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Quadrics on complex Riemannian spaces of constant curvature, separation of variables, and the Gaudin magnet

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Integrable systems that are connected with orthogonal separation of variables in complex Riemannian spaces of constant curvature are considered herein. An isomorphism with the hyperbolic Gaudin magnet, previously pointed out by one of the authors, extends to coordinates of this type. The complete classification of these separable coordinate systems is provided by means of the corresponding $L$ matrices for the Gaudin magnet. The limiting procedures (or $e$ calculus) which relate various degenerate orthogonal coordinate systems play a crucial role in the classification of all such systems.

I. CLASSICAL INTEGRABLE SYSTEMS ON COMPLEX CONSTANT CURVATURE SPACES AND THE COMPLEX GAUDIN MAGNET

Separation of variables in the Hamilton Jacobi equation

$$H(p_1,\ldots,p_n;x_1,\ldots,x_n) = \sum_{a,b=1}^{n} g^{ab} p_a p_b = E, \quad p_a = \frac{\partial W}{\partial x_a}, \quad \alpha = 1,\ldots,n$$

amounts to looking for a solution of the form

$$W = \sum_{a=1}^{n} W_a(x_a,h_1,\ldots,h_n), \quad h_n = E.$$  

The solution is said to be a complete integral if $\det(\partial^2 W/\partial x \partial h_j)_{n \times n} \neq 0$. The solution then describes that of free motion on the corresponding Riemannian space with contravariant metric $g^{ab}$. Indeed, if we require $b_j = (\partial W/\partial h_j) = -h_n \delta_{aj}, \quad j = 1,\ldots,n$ for parameters $b_j$, we find that the functions $x(b,h),p(b,h)$ satisfy Hamilton's equations

$$\dot{x}_a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial x_a}.$$  

In this article we allow the Riemannian space to be complex and we consider variable separation of Eq. (1.1) for the following two classes of spaces:

1. The $n$ dimensional complex sphere $S_nC$. This is commonly realized by the set of complex vectors $x = (x_1,\ldots,x_{n+1})$ which satisfy $\sum_{a=1}^{n+1} x_a^2 = 1$ and have infinitesimal distance $dx \cdot dx = \sum_{a=1}^{n+1} dx_a^2$.

2. The $n$ dimensional complex Euclidean space $E_nC$. This is the set of complex vectors $x = (x_1,\ldots,x_n)$ with infinitesimal distance $dx \cdot dx = \sum_{a=1}^{n} dx_a^2$.

A fundamental problem from the point of view of separation of variables on these manifolds is to find all "inequivalent" coordinate systems. As yet, this is an unsolved problem, principally
because many such coordinate systems are intrinsically nonorthogonal. For orthogonal coordinate systems the problem is completely solved and in this case the constants $h_i$ occurring in the complete integral can be chosen to be the values of an involutive set of constants of motion

$$A_j = \sum_{\alpha, \beta=1}^{n} a^{(j)}_{\alpha\beta} p_{\alpha} p_{\beta}, \quad j=1,...,n, \quad A_n = H, \quad \{A_j, A_k\} = 0, \quad (1.4)$$

where

$$\{F(x_{\alpha} p_{\beta}), G(x_{\alpha} p_{\beta})\} = \sum_{\gamma=1}^{n} \left( \frac{\partial F}{\partial x_{\alpha}} \frac{\partial G}{\partial x_{\gamma}} - \frac{\partial F}{\partial x_{\gamma}} \frac{\partial G}{\partial x_{\alpha}} \right), \quad \alpha, \beta = 1,...,n$$

is the Poisson bracket. These constants of the motion are such that

1. each of the tensors $a^{(j)}_{\alpha\beta}$ is a Killing tensor and satisfies Killing’s equations $V(a^{(j)}_{\alpha\beta}) = 0$, (Ref. 1), and
2. $A_j$ can be represented as a sum of quadratic elements of the enveloping algebra of the Lie algebra of symmetries of each of these two considered spaces.

The Lie algebras of these spaces have, respectively, bases of the form

1. $SO(n+1): M_{\alpha\beta} = x_{\alpha} p_{\beta} - x_{\beta} p_{\alpha}, \quad \alpha, \beta = 1,...,n+1$. Here

$$\{M_{\alpha\beta}, M_{\gamma\delta}\} = \delta_{\alpha\gamma} M_{\delta\beta} + \delta_{\alpha\delta} M_{\beta\gamma} + \delta_{\beta\gamma} M_{\alpha\delta} + \delta_{\beta\delta} M_{\alpha\gamma}.$$  

2. $E(n): M_{\alpha\beta} = p_{\alpha}, \quad \alpha, \beta = 1,...,n, \quad \alpha \neq \beta$. Here

$$\{M_{\alpha\beta}, P_{\gamma}\} = \delta_{\beta\gamma} P_{\alpha} - \delta_{\alpha\gamma} P_{\beta}, \quad \{P_{\alpha}, P_{\beta}\} = 0.$$

In this section the separable coordinate systems classified in Refs. 1–3 are given an algebraic interpretation. This is done using the complex analog of the isomorphism between all integrable systems connected with all possible separable systems and the $m$-site $SO(2,1)$ Gaudin magnet. The $m$-site complex Gaudin magnet can be realized as follows. Consider the direct sum of Lie algebras, each of rank $1$

$$\mathfrak{a} = \bigoplus_{\alpha=1}^{m} so_{3}(C). \quad (1.5)$$

The generators $s_{\alpha} \in C^{3}, \alpha = 1,...,m$ of $\mathfrak{a}$ satisfy the Poisson bracket relations

$$\{s^{(i)}_{\alpha}, s^{(j)}_{\beta}\} = -\delta_{\alpha\beta} e_{ij} s^{(k)}_{\alpha}. \quad (1.6)$$

The following metric will be used subsequently when norms and scalar products are calculated:

$$s^{2}_{\alpha} = (s_{\alpha}, s_{\alpha}) = (s^{1}_{\alpha})^{2} + (s^{2}_{\alpha})^{2} + (s^{3}_{\alpha})^{2}, \quad (s_{\alpha}, s_{\beta}) = s^{1}_{\alpha} s^{1}_{\beta} + s^{2}_{\alpha} s^{2}_{\beta} + s^{3}_{\alpha} s^{3}_{\beta}.$$  

If for each $\alpha$, $s^{2}_{\alpha} = c^{2}_{\alpha}$ then the variables $s_{\alpha}$ lie on the direct product of $n$ complex spheres in $C^{3}$. The complex Gaudin magnet is the integrable Hamiltonian system described by the $n$ integrals of motion $H_{\alpha}$ which are in involution with respect to the Poisson bracket

$$H_{\alpha} = 2 \sum_{\beta=1}^{m} \frac{(s_{\alpha}, s_{\beta})}{e_{\alpha} - e_{\beta}}, \quad \{H_{\alpha}, H_{\gamma}\} = 0. \quad (1.7)$$
(We will give a simple proof of this involution property later.) Here $e_{\alpha}$ are taken to be pairwise distinct. This integrable Hamiltonian system is called an $m$-site $SO(3,\mathbb{C})$-XXX Gaudin magnet. The $H_{\alpha}$ are all quadratic functions in the generators of the $\mathfrak{so}(3,\mathbb{C})$ algebra and the following identities are satisfied:

$$
\sum_{\alpha=1}^{m} H_{\alpha} = 0, \quad \sum_{\alpha=1}^{m} e_{\alpha} H_{\alpha} = \mathbf{J}^2 - \sum_{\alpha=1}^{m} e_{\alpha}^2,
$$

where we have introduced the variables

$$
\mathbf{J} = \sum_{\alpha=1}^{m} s_{\alpha}, \quad \mathbf{J}^2 = \langle \mathbf{J}, \mathbf{J} \rangle
$$

the total sum of the momenta $s_{\alpha}$. Indeed

$$
\{J^{i}, J^{j}\} = -\epsilon^{ijk} J^{k}, \quad \{J^{i}, H_{\alpha}\} = 0, \quad i, j, k = 1, 2, 3, \quad \alpha = 1, \ldots, m.
$$

