \geq \varepsilon > 0$. Then (again use Theorem VI.1.5 (Part I))

\begin{align*}
|1 - x_k y_k| &\geq |1 - \hat{x}_k y_k| - |y_k||x_k - \hat{x}_k| \\
&\geq \frac{1}{2} \varepsilon
\end{align*}

if $s$ is close to 1. This proves (16).

Part II. Assume (19) and $y_k = 1$. Because of $\varphi_k(w + \frac{1}{2}) = -1 + (1 - x_k)^{-1}$ it suffices to prove
\((**)*\) \[
\frac{1}{1-x_k} \in O(E(-w)) \text{ (uniformly in } w). 
\]

Put \(w' := \frac{1}{2} - w\), so \(x_k = f_k(-w')\). From Lemma VI.1.7 (with \(v=0\) and \(\mu=-\)) we know

\[
(a) \quad \frac{1}{1-x_k} = \frac{1}{1-c_k(s)} \cdot \frac{1}{\cos 2\pi w'} \cdot \frac{1 + S}{1 + \frac{R}{\cos 2\pi w'}}
\]

with \(R, S \in O(E(\frac{1}{2}-w')) = O(E(w))\) for \(s \to 1\) uniformly in \(w\); in particular

\[
(b) \quad 1 + S \in O(1).
\]

Use Aux. Property VI.1.6 to conclude that \(\frac{E(-w')}{\cos 2\pi w'}\) is bounded; hence

\[
(c) \quad \frac{1}{\cos 2\pi w'} \in O(E(w')) = O(E(\frac{1}{2}-w'))
\]

and

\[
(d) \quad \frac{R}{\cos 2\pi w'} \in O(E(\frac{1}{2})).
\]

From VI.1.(2)-(3)-(20) we know

\[
(e) \quad \frac{1}{1-c_k(s)} \in O(E(-\frac{1}{2})).
\]

Now use (b) through (e) to conclude from (a)

\[
\frac{1}{1-x_k} \in O(E(-w)) \frac{1 + o(E(\frac{1}{2}))}{1 + O(E(\frac{1}{2}))} \in O(E(-w)),
\]

which proves (18).
In the sequel \( i, j = 3 - i \) and \( \lambda \) are fixed. For the sign \( \mu_k \) see (6), for the constant \( m_k = m_k(y_k) \) see VI.2.(22).

**Lemma 2.** Assume VI.2.(0)-(1), and fix \( i \in \{1, 2\} \) and \( j := 3 - i \). Also assume (2A)-(2B) and (13). Then for \( k = 1, 2 \) the following holds. If

\[
(20) \quad w \in G^\lambda_j(\delta) + \frac{1}{2} \frac{1}{\cos 2\pi(\frac{1}{2} - w)}
\]

then

\[
(21) \quad |g_k(w)| < 1,
\]

\[
(22) \quad w \sim E(m_i, w) \psi_k(w, y_k) \text{ is bounded},
\]

\[
(23) \quad w \sim \frac{1}{\cos 2\pi(\frac{1}{2} - w)} \in O(1) \text{ for } s \to 1 \text{ (uniformly in } w).\]

**Proof.** Proof of (21). This follows from (cf. (15) and (11))

\[
(24) \quad G^\lambda_j(\delta) + \mu_k \psi_k \subset \mathcal{S}_k \quad \text{for } k = 1, 2.
\]

Proof of (22). Assume (20). With the notation \( x_k := g_k(w) \) it suffices to show (because of (21))

\[
(25) \quad w \sim E(m_i, w) (1 - x_k y_k)^{-1} \text{ is bounded.}
\]

First assume \( y_k \neq 1 \) (so \( m_k = 0 \)). Choose a sufficiently large positive number \( M' \), viz. so that

\[
|1 - x_k| \leq \frac{1}{2} |1 - y_k| \quad \text{if } |\text{Im } w| \geq M'.
\]

(cf. Proposition II.4.23, Part I).

On the one hand, if \( |\text{Im } w| \geq M' \) then

\[
|1 - x_k y_k| \geq |1 - y_k| - |y_k| \cdot |1 - x_k| \geq \frac{1}{2} |1 - y_k| > 0.
\]

On the other hand, if \( |\text{Im } w| \leq M' \) then (because of (21)) \( |x_k| \leq 1 - \eta < 1 \) for some positive \( \eta \) (compactness argument); hence,

\[
|1 - x_k y_k| \geq 1 - |x_k| \geq \eta > 0.
\]
So we have proved

(b) \( (1 - x_k y_k)^{-1} \) is bounded if \( y_k < 1 \).

Next consider the case \( y_k = 1 \) (so \( m_k = 1 \)). From

\[
(1 - x_k)^{-1} = \frac{\cos 2\pi w + b_k}{a_k + b_k}
\]

it follows

(c) \( E(w) (1 - x_k)^{-1} \) is bounded.

Combine (b) and (c) to conclude (a).

Proof of (23). Assume (20). Observe

\[
|\cos 2\pi \left( \frac{1}{2} - w \right)|^2 = \\
= \sinh^2 2\pi \text{Im} \left( \frac{1}{2} - w \right) + \cos^2 2\pi \text{Re} \, w.
\]

If \( w \in C_{\frac{1}{2}}(6) + \mu_k \frac{1}{2} \) (cf. (12)) then \( |\text{Im} \, w| \leq M' \) for some \( M' \), hence

\[
\sinh^2 2\pi \text{Im} \left( \frac{1}{2} - w \right) \leq \sinh^2 2\pi \left( \frac{1}{2} \text{Im} \, (1 - M') \right) \to \infty \text{ for } s + 1.
\]

If \( w \in R_{\frac{1}{2}}(6) + \mu_k \frac{1}{2} \) then \( w \in S_{\frac{1}{2}}(6) \) by (3), and so \( \cos^2 2\pi \text{Re} \, w \geq \cos^2 2\pi \left( \frac{1}{2} \text{Re} \, (6) \right) > 0 \) by (28). This proves (23).

In the sequel we use, besides the notation (6), also the notations (7) through (10).

**PROPERTY 3.** Assume VI.2.(0)-(1), and \( j \in \{1, 2\} \), \( \lambda \in \{-, +\} \). Then for \( k = 1, 2 \) the following holds. If

\[
(25) \quad w \in G_{\frac{1}{2}}^\lambda \quad \text{and} \quad \text{Im} \, w \leq \frac{1}{2} \text{Im} \, \tau
\]

then

\[
(26) \quad E(m_k w) \varphi_k(u_k) \in O(1) \quad \text{for } s + 1
\]

uniformly in \( w \).
(27) \( f_k(u_k) - g_k(\tilde{w}_k) \in O(E(\frac{1}{2})) \) for \( s + 1 \)
uniformly in \( w \).

PROOF. Let \( \delta \) satisfy (2A)-(2B) and (13).

In order to prove (26) and (27), assume (25), and also assume that \( s \) is sufficiently close to 1, viz. so that (cf. (10)) \( |\Delta_k(s)| \leq \delta \); then

(*) \[ [\tilde{w}_k, w_k] \subset G^\lambda_j(\delta) + \mu_k \frac{1}{2} \delta \]
for \( k=1,2 \) and all \( w \).

We prove (26). First assume either that \( y_k \neq 1 \) (so \( m_k = 0 \)) or that \( y_k = 1 \) (so \( m_k = 1 \)) and \( \text{Im} \ w \) is bounded. In both cases \( E(tm, w) \) is bounded, and we can use (21) and (22) to conclude from Lemma 1 (Part I) that (26) is true.

Next assume \( y_k = 1 \) (so \( m_k = 1 \)) and \( \text{Im} \ w \geq 0 \). Now we can use (23) to conclude from Lemma 1 (Part II) that (26) is true.

We prove (27). First, (*) implies (by Lemma 2) \( |g_k(w) - g_k(\tilde{w}_k)| < 1 \); hence, by Theorem VI.1.5 (Part I with \( w := w_k, v := 0 \) and \( \lambda := +1 \))

(i) \[ f_k(u_k) - g_k(w_k) \in O(E(\frac{1}{2})) \]
uniformly in \( w \).

Secondly, we claim

(ii) \[ g_k(w_k) - g_k(\tilde{w}_k) \in O(E(\frac{1}{2} + w)) \]
uniformly in \( w \).

To prove (ii), observe

(a) \[ |g_k(w_k) - g_k(\tilde{w}_k)| \leq |\Delta_k(s)| \cdot \sup |g_k'(\cdot)| \]

where the supremum is over the line segment \([\tilde{w}_k, w_k]\). From II.2.(20) we know

\[ g_k'(v) = (-2\pi) (a_k + b_k) \frac{\sin 2\pi v}{(\cos 2\pi v + b_k)^2} \]
Observe that

\[ v \sim [E(v) \land E(-v)] \quad \text{sin } 2\pi p \text{ is bounded,} \]

and that

\[ v \sim \frac{E(-v) \lor E(v)}{\cos 2\pi v + b_i} \quad \text{is bounded on } \hat{S}_k \]

(because \((\cos 2\pi v + b_i)^{-1} = (a_k + b_i)^{-1}(1-g_k(v))\) is bounded on \(\hat{S}_k\), and apply Aux. Property VI.1.6). Consequently,

\[ v \sim [E(-v) \lor E(v)] g_k^i(v) \quad \text{is bounded on } \hat{S}_k, \]

hence in particular (use (*) and (24))

\[ (b) \quad v \sim E(-v) g_k^i(v) \quad \text{is bounded on } [\hat{w}_k, w_k]. \]

From VI.1.(33) we know

(c) \[ \Delta_k(s) \in O(\frac{1}{2^r}). \]

From (a), (b), and (c) conclude (ii).

Finally, combine (i) and (ii) to conclude (27) because \(E(w)\) is bounded.

\[ \bullet \]

**Corollary 4.** Assumptions as in Property 3. Then for \(k=1,2\)

\[ \psi_k(u_k) - \psi_k(\hat{w}_k, y_k) \in \]

\( \in O(\frac{1}{2^7 - 2m_k w}) \)

uniformly in \(w\) provided (25) holds.

**Proof.** We use the abbreviations

\[ f_k := f_k(u_k), \quad g_k := g_k(\hat{w}_k), \]

and

\[ \varphi_k := \varphi_k(u_k), \quad \psi_k := \varphi_k(\hat{w}_k, y_k). \]
Observe
\[ \phi_k - \psi_k = \frac{f_k - g_k}{(1-f_k^* y_k)(1-g_k^* y_k)} = (f_k - g_k)(1+y_k \phi_k)(1+y_k^* \psi_k), \]
and use VI.2.(1) and the estimates (27), (26), and (22) to conclude (28).
\[ \Box \]

For the function \( F^\lambda_j(\cdot) \) (which depends also on \( y \)) see V.7.(8), for the function \( H^\lambda_j(\cdot, y) \) see II.2.(135), and for the constant \( n(y) \) see VI.2.(23).

**COROLLARY 5.** Assumptions as in Property 3. Then

\[ F^\lambda_j((w+\lambda^\sigma \tau+\frac{1}{2}) - H^\lambda_j(w+\lambda^\sigma \tau, y) \leq \epsilon \]
\[ \in \mathcal{O}(E(\frac{1}{3\tau} - n(y)w)) \]
uniformly in \( w \) provided (25) holds.

**PROOF.** Using the notations (29), (cf. (8), (7), (6)) we have

\[ F^\lambda_j((w+\lambda^\sigma \tau+\frac{1}{2}) = \phi_j \phi_j, \]
\[ H^\lambda_j(w+\lambda^\sigma \tau, y) = \psi_j^* \psi_j. \]

So the difference in (30) on the left equals

\[ (\phi_j - \psi_j)^* \psi_j + (\phi_j^* - \psi_j^*) \psi_j + (\phi_j^* - \psi_j)(\phi_j - \psi_j). \]

Now use the estimates (28) and (22) to conclude (30).
\[ \Box \]

**PROOF** of Proposition VI.2.4. The difference VI.2.(26) with \( \mu=+ \) coincides with the difference in (30) on the left.

Part I. Use (14B) and observe that Im \( w \) is bounded if \( w \in \Gamma_j^0 \), hence \( E(-n(y)w) \) is bounded for such \( w \). So the claim follows from (30).

