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Output photon statistics for a beam splitter with input squeezed light  
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Abstract  
The mixing of (quadrature-) squeezed states by a lossless beam splitter is studied. Two generating  
functions describing the outgoing-photon distribution are calculated. With the use of these generating  
functions the distribution is determined explicitly for some special cases.

1 Introduction  
The fact that the photon statistics can be modified in a non-trivial way by classical objects such  
as semi-transparent mirrors or beam splitters was realized a long time ago. Aharonov e. a. [1]  
showed that only a Glauber coherent input state splits in two uncorrelated output states.

More recently, the action of a beam splitter on non-classical light has been considered by  
several authors [2–9]. In these papers simultaneous input from both sides of the beam splitter is  
allowed for. Campos e. a. [6] expressed the distribution of the outgoing photons in terms of the  
incoming-photon distribution by means of sums over Jacobi polynomials. For specific examples  
of non-classical input, however, one expects that less formal expressions can be found.

In some of the papers mentioned above, beam splitters with incoming (quadrature-) squeezed  
states have been studied, mostly by determining the first few moments of the number distribution  
of the outgoing photons [4,5]. Kuznetsov [9] studied the particular case that the input states  
have matched squeeze and coherence parameters. He showed that also in this case uncorrelated  
outgoing states may be found.

For arbitrary squeeze and coherence parameters, however, full information on the statistics is  
still lacking. It is the aim of this paper to treat this general case. To that end we shall evaluate two  
generating functions describing the outgoing-photon distribution. As an illustration of the use of  
these generating functions we will discuss a few examples.

2 Transformation from squeezed input states to output Fock number states  

A semi-transparent mirror at $z = 0$ couples plane waves that move in the positive and negative  
z-direction. The reflection and transmission coefficients obey the usual relations:

$$|r|^2 + |t|^2 = 1, \quad rt^* + r^*t = 0. \quad (2.1)$$

For each value of the frequency $\omega$ there are two different modes:

$$U(z, t) = \begin{cases}  
e^{ikz-i\omega t} + r e^{-ikz-i\omega t} & \text{for } z < 0,  
\text{for } z > 0, \end{cases} \quad (2.2)$$
\[ V(z, t) = \begin{cases} te^{-ikz-i\omega t} & \text{for } z < 0, \\ e^{-ikz-i\omega t} + re^{ikz-i\omega t} & \text{for } z > 0, \end{cases} \] 

(2.3)

with \( k = \omega/c \). An annihilation operator is associated with each mode. The positive-frequency part of (a polarization component of) the electrical field is proportional to:

\[ \mathcal{E}^+(z, t) = aU(z, t) + bV(z, t) = \begin{cases} a e^{-ikz-i\omega t} + ce^{-ikz-i\omega t} & \text{for } z < 0, \\ b e^{-ikz-i\omega t} + de^{ikz-i\omega t} & \text{for } z > 0, \end{cases} \] 

(2.4)

with:

\[
\begin{pmatrix}
c \\
d
\end{pmatrix} = \begin{pmatrix}
r & t \\
t & r
\end{pmatrix} \begin{pmatrix}
a \\
b
\end{pmatrix}.
\] 

(2.5)

The annihilation operators \( a \) and \( b \) are associated with the incoming states, and \( c \) and \( d \) with the outgoing states. The operators \( c \) and \( d \) are independent due to the unitarity of the transformation matrix.

We consider a state of the system that factorizes as \(| \psi \rangle_a \otimes | \chi \rangle_b \equiv | \psi ; \chi \rangle_{ab} \). The state vector can be written in a basis of Fock number states of the outgoing photons:

\[ | \psi ; \chi \rangle_{ab} = \sum_{n,m=0}^{\infty} f_{nm} | n ; m \rangle_{cd}. \] 

(2.6)

The coefficients \( f_{nm} \) follow from:

\[ \mathcal{F}_{\psi \chi}(x, y) = \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{n! \, m!}} f_{nm} x^n y^m, \] 

(2.7)

where the generating function \( \mathcal{F}_{\psi \chi}(x, y) \) is defined by

\[ \mathcal{F}_{\psi \chi}(x, y) = \langle 0 ; 0 | e^{x \sigma} e^{y \tau} | \psi ; \chi \rangle_{ab}. \] 

(2.8)