The complete set of involutive integrals of motion is provided by $H_{\alpha}$, $\mathbf{J}^2$ and, for example, $J^3$. The integrals are generated by the $2 \times 2$ $L$ matrices

$$
L(u) = \sum_{\alpha=1}^{m} \frac{1}{u - e_{\alpha}} \begin{pmatrix} s_{\alpha}^3 & -s_{\alpha}^1 - is_{\alpha}^2 \\ -s_{\alpha}^1 + is_{\alpha}^2 & -s_{\alpha}^3 \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},
$$

where

$$
\det L(u) = -(A^2 + BC) = -\sum_{\alpha=1}^{m} \frac{H_{\alpha}}{u - e_{\alpha}} - \sum_{\alpha=1}^{m} \frac{e_{\alpha}^2}{(u - e_{\alpha})^2}.
$$

Furthermore, $L(u)$ satisfies the linear $r$ matrix algebra

$$
\{L(u), L(v)\} = \frac{i}{u - v} [P, L(u) + L(v)], \quad i = \sqrt{-1},
$$

where

$$
P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L(u) = L(u) \otimes \mathbb{I}, \quad L(v) = \mathbb{I} \otimes L(v).
$$

The algebra specified by Eq. (1.13) contains all the necessary Hamiltonian structure of the problem in question. Note that Eq. (1.13) is equivalent to the easily proved relations

$$
\{A(u), A(v)\} = \{B(u), B(v)\} = \{C(u), C(v)\} = 0,
$$

$$
\{A(u), B(v)\} = \frac{i}{u - v} (B(v) - B(u)), \quad \{A(u), C(v)\} = \frac{i}{u - v} (C(u) - C(v)),
$$

$$
\{B(u), C(v)\} = \frac{2i}{u - v} (A(v) - A(u)).
$$
From Eq. (1.15) and the Leibnitz property of the Poisson bracket it is straightforward to deduce that
\[
\{ \det L(u), \det L(v) \} = 0.
\]
In his article5 Kuznetsov has explicitly given the nature of the isomorphism between the XXX Gaudin magnet models and the separation of variables on the \( n \) dimensional real sphere \( S^\ast \). The purpose of this article is to extend these ideas to complex orthogonal coordinate systems on the complex \( n \) sphere \( S^\ast_{\mathbb{C}} \) and, of course, as a consequence complex Euclidean space \( E^\ast_{\mathbb{C}} \). Following Kuznetsov5,8 in the case of the sphere, we set \( c_\alpha = 0, \alpha = 1, \ldots, n + 1 \). The coordinates on the resulting cones are parametrized by

\[
s^\alpha = \frac{1}{4} (p^2 + x^2), \quad s^\alpha = \frac{i}{4} (p^2 - x^2), \quad s^3 = \frac{i}{2} p a x^\alpha.
\]

(1.16)

It follows from Eq. (1.6) that \( \{x^\alpha, x_\beta\} = \{p_a, p_\beta\} = 0, \{p_\alpha, x_\beta\} = \delta_{\alpha\beta} \). Introducing the new variables \( M_{\alpha\beta} = x_{a\beta} p_{\beta} - x_{\beta} p_{a} \) which are the generators of rotations we have

\[
(s^\alpha, s_\beta) = \frac{1}{4} M_{\alpha\beta}^2.
\]

These generators satisfy the commutations relations given previously. This equality establishes the simple quadratic connection between the generators \( s_\alpha \) of \( \mathfrak{so} \) and the \( M_{\alpha\beta} \) of \( SO(n + 1) \). Under this isomorphism the integrals given in Eq. (1.7) (and the subsequent discussion) transform into the following integrals for the free motion on the \( n \) sphere:

\[
\mathcal{H} = \sum_{\alpha=1}^{n+1} h_\alpha H_\alpha - \frac{1}{4} \sum_{\alpha<\beta} h_\alpha - h_\beta M_{\alpha\beta}^2.
\]

For \( h_\nu = \epsilon_\nu \) we obtain the Casimir element of the \( so(n + 1) \) algebra \( \Sigma_{\alpha<\beta} M_{\alpha\beta}^2 \). The total momentum \( \mathbf{J} \) takes the form

\[
J^1 = \frac{1}{4} (p \cdot p + x \cdot x), \quad J^2 = \frac{i}{4} (p \cdot p - x \cdot x), \quad J^3 = \frac{i}{2} p \cdot x,
\]

where the scalar product for the vectors \( x \) and \( p \) in \( \mathbb{C}^{n+1} \) is Euclidean. The quantities \( M \) and \( J \) form the direct sum \( so(n + 1) \oplus so(3) \) as a result of the commutation relations

\[
\{ M_{\alpha\beta}, J^i \} = 0, \quad \{ J^i, J^j \} = -\epsilon_{ijk} J^k.
\]

Therefore, in addition to the involutive set of integrals \( H_\alpha \) we can choose \( J^2 = \frac{i}{4} \Sigma_{\alpha<\beta} M_{\alpha\beta}^2 \) and \( 2(J^1 + iJ^2) = x \cdot x = c \), which gives the equation of the \( n \) sphere.

II. GENERIC ELLIPSOIDAL COORDINATES ON \( S^\ast_{\mathbb{C}} \) AND \( E^\ast_{\mathbb{C}} \)

Critical to the separation of variables on the \( n \) sphere \( S^\ast_{\mathbb{C}} \) is the system of ellipsoidal coordinates graphically pictured by the irreducible block

\[
(S_{\mathbb{C}}|e_1|e_2| \cdots |e_{n+1}),
\]

(2.1)

where in general \( e_\alpha \neq e_\beta \) for \( \alpha \neq \beta \). The separation variables are defined as zeros of the off diagonal element \( B(u) \) of the \( L \) matrix, i.e., \( B(u_j) = 0, j = 1, \ldots, n \). It follows that
\[
\sum_{\alpha=1}^{n+1} \frac{x^2_{\alpha}}{u-e_{\alpha}} = 0, \quad \text{for} \ u = u_j \quad \text{and} \quad x^2_{\alpha} = c \frac{\prod_{j=1}^{n+1} (u_j - e_{\alpha})}{\prod_{\beta \neq \alpha} (e_{\beta} - e_{\alpha})}.
\]

(2.2)

Each vector of momentum \( s_\alpha \) is associated with a cell \( e_{\alpha} \) of the block. Note that \( x^2_{\alpha} = 2(s^2_{\alpha} + i s^2_\beta) \).

For each \( u_j \) the conjugate variable \( v_j \) is defined according to

\[
v_j = -i A(u_j) = \frac{1}{2} \sum_{\alpha=1}^{n+1} \frac{x_{\alpha} p_{\alpha}}{u_j - e_{\alpha}}.
\]

(2.3)

From Eq. (1.15) one can show that the \( u_j, v_i \) satisfy the canonical relations

\[
\{u_j, u_i\} = \{v_j, v_i\} = 0, \quad \{v_j, u_i\} = \delta_{ij}.
\]

(2.4)

The change to the new variables \( v_j, u_j, c \) and \( J^3 \) is effectively the procedure of variable separation of Eq. (1.1) in ellipsoidal coordinates on the \( n \) sphere. Writing the \( L \) matrix in terms of the new variables we obtain

\[
L(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & -A(u) \end{pmatrix}, \quad B(u) = -c \frac{\prod_{j=1}^{n+1} (u - u_j)}{2 \prod_{\alpha=1}^{n+1} (u - e_{\alpha})},
\]

\[
A(u) = -\frac{2i}{c} B(u) \left( -i J^3 + \sum_{j=1}^{n} \frac{v_j}{u - u_j} \frac{\prod_{j=1}^{n+1} (u_j - e_{\gamma})}{\prod_{\alpha \neq j} (u - u_j)} \right).
\]

(2.5)

where \( A(u_j) = \lambda_j, j = 1, \ldots, n \), and \( A(u) \to (1/u) J^3 + \cdots \) as \( u \to \infty \).

