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DOI
10.1016/0030-4018(94)00491-9

Publication date
1994

Published in
Optics Communications

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August 17, 1994

Abstract

The mixing of (quadrature-) squeezed states by a lossless beam splitter is studied. Two generating functions describing the outgoing-photon distribution are calculated. With the use of these generating functions the distribution is determined explicitly for some special cases.

1 Introduction

The fact that the photon statistics can be modified in a non-trivial way by classical objects such as semi-transparent mirrors or beam splitters was realized a long time ago. Aharonov e. a. [1] showed that only a Glauber coherent input state splits in two uncorrelated output states.

More recently, the action of a beam splitter on non-classical light has been considered by several authors [2–9]. In these papers simultaneous input from both sides of the beam splitter is allowed for. Campos e. a. [6] expressed the distribution of the outgoing photons in terms of the incoming-photon distribution by means of sums over Jacobi polynomials. For specific examples of non-classical input, however, one expects that less formal expressions can be found.

In some of the papers mentioned above, beam splitters with incoming (quadrature-) squeezed states have been studied, mostly by determining the first few moments of the number distribution of the outgoing photons [4,5]. Kuznetsov [9] studied the particular case that the input states have matched squeeze and coherence parameters. He showed that also in this case uncorrelated outgoing states may be found.

For arbitrary squeeze and coherence parameters, however, full information on the statistics is still lacking. It is the aim of this paper to treat this general case. To that end we shall evaluate two generating functions describing the outgoing-photon distribution. As an illustration of the use of these generating functions we will discuss a few examples.

2 Transformation from squeezed input states to output Fock number states

A semi-transparent mirror at $z = 0$ couples plane waves that move in the positive and negative z-direction. The reflection and transmission coefficients obey the usual relations:

$$|r|^2 + |t|^2 = 1, \qquad rt^* + r^*t = 0. \tag{2.1}$$

For each value of the frequency ω there are two different modes:

$$U(z,t) = \begin{cases} e^{ikz-i\omega t} + re^{-ikz-i\omega t} & \text{for } z < 0, \\ te^{ikz-i\omega t} & \text{for } z > 0, \end{cases} \tag{2.2}$$

$$V(z,t) = \begin{cases} t e^{-ikz-i\omega t} & \text{for } z < 0, \\ e^{-ikz-i\omega t} + r e^{ikz-i\omega t} & \text{for } z > 0, \end{cases} \tag{2.3}$$

with $k = \omega/c$. An annihilation operator is associated with each mode. The positive-frequency part of (a polarization component of) the electrical field is proportional to:

$$\mathcal{E}^{+}(z,t) = aU(z,t) + bV(z,t) = \begin{cases} a e^{ikz-i\omega t} + c e^{-ikz-i\omega t} & \text{for } z < 0, \\ b e^{-ikz-i\omega t} + d e^{ikz-i\omega t} & \text{for } z > 0, \end{cases} \tag{2.4}$$

with:

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} r & t \\ t & r \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.5}$$

The annihilation operators a and b are associated with the incoming states, and c and d with the outgoing states. The operators c and d are independent due to the unitarity of the transformation matrix.

We consider a state of the system that factorizes as $\mid \psi \rangle_a \otimes \mid \chi \rangle_b \equiv \mid \psi ; \chi \rangle_{ab}$. The state vector can be written in a basis of Fock number states of the outgoing photons:

$$\mid \psi ; \chi \rangle_{ab} = \sum_{n,m=0}^{\infty} f_{nm} \mid n ; m \rangle_{cd}. \tag{2.6}$$

The coefficients f_{nm} follow from:

$$\mathcal{F}_{\psi\chi}(x,y) = \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{n!\,m!}} f_{nm} x^n y^m, \tag{2.7}$$

where the generating function $\mathcal{F}_{\psi\chi}(x,y)$ is defined by

$$\mathcal{F}_{\psi\chi}(x,y) = \langle 0;0 \mid e^{xc} e^{yd} \mid \psi ; \chi \rangle_{ab}. \tag{2.8}$$

We want to determine this function for the case that the incoming states are both squeezed:

$$\begin{aligned} \mid \psi ; \chi \rangle_{ab} = \mid \alpha, z ; \beta, w \rangle_{ab} &= \exp(\tfrac{1}{2} z a^{\dagger 2} - \tfrac{1}{2} z^* a^2) \exp(\alpha a^\dagger - \alpha^* a) \\ &\times \exp(\tfrac{1}{2} w b^{\dagger 2} - \tfrac{1}{2} w^* b^2) \exp(\beta b^\dagger - \beta^* b) \mid 0;0 \rangle. \end{aligned} \tag{2.9}$$