Part II. For \( \mu=+ \) and \( j=1,2 \) and \( \lambda=\pm \) the claim has been proved in Corollary 5. This implies the claim for \( \mu=- \) because of
\[ F_j^\lambda(-w+\lambda\frac{1}{2}v)i+\frac{1}{2}) \equiv F_j^\lambda(w+\lambda\frac{1}{2}v)i+\frac{1}{2}) \]

(use V.7.(9)-(10) and the period 1) and

\[ H_j^\lambda(-w+\lambda\frac{1}{2}v)i+\frac{1}{2}, y) \equiv H_j^\lambda(w+\lambda\frac{1}{2}v)i+\frac{1}{2}, y) \]

(use II.2.(137)-(138)).

\[ \text{PROOF of Property VI.2.5.} \]

Part I and Part II for \( \mu=+ \) follow from Property 3 (Part I) by means of (31). This implies Part II for \( \mu=- \) by means of (32).

\[ \text{VI.3.3 Proof of Proposition VI.2.6} \]

The following lemma has been proved in Chapter III, see Proposition III.2.8 (Part 1).

**LEMMA 6.** For \( \mu=\ast \) the following estimate holds:

\[ \frac{\theta'}{e^2} (\mu w + v + |^{-1}) + \pi \operatorname{tg}\pi (\mu w + v) \in O(E(t-w)) \]

for \( \operatorname{Im} t \to \infty \), uniformly in \( (\operatorname{Re} t, v, w) \) provided for some \( \varepsilon \in (0,1) \)

\[ \begin{cases} 
0 \leq \operatorname{Im} w \leq \varepsilon \operatorname{Im} t \\
\operatorname{Im} v \text{ is bounded} \\
\operatorname{tg}\pi (\mu w + v) \text{ is bounded}. 
\end{cases} \]

\[ \text{Notation. We put} \]

\[ \tilde{\varphi}(w) := \cos \frac{tw}{2\sigma(s)}, \text{ so } \frac{\tilde{\varphi}'}{\tilde{\varphi}}(w) = \frac{-\pi}{2\sigma(s)} \operatorname{tg}\frac{tw}{2\sigma(s)}. \]

**COROLLARY 7.** Assume VI.2.(0). Then for \( \mu=\ast \) the following estimate holds:

\[ \tau \tilde{\eta}(\mu w + v | 2\sigma t) - \frac{\tilde{\varphi}'}{\tilde{\varphi}}(\mu w + v) \in O(E(t-w)) \]

\[ \tilde{\eta}(\mu w + v | 2\sigma t) \to \tilde{\eta}(\mu w + v | 2\sigma t) \]

\[ \tilde{\varphi}(w) := \cos \frac{tw}{2\sigma(s)}, \text{ so } \frac{\tilde{\varphi}'}{\tilde{\varphi}}(w) = \frac{-\pi}{2\sigma(s)} \operatorname{tg}\frac{tw}{2\sigma(s)}. \]
for $s+1$, uniformly in $w$ and $v$ provided

$$\left\{ \begin{array}{l}
0 \leq \text{Im } w \leq \frac{1}{2} \text{ Im } \bar{\tau}, \\
\text{Im } v \text{ is bounded}, \\
\sigma' \overset{\sim}{\varphi}' (\mu w + v) \text{ is bounded.}
\end{array} \right.$$  

**PROOF.** Observe from IV.5.(34), IV.5.(36F) and VI.1.(14)

$$\tau.\overset{\sim}{\eta} (\cdot | 2\sigma) = \frac{1}{2\sigma} \frac{\theta(\cdot | 2\sigma)}{\bar{\sigma}(\cdot | 2\sigma)}$$

and apply Lemma 6 with $\varepsilon := \frac{1}{2} (\text{in (33})$ divide $w$, $v$ and $\bar{\tau}$ by $2\sigma$, and use VI.1.(14)-(33) and II.2.(50)).

For $\overset{\sim}{\varphi}(\cdot)$ see (34), for $\overset{\sim}{\varphi}' (\omega i)$ see VI.2.(16), for $\varphi(\cdot)$ see II.4.(23), and for $\varphi(\omega i)$ see II.4.(56).

**PROPOSITION 8.** Assume VI.2.(0).

I. Then for $s+1$

$$\sigma' \overset{\sim}{\varphi}' (\omega i) - \frac{\varphi'}{\varphi} (\omega i) \in \mathcal{O}(\mathcal{B} \left( \frac{1}{2} \tau \right)).$$

II. For $\mu = \tau$

$$\left[ \frac{\varphi'}{\varphi} (\mu w + \nu + t \sigma) - \mu \frac{\varphi'}{\varphi} (\omega i) \right] +$$

$$\left[ \frac{\varphi'}{\varphi} (\mu w + \nu + t \sigma) - \mu \frac{\varphi'}{\varphi} (\omega i) \right] \in$$

$$\mathcal{O} \left( (1 + |w|) \mathcal{B} \left( \frac{1}{2} \tau + \frac{w}{2\sigma} \right) \right)$$

for $s+1$, uniformly in $(w, v, \tau)$ provided:

(39A) $w \in i \mathbb{R}$ and $0 \leq \text{Im } w \leq \frac{1}{2} \text{ Im } \bar{\tau}$

(39B) $v$ is bounded
(39C) \[ t \in \mathbb{R} \]

(39D) \[ \frac{\omega'}{\partial} \left( \mu w + v + \tau \dot{\omega} \right) \text{ is bounded.} \]

**Proof.** Part I. Use the estimate (cf. VI.1.(33) and II.2.(50))

(a) \[ \frac{1}{\sigma} - \frac{1}{\hat{\sigma}} \in O(E(\frac{1}{\tau})). \]

Part II. We use the abbreviations

\[ u := \frac{w + \mu (v + t \omega)}{2\sigma}, \quad \hat{u} := \frac{w + \mu (v + \tau \hat{\omega})}{2\hat{\sigma}}, \]

so

\[ u - \hat{u} = \frac{w + \mu v}{2} \left[ \frac{1}{\sigma} - \frac{1}{\hat{\sigma}} \right]. \]

The difference in (38) on the left can be rewritten as follows (because the functions occurring are odd)

\[ (*) \quad \frac{\mu}{2} \left[ \frac{1}{\sigma} \left[ \tan \dot{u} - \tan u \right] + i \left[ \frac{1}{\sigma} - \frac{1}{\hat{\sigma}} \right] \left[ 1 + i \tan \dot{u} \right] \right]. \]

First, observe from (a)

(b) \[ \frac{1}{\sigma} \in O(1). \]

Next, we have from (39D) that \( \tan \dot{u} \) is bounded, hence,

(c) \[ E(-\dot{u}) \left[ 1 + i \tan \dot{u} \right] = 1 - i \tan \dot{u} = 2[1 + E(\dot{u})]^{-1} \]

is bounded.

From (39B) and (39C) it follows

(d) \[ E(-\frac{w}{2\sigma}) E(\dot{u}) \text{ is bounded.} \]

Finally, consider the difference (cf. the second equality in (c))
\[ t g \hat{u} - t g \hat{w} = 2i([1+E(\hat{u})]^{-1} - [1+E(\hat{w})]^{-1}) = \]
\[ = \frac{2i[1+E(\hat{u})]^{-2}}{1 + [1+E(\hat{u})]^{-1} \Delta}, \]
where \( \Delta := [E(\hat{u}) - 1] E(\hat{u}), \) hence (because of the inequality \(|\exp(z)-1| \leq \frac{1}{|z|} \exp(|z|) \))
\[ |\Delta| \leq 2\pi |u-\hat{u}| \exp(2\pi |u-\hat{u}|) |E(\hat{u})|. \]

Due to (39B) and (a) we have
\[ |u-\hat{u}| \in O((1+|w|) E(1), \)

therefore (39A) \( |u-\hat{u}| \in O \left( \frac{1}{2} E(1) \right) \rightarrow 0, \) and so
\[ \exp(2\pi |u-\hat{u}|) \in O(1). \]

Therefore (with (d)),
\[ (e) \quad \Delta \in O \left( (1+|w|) E \left( \frac{1}{2} \right) \frac{1}{\Delta} \right), \]

and so (by (39A)) \( \Delta \in O(E(1)) \rightarrow 0. \) Consequently (with (c)),
\[ (f) \quad t g \hat{u} - t g \hat{w} \in O(\Delta). \]

From (a) through (f) it now follows that the expression (*) belongs to \( O(\Delta), \) which proves (38).

\[ \square \]

**PROOF** of Proposition VI.2.6. Part I. We use the abbreviations
\[ u := w - z_{\hat{\zeta}} + \frac{1}{2} \zeta, \quad \hat{u} := w - z_{\hat{\zeta}} + \frac{1}{2} \zeta. \]

Assume that \( w \in (t_{\hat{\zeta}})^{*} \) and that \( z_{\hat{\zeta}} \) is sufficiently close to \( z_{\hat{\zeta}}. \) From Property VI.2.7 (Part I) we know that \( \phi'(\hat{u}) \) is bounded. Next use (38) (with \( w=0, v:=w-z_{\hat{\zeta}}, \) and \( t:=\frac{1}{2} \)) and (37) to conclude
\[ (i) \quad \frac{\phi'(u)}{\phi} - \phi'(\hat{u}) \in O \left( E \left( \frac{1}{2} \right) \right) \]

uniformly in \( w \) and \( z_{\hat{\zeta}}. \) Observe from (i) that \( \phi'(u) \in O(1) \) and
conclude from Corollary 7 (with \( w := 0 \) and \( v := u \)).

\[(ii) \quad \tau \hat{\eta}(ut|2\sigma t) - \frac{\hat{\psi}'}{\psi}(u) \in O\left(E\left(\frac{\tau}{2\sigma}\right)\right) \subseteq O\left(E\left(\frac{\tau}{\sigma}\right)\right)
\]

uniformly in \( w \) and \( z_{\hat{\tau}} \). From (i) and (ii) the claim follows.

Part II. We use the abbreviations

\[ u := \mu w - z_{\hat{\tau}} + \frac{1}{2} \hat{\tau}, \quad \hat{u} := \mu w - z_{\hat{\tau}} + \frac{1}{2} \tau. \]

Assume that \( w \) is in the set VI.2.27 and that \( z_{\hat{\tau}} \) is sufficiently close to \( z_{\hat{\tau}} \).

From Property VI.2.7 (Part II) it follows that \( \frac{\psi'}{\phi}(\hat{u}) \) is bounded. Next use (38) with \( v := -z_{\hat{\tau}} \) and \( t := \frac{1}{2} \) to conclude

\[(iii) \quad \left[ \frac{\hat{\phi}'}{\phi}(u) - \mu \frac{\hat{\phi}'}{\phi}(w) \right] \in O\left((1+|w|) E\left(\frac{1}{2} + \frac{w}{2\sigma}\right)\right) \subseteq O\left(\tau E\left(\frac{1}{2} + \frac{w}{2\sigma}\right)\right)
\]

uniformly in \( w \) and \( z_{\hat{\tau}} \). Observe from (iii) that \( \frac{\hat{\phi}'}{\phi}(u) \in O(1) \), and use Corollary 7 with \( v := -z_{\hat{\tau}} + \frac{1}{2} \sigma \) to conclude

\[(iv) \quad \tau \hat{\eta}(ut|2\sigma t) - \frac{\hat{\phi}'}{\phi}(u) \in O\left(E\left(\frac{\tau-w}{2\sigma}\right)\right)
\]

uniformly in \( w \) and \( z_{\hat{\tau}} \). From (iii) and (iv) the claim follows. \( \Box \)
VI.4 Limits for $s \to 1$ in the case of skipfree, all-sided, driftless, even random walk

Assumptions:

\[ \begin{align*}
(0) \quad &\begin{cases}
  &\text{the random walk is skipfree, all-sided, and driftless} \\
  &\text{the random walk is even} \\
  &0 < s < 1
\end{cases}
\end{align*} \]

These assumptions imply, in particular,

\[ -1 < \rho < 1 \]

where $\rho$ is the correlation coefficient of the two components of the step.

VI.4.1 Limit behaviour of the zeros of the discriminants

The constant $c$ has been defined in V.3.(9)-(10), the constant $d$ in V.3.(11)-(12). Explicit expressions are given in V.3.(15)-(16).

**Proposition 1** Under the assumptions (0) the following estimates hold:

\[ \begin{align*}
(2) \quad &1 - c(s) \in \sqrt{\frac{2}{1 - \rho^2}} \sqrt{1 - s} \left[ 1 + O(\sqrt{1 - s}) \right] \quad (s \to 1) \\
(3) \quad &\tilde{c}(s) := \frac{1 - c}{1 + c} \in \frac{1}{\sqrt{2(1 - \rho^2)}} \sqrt{1 - s} \left[ 1 + O(1 - s) \right] \quad (s \to 1) \\
(4) \quad &\left( \frac{1 - c(s)^2}{1 + c(s)^2} \right)^2 \in \frac{2}{1 - \rho^2} (1 - s) \left[ 1 + O(1 - s) \right] \quad (s \to 1)
\end{align*} \]

in fact, $c(s)$ is a regular function of $\sqrt{1 - s}$ near $s = 1$;

in fact, $\tilde{c}(s)/\sqrt{1 - s}$ is a regular function of $1 - s$ near $s = 1$;

in fact, the cross ratio on the left is a regular function of $1 - s$ near $s = 1$.