To obtain \( C(u) \) we first notice that equating residues at \( e_{\alpha} \) on the right- and left-hand side of \( A(u) \) gives

\[
p_{\alpha} = \frac{2 x_{\alpha}}{c} \left( -i J^3 + \sum_{j=1}^{n} \frac{v_j}{e_{\alpha} - u_j} \frac{\prod_{j=1}^{n+1} (u_j - e_{\gamma})}{\prod_{\alpha \neq j} (u_j - u_i)} \right).
\]

This together with the expression (2.2) for \( x^2_{\alpha} \) in terms of \( u_j \) gives \( C(u) \) in the new variables.

Three other useful formulas are

\[
M_{\alpha \beta} = \frac{2 x_{\alpha} x_{\beta}}{c} (e_{\alpha} - e_{\beta}) \sum_{j=1}^{n} v_j \frac{\prod_{j=1}^{n+1} (u_j - e_{\gamma})}{\prod_{\alpha \neq j} (u_j - u_i)},
\]

(2.6)

where the hat in Eq. (2.6) means that the product terms with \( \gamma = \alpha \) and \( \gamma = \beta \) are omitted, and

\[
J^2 = -\sum_{j=1}^{n} v^2_j \frac{\prod_{j=1}^{n+1} (u_j - e_{\gamma})}{\prod_{\alpha \neq j} (u_j - u_i)} = \sum_{j=1}^{n} c v^2_j \left( \sum_{\alpha=1}^{n+1} \frac{x^2_{\alpha}}{(u_j - e_{\alpha})^2} \right)^{-1},
\]

(2.7)

These relations together with Eq. (2.2) establish the explicit connection between the two sets of \( 2n + 2 \) variables \( p_{\alpha}, x_{\alpha} \) and \( u_j, v_j, c, J^3 \). The equation for the eigenvalue curve \( \Gamma: \det(L(u) - \lambda I) = 0 \) has the form

\[
-\lambda^2 - A(u)^2 - B(u) C(u) = 0.
\]

If we put \( u = u_j \) into this equation then \( \lambda = \pm v_j \). Thus variables \( u_j \) and \( v_j \) lie on the curve \( \Gamma \).
Equations (2.8) are the separation equations for each of the $n$ degrees of freedom connected with the values of the integrals $H_\alpha$. For the sphere $e_\alpha=0$ these have the form

$$H_\alpha = \frac{1}{4} \sum_{\beta} \frac{M_{\alpha\beta}}{e_\alpha - e_\beta}, \quad \sum_\alpha H_\alpha = 0, \quad \sum_\alpha e_\alpha H_\alpha = \frac{1}{4} \sum_{\alpha<\beta} M_{\alpha\beta}^2 = S^2.$$  

The Hamilton Jacobi equation (1.1) when parametrized by these variables has the form

$$\frac{1}{4} \sum_{\alpha<\beta} M_{\alpha\beta}^2 = S^2 = - \sum_{j=1}^n \left( \frac{\partial W}{\partial u_j} \right)^2 \frac{\Pi_{\gamma=1}^{n-1} \left( u_j - e_\gamma \right)}{\Pi_{\gamma \neq j} \left( u_j - u_\gamma \right)} = E, \quad v_j = \frac{\partial W}{\partial u_j},$$  

which can be solved by the separation of variables ansatz

$$W = \sum_{j=1}^n W_j(u_j, H_1, \ldots, H_n) = \sum_{j=1}^n v_j du_j. \quad (2.10)$$

It will be convenient to employ an alternative form of the $L$ matrix. If we use the vector

$$L(u) = \sum_{\alpha=1}^{\pi+1} \frac{s_\alpha}{u-e_\alpha}, \quad L(u) = (s, L(u)), \quad \det L(u) = -L(u) \cdot L(u),$$  

where

$$s^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad s^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we see that $L$ satisfies

$$\{ L^i(u), L^j(v) \} = \frac{\epsilon_{ijk}}{u-v} (L^k(u) - L^k(v)). \quad (2.12)$$

At this point we must consider a crucial difference between the real sphere and its complex counterpart. In the case of the complex sphere the generic ellipsoidal coordinates can admit multiply degenerate forms: the restriction $e_\alpha \neq e_\beta$, for $\alpha \neq \beta$ can be lifted. The resulting coordinates can be denoted by the block form

$$(S_{NC} | e_1^1 e_2^2 \cdots e_q^q), \quad \lambda_1 + \cdots + \lambda_q = n + 1,$$

where the $\lambda_\alpha$ denote the multiplicities of $e_\alpha$. To understand how the previous analysis applies to these types of coordinates we first illustrate with an example corresponding to the coordinates with diagram

$$(S_{NC} | e_1^2 e_3^3 \cdots e_{n+1}^1).$$

In this case we write

$$L(u) = \frac{a_1 s_1}{u-e_1} + \frac{a_2 s_2}{u-e_2} + \sum_{\alpha=3}^{\pi+1} \frac{s_\alpha}{u-e_\alpha}. \quad (2.13)$$
Putting
\[ x_2 \to x'_2 + \varepsilon x'_2, \quad x_1 \to x'_1, \quad p_2 \to p'_1 + \varepsilon p'_1, \quad p_1 \to p'_1, \]
(2.14)

then in the limit as \( \varepsilon \to 0 \) we find

\[ L(u) = \sum_{\alpha=3}^{\gamma} \sum_{\alpha=3}^{\gamma} \frac{z_\alpha}{u - e_\alpha}, \]
where

\[ z_1 = \left( \frac{1}{4} (p_i^2 + x_i^2), \frac{i}{2} (p_i^2 - x_i^2), \frac{i}{2} x_i p_i \right), \]
\[ z_2 = \left( \frac{1}{2} (p_i p_j + x_i x_j), \frac{i}{2} (p_i p_j - x_i x_j), \frac{i}{2} (p_i x_j + p_j x_i) \right), \]
(2.15)
\[ z_\alpha = s_\alpha, \quad \alpha = 3, \ldots, n + 1. \]

The components of \( z_1, z_2 \) satisfy the \( E(3,3) \) algebra relations

\[ \{z'_i, z'_i\} = 0, \quad \{z'_i, z'_j\} = -\varepsilon_{ijk} x'^k, \quad \{z'_i, z'_j\} = -\varepsilon_{ijk} x'^k, \]
where the Poisson bracket (expressed in the primed coordinates) is

\[ \{F, G\} = \sum_{j,k=1}^{n+1} B_{jk} \left( \frac{\partial F}{\partial p'_j} \frac{\partial G}{\partial x'_k} - \frac{\partial F}{\partial x'_j} \frac{\partial G}{\partial p'_k} \right) \]

and

\[ B = (B_{jk}) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix} \]

It can easily be verified from these relations that relations (2.12) are again satisfied. Thus, \( \{L \cdot L(u), L \cdot L(u)\} = 0 \) so the coefficients of the various powers \( (u - e_j)^{-k} \) in the expansion of \( L \cdot L(u) \) form an involutive set of integrals of motion.