We want to determine this function for the case that the incoming states are both squeezed:

\[ | \psi ; \chi \rangle_{ab} = | \alpha, z ; \beta, w \rangle_{ab} = \exp(\frac{i}{2} za\dagger^2 - \frac{i}{2} z^* a^2) \exp(\alpha a\dagger - \alpha^* a) \times \exp(\frac{i}{2} wb\dagger^2 - \frac{i}{2} w^* b^2) \exp(\beta b\dagger - \beta^* b) | 0 ; 0 \rangle. \] 

(2.9)

The exponential operators in (2.8) with (2.9) can be rewritten into normally-ordered form with the use of the method of parameter differentiation [10]. This leads to:

\[ \mathcal{F}_{\alpha, z, \beta, w}(x, y) = [\cosh(|z|) \cosh(|w|)]^{-\frac{1}{2}} \exp \left[ (xt + yt)^2 e^{i\phi} \frac{1}{2} \tanh(|z|) + \frac{(xt + yt) \alpha}{\cosh(|z|)} \right] \times \exp \left[ (yt + xt)^2 e^{i\psi} \frac{1}{2} \tanh(|w|) + \frac{(yt + xt) \beta}{\cosh(|w|)} \right] \times \exp \left[ -\frac{1}{2} \alpha^2 e^{-i\theta} \tanh(|z|) - \frac{1}{2} \beta^2 e^{-i\theta} \tanh(|w|) - \frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 \right], \] 

(2.10)
The generating function $F$ can be used to determine the amplitudes $f_{nm}$ and the probabilities $P_{nm}$. The marginal probabilities can be found by summing $P_{nm}$ over one of the indices, although it may be quite complicated to carry out the summation over $n$ or $m$ explicitly. To deal with the marginal probabilities more efficiently, we will derive another generating function in the next section.

3 The generating function for the outgoing-photon probabilities

We will now calculate the generating function for the probabilities $P_{nm} = |f_{nm}|^2$. This generating function is defined by:

$$G_{\alpha, z; \beta, w}(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_{nm} x^n y^m.$$  \hfill (3.1)

Using the completeness relation of the coherent states and their expansion in Fock states, we can perform the summations:

$$G_{\alpha, z; \beta, w}(x, y) = \int \frac{d^2\mu}{\pi} \int \frac{d^2\nu}{\pi} \int \frac{d^2\rho}{\pi} \int \frac{d^2\sigma}{\pi} \exp \left[ -|\mu|^2 - |\nu|^2 - |\rho|^2 - |\sigma|^2 + x\mu\rho^* + y\nu\sigma^* \right]$$
$$\times F_{\alpha, z; \beta, w}(\mu^*, \nu^*) \left[ F_{\alpha, z; \beta, w}(\rho^*, \sigma^*) \right]^*.$$  \hfill (3.2)

The integrals are all Gaussian, and can therefore be carried out. If we define:

$$c_1 = \frac{r\alpha}{\cosh(|z|)} + \frac{t\beta}{\cosh(|w|)}$$
$$c_2 = \frac{t\alpha}{\cosh(|z|)} + \frac{r\beta}{\cosh(|w|)}$$
$$c_3 = \frac{r\alpha e^{-i\phi}}{\sinh(|z|)} - \frac{t\beta e^{-i\theta}}{\sinh(|w|)}$$
$$c_4 = \frac{t\alpha e^{-i\phi}}{\sinh(|z|)} - \frac{r\beta e^{-i\theta}}{\sinh(|w|)}$$  \hfill (3.3)

then we have:

$$G_{\alpha, z; \beta, w}(x, y) = \frac{\exp(\mathcal{N}/\mathcal{D} + A)}{\cosh(|z|) \cosh(|w|) \sqrt{\mathcal{D}}}$$  \hfill (3.4)

with

$$\mathcal{N}(x, y) = -x^2 y^2 \tanh(|z|) \tanh(|w|) \left\{ R \left( (t^* c_1 + t^* c_2)e^{-i\phi} \right) \tanh(|w|) + R \left( (t^* c_1 + t^* c_2)e^{-i\theta} \right) \tanh(|z|) \right\}$$
$$+ R \left( (t^* x c_1 + t^* y c_2)e^{-i\phi} \right) \tanh(|z|) + R \left( (t^* x c_1 + t^* y c_2)e^{-i\theta} \right) \tanh(|w|)$$
$$+ x|c_1|^2 + y|c_2|^2 - xy^2 \tanh^2(|z|) \tanh^2(|w|)|c_3|^2 - x^2 y \tanh^2(|z|) \tanh^2(|w|)|c_4|^2,$$  \hfill (3.5)