**Proof.** Proof of (3). Using the abbreviation

\[ a(\lambda) := \frac{s \sqrt{1 - \rho^2} - \lambda \sqrt{1 - s^2 \rho^2} + \sqrt{1 - s^2}}{s \sqrt{1 - \rho^2} + \sqrt{1 - s^2 \rho^2} - \lambda \sqrt{1 - s^2}} \quad (\lambda = \pm) \]

one computes from V.3.(15)

\[ \tilde{c}(s) = a(+) = \prod_{\lambda = \pm} a(\lambda) = \frac{\sqrt{1 - s} \sqrt{1 + s}}{s \sqrt{1 - \rho^2} + \sqrt{1 - s^2 \rho^2}} , \]

which implies (3). With $c = (1 - \tilde{c})/(1 + \tilde{c}) = 1 - 2\tilde{c}/(1 + \tilde{c})$ this easily implies (2). Finally, for (4) use $(1 - c^2)/(1 + c^2) = 2\tilde{c}/(1 + \tilde{c}^2)$. 

$\square$
Proposition 2  Under the assumptions (0) the following estimates hold:

\[ 1 - d(s) \in -\rho \sqrt{\frac{2}{1 - \rho^2}} \sqrt{1 - s} \left[ 1 + O(\sqrt{1 - s}) \right] \quad (s \to 1) \]

in fact, \( d(s) \) is a regular function of \( \sqrt{1 - s} \) near \( s = 1 \);

\[ \tilde{d}(s) := \frac{1 - d(s)}{1 + d} \in \frac{-\rho}{\sqrt{2(1 - \rho^2)}} \sqrt{1 - s} \left[ 1 + O(1 - s) \right] \quad (s \to 1) \]

in fact, \( \tilde{d}(s)/\sqrt{1 - s} \) is a regular function of \( 1 - s \) near \( s = 1 \);

\[ \left( \frac{1 - d(s)^2}{1 + d(s)^2} \right)^2 \in \frac{2\rho^2}{1 - \rho^2} (1 - s) \left[ 1 + O(1 - s) \right] \quad (s \to 1) \]

in fact, the cross ratio on the left is a regular function of \( 1 - s \) near \( s = 1 \).

**Proof.** Proof of (6). Using the abbreviation

\[ b(\lambda) := \frac{\sqrt{1 - \rho^2} - \sqrt{1 - s^2}\rho^2 - \lambda \rho \sqrt{1 - s^2}}{\sqrt{1 - \rho^2} + \lambda \sqrt{1 - s^2}\rho^2 + \rho \sqrt{1 - s^2}} \quad (\lambda = \pm) \]

one computes from V.3.(16)

\[ \tilde{d}(s) = b(+) = \prod_{\lambda = \pm} b(\lambda) = \frac{-\rho \sqrt{1 - s} \sqrt{1 + s}}{\sqrt{1 - \rho^2} + \sqrt{1 - s^2}\rho^2}, \]

which implies (6).

Proof of (5) and (7): similar to the proof of (2) and (4).

\[ \square \]

**VI.4.2 Limit behaviour of \( \tau(s) \)**

In V.5.(1) the value \( \tau \) has been defined by

\[ \tau \in \mathbb{R} i, \quad \text{Im} \tau > 0, \quad m(0|\tau) = \left( \frac{1 - c(s)^2}{1 + c(s)^2} \right)^2. \]

We define \( q, \tilde{\tau}, \) and \( \tilde{q} \) by (for \( E(\cdot) \) see VI.1.(13))

\[ q = E(\frac{1}{2}i\tau) \]

\[ \tau \tilde{\tau} = -1 \quad \text{(Imaginary Transformation of Jacobi)} \]

\[ \tilde{q} = E(\frac{1}{2}i\tilde{\tau}) \]

**Remark 1** From III.1.(48) we know \( m(0|\tau) + m(0|\tilde{\tau}) = 1 \), hence,

\[ m(0|\tilde{\tau}) = \left( \frac{2c(s)}{1 + c(s)^2} \right)^2. \]
**Proposition 3** Under the assumptions (0) the following hold:

\[(13) \quad \tau(s) \to 0 \quad \tilde{\tau}(s) \to \infty i \quad (s \to 1),\]
\[(14) \quad q(s) \to 1 \quad \tilde{q}(s) \to 0 \quad (s \to 1),\]
\[(15) \quad \tilde{q}(s) \in \frac{1}{8(1 - \rho^2)} (1 - s) [1 + O(1 - s)] \quad (s \to 1).\]

**Proof.** [Cf. the proof of Proposition VI.1.3] From (8) and (2) we have \(m(0|\tau) \to 0\) hence \(\tau(s) \to 0\) by Lemma III.1.21. This implies (13) and (14).

We prove (15). From III.2.(65)-(42)-(43) it follows that (with the abbreviation \(m = m(0|\tau)\)) the mapping \(\tilde{q} \mapsto m\) has the form

\[m = 16 \tilde{q} F(\tilde{q})\]

where \(F(\cdot)\) is regular in the open unit disc with \(F(0) = 1\). By the Implicit Function Theorem, the inverse mapping has the form

\[\tilde{q} = \frac{1}{16} m G(m)\]

where \(G(m)\) is regular near \(m = 0\) and \(G(0) = 1\). With (8) and (4) this implies (15). In particular, \(\tilde{q}(s)\) is a regular function of \(s\) near \(s = 1\).

\(\square\)

**Remark 2** From (15) it follows

\[(16) \quad 2 \tilde{q}(s)^{1/2} \in \frac{1}{\sqrt{2(1 - \rho^2)}} \sqrt{1 - s} [1 + O(1 - s)] \quad (s \to 1)\]

and so

\[(17) \quad O(\tilde{q}(s)) = O(1 - s), \quad O(\tilde{q}(s)^{1/2}) = O\left(\sqrt{1 - s}\right) \quad (s \to 1).\]

**VI.4.3 Limit behaviour of the function \(f_{\tilde{q}}(\cdot)\)**

Under the present assumptions the function \(f_{\tilde{q}}(\cdot)\) is identical with the function \(f(\cdot|\tau)\), see Property V.5.24.

**Proposition 4** Under the assumptions (0) the following estimates hold \((\mu = \pm 1)\), with \(\tilde{\tau}\) given in (10) and \(\delta\) a fixed number, \(0 \leq \delta < 1\):

\[(18) \quad f(\mu w + v|\tilde{\tau}) = E\left(\frac{1}{4} e^{\tilde{\tau}}\right) \frac{\cos 2\pi(\mu w + v) + R}{1 + S}\]

where

\[(19) \quad R \in (E(2\tilde{\tau} - 3w)), \quad S \in O(E(\tilde{\tau} - 2w)) \quad (s \to 1)\]

uniformly in \((w, v)\) provided

\[(20) \quad 0 \leq \text{Im } w \leq \delta^3 \tilde{\tau} \quad \text{and} \quad \text{Im } v \text{ is bounded}.\]
Proof. Using the expression V.5.(96 A) and the estimates from Proposition III.2.7 and the connecting Remark III.2.8 (with \( t = 2 \)) one obtains
\[
\begin{align*}
    f(\mu w + v|\tilde{\tau}) &= \frac{\theta_2}{\theta_3} (\mu 2 w + 2 v|2\tilde{\tau}) \\
    &\leq 2 E(\frac{1}{4} \tilde{\tau}) \cos 2\pi (\mu w + v) + E(2\tilde{\tau} - 3w) \left( 1 + O(E(\tilde{\tau} - 2w)) + O(E(4\tilde{\tau} - 4w)) \right) \quad (s \to 1)
\end{align*}
\]
uniformly in \((w, v)\) provided \(0 \leq \text{Im} \ w \leq 2\text{Im} \tilde{\tau}\) and \(\text{Im} \ v\) is bounded; in the denominator the second estimate is negligible under the assumption (20).

Next we show that \(\sigma\) has the limit \(\hat{\sigma}\) (defined in II.1.(16) ). The constant \(\sigma\) has been defined in V.5.(5)-(6) in combination with V.3.(12).

Corollary 5 Under the assumptions (0) the following estimates hold:
\[(21) \quad \cos 2\pi \sigma \in \cos 2\pi \hat{\sigma} + O( E(\frac{1}{2} \tilde{\tau})) \quad (s \to 1),\]
\[(22) \quad \sigma \in \hat{\sigma} + O( E(\frac{1}{2} \tilde{\tau})) \quad (s \to 1).\]

Proof. We know \(f(\sigma\tau|\tau) = d\), hence, by V.5.(98 H-B) and (6),
\[
\begin{align*}
    f(\sigma|\tilde{\tau}) &= f(-\sigma\tau|\tilde{\tau}) = 1 - f(\sigma\tau|\tau) = 1 - \frac{d}{1 + d} = \tilde{d}.
\end{align*}
\]
Apply (18) with \(w = 0\) and \(v = \sigma\) to obtain, with (11),
\[
\tilde{d} = 2\tilde{q}^{1/2} \frac{\cos 2\pi \sigma + R}{1 + S}
\]
with \(R\) and \(S\) satisfying (19) with \(w = 0\), and so
\[
\cos 2\pi \sigma = \frac{\tilde{d}}{2\tilde{q}^{1/2}} [1 + S] - R.
\]
Hence, by (6) and (16)-(17),
\[
\cos 2\pi \sigma \in -\rho [1 + O(\tilde{q})] [1 + S] - R \subset -\rho + O(\tilde{q}),
\]
which, with (11), proves (21).
This also proves (22) because the function \(z \mapsto \cos 2\pi z\) is regular and injective with nonvanishing derivative for \(z \in (0, \frac{1}{2})\).

\(\square\)

Corollary 6 Under the assumptions (0) the following estimates hold \((\mu = \pm 1)\):
\[(23) \quad 1 - f( (\mu w + v)\tau|\tau) = 4 E(\frac{1}{4} \tilde{\tau}) \frac{\cos 2\pi (\mu w + v) + R}{1 + T}\]
where
\[(24) \quad R \in O(E(2\tilde{\tau} - 3w)), \quad T \in O(E(\frac{1}{4} \tilde{\tau} - w)) \quad (s \to 1)\]
uniformly in \((w, v)\) provided
\[(25) \quad 0 \leq \text{Im} \ w \leq \frac{3}{4} \text{Im} \tilde{\tau} \quad \text{and} \quad \text{Im} \ v \text{ is bounded}.\]
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Proof. Use V.5.(98 H) to obtain from Proposition 4

\[
1 - f((\mu w + v)\tau|\tau) = \frac{2f}{1 + f} (\mu w + v|\tilde{\tau}) = 4E(\frac{1}{4}\tilde{\tau}) \frac{\cos 2\pi(\mu w + v) + R}{1 + T}
\]

with \(R\) as in (19) and (cf. III.2.(57))

\[
T = S + 2E(\frac{1}{4}\tilde{\tau}) \cdot [\cos 2\pi(\mu w + v) + R] 
\in O(E(\tilde{\tau} - 2w)) + O(E(\frac{1}{4}\tilde{\tau} - w)) + O(E(\frac{9}{4}\tilde{\tau} - 3w))
\]

where the first term and the last one are dominated by the term in the middle if (25) holds, because of

\[
\text{Im} (\frac{9}{4}\tilde{\tau} - w) \geq \text{Im} (\tilde{\tau} - 2w) \geq \text{Im} (\frac{1}{4}\tilde{\tau} - w).
\]

\[
\square
\]

The following corollary is used in the proof of Lemma VI.6.5.