The exponential operators in (2.8) with (2.9) can be rewritten into normally-ordered form with the use of the method of parameter differentiation [10]. This leads to:

$$\begin{aligned} \mathcal{F}_{\alpha,z;\beta,w}(x,y) &= [\cosh(|z|)\cosh(|w|)]^{-\frac{1}{2}} \exp\left[(xr+yt)^2 e^{i\phi} \tfrac{1}{2} \tanh(|z|) + \frac{(xr+yt)\,\alpha}{\cosh(|z|)} \right] \\ &\times \exp\left[(yr+xt)^2 e^{i\theta} \tfrac{1}{2} \tanh(|w|) + \frac{(yr+xt)\,\beta}{\cosh(|w|)} \right] \\ &\times \exp\left[-\tfrac{1}{2}\alpha^2 e^{-i\phi} \tanh(|z|) - \tfrac{1}{2}\beta^2 e^{-i\theta} \tanh(|w|) - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2 \right], \end{aligned} \tag{2.10}$$

where ϕ and θ are the phases of z and w, respectively.

The generating function $\mathcal{F}$ can be used to determine the amplitudes f_{nm} and the probabilities P_{nm}. The marginal probabilities can be found by summing P_{nm} over one of the indices, although it may be quite complicated to carry out the summation over n or m explicitly. To deal with the marginal probabilities more efficiently, we will derive another generating function in the next section.

3 The generating function for the outgoing-photon probabilities

We will now calculate the generating function for the probabilities $P_{nm} = |f_{nm}|^2$. This generating function is defined by:

$$\mathcal{G}_{\alpha,z;\beta,w}(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{nm}x^n y^m. \tag{3.1}$$

Using the completeness relation of the coherent states and their expansion in Fock states, we can perform the summations:

$$\begin{aligned}\mathcal{G}_{\alpha,z;\beta,w}(x,y) = &\int\frac{d^2\mu}{\pi}\int\frac{d^2\nu}{\pi}\int\frac{d^2\rho}{\pi}\int\frac{d^2\sigma}{\pi}\exp\left[-|\mu|^2-|\nu|^2-|\rho|^2-|\sigma|^2+x\mu\rho^*+y\nu\sigma^*\right]\\ &\times\mathcal{F}_{\alpha,z;\beta,w}(\mu^*,\nu^*)\left[\mathcal{F}_{\alpha,z;\beta,w}(\rho^*,\sigma^*)\right]^*.\end{aligned} \tag{3.2}$$

The integrals are all Gaussian, and can therefore be carried out. If we define:

$$\begin{aligned} c_1 &= \frac{r\alpha}{\cosh(|z|)}+\frac{t\beta}{\cosh(|w|)}, & c_2 &= \frac{t\alpha}{\cosh(|z|)}+\frac{r\beta}{\cosh(|w|)},\\ c_3 &= \frac{r\alpha e^{-i\phi}}{\sinh(|z|)}-\frac{t\beta e^{-i\theta}}{\sinh(|w|)}, & c_4 &= \frac{t\alpha e^{-i\phi}}{\sinh(|z|)}-\frac{r\beta e^{-i\theta}}{\sinh(|w|)},\end{aligned} \tag{3.3}$$

then we have:

$$\mathcal{G}_{\alpha,z;\beta,w}(x,y) = \frac{\exp\left(\mathcal{N}/\mathcal{D}+\mathcal{A}\right)}{\cosh(|z|)\cosh(|w|)\sqrt{\mathcal{D}}}, \tag{3.4}$$

with

$$\begin{aligned}\mathcal{N}(x,y) = &-x^2y^2\tanh(|z|)\tanh(|w|)\left\{\Re\left[(r^*c_1+t^*c_2)^2e^{-i\phi}\right]\tanh(|w|)+\Re\left[(t^*c_1+r^*c_2)^2e^{-i\theta}\right]\tanh(|z|)\right\}\\ &+\Re\left[(r^*xc_1+t^*yc_2)^2\,e^{-i\phi}\right]\tanh(|z|)+\Re\left[(t^*xc_1+r^*yc_2)^2e^{-i\theta}\right]\tanh(|w|)\\ &+x|c_1|^2+y|c_2|^2-xy^2\tanh^2(|z|)\tanh^2(|w|)|c_3|^2-x^2y\tanh^2(|z|)\tanh^2(|w|)|c_4|^2,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{D}(x,y) = &\left|1-xye^{i(\phi-\theta)}\tanh(|z|)\tanh(|w|)\right|^2\\ &-\left|e^{i(\phi-\theta)}\tanh(|z|)\left(x|r|^2+y|t|^2\right)-\tanh(|w|)\left(x|t|^2+y|r|^2\right)\right|^2,\end{aligned} \tag{3.6}$$

$$\mathcal{A} = -\cosh^2(|z|)\,|r^* c_1 + t^* c_2|^2 - \tanh(|z|)\cosh^2(|z|)\Re\left[(r^* c_1 + t^* c_2)^2 e^{-i\phi}\right]$$
$$- \cosh^2(|w|)\,|t^* c_1 + r^* c_2|^2 - \tanh(|w|)\cosh^2(|w|)\Re\left[(t^* c_1 + r^* c_2)^2 e^{-i\theta}\right]. \tag{3.7}$$