We obtain

\[ L \cdot L = \frac{2}{(u - e_1)^2} z_1 \cdot z_1 + \frac{2}{(u - e_1)^2} z_2 \cdot z_2 + \frac{1}{(u - e_1)^2} z_1 \cdot z_2 + \sum_{\alpha \geq 3} \frac{2 z_1 \cdot z_\alpha}{(u - e_1)^2 (u - e_\alpha)} + \sum_{a \geq 3} \frac{2 z_1 \cdot z_\alpha}{(u - e_1) (u - e_\alpha)} + \sum_{\alpha \neq \beta \geq 3} \frac{1}{(u - e_\alpha) (u - e_\beta)} (z_\alpha, z_\beta) \]
\[ \begin{align*}
\mathbf{L} \cdot \mathbf{L} &= \frac{1}{(u-e_i)^2} \left[ -\frac{1}{4} (p_i^2 x_j^2 - p_j^2 x_i^2)^2 + \sum_{a=3}^{n+1} \frac{1}{4(u-e_a)^2(u-e_a)} (p_i^2 x_a^2 - p_a^2 x_i^2)^2 
+ \frac{1}{2(u-e_i)(u-e_a)} (p_i^2 x_a^2 - p_a^2 x_i^2) (p_j^2 x_a^2 - p_a^2 x_j^2) 
+ \sum_{\alpha, \beta = 3, \alpha \neq \beta}^{n} \frac{1}{8(u-e_{\alpha})(u-e_{\beta})} (p_{\alpha} x_{\alpha}^2 - p_{\alpha}^2 x_i^2)^2 \right].
\end{align*} \]

In particular

\[ z_1 \cdot z_1 = 0, \quad z_1 \cdot z_2 = 0, \quad z_\alpha \cdot z_1 = (p_i^2 x_\alpha^2 - p_{\alpha}^2 x_i^2)^2 / 8, \quad z_2 \cdot z_2 = -(p_i^2 x_2^2 - p_2^2 x_i^2)^2 / 4, \]

\[ z_\alpha \cdot z_2 = (p_i^2 x_\alpha^2 - p_{\alpha}^2 x_2^2)^2 / 4. \]

To relate this to the projective coordinates on the complex \( n \) sphere we recall that under the transformation (2.14) the fundamental quadratic forms \( X = x_1 x_2 + \sum_{\alpha=3}^{n+1} x_\alpha^2 \) and \( P = p_1 p_2 + \sum_{\alpha=3}^{n+1} p_\alpha^2 \) transform to \( X = 2x_1 x_2 + \sum_{\alpha=3}^{n+1} x_\alpha^2 \), \( P = 2p_1 p_2 + \sum_{\alpha=3}^{n+1} p_\alpha^2 \). Therefore if we take the coordinates

\[ x_1 = \frac{1}{\sqrt{2}} (X_1 + iX_2), \quad x_2 = \frac{1}{\sqrt{2}} (X_1 - iX_2), \quad x_\alpha = x_\alpha, \quad \alpha = 3, \ldots, n+1, \]

\[ p_1 = \frac{1}{\sqrt{2}} (P_1 + iP_2), \quad p_2 = \frac{1}{\sqrt{2}} (P_1 - iP_2), \quad p_\alpha = p_\alpha, \quad \alpha = 3, \ldots, n+1 \]

we then can write

\[ p_i^2 x_\alpha^2 - p_{\alpha}^2 x_i^2 = \frac{1}{\sqrt{2}} (M_{1\alpha} + iM_{2\alpha}), \quad p_2 x_\alpha^2 - p_{\alpha}^2 x_2^2 = \frac{1}{\sqrt{2}} (M_{1\alpha} - iM_{2\alpha}), \]

\[ p_i^2 x_2^2 - p_{2}^2 x_i^2 = \sqrt{2} M_{12}, \quad X = \sum_{j=1}^{n+1} X_j^2, \quad P = \sum_{j=1}^{n+1} P_j^2, \]

where \( M_{jk} = X_j P_k - X_k P_j \).

The integrals of motion \( H_{\alpha}, Z \) in this case have, using partial fractions, the form

\[ \mathbf{L} \cdot \mathbf{L} = \frac{1}{(u-e_i)^2} \left[ -\frac{1}{4} (p_i^2 x_2^2 - p_2^2 x_i^2)^2 + \sum_{a=3}^{n+1} \frac{1}{4(u-e_a)^2(u-e_a)} (p_i^2 x_a^2 - p_{\alpha}^2 x_i^2)^2 \right.
+ \sum_{\alpha=3}^{n} \frac{1}{u-e_a} \left[ \frac{1}{4(e_1-e_a)^2} (p_i^2 x_a^2 - p_{\alpha}^2 x_i^2)^2 - \frac{1}{2(e_1-e_a)} (p_i^2 x_a^2 - p_{\alpha}^2 x_i^2) (p_j^2 x_a^2 - p_{\alpha}^2 x_j^2) \right.
+ \sum_{\beta=3, \beta \neq \alpha}^{n} \frac{1}{4(e_\alpha-e_{\beta})} (p_{\alpha} x_a^2 - p_{\alpha}^2 x_i^2)^2 \left( p_j^2 x_{\beta}^2 - p_{\alpha}^2 x_j^2 \right) \right] \]

\[ + \frac{1}{2(e_1-e_a)} (p_i^2 x_a^2 - p_{\alpha}^2 x_i^2) (p_j^2 x_a^2 - p_{\alpha}^2 x_j^2) \]
\begin{equation}
\sum_{\alpha=3}^{n+1} \frac{H_{\alpha}}{u-e_{\alpha}} + \frac{Z}{u-e_{1}^{2}} + \frac{Y}{u-e_{1}},
\end{equation}

where $Y = \sum_{\alpha=3}^{n+1} H_{\alpha}$.

The analysis presented so far could have been deduced from Kuznetsov's work where the double root is essentially contained in the $s$ systems of type $C$ on the real hyperboloid. Furthermore, the threefold root is contained in Kuznetsov's type $D$ systems. The question we now answer is how to use these techniques on the case of ellipsoidal coordinates corresponding to multiply degenerate roots. For this we use the limiting procedures developed by Kalnins, Miller, and Reid. We recall that the process of using these limiting procedures amounts to altering the elementary divisors of the two quadratic forms

\begin{equation}
\sum_{\alpha=1}^{n+1} \frac{x_{\alpha}}{u-e_{\alpha}} = 0, \quad \sum_{\alpha=1}^{n+1} x_{\alpha}^2 = c^2.
\end{equation}

**Theorem 1:** Let $u_i$ be the generic ellipsoidal coordinates on the $n$ sphere, viz.,

\begin{equation}
a_{\alpha}x_{\alpha} = \frac{c^2\Pi_{j=1}^{n}(u_j-e_{\alpha})}{\Pi_{\beta \neq \alpha}(e_{\beta}-e_{\alpha})}, \quad \alpha = 1, \ldots, n+1,
\end{equation}

with corresponding infinitesimal distance

\begin{equation}
ds^2 = -\frac{1}{4} \sum_{i=1}^{n} \Pi_{j \neq i}(u_i-u_j) \frac{(du_i)^2}{\Pi_{j=1}^{n+1}(u_i-e_j)}
\end{equation}

and coordinate curves

\begin{equation}
\sum_{\alpha=1}^{n+1} \frac{x_{\alpha}^2}{u-e_{j}} = 0, \quad \sum_{\alpha=1}^{n+1} x_{\alpha}^2 = c^2.
\end{equation}