Corollary 7 Under the assumptions (0) the following estimates hold (\(\mu = \pm 1\)):

\[
(26) \quad [1 - f((\mu w + v)\tau + \mu^1|\tau)]^{-1} \in O(E(-w)) \quad (s \to 1)
\]

uniformly in \((w, v)\) provided

\[
(27) \quad 0 \leq \text{Im} w \leq \frac{1}{4} \text{Im} \tilde{\tau} \quad \text{and} \quad \text{Im} v \text{ is bounded}
\]

\[
(28) \quad [\cos 2\pi(\mu w + v - \mu^\frac{1}{4}\tilde{\tau})]^{-1} \text{ is bounded.}
\]

Proof. We use the abbreviations \(\tilde{w} := \frac{1}{4}\tilde{\tau} - w\) and \(\tilde{v} := -v\), and have

\[
(\mu w + v)\tau + \mu^1 = -(\mu \tilde{w} + \tilde{v})\tau \\
[\cos 2\pi(\mu \tilde{w} + \tilde{v})]^{-1} \in O(E(\tilde{w})) = O(E(\frac{1}{4}\tilde{\tau} - w)) \quad \text{(Aux. Prop. VI.1.6)} \\
\frac{1}{4}\tilde{\tau} - \tilde{w} = w \\
2\tilde{\tau} - 2\tilde{w} = \frac{3}{2}\tilde{\tau} + 2w \quad \text{so} \quad O(E(2\tilde{\tau} - 2\tilde{w})) \in O(E(\frac{3}{2}\tilde{\tau})) \to 0.
\]

Then it follows from Corollary 6 using V.5.(98 B) that the term in (26) on the left equals

\[
= [1 - f((\mu \tilde{w} + \tilde{v})\tau|\tau)]^{-1} \in \frac{1}{4} E(\frac{1}{4}\tilde{\tau}) [\cos 2\pi(\mu \tilde{w} + \tilde{v}) + O(E(2\tilde{\tau} - 3\tilde{w}))] 
\in \frac{1}{4} E(\frac{1}{4}\tilde{\tau}) \cos 2\pi(\mu \tilde{w} + \tilde{v}) \quad \text{where the first term and the last one are dominated by the term in the middle if (25) holds, because of}
\]

\[
\text{Im} (\frac{9}{4}\tilde{\tau} - w) \geq \text{Im} (\tilde{\tau} - 2w) \geq \text{Im} (\frac{1}{4}\tilde{\tau} - w).
\]

\[
\square
\]
**Proposition 8** Under the assumptions (0) the following estimates hold \((\mu = \pm 1, \nu = \pm 1)\):

\[
(29) \quad f(\nu(\mu w + v)\tau + \frac{1}{4}|\tau|) = -\frac{1}{2} \sin \pi \nu (\mu w + v) + R \quad \cos \pi \nu (\mu w + v) + S
\]

where

\[
(30) \quad R, S \in O(E^{\frac{1}{2}} - \frac{3}{2} w) \quad (s \to 1)
\]

uniformly in \((w, v)\) provided

\[
(31) \quad 0 \leq \text{Im} w \leq \delta \frac{1}{2} \text{Im} \tilde{\tau} \quad \text{and} \quad \text{Im} v \text{ is bounded}.
\]

**Proof.** For \(v \in \mathbb{C}\), using V.5.(96A) and III.2.(11)-(18)-(37)-(38)-(39) and III.2.(49), obtain

\[
(32) \quad f(\nu \tau + \frac{1}{4}|\tau|) = \frac{\theta_2}{\theta_3} (2 \nu \tau + \frac{1}{2} |2\tau|) = -\frac{\theta_1}{\theta_4} (2 \nu v |2\tau|) = -\frac{\theta_1}{\theta_2} (\nu v |\frac{1}{2} \tilde{\tau})
\]

[or check the periods \(1, \tilde{\tau}\), the zeros \(\pm \frac{1}{4} \tilde{\tau}\), the poles \(\frac{1}{2} \pm \frac{1}{4} \tilde{\tau}\), and the value \(-\nu i\)]. From Proposition III.2.7 one obtains the following estimates:

\[
\theta_1(\nu(\mu w + v)|\frac{1}{2} \tilde{\tau}) \in 2q^{1/8} [\sin \pi \nu (\mu w + v) + R]
\]

\[
\theta_2(\nu(\mu w + v)|\frac{1}{2} \tilde{\tau}) \in 2q^{1/8} [\cos \pi \nu (\mu w + v) + S]
\]

with \(R\) and \(S\) satisfying (30) provided \((w, v)\) satisfies (31).

Recall that \(f(\cdot|\tau)\) has the properties given in V.5.(98A)...(98I).

From II.1.(12) recall that the function \(g(v) := g_1(v) = g_2(v) = -i \tan \pi v\) has the properties:

- **periods:** \(\mathbb{Z}\) \quad \(g(\cdot)\) is odd
- **zeros:** \(\mathbb{Z}\) \quad \(g(\cdot + \frac{1}{2}) = 1/g(\cdot)\)
- **poles:** \(\frac{1}{2} + \mathbb{Z}\) \quad \(g(\tau) = g(\cdot)\)
- **value i:** \(-\frac{1}{4} + \mathbb{Z}\) \quad the values \(\pm 1\) are not assumed.

**Proposition 9** Under the assumptions (0) the following estimates hold for \(\lambda = \pm 1, \mu = \pm 1, \nu = \pm 1, \) and fixed \(\delta \in (0, 1)\):

\[
(33) \quad f(\nu(\mu w + v)\tau + \frac{1}{4}|\tau|^\lambda - g(\nu(\mu w + v))|^\lambda) \in O(E^{\frac{1}{2} \tilde{\tau} - w}) \quad (s \to 1)
\]

uniformly in \((w, v)\) provided

\[
(34) \quad 0 \leq \text{Im} w \leq \delta \frac{1}{2} \text{Im} \tilde{\tau} \quad \text{and} \quad \text{Im} v \text{ is bounded},
\]

\[
(35) \quad g(\nu(\mu w + v))|^\lambda \text{ is bounded}.
\]
PROOF. Take $\lambda = +1$. Observe

$$g(\mu w + v) = \mu g(w + \mu v)$$

is bounded iff \( [\cos \pi (w + \mu v)]^{-1} \) is bounded.

Hence, by Aux. Property VI.1.6, (35) is equivalent to

$$E(-\frac{1}{2}(w + \mu v)) [\cos \pi (w + \mu v)]^{-1}$$

is bounded

or,

$$E(-\frac{1}{2}w) [\cos \pi (\mu w + v)]^{-1}$$

is bounded.

So, assuming (34) and (35), we obtain from (29)

$$f(\nu(\mu w + v)\tau + \frac{1}{2}|\tau) \in \left\{ \begin{array}{l}
\sim -i \frac{\sin \pi \nu(\mu w + v) + O(E(\frac{1}{2}\tau - \frac{3}{2}w))}{\cos \pi \nu(\mu w + v)[1 + O(E(\frac{1}{2}\tau - w))]} \\
= \frac{g(\nu(\mu w + v)) + O(E(\frac{1}{2}\tau - w))}{1 + O(E(\frac{1}{2}\tau - w))} \\
= [g(\nu(\mu w + v)) + O(E(\frac{1}{2}\tau - w))] [1 + O(E(\frac{1}{2}\tau - w))] \\
= g(\nu(\mu w + v)) + O(E(\frac{1}{2}\tau - w))
\end{array} \right.$$

For $\lambda = -1$ use

$$1/g(\cdot)$$

is bounded iff $1/\sin(\cdot)$ is bounded,

or apply the $\lambda = +1$ case with $v = \frac{1}{2}$ using V.5.(98 D) and the formula $g(\cdot + \frac{1}{2}) = 1/g(\cdot).$

\[\square\]
VI.5 Continuation of Sec. VI.4 — proof of Basic Theorem II.4.55 for even walk

The assumptions are the same as in the previous section, viz.:

\[
\begin{align*}
- & \text{the random walk is skipfree, all-sided, and driftless} \\
- & \text{the random walk is even} \\
- & 0 < s < 1
\end{align*}
\]

In addition we assume (for the notations see II.4.(101)-(96)-(83))

\[
\begin{align*}
(1) & \quad y \in \mathcal{Y}(\hat{\sigma}) \\
(2) & \quad x \in \mathcal{U}^2.
\end{align*}
\]

Furthermore we define \( \hat{\zeta} = (\hat{\zeta}_1, \hat{\zeta}_2) \) by (cf. II.1.(26))

\[
(3) \quad g_k(\hat{\zeta}_k) = x_k \quad \text{and} \quad \hat{\zeta}_k \in \hat{S}_k \quad (k = 1, 2).
\]

**Remark 1** In the case of even walk all “generalized open horizontal strips” are ordinary ones, viz.

\[
(4) \quad \hat{S}_k = \{ \hat{\omega} : |\text{Re} \hat{\omega}| < \frac{1}{4} \} \quad (\text{cf. II.1.(24)})
\]

\[
(5) \quad S_k = \{ w : |\text{Im} w| < \frac{1}{4} \text{Im} \tau(s) \} \quad (\text{cf. V.5.(93)})
\]

and so we have, using \((\text{Re} \hat{\omega}) \cdot (\text{Im} \tau) = \text{Im} (\hat{\omega} \tau + \frac{1}{4})\),

\[
(6) \quad \hat{\omega} \in \hat{S}_k \iff \hat{\omega} \tau + \frac{1}{4} \in S_k.
\]

VI.5.1 The function \((SF)(z_1 \tau + \frac{1}{4}, z_2 \tau + \frac{1}{4})\) for \(s\) close to 1 and \(z\) close to \(\hat{\zeta}\)

As in Sec.VI.2 we have the following. If

\[
(7) \quad z_k \tau + \frac{1}{4} \in S_k \quad (k = 1, 2)
\]

then

\[
(8) \quad (SF)(z_1 \tau + \frac{1}{4}, z_2 \tau + \frac{1}{4}) = \sum_{i=1,2} \sum_{\lambda=\pm} -\lambda I^\lambda_j(z_i)
\]

where (for the function \(F^\lambda_j(\cdot)\) see V.7.(7)-(8))

\[
(9) \quad I^\lambda_j(z_i) = \int_{\Lambda^\lambda_j} \frac{du}{2\pi i} \tilde{\eta}(z_i \tau + \frac{1}{4} - u|2\sigma \tau) F^\lambda_j(u)
\]

and \(\Lambda^\lambda_j\) is a path satisfying VI.2.(5 A)-(5 B) with \(v = z_i \tau + \frac{1}{4}\); in other words,

\[
(10) \quad \begin{cases} 
- \text{the initial point } f^\lambda_j \text{ must satisfy IV.5.(38 A)-(38 B)}, \\
- \text{the path } \Lambda^\lambda_j \text{ must satisfy IV.5.(39 A)-(39 B) with } v = z_i \tau + \frac{1}{4}.
\end{cases}
\]
Choice of the integration paths

Property 1 There exists a neighbourhood of \( \hat{z} \) (not depending on \( s \)) so that (7) is satisfied for all \( z \) in this neighbourhood and all \( s \in (0, 1) \).

Proof. From (6) we know that (7) is equivalent to \( \hat{z}_k \in \hat{S}_k \) \((k = 1, 2)\), where \( \hat{S}_k \) is open and doesn’t depend on \( s \). So the claim is a trivial consequence of the assumption (3).

In addition to (0)-(3) we assume \( M \) is a fixed sufficiently large positive number, viz. so that
\[
\left| \text{Im} \hat{z}_k \right| < M \quad (k = 1, 2)
\]
and we assume \( s \) is sufficiently close to 1, viz. so that
\[
M \leq \frac{1}{4} \text{Im} \bar{\tau}(s).
\]

Then, in particular,
\[
\left| \text{Im} \hat{z}_k \right| < \frac{1}{4} \text{Im} \bar{\tau}(s), \quad \text{or} \quad 0 < \text{Re} (\hat{z}_k \tau(s) + \frac{1}{4}) < \frac{1}{2} \quad (k = 1, 2).
\]
The integration path \( \gamma_j^\lambda(\cdot) \) is the chain composed of \( \gamma_j^\lambda_\mu(\cdot) \), \( \mu = \pm \), where:
\[
\gamma_j^\lambda_\mu^-(\cdot) \text{ is the straight line segment from } -\frac{3}{4} \bar{\tau}(s) \text{ to } -\frac{1}{4} \bar{\tau}(s),
\]
\[
\gamma_j^\lambda_\mu^+(\cdot) \text{ consists of:}
\]
- the straight line segment from \( -\frac{3}{4} \bar{\tau}(s) \) to \( -M i \),
- a path \( \Gamma_j^\lambda(\cdot) \) from \( -M i \) to \( +M i \) satisfying:
\[
\left( \Gamma_j^\lambda \right)^* + \lambda_2^\frac{1}{2} \hat{\sigma} \subset \mathcal{H}_j^\lambda \quad (\text{cf. II.4.(3)})
\]
\[
\left( \Gamma_j^\lambda \right)^* + \lambda_2^\frac{1}{2} \hat{\sigma} \subset D(\hat{z}_i) \quad (\text{cf. II.4.(18)-(17)})
\]
- the straight line segment from \( +M i \) to \( +\frac{4}{4} \bar{\tau}(s) \).