$$\mathcal{D}(x, y) = \left| 1 - xy e^{i(\phi - \theta)} \tanh(|z|) \tanh(|w|) \right|^2$$
$$- \left| e^{i(\phi - \theta)} \tanh(|z|) \left( x|r|^2 + y|t|^2 \right) - \tanh(|w|) \left( x|t|^2 + y|r|^2 \right) \right|^2,$$  \hfill (3.6)
\[ A = - \cosh^2(|z|) |r^* c_1 + t^* c_2|^2 - \tanh(|z|) \cosh^2(|z|) \Re \left[ (r^* c_1 + t^* c_2)^2 e^{-i\phi} \right] \]
\[ - \cosh^2(|w|) |t^* c_1 + r^* c_2|^2 - \tanh(|w|) \cosh^2(|w|) \Re \left[ (t^* c_1 + r^* c_2)^2 e^{-i\theta} \right]. \]  (3.7)

The marginal probabilities can be obtained directly by differentiating the generating function \( G \) to \( x \) (or \( y \)) at \( x = 0, y = 1 \) (or \( x = 1, y = 0 \)), whereas the probabilities \( P_{nm} \) follow by differentiation at \( x = y = 1 \). Note that the latter can also be found from \( F \). Differentiation of \( G \) at \( x = y = 1 \) yields the normally-ordered moments.

4 Special cases

We will study a few special cases with the help of the generating functions defined in the previous sections.

The case \( w = -z \).

Both \( F \) and \( G \) factorize in functions of \( x \) and \( y \) if and only if the squeezing parameters \( z \) and \( w \) have opposite values. The factorization of \( F \) implies that the output states are uncorrelated. In fact, one finds:
\[ |\alpha, z; \beta, -z\rangle_{ab} = |\alpha', z'; \beta', -z'\rangle_{cd}, \]  (4.1)
where \( \alpha' = r \alpha + t \beta; \beta' = t \alpha + r \beta \) and where \( z' = (r^2 - t^2)z \), as can be verified from (2.10). Consequently, \( f_{nm} \) and \( P_{nm} \) are both easily found. Note that the marginal probabilities follow directly from \( P_{nm} \), as \( P_{nm} \) factorizes in \( \sum_m P_{nm} \) and \( \sum_n P_{nm} \). Hence, there is no need for \( G \).

Equation (4.1) generalizes a result by Kuznetsov [9], who showed that for \( w = -z \) and \( |\alpha| = |\beta| \) a factorization occurs, and that for \( |r| = |t| \) a squeezed vacuum is then produced at one of the output ports. Note that in [9] extra phase factors are introduced in the transformation matrix (2.5). These phase factors can be absorbed into the definition of the ingoing states, as has been done here.

The case \( \alpha = \beta = 0 \).

If the two incoming states are both squeezed vacua, the generating functions get the form:
\[ F_{0,0,0,0}(x, y) = \left[ \cosh(|z|) \cosh(|w|) \right]^{-\frac{1}{2}} \exp \left\{ \frac{x^2}{2} \left[ r^2 e^{i\phi} \tanh(|z|) + t^2 e^{i\theta} \tanh(|w|) \right] \right\} \]
\[ \times \exp \left\{ \frac{y^2}{2} \left[ t^2 e^{i\phi} \tanh(|z|) + r^2 e^{i\theta} \tanh(|w|) \right] \right\} \]
\[ \times \exp \left\{ x y r t \left[ e^{i\phi} \tanh(|z|) + e^{i\theta} \tanh(|w|) \right] \right\}, \]  (4.2)
\[ G_{0,0,0,0}(x, y) = \frac{1}{\cosh(|z|) \cosh(|w|) \sqrt{D}}, \]  (4.3)
where \( D \) is given by (3.6).