Then the degenerate ellipsoidal coordinates having the infinitesimal distance

\begin{equation}
ds^2 = -\frac{1}{4} \sum_{i=1}^{n} [\Pi_{j \neq i}(u_i-u_j)] \frac{(du_i)^2}{\Pi_{j=1}^{p}(u_i-e_j)^N_j}
\end{equation}

can be obtained from generic ellipsoidal coordinates via the transformations

\begin{equation}
e_{j}^{i} \rightarrow e_{j}^{i}+e_{j-1}^{i}, \quad j = 1, \ldots, N_J, \quad J = 1, \ldots, p,
\end{equation}

\begin{equation}
p_{j}^{i} \rightarrow p_{j}^{i} + \sum_{i=2}^{N_J} I_{j+1-i}^{i} p_{j-1}^{i},
\end{equation}

where

\begin{equation}
I_{j+1-i}^{i} = \Pi_{k=2}^{i} (\epsilon_{j-1}^{k}-\epsilon_{j-2}^{k}), \quad a_{j}^{i} = 1/[\Pi_{k \neq j}(\epsilon_{j-1}^{k}-\epsilon_{j-1})], \quad k = 1, \ldots, N_J
\end{equation}

and $N_1 + \cdots + N_p = n+1$. (We require $\epsilon_0^{i} = 0$ and take the limit as $\epsilon_{h}^{i} \rightarrow 0$ for $h = 1, \ldots, N_J - 1$.) In particular
The \( \mathbf{L}(u) \) satisfies the Poisson bracket relations

\[
\{ \mathbf{L}(u), \mathbf{L}(v) \} = 0.
\]

The constants of the motion can be read off from the partial fraction decomposition of Eq. (2.25). This clearly illustrates the compactness of the \( r \) matrix formulation for the operators describing the integrable systems examined so far.

In dealing with the case of Euclidean space \( \mathbb{R}_n \mathbb{C} \) the most transparent way to proceed is as follows. The generic ellipsoidal coordinates in \( n \) dimensional complex Euclidean space are given by

\[
{x_a}^2 = \frac{\prod_{j=1}^{n} (u_j - e_{a})}{\prod_{j=1}^{n} (u_j - e_{a})}, \quad j, \alpha, \beta = 1, \ldots, n,
\]

with coordinate curves

\[
\sum_{a=1}^{n} \frac{{x_a}^2}{(u - e_a)} = 1, \quad u = u_j, \quad j = 1, \ldots, n.
\]

Proceeding in analogy with Eqs. (2.2)–(2.10) we can obtain the \( r \) matrix algebra. The corresponding \( \mathbf{L}(u) \) operator is

\[
\mathbf{L}(u) = \sum_{a=1}^{n} \frac{s_a}{u - e_a} + \frac{1}{4} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}.
\]

[Indeed the equation \( L^1(u) + iL^2(u) = 0 \) is just Eq. (2.27). Moreover it is obvious, due to the fact that expressions (2.11) satisfy (2.12), that expressions (2.28) also satisfy (2.12).] The conjugate variables \( \psi_j \) are defined by \( \psi_j = -iL^\gamma(u_j) \) and they must satisfy the canonical relations (2.4). The integrals of motion \( \hat{H}_\alpha \) are determined from...
$$L^2(u) = \sum_{\alpha=1}^{n} \frac{H_\alpha}{u - e_\alpha},$$  \hspace{1cm} (2.29)  

where

$$H_\alpha = 2 \sum_{\beta=1}^{n} \frac{(s_\alpha, s_\beta)}{e_\alpha - e_\beta} - \frac{1}{2} (s_\alpha^2 + s_\beta^2) = \frac{1}{4} \left( \sum_{\beta=1}^{n} \frac{M_{\alpha\beta}^2}{e_\alpha - e_\beta} - p_\alpha^2 \right),$$

with $\Sigma_\alpha H_\alpha = -p^2/4$. The separation equations are of the form

$$v_j^2 + L^2(u_j) = 0, \hspace{0.5cm} j = 1, \ldots, n$$

as in Eq. (2.8).

### III. CYCLIDIC COORDINATES

Associated with the separation of variables problem for the Hamilton-Jacobi equation (1.1) with $E \neq 0$ is the corresponding $E=0$ problem. In this case the equation is

$$\sum_{\alpha, \beta=1}^{n} g^{\alpha \beta} p_\alpha p_\beta = 0, \quad p = \frac{\partial W}{\partial x_\alpha}, \hspace{0.5cm} \alpha = 1, \ldots, n$$  \hspace{1cm} (3.1)  

and we consider only complex Euclidean space. While it is true that all the coordinate systems discussed for $E \neq 0$ with $E \neq 0$ will provide a separation of variables of this equation, there are coordinates that provide an additive separation of variables only when $E=0$. This is related to the fact that the $E=0$ equation admits a conformal symmetry algebra.$^{11,12,13}$

The most convenient way to proceed is to introduce hyperspherical coordinates

$$x_1 = r^2 \left( \sum_{j=1}^{n} x_j^2 - 1 \right), \quad x_2 = i r^2 \left( \sum_{j=1}^{n} x_j^2 - 1 \right), \quad x_{k+2} = 2 r x_k^2, \hspace{0.5cm} k = 1, \ldots, n$$

related to the usual Cartesian coordinates $\{x_1, \ldots, x_n\}$ according to

$$x_{k+2} = -\frac{x_k}{x_1 + i x_2}, \hspace{0.5cm} k = 1, \ldots, n, \quad \sum_{j=1}^{n+2} x_j^2 = 0.$$  

We consider the system of Sec. I in $n+2$ dimensions, where $J = 0$ and the $c_\alpha = 0$. The general (separable) cyclidic coordinates are specified by

$$\Omega = \sum_{j=1}^{n+2} \frac{x_j^2}{\lambda - e_i} = 0, \quad \Phi = \sum_{j=1}^{n+2} x_j^2 = 0, \quad \lambda = u_1, \ldots, u_n, \hspace{0.5cm} e_k \neq e_j.$$  

Furthermore

$$\sigma x_\alpha^2 = \frac{\prod_{j=1}^{n} (u_j - e_\alpha)}{\prod_{\beta \neq \alpha} (e_\beta - e_\alpha)}, \quad \sigma = -\left[ \sum_{j=1}^{n+2} e_j x_j^2 \right]^{-1}. $$

The Hamilton-Jacobi equation is given by

$$J^2 = -\sigma^2 \sum_{j=1}^{n} \left( \frac{\partial W}{\partial u_j} \right)^2 \frac{\prod_{j=1}^{n+2} (u_j - e_\alpha)}{\prod_{j \neq j} (u_j - u_i)} = 0. $$  \hspace{1cm} (3.2)
The quadratic forms \( \Omega \) and \( \Phi \) have elementary divisors \([11 \cdots 1]\), see Refs. 2, 12. It is known that the geometry of these fourth order coordinate curves is unchanged under birational transformations of the form

\[
e_k \rightarrow \frac{\alpha e_k + \beta}{\gamma e_k + \delta}, \quad u_j \rightarrow \frac{\alpha u_j + \beta}{\gamma u_j + \delta}, \quad \lambda \rightarrow \frac{\alpha \lambda + \beta}{\gamma \lambda + \delta}
\]

for \( \alpha \delta - \beta \gamma \neq 0 \) and \( k = 1, \ldots, n + 2, j = 1, \ldots, n \).