Notice that \( \gamma_j^\lambda(\cdot) \) depends on \( s \) only through its initial point and its end point.

Next, we define the integration path \( \Lambda_j^\lambda(\cdot) \) by
\[
\Lambda_j^\lambda(\cdot) \text{ is the opposite path to } [\gamma_j^\lambda(\cdot) + \lambda_2^\frac{1}{2} \hat{\sigma}(s)] \tau(s) + \frac{1}{4}.
\]

Remark 2 We investigate whether the path \( \Gamma_j^\lambda(\cdot) \) can be a straight line segment from \( -M i \) to \( +M i \). Clearly, because of \( 0 < \hat{\sigma} < \frac{1}{2} \), the straight line segment always satisfies the condition (16). The condition (17) guarantees that the path passes between the points \( z_i \pm \hat{\sigma} \), and the straight line segment satisfies this condition if
\[
\left| \text{Re} z_i - \lambda_2^\frac{1}{2} \hat{\sigma} \right| < \hat{\sigma}.
\]
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Hence, under the condition (19):

\[ \gamma_j^\lambda \] is the straight line segment from \(-\frac{3}{4}\tilde{\tau}(s)\) to \(+\frac{1}{4}\tilde{\tau}(s)\),

\[ \Lambda_j^\lambda \] is the straight line segment from \(\lambda_2^\frac{1}{2}\sigma\tau\) to \(\lambda_2^\frac{1}{2}\sigma\tau + 1\).

Consequently, all integration paths are straight line segments iff

\[ |\text{Re} \, z_i| < \frac{1}{2}\hat{\sigma} \quad (i = 1, 2). \]

**Proposition 2** There exist a neighbourhood of \(s = 1\) and a neighbourhood of \(z = \hat{z}\) with the following properties. If \(s\) is in the neighbourhood of 1 and \(z\) is in the neighbourhood of \(\hat{z}\), then the above defined path \(\Lambda_j^\lambda\) satisfies the conditions (10) with \(f_j^\lambda = \lambda_2^\frac{1}{2}\sigma\tau\).

The proof is postponed to Sec. VI.6.

**Property 3** Assume (11). If \(s\) is in the neighbourhood of 1 and \(z\) is in the neighbourhood of \(\hat{z}\) (cf. Proposition 2) then

\[ I_j^\lambda M H_j^\lambda(z_i, y) = \int_{\Gamma_j^\lambda} \frac{dw}{2\pi i} \frac{\phi'(w + \lambda_2^\frac{1}{2}\hat{\sigma} - z_i)}{\phi(w + \lambda_2^\frac{1}{2}\hat{\sigma}, y)} H_j^\lambda(w + \lambda_2^\frac{1}{2}\hat{\sigma}, y) \]

(see II.4.(28)).

**Proof.** Trivial.

Transformation of the sum of integrals (8) In the integral (9) introduce the new integration variable

\[ w = -\tilde{\tau}(u - \frac{1}{4}) - \lambda_2^\frac{1}{2}\sigma \quad (\text{so} \quad u = (w + \lambda_2^\frac{1}{2}\sigma)\tau + \frac{1}{4}) \]

to find, because \(\tilde{\eta}(\cdot|2\sigma\tau)\) is odd,

\[ I_j^\lambda(z_i) = \int_{\frac{3}{4}\tilde{\tau}}^{\frac{1}{4}\tilde{\tau}} \frac{dw}{2\pi i} \cdot \tau\tilde{\eta}((w + \lambda_2^\frac{1}{2}\sigma - z_i)\tau|2\sigma\tau) \cdot F_j^\lambda((w + \lambda_2^\frac{1}{2}\sigma)\tau + \frac{1}{4}) \]

where the integration path between \(\pm M\) is a special one, see (16)-(17). The integration path is divided in two parts, giving

\[ I_j^\lambda(z_i) = \sum_{\nu=\pm} I_{j,\nu}^\lambda(z_i) \]

\[ I_{j,+}^\lambda(z_i) := \int_{\frac{3}{4}\tilde{\tau}}^{\frac{1}{4}\tilde{\tau}} \frac{dw}{2\pi i} \cdot \tau\tilde{\eta}((w + \lambda_2^\frac{1}{2}\sigma - z_i)\tau|2\sigma\tau) \cdot F_j^\lambda((w + \lambda_2^\frac{1}{2}\sigma)\tau + \frac{1}{4}) \]

(where the integration path is a special one) and (with the substitution \(w' := w + \frac{1}{4}\tilde{\tau}\))

\[ I_{j,-}^\lambda(z_i) := \int_{\frac{3}{4}\tilde{\tau}}^{\frac{1}{4}\tilde{\tau}} \frac{dw'}{2\pi i} \cdot \tau\tilde{\eta}((w' + \lambda_2^\frac{1}{2}\sigma - z_i)\tau + \frac{1}{4}|2\sigma\tau) \cdot F_j^\lambda((w' + \lambda_2^\frac{1}{2}\sigma)\tau - \frac{1}{4}) \]
because \( F^\lambda_j(\cdot) \) has period 1. We introduce the following notations:

\[
\begin{align*}
(22) \quad \tilde{\eta}_+(v|2\sigma \tau) & := \tilde{\eta}(v|2\sigma \tau) = \\
& = \frac{1}{2\sigma} \theta_2 \left( \frac{v}{-2\sigma \tau} \right) - 2\sigma \tau \theta_3 \left( \frac{v}{-2\sigma \tau} \right) \\
(23) \quad \tilde{\eta}_-(v|2\sigma \tau) & := \frac{1}{2\sigma} \theta_2 \left( \frac{v}{-2\sigma \tau} \right) - 2\sigma \tau \theta_3 \left( \frac{v}{-2\sigma \tau} \right)
\end{align*}
\]

These two functions are related as follows:

\[
(24) \quad \tau \tilde{\eta}_+(v + \frac{1}{2}|2\sigma \tau) = \frac{\pi i}{2\sigma} + \tau \tilde{\eta}_-(v|2\sigma \tau).
\]

This follows from the relationship \((\theta_2'/\theta_2)(v + \frac{1}{2}|\tau) = -\pi i + (\theta_3'/\theta_3)(v|\tau)\) which in turn follows from \(\theta_2(v + \frac{1}{2}|\tau) = E(-\frac{1}{2}v - \frac{1}{8}\tau)\theta_3(v|\tau)\), cf. III.2.(38)-(12)-(11)-(18)-(39).] From V.7.(9) it follows

\[
F^\lambda_j(\cdot + \lambda \frac{1}{2}\sigma \tau) - F^\lambda_i(\cdot - \lambda \frac{1}{2}\sigma \tau) \equiv 0
\]

hence (multiply by \(\lambda\) and sum)

\[
(25) \quad \sum_{i=1,2} \sum_{\lambda=\pm} \lambda F^\lambda_j((\cdot + \lambda \frac{1}{2}\sigma \tau) \tau \pm \frac{1}{4}) \equiv 0.
\]

Consequently, we can rewrite (8) as follows:

\[
(SF)(z_1 \tau + \frac{1}{4}, z_2 \tau + \frac{1}{4}) = \sum_{i=1,2} \sum_{\lambda=\pm} \sum_{\nu=\pm} -\lambda J^\lambda_j^\nu(z_i)
\]

with

\[
J^\lambda_j^0(z_i) := I_j^\lambda(z_i)
\]

and, using (24) and (25),

\[
J^\lambda_j^{-}(z_i) := \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{dw}{2\pi i} \cdot \tau \tilde{\eta}_-(w + \lambda \frac{1}{2}\sigma - z_i|2\sigma \tau) \cdot F^\lambda_j((w + \lambda \frac{1}{2}\sigma) \tau - \frac{1}{4}).
\]

Observe (because \(f_k(\cdot) = f(\cdot|\tau)\) is even) from V.7.(8)

\[
(26) \quad F^\lambda_j((\mu + \lambda \frac{1}{2}\sigma) \tau + \nu \frac{1}{4}) = F^\lambda_j^\nu(\nu(\mu + \lambda \frac{1}{2}\sigma) \tau + \frac{1}{4}).
\]

so we obtain (with in the \(\nu = +\) case the special integration path)

\[
J^\lambda_j^\nu(z_i) := \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{dw}{2\pi i} \cdot \tau \tilde{\eta}_+(w + \lambda \frac{1}{2}\sigma - z_i|2\sigma \tau) \cdot F^\lambda_j^\nu((w + \lambda \frac{1}{2}\sigma) \tau + \frac{1}{4}).
\]

We have, with the substitution \(w' := \frac{1}{4}\tau - w\) if \(w\) is on the negative imaginary axis,

\[
(27) \quad (SF)(z_1 \tau + \frac{1}{4}, z_2 \tau + \frac{1}{4}) = \sum_{i=1,2} \sum_{\lambda=\pm} -\lambda \left( K^\lambda_j^{0,+}(z_i) + \sum_{\mu=\pm} \sum_{\nu=\pm} K^\lambda_j^{\mu,\nu}(z_i) \right)
\]
with (with the notation VI.2.(16) for $\tilde{\phi}'/\tilde{\phi}(\infty i)$)

\begin{align}
K_{j}^{\lambda,0,+}(z_{i}) & := \int_{\Gamma_{j}^{\lambda}} \frac{dw}{2\pi i} \cdot \tau \tilde{\eta}((w + \lambda_{2}^{1/2}\sigma - z_{i})\tau|2\sigma\tau) \cdot \\
& \quad \cdot F_{j}^{\lambda}((w + \lambda_{2}^{1/2}\sigma)\tau + \frac{1}{4})
\end{align}

\begin{align}
K_{j}^{\lambda,\mu,+}(z_{i}) & := \int_{\Gamma_{j}^{\lambda}}^{+ \frac{1}{4}\tau} \frac{dw}{2\pi i} \cdot \tau \tilde{\eta}((\mu w + \lambda_{2}^{1/2}\sigma - z_{i})\tau|2\sigma\tau) - \mu \frac{\tilde{\phi}'}{\tilde{\phi}}(\infty i) \cdot \\
& \quad \cdot F_{j}^{+\lambda}(+(\mu w + \lambda_{2}^{1/2}\sigma)\tau + \frac{1}{4})
\end{align}

\begin{align}
K_{j}^{\lambda,\mu,-}(z_{i}) & := \int_{0}^{\frac{1}{4}\tau} \frac{dw}{2\pi i} \cdot \tau \tilde{\eta}_-(((\mu w + \lambda_{2}^{1/2}\sigma - z_{i})\tau|2\sigma\tau) \cdot \\
& \quad \cdot F_{j}^{-\lambda}(-(\mu w + \lambda_{2}^{1/2}\sigma)\tau + \frac{1}{4}).
\end{align}

### VI.5.2 Limit of $(SF)(z_{1}\tau + \frac{1}{4}, z_{2}\tau + \frac{1}{4})$ for $s \to 1$

**Notations** For fixed $y$, fixed $\nu = \pm$ and $k = 1, 2$: (cf. II.4.(83))

\begin{align}
m_{k}^{\nu} = m_{k}^{\nu}(y_{k}) & := \begin{cases} 0 & \text{if } y_{k} \neq \ell_{\nu} \text{ (i.e. } y_{k} \in U_{1}^{\nu}) \\ 1 & \text{if } y_{k} = \ell_{\nu} \end{cases} \\
n^{\nu} = n^{\nu}(y) & := \max_{i=1,2} \{ 2m_{j}^{\nu} + m_{k}^{\nu} \} \\
& = \begin{cases} 0 & \text{if } m_{1}^{\nu} + m_{2}^{\nu} = 0 \quad \text{(case 1)} \\ 2 & \text{if } m_{1}^{\nu} + m_{2}^{\nu} = 1 \quad \text{(case 2)} \\ 3 & \text{if } m_{1}^{\nu} + m_{2}^{\nu} = 2 \quad \text{(case 3).} \end{cases}
\end{align}

**Proposition 4** Consider the difference

\begin{align}
F_{j}^{\lambda \nu}(\nu(\mu w + \lambda_{2}^{1/2}\sigma)\tau + \frac{1}{4}) - H_{j}^{\lambda \nu}(\nu(\mu w + \lambda_{2}^{1/2}\sigma), y).
\end{align}

I. Assume $\mu = \nu = +$. Then this difference is of the order

\begin{align}
O(E(\frac{1}{2}\tau)) \quad (s \to 1)
\end{align}

uniformly in $w$ provided $w \in (\Gamma^{\lambda}_{j})^{*}$.