The marginal probabilities can be obtained directly by differentiating the generating function $\mathcal{G}$ to x (or y) at $x = 0, y = 1$ (or $x = 1, y = 0$), whereas the probabilities P_{nm} follow by differentiation at $x = y = 0$. Note that the latter can also be found from $\mathcal{F}$. Differentiation of $\mathcal{G}$ at $x = y = 1$ yields the normally-ordered moments.

4 Special cases

We will study a few special cases with the help of the generating functions defined in the previous sections.

The case $w = -z$.

Both $\mathcal{F}$ and $\mathcal{G}$ factorize in functions of x and y if and only if the squeezing parameters z and w have opposite values. The factorization of $\mathcal{F}$ implies that the output states are uncorrelated. In fact, one finds:

$$|\, \alpha, z\,; \beta, -z\,\rangle_{ab} = |\, \alpha', z'\,; \beta', -z'\,\rangle_{cd}, \tag{4.1}$$

where $\alpha' = r\alpha + t\beta$; $\beta' = t\alpha + r\beta$ and where $z' = (r^2 - t^2)z$, as can be verified from (2.10). Consequently, f_{nm} and P_{nm} are both easily found. Note that the marginal probabilities follow directly from P_{nm}, as P_{nm} factorizes in $\sum_m P_{nm}$ and $\sum_n P_{nm}$. Hence, there is no need for $\mathcal{G}$.

Equation (4.1) generalizes a result by Kuznetsov [9], who showed that for $w = -z$ and $|\alpha| = |\beta|$ a factorization occurs, and that for $|r| = |t|$ a squeezed vacuum is then produced at one of the output ports. Note that in [9] extra phase factors are introduced in the transformation matrix (2.5). These phase factors can be absorbed into the definition of the ingoing states, as has been done here.

The case $\alpha = \beta = 0$.

If the two incoming states are both squeezed vacua, the generating functions get the form:

$$\mathcal{F}_{0,z;0,w}(x, y) = \left[\cosh(|z|)\cosh(|w|)\right]^{-\frac{1}{2}} \exp\left\{\frac{x^2}{2}\left[r^2 e^{i\phi}\tanh(|z|) + t^2 e^{i\theta}\tanh(|w|)\right]\right\}$$
$$\times \exp\left\{\frac{y^2}{2}\left[t^2 e^{i\phi}\tanh(|z|) + r^2 e^{i\theta}\tanh(|w|)\right]\right\}$$
$$\times \exp\left\{xyrt\left[e^{i\phi}\tanh(|z|) + e^{i\theta}\tanh(|w|)\right]\right\}, \tag{4.2}$$

$$\mathcal{G}_{0,z;0,w}(x, y) = \frac{1}{\cosh(|z|)\cosh(|w|)\sqrt{\mathcal{D}}}, \tag{4.3}$$

where $\mathcal{D}$ is given by (3.6).

The coefficients f_{nm}, and hence the probabilities P_{nm}, can be found directly by expanding $\mathcal{F}$.

For $m \geq n$, $m + n$ even, we have:

$$f_{nm} = \Big[\cosh(|z|)\cosh(|w|)\Big]^{-\frac{1}{2}} \sqrt{\frac{n!}{m!}}(m-n-1)!!\frac{r^m t^n}{|r|^m |t|^n}\,(a_1)^{\frac{n}{2}}\, a_2^{\frac{1}{2}(m-n)} C_n^{(\frac{m-n+1}{2})}\,(a_3)\,. \tag{4.4}$$

Here $C_n^{\lambda}(z)$ are Gegenbauer polynomials, and a_1, a_2 and a_3 are given by:

$$\begin{aligned} a_1 &= e^{i(\phi+\theta)}\tanh(|z|)\tanh(|w|), \qquad a_2 = -|t|^2 e^{i\phi}\tanh(|z|) + |r|^2 e^{i\theta}\tanh(|w|), \\ a_3 &= |r|\,|t|\left[e^{i\phi}\tanh(|z|) + e^{i\theta}\tanh(|w|)\right] / \sqrt{a_1}. \end{aligned} \tag{4.5}$$

Note that as a consequence of the parity properties of the Gegenbauer polynomials the ambiguity in the sign of the root of a_1 is irrelevant. For f_{mn}, with $m \geq n$ and $m + n$ even, a similar formula, with $r \leftrightarrow t$, is found. For odd $m + n$ the coefficients f_{nm} are zero.