Now we can mimic the exposition given for the Gaudin magnet integrable systems using the hyperspherical coordinates and the Poisson bracket

\[
\{F(x_i, p_j), G(x_i, p_j)\}_h = \sum_{k=1}^{n+2} \left( \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial x_k} - \frac{\partial F}{\partial x_k} \frac{\partial G}{\partial p_k} \right)
\]  

(3.3)

the \( x_i \), \( p_j \) now being regarded as independent. The analysis then proceeds much as in the construction (2.2)–(2.12), but with \( n \) replaced by \( n + 1 \); indeed the coordinates are defined as zeros of the off diagonal element

\[
B(u) = \sum_{a=1}^{n+2} \frac{x_a^2}{u - e_a}
\]

subject to the restriction \( \Phi = 0 \). The crucial difference is that \( J = 0 \). The expressions for \( p_a \) and \( \mathcal{M}_{ab} \) are altered by a factor \( \sigma \)

\[
p_a = 2\sigma x_a \left( \sum_{j=1}^{n} \frac{v_j \Pi_{\gamma=1}^{n+2}(u_j - e_\gamma)}{e_\alpha - u_j \Pi_{i \neq j}(u_j - u_i)} \right),
\]

which together with \( x_a^2 \) in terms of \( u_j \) gives \( C(u) \) in the new variables. Another useful formula is

\[
\mathcal{M}_{ab} = 2\sigma x_a x_b (e_a - e_b) \sum_{j=1}^{n} \frac{v_j \Pi_{\gamma=1}^{n+2}(u_j - e_\gamma)}{\Pi_{i \neq j}(u_j - u_i)}.
\]

In fact the Poisson bracket \( \{\cdot,\cdot\}_h \) can be identified with the Poisson bracket \( \{\cdot,\cdot\} \) for functions defined in the \( n \)-dimensional space spanned by \( \{z_1, \ldots, z_n, p_1, \ldots, p_n\} \). This can readily be seen by noting that \( F(z_1, \ldots, z_n, p_1, \ldots, p_n) = F(-x_3/(x_1 + ix_2), \ldots, -x_{n+2}/(x_1 + ix_2), -(x_1 + ix_2)p_{x_3} + x_3(p_{x_1} + ip_{x_2}), \ldots, -(x_1 + ix_2)p_{x_{n+2}} + x_{n+2}(p_{x_1} + ip_{x_2})) \) from which the equality of the Poisson brackets follows identically. The infinitesimal distance for general cyclidic coordinates is

\[
\text{d}s^2 = \frac{1}{4} (x_1 + ix_2)^{-2} \left( \sum_{i=1}^{n} \Pi_{j \neq i}(u_i - u_j) \frac{(du_i)^2}{\Pi_{j=1}^{n+2}(u_i - e_j)} \right).
\]

(3.4)

We denote the coordinates defined by this graph as

\[
[\mathcal{E}_{nC}, E = 0 | e_1 | \cdots | e_{n+2}].
\]

Other coordinates of this type are those corresponding to the graphs

\[
(S_{pC} | e_1 | \cdots | e_{p+1} | \oplus (S_{qC} | f_1 | \cdots | f_{q+1}),
\]

where \( p + q = n + 2 \). These coordinates are given by
\[ \alpha \chi_k^2 = \frac{\prod_{j=1}^{p} (u_j - e_\alpha)}{\prod_{\beta \neq \alpha} (e_\beta - e_\alpha)}, \quad \alpha, \beta = 1, \ldots, p+1, k = 1, \ldots, p, \]

\[ \alpha \chi_k^2 = \frac{\prod_{j=1}^{q} (v_j - f_\alpha)}{\prod_{\beta \neq \alpha} (f_\beta - f_\alpha)}, \quad \alpha, \beta = 1, \ldots, q+1, k = p+1, \ldots, p+q+2, \quad (3.5) \]

We note here that the \( e_i \) and the \( f_j \) are pairwise distinct, for if they were not then for any of these quantities which occurred with multiplicity more than 1 a birational transformation could transform it to \( \infty \) and hence to a graph corresponding to \( E_{nC} \).

It can happen just as in the case of generic ellipsoidal integrable systems on the sphere that some of the \( e_i \) in Eq. (3.5) are equal to some of the \( f_j \) also. In this case the rules for obtaining the corresponding \( L \) matrix are summarized in the following theorem.

**Theorem 2:** Denote the generic ellipsoidal coordinates by the graph

\[ [E_{nC}|e_1| \cdots |e_n], \quad e_i \neq e_j \]

and generic cyclidic coordinates by the graph

\[ \{CE_{nC}|e_1| \cdots |e_{n+2}\}, \quad e_i \neq e_j. \]

Separable coordinates for the Hamilton Jacobi equation (3.1) corresponding to generic graphs with multiplicities

\[ [E_{nC}|e_1^n| \cdots |e_p^n], \quad n_1 + \cdots + n_p = n, \]

\[ \{CE_{nC}|e_1^n| \cdots |e_q^n\}, \quad n_1 + \cdots + n_q = n+2, \]

(Ref. 2) can be obtained via the transformations of Theorem 1 applied to the quadratic forms

\[ \Omega = \sum_{i=1}^{m} a_i \chi_i^2 = 0, \quad \Phi = \sum_{i=1}^{m} a_i \chi_i^2 = 0, \]

where \( m = n \) for generic ellipsoidal coordinates in \( E_{nC} \) and \( m = n+2 \) for the corresponding cyclidic coordinates.

For coordinates corresponding to the direct sum of two spheres, viz., Eq. (3.5), we can merely apply the result of the previous theorem for spherical coordinates to each of the pairs of quadratic forms

\[ \Omega'_1 = \sum_{\alpha=1}^{p+1} \frac{a_\alpha \chi_\alpha^2}{u - e_\alpha}, \quad \Phi'_1 = \sum_{\alpha=1}^{p+1} a_\alpha \chi_\alpha^2, \]

\[ \Omega'_2 = \sum_{\alpha=p+2}^{n+2} \frac{a_\alpha \chi_\alpha^2}{u - e_\alpha}, \quad \Phi'_2 = \sum_{\alpha=p+2}^{n+2} a_\alpha \chi_\alpha^2. \]

Indeed the freedom to subject the coordinates \( u_j \) and \( e_i \) to birational transformations in the expressions for generic cyclidic coordinates allows us to let \( e_1 \to \infty \). (In this particular case the resulting coordinates can be identified with generic elliptical coordinates on the \( n \) sphere.) The process described in Theorem 1 enables one to pass from the elementary divisors \([11 \cdots 1]\) to
\[ [N_1, N_2, \ldots, N_p], N_i \geq 1, i = 1, \ldots, p, \text{ see Refs. 2,12.} \] It is then possible (via Theorem 2) to take \( e_1 \to \infty \) in which case \([N_1, N_2, \ldots, N_p]\) corresponds to the various generic coordinate systems in Euclidean space if \( N_1 > 1 \). To illustrate how this works for the Gaudin XXX magnet model consider the quadratic forms \( \Omega, \Phi \) corresponding to elementary divisors \([21, \ldots, 1]\), viz.,

\[
\Omega = \frac{x_1^2}{(\lambda - e_1 - \lambda e_1)^2} + \frac{2x_1x_2}{\lambda - e_1} + \cdots + \frac{x_{n+2}^2}{\lambda - e_{n+2}} = 0, \quad \Phi = 2x_1x_2 + x_3^2 + \cdots + x_{n+2}^2 = 0.
\]

Putting \( \lambda \to 1/\lambda, e_i \to 1/e_i \) we find (with the use of \( \Phi = 0 \))

\[
\hat{\Omega} = \lambda^2 \frac{e_1^2x_1^2}{(e_1 - \lambda)^2} + \sum_{k=3}^{n+2} \frac{(e_1 - e_k)}{(e_1 - \lambda)(e_k - \lambda)} x_k^2
\]

or, in the limit as \( e_1 \to \infty \)

\[
x_1^2 - \sum_{k=3}^{n+2} \frac{x_k^2}{\lambda - e_k} = 0.
\]

Now if we perform the limiting procedure \((2.13)-(2.15)\) on \( L(u) \) (in \( n + 2 \) dimensions), let \( u \to 1/u, e_1 \to 1/e_1 \), let \( e_1 \to \infty \), and then evaluate at \( p_1 = 0, x_1 = 1 \) we find (up to a common factor \( u \))

\[
L(u) = \sum_{\alpha=3}^{n+2} \frac{s_\alpha}{u - e_\alpha} + \frac{1}{4} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.
\]

Here, we have made use of the fact that \( J = 0 \). This agrees with Eq. (2.28) and with the \( L(u) \) operator given by Kuznetsov, and corresponds to elementary divisors \([2,1,\ldots,1]\).