II. This difference is of the order

\begin{align}
O(E(\frac{1}{2}\tau - [1 + n^{\nu}]w)) \quad (s \to 1)
\end{align}

uniformly in $w$ provided

\begin{align}
w \in \mathbb{R} i \quad \text{and} \quad 0 \leq \text{Im } w \leq \frac{1}{4}\text{Im } \tilde{\tau}.
\end{align}

For the proof see Sec. VI.6.

**Proposition 5** Consider the function (not dependent on $s$)

\begin{align}
H_{j}^{\lambda \nu}(\nu(\mu w + \lambda_{2}^{1/2}\sigma), y).
\end{align}
I. Assume \( \mu = \nu = + \). Then this function, as a function of \( w \) in the set \((\Gamma^\lambda_j)^*\), is bounded.

II. This function is of the order

\[
\mathcal{O}(E(-[m_1^{\mu \nu} + m_2^{\mu \nu}]w)) \quad (s \to 1)
\]
uniformly in \( w \) provided

\[
w \in \mathbb{R} i \quad \text{and} \quad 0 \leq \Im w.
\]

For the proof see Sec. VI.6.

**Corollary 6** Using \( F^\lambda_j = H^\lambda_j + [F^\lambda_j - H^\lambda_j] \) one easily concludes (because (35)
is at most of the same order as (38)) that

\[
F^\lambda_j(\nu(\mu w + \lambda^{\frac{1}{2}} \sigma)\tau + \frac{1}{2})
\]
also can be estimated by (38) uniformly in \( w \) provided (36) holds.

**Proposition 7** For \( \mu = \pm \) and fixed \( \delta \in (0,1) \) the function

\[
\tilde{\tau}_{\tilde{\eta}}((\mu w + v)\tau|2\sigma \tau)
\]
is of the order

\[
\mathcal{O}\left(E\left(\frac{1}{2} \tilde{\tau} - \frac{w}{2\sigma}\right)\right) \quad (s \to 1)
\]
uniformly in \((w,v)\) provided

\[
0 \leq \Im w \leq \delta \frac{1}{2} \Im \tilde{\tau} \quad \text{and} \quad \Im v \text{ is bounded.}
\]

**PROOF.** By (23) the function in (41) is identical with

\[
\frac{1}{-2\sigma} \frac{\theta'_3}{\theta_3} \left( \frac{\mu w + v}{-2\sigma} | \tilde{\tau} \right)
\]
(where the minus signs cancel out), and apply Proposition III.2.8 (Part II), using

\((0,1/2) \ni \sigma(s) \to \tilde{\sigma} \in (0,1/2).
\]

In the following proposition we use the notations \( \phi \) and \( \tilde{\phi} \), see II.4.(23)-(56), and VI.3.(34)) and VI.2.(16).

**Proposition 8**

I. For \( \lambda = \pm \) the difference

\[
\tilde{\tau}_{\tilde{\eta}}((w + \lambda^{\frac{1}{2}} \sigma - \frac{1}{i})\tau|2\sigma \tau) - \frac{\phi'}{\phi}(w + \lambda^{\frac{1}{2}} \tilde{\sigma} - \frac{1}{i})
\]
is of the order (34) uniformly in \( w \) provided \( w \in (\Gamma^\lambda_j)^* \) and uniformly in \( z_i \) in some (sufficiently small) neighbourhood of \( \tilde{z}_i \).

II. Assume (11). For \( \lambda = \pm, \mu = \pm \), the difference

\[
(\tau \tilde{\eta}_2((\mu w + \lambda \frac{1}{2} \sigma - z_i)\tau|2\sigma\tau) - \frac{\tilde{\phi}'}{\phi} (\infty i)) + \left[ \frac{\phi'}{\phi}(\mu w + \lambda \frac{1}{2} \sigma - z_i) - \mu \frac{\phi'}{\phi} (\infty i) \right]
\]

is of the order \( O \left( 1 + |w| E \left( \frac{1}{2} \tau^2 + \frac{w}{2\sigma} \right) \right) \), hence,

\[
O \left( \frac{\tilde{\tau}}{E} E \left( \frac{1}{2} \tilde{\tau} + \frac{w}{2\sigma} \right) \right) \quad (s \to 1)
\]

uniformly in \( w \) provided

\[
(47) \quad w \in \mathbb{R} i \quad \text{and} \quad M \leq \text{Im} w \leq \frac{1}{4} \text{Im} \tilde{\tau}.
\]

**Proof.** See Proposition VI.2.6. For Part II observe that

\[
O \left( E \left( \frac{1}{2} \tilde{\tau} - \frac{w}{2\sigma} \right) \right) \quad \text{is negligible in comparison with} \quad O \left( 1 + |w| E \left( \frac{1}{2} \tilde{\tau} + \frac{w}{2\sigma} \right) \right)
\]

because of (47) using VI.4.(22).

\( \Box \)

**Proposition 9** [Cf. Proposition VI.2.7]

I. The function

\[
(48) \quad \frac{\phi'}{\phi}(w + \lambda \frac{1}{2} \sigma - z_i)
\]

is bounded provided \( w \in (\Gamma^\lambda_j)^* \) and \( z_i \) in some (sufficiently small) neighbourhood of \( \tilde{z}_i \).

II. Assume (11). Then the difference

\[
(49) \quad \frac{\phi'}{\phi}(\mu w + \lambda \frac{1}{2} \sigma - z_i) - \mu \frac{\phi'}{\phi} (\infty i)
\]

is bounded by a constant times

\[
(50) \quad E \left( \frac{w}{2\sigma} \right)
\]

provided

\[
(51) \quad \text{Im} w \geq M \quad \text{and} \quad z_i \text{ is sufficiently close to } \tilde{z}_i.
\]
Limits of the integrals $K_{j}^{\lambda,\mu,\nu}(z_i)$
First take $\nu = -$ and consider (30).

**Proposition 10** The integral $K_{j}^{\lambda,\mu,-}(z_i)$ is of the order

$$O\left( E \left( \frac{1}{4} \tilde{\tau} \left[ \frac{1}{2\bar{\sigma}} - (m_1^{\mu} + m_2^{\mu}) \right] \right) \right) \quad (s \to 1)$$

uniformly in $z_i$ in some (sufficiently small) neighbourhood of $\tilde{z}_i$.

**Proof.** With Proposition 7 and Corollary 6 the integrand in (30) can be estimated by (42) · (38) (with $\nu = -$). This implies the claim because we can do the integration.

Next take $\nu = +$ and consider (28).

**Proposition 11** The difference

$$K_{j}^{\lambda,0,+}(z_i) - I_{\phi}^{\lambda,M}H_{j}^{\lambda}(z_i, y)$$

is of the order (34) uniformly in $z_i$ in some (sufficiently small) neighbourhood of $\tilde{z}_i$.

**Proof.** With Property 3, the difference (53) can be rewritten as follows:

$$\int_{\Gamma_j^{\lambda}} \frac{dw}{2\pi i} \cdot [(44) \cdot (37)]_{\mu=\nu=+} + [(48) + (44)]_{\mu=\nu=+}$$

where the integrand can be estimated as follows:

$$(34) \cdot O(1) + [O(1) + (34)] \cdot (34) ,$$

which proves the claim.

Next take $\nu = +$ and consider (29). We put

$$M' := \text{Im} \frac{1}{4} \tilde{\tau}(s).$$

With the abbreviation (54) have from II.4.(104)-(115)

$$\tilde{R}_{\phi,\mu}^{\lambda,M,M'}H_{j}^{\lambda}(z_i, y) =$$

$$= \int_{M_i}^{M'} \frac{dw}{2\pi i} \cdot \left[ \frac{\phi'}{\phi}(\mu w + \lambda \frac{1}{2}\bar{\sigma} - z_i) - \mu \frac{\phi'}{\phi}(\infty i) \right] \cdot H_{j}^{\lambda}(\mu w + \lambda \frac{1}{2}\bar{\sigma}, y) =$$

$$= \int_{M_i}^{M'} \frac{dw}{2\pi i} \cdot (49) \cdot [ (37) \text{ with } \nu = + ].$$

From II.4.(99)-(95) we recall

$$\gamma^{\mu}(\bar{\sigma}) = \begin{cases} \langle U \rangle^2 & \text{if } 0 < \bar{\sigma} < 1/4, \\ \langle U \rangle^2 \setminus \{(\ell^\mu, \ell^\mu)\} & \text{if } 1/4 \leq \bar{\sigma} < 1/2. \end{cases}$$
Proposition 12 Assume \( y \in \mathcal{Y}_\mu(\hat{\sigma}) \). The difference

\[
K_{\lambda j}^{\lambda \mu}(z_i) - \tilde{R}_{\phi,\mu}^{\lambda,M,M'} H_j^\lambda(z_i, y)
\]

is of the order (57) + (58) uniformly in \( z_i \) in some (sufficiently small) neighbourhood of \( \hat{\tau}_i \):

\[
O\left( \tilde{\tau} E \left( \frac{1}{2} \tilde{\tau} + a w \right) \right) \quad (s \to 1)
\]

\[
O\left( E \left( \frac{1}{4} \tilde{\tau} \left[ \frac{1}{2\tilde{\sigma}} + 1 - n'^\mu(y) \right] \right) \right) \quad (s \to 1).
\]

PROOF. Writing \( \eta \cdot F \) for the integrand in (29), \( \phi \cdot H \) for the integrand in (55), we have

\[
\eta \cdot F - \phi \cdot H = (\eta - \phi)H + \phi(F - H) + (\eta - \phi)(F - H)
\]

\[
= (45) (37)_{\nu=+} + (49) (33)_{\nu=+} + (45) (33)_{\nu=+}
\]

which can be estimated by

\[
(46) (38)_{\nu=+} + (50) (35)_{\nu=+} + (46) (35)_{\nu=+}
\]

In the sum on the right the last term is negligible, because (46) is negligible compared to (50). The first term is of the order

\[
O\left( \tilde{\tau} E \left( \frac{1}{2} \tilde{\tau} + aw \right) \right) \quad \text{with} \quad a := \frac{1}{2\tilde{\sigma}} - [m'^\mu(y_1) + m'^\mu(y_2)]
\]

where \( a > 0 \) because of \( y \in \mathcal{Y}_\mu(\hat{\sigma}) \). Hence, the first integral (integrate over \([0, \frac{1}{4} \tilde{\tau}]\)) is of the order \( O\left( \tilde{\tau} E \left( \frac{1}{2} \tilde{\tau} \right) \right) \). The second term is of the order

\[
O\left( E \left( \frac{1}{2} \tilde{\tau} + bw \right) \right) \quad \text{with} \quad b := \frac{1}{2\tilde{\sigma}} - [1 + n'^\mu(y)]
\]

where \( b > -2 \) because of \( y \in \mathcal{Y}_\mu(\hat{\sigma}) \). Hence, the second integral (perform the integration) is of the order

\[
O\left( E \left( \frac{1}{2} \tilde{\tau} \right) \right) + O\left( E \left( \frac{1}{2} \tilde{\tau} + b \frac{1}{4} \tilde{\tau} \right) \right) = O\left( E \left( \frac{1}{2} \tilde{\tau} \right) \right) + O\left( E \left( \frac{1}{4} \tilde{\tau}(2 + b) \right) \right)
\]

where \( 2 + b = 1/(2\hat{\sigma}) + 1 - n'^\mu(y) \). This implies the claim.

We put (cf. II.4.(122) and (55), see VI.2.(43))

\[
\tilde{R}_{\phi,\mu}^{\lambda,M} H_j^\lambda(z_i, y) = \lim_{M' \to \infty} \tilde{R}_{\phi,\mu}^{\lambda,M,M'} H_j^\lambda(z_i, y).
\]

This limit exists because of (11) and \( y \in \mathcal{Y}_\mu(\hat{\sigma}) \) (cf. Proposition II.4.26); this also follows from the following proposition, using

\[
(59) \left( \tilde{R}_{\phi,\mu}^{\lambda,M} - \tilde{R}_{\phi,\mu}^{\lambda,M,M'} \right) H_j^\lambda(z_i, y) = \tilde{R}_{\phi,\mu}^{\lambda,M'} H_j^\lambda(z_i, y).
\]
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Proposition 13 The integral

\[ \tilde{R}^{\lambda,M'}_\phi \mu \ H^\lambda_j(z_i, y) \quad \text{with } M' \text{ given in (54)} \]

is of the order

\[ O \left( E \left( \frac{1}{4} \tilde{\tau} \left[ \frac{1}{2\sigma} - (m^\mu_1 + m^\mu_2) \right] \right) \right) \quad (s \to 1) \]

uniformly in \( z_i \) in some (sufficiently small) neighbourhood of \( \hat{z}_i \).