In the present case the marginal probabilities P_n (or P_m) are not easily derived by summing P_{nm}, as the resulting expression becomes quite complicated. However, the marginal distribution is conveniently calculated from the generating function $\mathcal{G}$. We get:

$$P_n = \frac{1}{\cosh(|z|)\cosh(|w|)}\,\frac{(b_3 b_4)^{\frac{1}{2}n}}{(b_1 b_2)^{\frac{1}{2}(n+1)}}\,C_n^{(\frac{1}{2})}\left(\frac{b_1 b_4 + b_2 b_3}{2\sqrt{b_1 b_2 b_3 b_4}}\right), \tag{4.6}$$

where the constants b_1, b_2, b_3 and b_4 are defined by the fact that $\mathcal{D}(x, y = 1)$ can be written as $(b_1 - b_3 x)(b_2 - b_4 x)$. For the special case $e^{i(\phi-\theta)} = \pm 1$ we have

$$\begin{aligned} b_1 &= 1 \mp |t|^2\tanh(|z|) + |r|^2\tanh(|w|), \qquad b_2 = 1 \pm |t|^2\tanh(|z|) - |r|^2\tanh(|w|), \\ b_3 &= \pm\tanh(|z|)\tanh(|w|) \pm |r|^2\tanh(|z|) - |t|^2\tanh(|w|), \\ b_4 &= \pm\tanh(|z|)\tanh(|w|) \mp |r|^2\tanh(|z|) + |t|^2\tanh(|w|). \end{aligned} \tag{4.7}$$

For general values of $e^{i(\phi-\theta)}$ the expressions for b_i are somewhat more complicated.

It is easily verified that for $z = w$ and $|r|^2 = 1/2$ one gets the geometrical distribution $P_n = (\tanh(|z|))^{2n} / \cosh^2(|z|)$. For $z = -w$ (see previous special case) or $|r|^2$ equal to 0 or 1 the distribution is that of the squeezed vacuum. The change-over from the squeezed vacuum distribution to the geometrical distribution as $|r|^2$ increases is shown for $z = w = 1.5$ in Figure 1. Note that already for $|r|^2 = 0.025$ the distribution closely resembles the geometrical distribution.

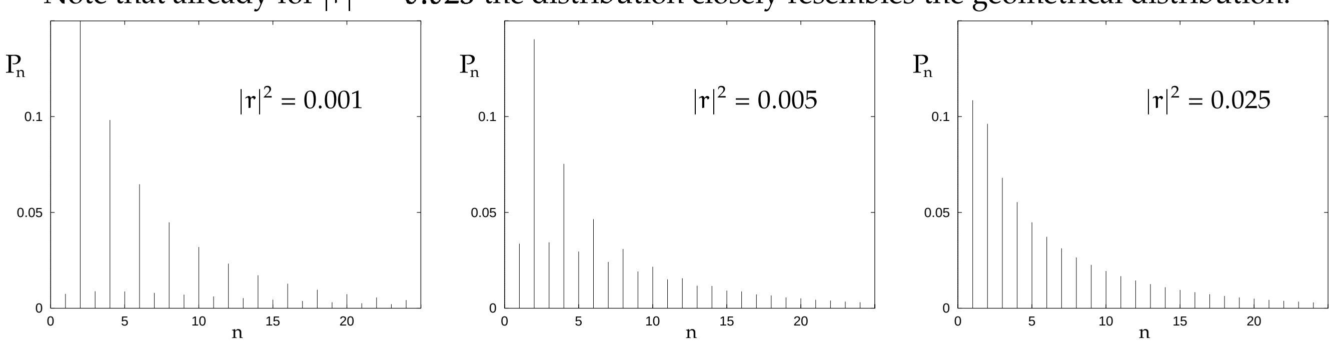

Figure 1: The marginal distribution P_n for $z = w = 1.5$ and $|r|^2 = 0.001, 0.005$ or 0.025, respectively.

In principle, the method outlined in [6] could have been used as a starting-point to treat this special case. However, the derivation of (4.4) and (4.6) along these lines involves complicated multiple sums for which no reduction formulas seem to be available.

5 Conclusions

We have calculated the generating functions for the outgoing-photon distribution in the case of squeezed input states. We discussed the merits of these generating functions, and studied two cases for which their specific usefulness becomes manifest. In doing this, we derived the condition for uncorrelated output states. For the case of incoming squeezed vacua we obtained the complete photon distribution.

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