This process can be generalized for quadratic forms corresponding to elementary divisors \([N,1,\ldots,1]\), \( N > 1 \). We have that

\[
\hat{\Omega} = u^{N-2}x_1^2 + u^{N-3}2x_1x_2 + \sum_i x_i x_{N-i} = \sum_{k=N+1}^{n+2} \frac{x_k^2}{u - e_k}, \quad \hat{\Phi} = \sum_{i=N+1}^{n+2} x_i x_{N+1-i} + \sum_{k=N+1}^{n+2} x_k^2
\]

The corresponding \( L(u) \) operator is

\[
L(u) = \sum_{\alpha=N+1}^{n+2} \frac{s_\alpha}{u - e_\alpha} + s_1,
\]

where

\[
\begin{align*}
  s_1 &= \frac{1}{4}[\hat{\Omega}(0,p_2,\ldots,p_{n+2}) - \hat{\Omega}(1,x_2,\ldots,x_{n+2})], \\
  s_2^1 &= \frac{1}{4}[\hat{\Omega}(0,p_2,\ldots,p_{n+2}) + \hat{\Omega}(1,x_2,\ldots,x_{n+2})], \\
  s_3^1 &= \frac{1}{4}[S'(0,p_2,\ldots,p_{n+2}; 1, x_2, \ldots, x_{n+2})], \\
  S' &= \sum_{j=2}^{N-1} \sum_{\ell+m=j} x_\ell p_m.
\end{align*}
\]
IV. BRANCHING RULES FOR THE CONSTRUCTION OF ORTHOGONAL NONGENERIC COMPLEX INTEGRABLE SYSTEMS ON $S_{nc}$ AND $E_{nc}$

To deal with the nongeneric separable coordinate systems in $E_{nc}$ and $S_{nc}$ we must combine coordinate systems for both manifolds.² (See Refs. 13,14 for tabulations of all cases for small values of $n$.) The branching laws for graphs on these manifolds are summarized below

\[(S_{nc}| \cdots | e_i | \cdots |)_{S_{nc}} \]

\[(S_{nc}| \cdots | e_i^\lambda | \cdots |), \quad \lambda_i > 1 \]_{E_{nc}}

\[(E_{nc}| \cdots | e_i | \cdots |)_{S_{nc}} \]

\[(E_{nc}| \cdots | e_i^\lambda | \cdots |), \quad \lambda_i > 1. \]_{E_{nc}}

As an example consider the coordinate system given by the graph on $S_{4c}$

\[(S_{4c}| e_1^2 | e_5 |), \]

\[[E_{4c}| f_3 | f_4 |].\]

The coordinates for this graph can be obtained from those of the generic graph

\[(S_{4c}| e_1^2 | e_3 | e_4 | e_5 |)\]

via the limiting transformations

\[e_j = e_1 + \epsilon + \epsilon^2 f_j, \quad j = 3,4, \quad f_3 \neq f_4,\]

\[u_j = e_1 + \epsilon + \epsilon^2 \bar{u}_j, \quad j = 3,4,\]

where $\epsilon \to 0$.

The corresponding coordinates are then given implicitly by

\[-x_1^2 = \frac{(u_1 - e_1)(u_2 - e_1)}{e_5 - e_1} \quad x_2^2 = \frac{(u_1 - e_5)(u_2 - e_3)}{(e_1 - e_5)^2}, \]

\[-2x_1x_2 = \frac{(u_1 - e_1)}{e_5 - e_1} \cdot \frac{(u_2 - e_1)}{e_5 - e_1} \cdot \frac{(u_1 - e_1)(u_2 - e_1)}{(e_5 - e_1)^2} \cdot \frac{(u_1 - e_1)(u_2 - e_1)}{e_5 - e_1} \cdot \frac{(f_3 \neq f_4 \quad \bar{u}_3 \quad \bar{u}_4)}, \]

\[-x_2^2 = \frac{(u_1 - e_5)(u_2 - e_1)}{(e_5 - e_1)} \cdot \frac{(f_3 \neq f_4 \quad \bar{u}_3 \quad \bar{u}_4)}, \]

\[-x_4^2 = \frac{(u_1 - e_1)(u_2 - e_1)}{(e_5 - e_1)} \cdot \frac{(f_3 \neq f_4 \quad \bar{u}_3 \quad \bar{u}_4)}.\]
Using the nomenclature given previously we have that $L(u)$ has the two different forms

$$L_1(u) = \frac{z_1}{u-e_1} + \frac{z_2}{u-e_2} + \frac{z_3+z_4}{u-e_3} + \frac{z_5}{u-e_5}, \quad u = u_1, u_2,$$

$$L_2(u) = \frac{z_3}{u-f_3} + \frac{z_4}{u-f_4}, \quad u = \bar{u}_3, \bar{u}_4.$$ 

As usual, the separation variables are the zeros of the equation $L_1(u) + iL_2(u) = 0$ with $u = u_1, u_2$ when $\lambda = 1$ and $u = \bar{u}_3, \bar{u}_4$ when $\lambda = 2$. Each of the $L_{\lambda}$ separately satisfy Eq. (2.12) for $\lambda = 1, 2$. In addition we have

$$\{L_1 \cdot L_1(u), L_2 \cdot L_2(v)\} = 0.$$

This example illustrates how to derive the substitutions that enable the various branching laws to be obtained from a generic form. For the sphere $S_{nc}$ and generic coordinates

$$(S_{nc}) \cdots |e_0^q + e_s|e_1|\cdots|e_s| \cdots|, \quad q > s, \quad \lambda > 1,$$

if we make the substitutions

$$e_k = e_0 + e^{s-t} + e^{s-1} + e^{s-t} f_k, \quad k = 1 \cdots s,$$

$$u_j = e_0 + e^{s-t} + e^{s-1} + e^{s-t} \bar{u}_j, \quad j = 1 \cdots q,$$

where $\epsilon \to 0$, then the graph illustrated transforms into

$$(S_{nc}) \cdots |e_0^q| \cdots|,$$

$$[E_{qc}] f_1| \cdots| f_s|.$$

The remaining branching rule is obtained from a graph of the type

$$(S_{nc}) \cdots |e_0|e_1|\cdots|e_s|.$$

By means of the substitutions

$$e_j = e_0 + e(f_0 - f_j), \quad j = 1 \cdots p + 1, \quad u_k = e_0 + e(\bar{u}_k - f_0), \quad k = 1 \cdots p$$

we obtain the coordinate system coming from the graph

$$(S_{nc}) \cdots |e_0| \cdots|,$$

$$[E_{qc}] f_0| \cdots| f_p|.$$

The corresponding substitutions for the analogous Euclidean space branching rules are essentially identical. To completely specify the coordinate systems on these manifolds we need a few more substitution rules. Firstly consider the graph

$$[E_{qc}] h_1| \cdots| h_p| + [E_{qc}] f_1| \cdots| f_q|.$$

This graph can be obtained from the generic graph for $E_{(p+q)c}$ via the substitutions
The graph $[E_{qC} | h_1 | \cdots | h_q]$, $q < n$ can be obtained from $[E_{nC} | h_1 | \cdots | h_n]$ via the substitutions

$$u_i = e_1 + e^{q-n}u_i, \quad i = 1, \ldots, n,$$

$$e_i = e^{q-n-1} + e^{q-n}u_i, \quad i = p + 1, \ldots, p + q, \quad K_1 \neq K_2.$$

Given these substitution rules all the corresponding graphs for $E_{nC}$ and $S_{nC}$ can be constructed together with their corresponding $L$ matrices.