PROOF. The difference (59) equals (cf. (55))

\[ \int_{\frac{1}{4}\tau}^{\infty} \frac{dw}{2\pi i} \cdot (49) \cdot \left[ (37) \text{ with } \nu = + \right] \]

and use the estimates (50) and (38).

\[ \square \]

Corollary 14 Assume \( y \in \mathcal{Y}^\mu(\tilde{\sigma}) \). The difference

\[ K^\lambda_\mu,+_j(z_i) = \tilde{R}^{\lambda,M}_\phi \mu \ H^\lambda_j(z_i, y) \]

is of the order (61) + (57) uniformly in \( z_i \) in some (sufficiently small) neighbourhood of \( \hat{z}_i \).

PROOF. Combine the propositions 12 and 13, using \( 1 - n^\mu \geq -(m^\mu_1 + m^\mu_2) \).

\[ \square \]

From VI.2.(51) we know

\[ (\Lambda H_0)(z, y) = \sum_{i=1,2} \sum_{\lambda = \pm} -\lambda \left( I^{\lambda,M}_\phi \mu + \sum_{\mu = \pm} \tilde{R}^{\lambda,M}_\phi \mu \right) H^\lambda_j(z_i, y) \]

Proposition 15 The difference

\[ (SF)(z_1 \tau + \frac{1}{4}, z_2 \tau + \frac{1}{4}) - (\Lambda H_0)(z, y) \]

is of the order (57) + (65) uniformly in \( z \) in some (sufficiently small) neighbourhood of \( \hat{z} \), where

\[ O \left( E \left( \frac{1}{4} \tilde{\tau} \left[ \frac{1}{2\sigma} - m \right] \right) \right) + O \left( E \left( \frac{1}{4} \tilde{\tau} \left[ \frac{1}{2\sigma} - m \right] \right) \right) \quad (s \to 1) \]

\[ m = m(y) := \max_{\nu = \pm} \left( m^\nu_1(y_1) + m^\nu_2(y_2) \right). \]

PROOF. From (27) and (63) using Proposition 10, Proposition 11, and Corollary 14.

\[ \square \]
VI.5.3 Proof of Basic Theorem II.4.55 for even walk

Proposition 16 There exists a continuous function \( s \in (0, 1] \mapsto \zeta_k \in \mathbb{C} \) satisfying:

\begin{align*}
(67) & \quad \zeta_k(1) = \bar{z}_k \quad (\text{cf. (3)}) \\
(68) & \quad \zeta_k(s) \tau(s) + \frac{1}{4} = w_k(s) \quad (0 < s < 1) \\
(69) & \quad f_k(w_k(s)) = x_k \quad (0 < s < 1) \\
(70) & \quad w_k(s) \in S_k \quad (0 < s < 1)
\end{align*}

Proof. [Cf. the proof of Property VI.2.11] Notice that by II.1.(26) the constant \( \hat{z}_k \) is determined by the requirement (3). For \( 0 < s < 1 \) define \( w_k(s) \) by (69)-(70) and (71):

\begin{align*}
(71) & \quad 0 \leq \text{Re} \ w_k(s) \leq \frac{1}{2},
\end{align*}

and next define \( \zeta_k(s) \) by means of (68). Observe that \( w_k(s) \) is uniquely defined by (69)-(70) and (71) because of V.5.(55) (with \( \lambda = + \)) and V.5.(13). Notice that \( \zeta_k(s) \in \tilde{S}_k \) and \( |\text{Im} \zeta_k(s)| \leq \frac{1}{4} \text{Im} \tilde{\tau}(s) \).

Next we prove that \( \zeta_k(\cdot) \) is continuous. For \( 0 < s < 1 \) this follows from Proposition V.5.4 (part II; cf. V.5.(34 A) and Property V.5.3). The continuity in \( s = 1 \) remains to be shown. Use Proposition VI.4.9 with \( \lambda = \mu = \nu = +, \ v = 0 \) and \( w = \zeta_k(s) \) to conclude [use \( f_k(\zeta_k(s)) \tau(s) + \frac{1}{4} = x_k = g_k(\zeta_k(1)) \)]

\begin{align*}
(72) & \quad g_k(\zeta_k(1)) - g_k(\zeta_k(s)) \in \mathcal{O}\left( E\left( \frac{1}{4} \text{Im} \tilde{\tau}(s) \right) \right) \quad (s \to 1)
\end{align*}

where \( \zeta_k(1), \zeta_k(s) \in \tilde{S}_k \). The function \( g_k(\cdot) \) is a bicontinuous bijection \( \tilde{S}_k \mapsto \mathcal{U} \). This proves the claim.

\[ \square \]

Corollary 17 The difference \( \zeta_k(1) - \zeta_k(s) \) is of the order

\begin{align*}
(73) & \quad \mathcal{O}\left( E\left( \frac{1}{4} \text{Im} \tilde{\tau}(s) \right) \right) \quad (s \to 1).
\end{align*}

Proof. This follows from (72), because \( g_k'(\zeta_k) \neq 0 \) [use, with \( z_k = \zeta_k(s) \), that \( g_k(z_k) - g_k(\zeta_k) = g_k'(\zeta_k + \theta(z_k - \zeta_k)) \cdot (z_k - \zeta_k) \)].

\[ \square \]

Proposition 18 Assume \( y \in \mathcal{Y}(\tilde{\sigma}) \). The difference

\begin{align*}
(\lambda H_0)(\zeta(s), y) - (\lambda H_0)(\zeta, y)
\end{align*}

where \( \zeta(s) = (\zeta_1(s), \zeta_2(s)) \), is of the order (73).
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Proof. From Property II.4.40 (Part III) we know that the function $z \mapsto (\Lambda H_0)(z, y)$ is regular in a neighbourhood of $\hat{z} = \zeta(1)$. So the claim follows from Corollary 17.

Combining the propositions 15 and 18 we obtain the following.

Theorem 19 Assume (0), (1), and (2). Let $\hat{z}$ satisfy (3), and let $\zeta(s)$ satisfy (67),..., (71). Then the difference

$$(SF)(\zeta_1(s)\tau(s) + \frac{1}{4}, \zeta_2(s)\tau(s) + \frac{1}{4}) - (\Lambda H_0)(\hat{z}, y)$$

is of the order (65) + (73).

Proof of Basic Theorem II.4.55 for even walk Assume (0), (1), and (2). From Theorem 19 we know

$$(SF)(\zeta_1(s)\tau(s) + \frac{1}{4}, \zeta_2(s)\tau(s) + \frac{1}{4}) \rightarrow (\Lambda H_0)((\zeta_1(1), \zeta_2(1)), y) \quad (s \rightarrow 1).$$

Because of (3), (67)-(70) ($k = 1, 2$) this implies (see V.7.(19) and II.4.(141))

$$(LP_0)(x, y, s) \rightarrow (LP_0)(x, y) \quad (s \rightarrow 1).$$

If we assume in addition: $Q(x, 1) \neq 0$, then it follows

$$\frac{(LP_0)(x, y, s) - P_0(x, y)}{Q(x, s)} \rightarrow \frac{(LP_0)(x, y) - P_0(x, y)}{Q(x, 1)} \quad (s \rightarrow 1),$$

i.e. (see V.7.(21) and II.4.(150))

$$\tilde{P}(x, y, s) \rightarrow \tilde{P}(x, y) \quad (s \rightarrow 1),$$

i.e., because of Theorem V.7.5,

$$P(x, y, s) \rightarrow \tilde{P}(x, y) \quad (s \rightarrow 1).$$

$\Box$
VI.6 Proofs of the Propositions VI.5.2, VI.5.4, and VI.5.5

VI.6.1 Proof of Proposition VI.5.2

The conditions IV.5.(38 A)-IV.5.(38 B) are trivially satisfied. We prove IV.5.(39 A). Obviously, \( \Lambda_{\lambda j}(\cdot) \) has initial point \( \lambda_1^{\sigma}(s) \tau(s) \) and end point \( \lambda_2^{\sigma}(s) \tau(s) + 1 \). Because of V.5.(14 A), if \( \sigma(s) - \tilde{\sigma} \) is sufficiently small then

\[
(\Gamma_{\lambda j}^*)^* + \lambda_1^{\sigma}(s) \subset \tilde{S}_j \cap (\tilde{S}_i + \lambda \sigma(s)),
\]

hence (c.f. VI.5.(4))

\[
(\gamma_{\lambda j}^*)^* + \lambda_1^{\sigma}(s) \subset \tilde{S}_j \cap (\tilde{S}_i + \lambda \sigma(s)),
\]

hence,

\[
(\Lambda_{\lambda j}^*)^* \subset S_j \cap (S_i + \lambda \sigma(s) \tau(s))
\]

by VI.5.(18)-(6). This proves IV.5.(39 A).

We prove IV.5.(39 B) with \( v := z_i \tau(s) + \frac{1}{4}, \) for \( s \) sufficiently close to 1 and \( z_i \) sufficiently close to \( \tilde{z}_i \). We must prove:

(I) \[
(\Gamma_{\lambda j}^*)^* \cap \left(z_i \tau(s) + \frac{1}{4} + \sigma(s) \tau(s) + \mathbb{Z} 2 \sigma(s) \tau(s)\right) = \emptyset,
\]

(II) \[
\text{Ind}^*(w | (\Lambda_{\lambda j}^*)^*) = \begin{cases} 0 & \text{if } w = z_i \tau(s) + \frac{1}{4} - (2k + 1) \sigma(s) \tau(s) \\ 1 & \text{if } w = z_i \tau(s) + \frac{1}{4} + (2k + 1) \sigma(s) \tau(s) & (k \geq 0). \end{cases}
\]

We prove (I). Because of VI.5.(18), (I) is equivalent to

\[
(\gamma_{\lambda j}^*)^* \cap \left(z_i + \sigma(s) + \mathbb{Z} 2 \sigma(s)\right) = \emptyset.
\]

For this, a sufficient condition is, obviously,

\[
(\gamma_{\lambda j}^*)^* \cap (z_i \pm \sigma(s), \infty) = \emptyset.
\]

Hence, assuming VI.5.(12) and \( |\text{Im} z_i| < M \), we need to show only

\[
(\Gamma_{\lambda j}^*)^* \cap (z_i \pm \sigma(s), \infty) = \emptyset.
\]

Because of VI.5.(17), this is true if both \( \sigma(s) - \tilde{\sigma} \) and \( z_i - \tilde{z} \) are sufficiently small, which proves (I).

We prove (II). We use the notation

\[
L^\mu(z_i) := z_i \tau(s) + \frac{1}{4} + \mu \tau(s)[\sigma(s), \infty) \quad (\mu = \pm)
\]

which is a half line (a similar notation has been used in the proof of Proposition VI.2.3, see Subsec. VI.3.1). For \( \nu = \pm \) the set \( E(\nu L^\mu(z_i)) \) is connected and (by
VI.6.2 Proofs of the Propositions VI.5.4 and VI.5.5

Notations For fixed $i, j \lambda, \mu, \nu$ we put ($k = 1, 2$)

1. $\lambda_j := \lambda$ $\quad \lambda_i := -\lambda$
2. $\hat{w}_k := \nu(\mu w + \lambda_k \frac{1}{2} \hat{\sigma})$
3. $w_k := \nu(\mu w + \lambda_k \frac{1}{2} \sigma(s))$
4. $u_k := w_k \tau(s) + \frac{1}{4}$

Also we use the abbreviations ($k = 1, 2$)

5. $\hat{g}_k := g_k(\hat{w}_k)$ $\quad g_k := g_k(w_k)$ $\quad f_k := f_k(u_k)$
6. $\hat{\psi}_k := \frac{\hat{g}_k}{1 - y_k \hat{g}_k}$ $\quad \psi_k := \frac{g_k}{1 - y_k g_k}$ $\quad \phi_k := \frac{f_k}{1 - y_k f_k}$
7. $\hat{H}_j^{\lambda\nu} := \hat{\psi}_j \hat{\psi}_i (= \hat{\psi}_1 \hat{\psi}_2)$ $\quad H_j^{\lambda\nu} := \psi_j \psi_i (= \psi_1 \psi_2)$ $\quad F_j^{\lambda\nu} := \phi_j \phi_i (= \phi_1 \phi_2)$

Then we have (see II.1.(32) and V.7.(7)) ($k = 1, 2$)

8. $\hat{\psi}_k = \psi_k(\hat{w}_k, y_k)$ $\quad \psi_k = \psi_k(w_k, y_k)$ $\quad \phi_k = \phi_k(u_k)$

and (see II.1.(39) and V.7.(8))

9. $\hat{H}_j^{\lambda\nu} = H_j^{\lambda\nu}(\nu(\mu \hat{w} + \lambda_j \frac{1}{2} \hat{\sigma}), y)$ $\quad F_j^{\lambda\nu} = F_j^{\lambda\nu}(\nu(\mu w + \lambda_j \frac{1}{2} \sigma(s)) \tau(s) + \frac{1}{4})$.