The coefficients \( f_{nm} \), and hence the probabilities \( P_{nm} \), can be found directly by expanding \( F \).
For $m \geq n$, $m + n$ even, we have:

$$f_{nm} = \left[\cosh(\vert z\vert) \cosh(\vert w\vert)\right]^{-\frac{1}{2}} \sqrt{\frac{n!}{m!} (m - n)!} \frac{r^m t^n}{\vert r \vert \vert t \vert^n} (a_1)^{\frac{n}{2}} a_2^{\frac{(m-n)}{2}} C_n^{(\frac{m-n+1}{2})}(a_3). \quad (4.4)$$

Here $C_n^\lambda(z)$ are Gegenbauer polynomials, and $a_1$, $a_2$ and $a_3$ are given by:

$$a_1 = e^{i(\phi_0 \pm \theta)} \tanh(\vert z\vert) \tanh(\vert w\vert), \quad a_2 = -|t|^2 e^{i\theta} \tanh(\vert z\vert) + |r|^2 e^{i\theta} \tanh(\vert w\vert),$$

$$a_3 = \vert r \vert \vert t \vert \left[ e^{i\theta} \tanh(\vert z\vert) + e^{i\theta} \tanh(\vert w\vert) \right] / \sqrt{a_1}. \quad (4.5)$$

Note that as a consequence of the parity properties of the Gegenbauer polynomials the ambiguity in the sign of the root of $a_1$ is irrelevant. For $f_{mn}$, with $m \geq n$ and $m + n$ even, a similar formula, with $r \leftrightarrow t$, is found. For odd $m + n$ the coefficients $f_{nm}$ are zero.

In the present case the marginal probabilities $P_n$ (or $P_m$) are not easily derived by summing $P_{nm}$, as the resulting expression becomes quite complicated. However, the marginal distribution is conveniently calculated from the generating function $G$. We get:

$$P_n = \frac{1}{\cosh(\vert z\vert) \cosh(\vert w\vert)} \frac{(b_3 b_4)^{\frac{n}{2}}}{(b_1 b_2)^{\frac{n}{2} + 1}} C_n^{(\frac{1}{2})} \left( \frac{b_1 b_4 + b_2 b_3}{2\sqrt{b_1 b_2 b_3 b_4}} \right), \quad (4.6)$$

where the constants $b_1, b_2, b_3$ and $b_4$ are defined by the fact that $D(x, y = 1)$ can be written as $(b_1 - b_3 x)(b_2 - b_4 x)$. For the special case $e^{i(\phi_0 - \theta)} = \pm 1$ we have

$$b_1 = 1 \mp |t|^2 \tanh(\vert z\vert) + |r|^2 \tanh(\vert w\vert), \quad b_2 = 1 \pm |t|^2 \tanh(\vert z\vert) - |r|^2 \tanh(\vert w\vert),$$

$$b_3 = \pm \tanh(\vert z\vert) \tanh(\vert w\vert) \pm |t|^2 \tanh(\vert z\vert) - |r|^2 \tanh(\vert w\vert),$$

$$b_4 = \pm \tanh(\vert z\vert) \tanh(\vert w\vert) \mp |t|^2 \tanh(\vert z\vert) + |r|^2 \tanh(\vert w\vert). \quad (4.7)$$

For general values of $e^{i(\phi_0 - \theta)}$ the expressions for $b_i$ are somewhat more complicated.

It is easily verified that for $z = w$ and $|r|^2 = 1/2$ one gets the geometrical distribution $P_n = (\tanh(\vert z\vert))^2^n / \cosh^2(\vert z\vert)$. For $z = -w$ (see previous special case) or $|r|^2$ equal to 0 or 1 the distribution is that of the squeezed vacuum. The change-over from the squeezed vacuum distribution to the geometrical distribution as $|r|^2$ increases is shown for $z = w = 1.5$ in Figure 1. Note already for $|r|^2 = 0.025$ the distribution closely resembles the geometrical distribution.

**Figure 1:** The marginal distribution $P_n$ for $z = w = 1.5$ and $|r|^2 = 0.001, 0.005$ or 0.025, respectively.
In principle, the method outlined in [6] could have been used as a starting-point to treat this special case. However, the derivation of (4.4) and (4.6) along these lines involves complicated multiple sums for which no reduction formulas seem to be available.

5 Conclusions

We have calculated the generating functions for the outgoing-photon distribution in the case of squeezed input states. We discussed the merits of these generating functions, and studied two cases for which their specific usefulness becomes manifest. In doing this, we derived the condition for uncorrelated output states. For the case of incoming squeezed vacua we obtained the complete photon distribution.

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