We need to estimate $F_j^{\lambda\nu} - \hat{H}_j^{\lambda\nu}$ and $\hat{H}_j^{\lambda\nu}$. From (7) we have

10. $F_j^{\lambda\nu} - \hat{H}_j^{\lambda\nu} = (\phi_j - \hat{\psi}_j) \hat{\psi}_i + (\phi_i - \hat{\psi}_i) \hat{\psi}_j + (\phi_j - \hat{\psi}_j)(\phi_i - \hat{\psi}_i)$

and from (6)

11. $\phi_k - \hat{\psi}_k = (f_k - \hat{g}_k)(1 + y_k \phi_k)(1 + y_k \hat{\psi}_k).$
So it suffices to estimate \((f_k - \hat{g}_k), \phi_k,\) and \(\hat{\psi}_k\).

In the following proofs, \(\delta\) is a fixed (sufficiently small) positive number satisfying

\[
\hat{w}_k - \delta, \hat{w}_k + \delta \subset \hat{S}_k \quad (w \in (\Gamma_j) \times \mathbb{R}^i).
\]

If we choose \(s\) close to 1, viz. so that

\[
|\sigma(s) - \hat{\sigma}| \leq 2\delta
\]

then

\[
[w_k, \hat{w}_k] \subset [\hat{w}_k - \delta, \hat{w}_k + \delta] \subset \hat{S}_k.
\]

**Lemma 1** For \(s \to 1\), uniformly in \(w\) (with the abbreviations (5)):

\[
\hat{g}_k, g_k \in O(1) \quad (w \in (\Gamma_j) \times \mathbb{R}^i),
\]

\[
g_k - \hat{g}_k \in \begin{cases} O\left(\frac{1}{2} E(\frac{1}{\tilde{\tau}})\right) & (w \in (\Gamma_j) \times \mathbb{R}^i), \\ O\left(\frac{1}{2} E(\frac{1}{\tilde{\tau}} + w)\right) & (w \in (\mathbb{R}^i) ) \end{cases}
\]

\[
f_k - g_k \in \begin{cases} O\left(\frac{1}{2} E(\frac{1}{\tilde{\tau}})\right) & (w \in (\Gamma_j) \times \mathbb{R}^i), \\ O\left(\frac{1}{2} E(\frac{1}{\tilde{\tau}} - w)\right) & (w \in [0, \frac{1}{\tilde{\tau}}] ) \end{cases}
\]

**Proof.** Proof of (15). Assume (12) and (13), so (14) holds. Then, because of (12),

\[
|g_k(\hat{w}_k + t)| < 1 \quad (w \in (\Gamma_j) \times \mathbb{R}^i) , \, t \in [-\delta, +\delta) ,
\]

which implies (15), cf. (14). Proof of (16). We have, also,

\[
|g'_k(\hat{w}_k + t)| < 2\pi \quad (w \in (\Gamma_j) \times \mathbb{R}^i) , \, t \in [-\delta, +\delta) ,
\]

because of \(g'_k(z) = -\pi i / \cos^2(\pi z) = -\pi i [1 - g_k^2(z)]\). Hence (Aux. Prop. VI.1.6)

\[
|g'_k(\hat{w}_k + t)| \in O(E(w)) \quad (w \in (\Gamma_j) \times \mathbb{R}^i) , \, t \in [-\delta, +\delta) .
\]

Use this and VI.4.(22) in the formula \(g_k - \hat{g}_k \leq \frac{1}{2} |\sigma(s) - \hat{\sigma}| \cdot \|g'_k\|_{\infty}\) to conclude (16).

Proof of (17). Observe that \(g_k\) is bounded (because of (18)), so Proposition VI.4.9 can be applied.

\[\square\]

**Corollary 2** For \(s \to 1\), uniformly in \(w\) (with the abbreviations (5)):

\[
f_k - \hat{g}_k \in \begin{cases} O\left(\frac{1}{2} E(\frac{1}{\tilde{\tau}})\right) & (w \in (\Gamma_j) \times \mathbb{R}^i), \\ O\left(\frac{1}{2} E(\frac{1}{\tilde{\tau}} - w)\right) & (w \in [0, \frac{1}{\tilde{\tau}}] ) \end{cases}
\]

For the notation \(m_k^\nu = m_k^\nu(y_k)\) see VI.5.(31).
Lemma 3  Consider for fixed $y_k$, $|y_k| \leq 1$, the function
\[(w, t) \mapsto \psi_k(\hat{w}_k + t, y_k)\]
Let $\delta$ satisfy (12). Then for $s \to 1$, this function is of the order (uniformly in $(w, t)$)
\[
\begin{cases}
O(1) & (w, t) \in \left(\Gamma_{\lambda}^{\beta}\right)^* \times [-\delta, +\delta]. \\
O(E(-m_{k}^{\mu\nu} w)) & (w, t) \in [0, \infty i) \times [-\delta, +\delta].
\end{cases}
\]

Proof. Part I. The function is continuous on the compact set $\left(\Gamma_{\lambda}^{\beta}\right)^* \times [-\delta, +\delta]$.

Part II. First assume $y_k \neq \ell^{\mu\nu}$, or $m_{k}^{\mu\nu} = 0$. We use the expression (6).
We can find a positive $M$ satisfying
\[|1 - g_k(z)| \leq \frac{1}{2}|1 - y_k\ell^{\mu\nu}| \quad (\text{Im } z \geq M)\]
cf. II.4.(85). With the abbreviation $\tilde{g}_k := g_k(\hat{w}_k + t)$ we obtain, because $g_k(\cdot)$ is odd,
\[|\ell^{\mu\nu} - \tilde{g}_k| = |1 - \mu\nu\tilde{g}_k| = |1 - g_k(w + \mu\lambda_k\frac{1}{2} \hat{\sigma} + \mu\nu t)| \leq \frac{1}{2}|1 - y_k\ell^{\mu\nu}| \quad (\text{Im } w \geq M; t \in \mathbb{R}),\]
hence, on the one hand,
\[|1 - y_k\tilde{g}_k| \geq |1 - y_k\ell^{\mu\nu}| - |y_k| |\ell^{\mu\nu} - \tilde{g}_k| \geq |1 - y_k\ell^{\mu\nu}| (1 - \frac{1}{2}\gamma_k) \geq \frac{1}{2}|1 - y_k\ell^{\mu\nu}| > 0 \quad (\text{Im } w \geq M; t \in \mathbb{R}).\]
On the other hand, we put
\[\gamma_k := \max |\tilde{g}_k| \quad ((w, t) \in [0, M i] \times [-\delta, +\delta])\]
and obtain, assuming (13),
\[|1 - y_k\tilde{g}_k| \geq 1 - |\tilde{g}_k| \geq 1 - \gamma_k > 0 \quad ((w, t) \in [0, M i] \times [-\delta, +\delta]).\]
Together with (18) this proves the claim.
Next assume $y_k = \ell^{\mu\nu}$, or $m_{k}^{\mu\nu} = 1$. In this case we know from II.1.(48)-(49)
\[\psi_k(\hat{w}_k + t, \ell^{\mu\nu}) = \mu\nu\frac{1}{2}[E(-\mu\nu(\hat{w}_k + t)) - 1],\]
which implies the claim.
\[\square\]

Proposition 4  Assume $|y_k| \leq 1$. Then for $s \to 1$ (with the abbreviations (6)):
\[
(20) \quad \psi_k, \hat{\psi}_k \in \begin{cases}
O(1) & (w \in \left(\Gamma_{\lambda}^{\beta}\right)^*), \\
O(E(-m_{k}^{\mu\nu} w)) & (w \in [0, \infty i]).
\end{cases}
\]
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Proof. The claim with respect to $\psi_k$ resp. $\hat{\psi}_k$ follows from Lemma 3 with $t = w_k - \hat{w}_k = \lambda_k \frac{1}{2}(\sigma(s) - \hat{\sigma})$ resp. $t = 0$.

\[ \text{Lemma 5} \]

I. Assume $|y_k| \leq 1$. Then (with the abbreviations (6)):

\[ \phi_k \in O(1) \quad (s \to 1) \]

uniformly in $w$ provided

\[ w \in \left( \Gamma_j \right)^* \cup [0, \frac{1}{4} \tilde{\tau}], \; g_k \text{ is bounded}, \; \psi_k \text{ is bounded}. \]

II. If $y_k = \ell^w$ then

\[ \phi_k \in O(E(-w)) \quad (s \to 1) \]

uniformly in $w$ provided

\[ w \in [0, \frac{1}{4} \tilde{\tau}]. \]

Proof. Part I. We use the expression (6). Because of (19) and (18), the numerator $f_k$ is bounded. In order to see that the denominator is bounded away from zero observe that $[1 - y_k g_k]^{-1} = 1 + y_k \psi_k$ is bounded, say $|1 - y_k g_k| \geq \epsilon > 0$, and so

\[ |1 - y_k f_k| \geq |1 - y_k g_k| - |y_k| |f_k - g_k| \geq \frac{1}{2} \epsilon, \]

if $s$ is close to zero (cf. (17)).

Part II. Observe $y_k \phi_k = -1 + [1 - y_k f_k]^{-1}$ where

\[ 1 - y_k f_k = 1 - \mu \nu f_k \]

\[ = 1 - f(u_k + \mu \nu \frac{1}{4} - \frac{1}{4} |\tau|) \]

\[ = 1 - f(w_k \tau + \mu \nu \frac{1}{4} |\tau|), \]

and so the claim is an application of Corollary VI.4.7 (use $|\cos(x + yi)| \geq |\cos(x)|$, so $|\cos 2\pi (w_k - \mu \nu \frac{1}{4} \tilde{\tau})| \geq \cos \pi \sigma$).

\[ \text{Proposition 6} \]

Assume $|y_k| \leq 1$. Then for $s \to 1$ (with the abbreviations (6)):

\[ \phi_k \begin{cases} O(1) & (w \in \left( \Gamma_j \right)^*), \\ O(E(-m_k^w w)) & (w \in [0, \frac{1}{4} \tilde{\tau}]). \end{cases} \]

Proof. Use Lemma 5. In the case $w \in \left( \Gamma_j \right)^*$ or $y_k \neq \ell^w$ apply Part I, using (15) and (20). In the case $y_k = \ell^w$ and $w \in [0, \frac{1}{4} \tilde{\tau}]$ apply Part II.

Corollary 7 Assume $|y_k| \leq 1$. Then for $s \to 1$ (with the abbreviations (6)):

\[ \phi_k - \hat{\psi}_k \begin{cases} O(E^{(\frac{1}{2} \tilde{\tau})}) & (w \in \left( \Gamma_j \right)^*), \\ O(E^{(\frac{1}{2} \tilde{\tau} - [1 + 2m_k^w] w}) & (w \in [0, \frac{1}{4} \tilde{\tau}]). \end{cases} \]

Proof. Use (11) and the estimates (17), (20), and (21).

\[ \square \]
Proof of Proposition VI.5.4  Use (10), and the estimates (20) and (22).
In the case of Part I the proof is immediate.
In the case of Part II we obtain
\[ F_j^\lambda - \tilde{H}_j^\lambda \in O(E(\frac{1}{2} \tilde{\tau} - [1 + 2m_j^{\mu\nu} + m_i^{\mu\nu}]w)) \]
\[ + O(E(\frac{1}{2} \tilde{\tau} - [1 + 2m_j^{\mu\nu} + m_i^{\mu\nu}]w)) \]
\[ + O(E(\tilde{\tau} - 2[1 + m_i^{\mu\nu} + m_j^{\mu\nu}]w)) \]
where the last O-term is negligible because of \( w \in [0, \frac{1}{4} \tilde{\tau}] \). This proves
\[ F_j^\lambda - \tilde{H}_j^\lambda \in O(E(\frac{1}{2} \tilde{\tau} - [1 + n^{\mu\nu}]w)) \].

\[ \square \]

Proof of Proposition VI.5.5  Trivial from the expression (7) and the estimate (20).
\[ \square \]