V. QUANTUM INTEGRABLE SYSTEMS ON COMPLEX CONSTANT CURVATURE SPACES AND THE QUANTUM GAUDIN MAGNET

To deal with the quantum version of this description of separation of variables we consider the Schrödinger or Helmholtz equation:

$$\mathcal{H}\Psi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y_\alpha} \left[ \sqrt{g} a^{ab} \frac{\partial}{\partial y_\beta} \right] \Psi = E\Psi, \quad (5.1)$$

where $E \neq 0$ for the moment. Separation of variables means (roughly) the solution of this equation of the form

$$\Psi = \Pi_{\alpha=1}^n \psi_\alpha(y_\alpha; h_1, \ldots, h_n), \quad (5.2)$$

where the quantum numbers $h_j$ are the eigenvalues of mutually commuting operators

$$\mathcal{A}_f = \sum_{a,b=1}^n a_{ab}^{(j)} \frac{\partial^2}{\partial y_\alpha \partial y_\beta} + \sum_{i,j=1}^n b_\beta \frac{\partial}{\partial y_\beta}, \quad f = 1, \ldots, n, \quad (5.3)$$

$$[\mathcal{A}_i, \mathcal{A}_j] = 0, \quad i, j = 1, \ldots, n, \quad \mathcal{A}_n = \mathcal{H}, \quad \mathcal{A}_j \Psi = h_j \Psi.$$

Furthermore these operators can be represented as symmetric quadratic elements in the enveloping algebra of the symmetry algebra of Eq. (5.1) in the case of $S_{nC}$ and $E_{nC}$. The standardized representations of these symmetry algebras are

$$\text{SO}(n+1) : \mathcal{M}_{a\beta} = \delta_{a\beta} - \delta_{a\beta}, \quad \mathring{\mathcal{P}}_{\alpha} = \frac{\partial}{\partial x_\alpha}, \quad \alpha, \beta = 1, \ldots, n + 1, \quad (5.4)$$

$$[\mathcal{M}_{a\beta}, \mathcal{M}_{\gamma\delta}] = \delta_{a\gamma} \mathcal{M}_{\delta\beta} + \delta_{a\beta} \mathcal{M}_{\gamma\delta} + \delta_{\beta\gamma} \mathcal{M}_{a\delta} + \delta_{\beta\delta} \mathcal{M}_{a\gamma},$$

$$E(n) : \mathcal{M}_{a\beta} = \frac{\partial}{\partial x_\gamma},$$

$$[\mathcal{M}_{a\beta}, \mathcal{P}_\gamma] = \delta_{\beta\gamma} \mathcal{P}_a - \delta_{a\gamma} \mathcal{P}_\beta, \quad [\mathcal{P}_a, \mathcal{P}_\beta] = 0, \quad (5.5)$$

where $[,]$ is the commutator bracket. Much of the analysis goes through as it did in the classical case with, of course, some critical differences. For the quantum Gaudin magnet one considers...
the sum of rank 1 Lie algebras \( \mathcal{A} = \bigoplus_{a=1}^{m} \mathfrak{so}_a(3) \) where the generators of the algebra satisfy the commutation relations (1.6) and the inner product is defined as in Sec. I. The Casimir elements of \( \mathcal{A} \) have the form

\[
(s_{a}, s_{a}) = k_{a}(k_{a} + 1),
\]

where \( k_{a} \) is a constant when the generators of \( \mathfrak{so}_a(3) \) determine an irreducible representation. The quantum Gaudin magnet is the quantum integrable Hamiltonian system on \( \mathcal{A} \) given by \( m \) commuting integrals of motion

\[
\mathcal{H}_a - 2 \sum_{\beta=1}^{m} (s_{a}, s_{\beta}) \quad e_{a} - e_{\beta}, \quad [\mathcal{H}_a, \mathcal{H}_\beta] = 0.
\]

This is the \( m \)-site \( SO(3,\mathbb{C}) \)-XXX quantum Gaudin magnet. These integrals (operators) satisfy

\[
\sum_{a=1}^{m} \mathcal{H}_a = 0, \quad \sum_{a=1}^{m} e_{a} \mathcal{H}_a = J^{2} - \sum_{a=1}^{m} k_{a}(k_{a} + 1),
\]

where \( J = \sum_{a=1}^{m} s_{a} \) is the total momentum operator. The complete set of commuting operators consists of \( \mathcal{H}_a, J^2, \) and \( J^3. \) The integrals are generated by the \( 2 \times 2 \) operator \( L(u) \) given by Eq. (1.11) understood in the operator sense. The quantum determinant is

\[
q\text{-det} L(u) = -A(u)^{2} - \frac{1}{2} [B(u), C(u)] = - \sum_{a=1}^{m} \frac{\mathcal{H}_a}{u - e_{a}} - \sum_{a=1}^{m} \frac{k_{a}(k_{a} + 1)}{(u - e_{a})^{2}},
\]

with \( L \) operator satisfying the \( r \) matrix algebra

\[
[\frac{1}{2} L(u), \frac{2}{2} L(v)] = i \frac{1}{u - v} [P, \frac{2}{2} L(v)], \quad i = \sqrt{-1},
\]

[compare with (1.13)]. The total momentum \( J \) has components

\[
J^1 = \frac{1}{4} (\hat{p} \cdot \hat{p} + \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}), \quad J^2 = \frac{i}{4} (\hat{p} \cdot \hat{p} - \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}), \quad J^3 = \frac{i}{4} (\hat{p} \cdot \hat{\mathbf{x}} + \hat{\mathbf{x}} \cdot \hat{p}).
\]

The Lie symmetries \( \mathcal{M}_{ab} \) and \( J \) form the direct sum \( \mathfrak{so}(m) \oplus \mathfrak{so}(3) \) with commutation relations

\[
[\mathcal{M}_{ab}, J^1] = 0, \quad [J^1, J^1] = i\epsilon_{jkl} J^j
\]

subject to the constraints

\[
J^2 = \frac{1}{4} \sum_{a < b} \mathcal{M}_{ab}^{2} + \frac{1}{16} m(m - 4), \quad 2(J^1 + iJ^2) = \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}.
\]

Considering again separation of variables using the coordinates of the irreducible block (2.1), the separation variables are defined, as before, as the zeros of the off diagonal elements \( B(u) \) of the \( L \) matrix. The \( q \)-determinant is the generating function of the commuting integrals of motion

\[
q\text{-det} L(u), q\text{-det} L(v) = 0.
\]

On the \( n \) sphere the algebra \( \mathcal{A} \) is realized by taking the canonical operators.

...
\( s_a^1 = \frac{1}{4} (\beta_a^2 + \dot{x}_a^2), \quad s_a^2 = \frac{i}{4} (\beta_a^2 - \dot{x}_a^2), \quad s_a^3 = \frac{i}{4} (\dot{\beta}_a, \dot{x}_a), \quad \alpha = 1, \ldots, n + 1, \) \hfill (5.14)

where \((s_\alpha, s_\alpha) = -\frac{3}{16} \). Furthermore
\[
(s_\alpha, s_\beta) = \frac{1}{8} (\mathcal{M}_{\alpha\beta}^2 + \frac{1}{2}) \hfill (5.15)
\]
and this establishes the relationship between the \(s_\alpha\) of \(\mathcal{A}\) and the \(\mathcal{M}_{\alpha\beta}\) of so\((n+1)\). The integrals transform into the family of integrals
\[
\mathcal{H}_u = \sum_{\alpha=1}^{n+1} h_\alpha \mathcal{H}_u = \frac{1}{4} \sum_{\alpha < \beta} h_\alpha - h_\beta \left( \mathcal{M}_{\alpha\beta}^2 + \frac{1}{2} \right) \hfill (5.16)
\]
and the coordinates are given by
\[
B(u_j) = 0; \quad \sum_{\alpha=1}^{n+1} \frac{x_\alpha^2}{u - e_\alpha} = 0.
\]

For each \(u_j\) variable there is defined the conjugate variable (operator)
\[
v_j = -iA(u_j) = \frac{1}{4} \sum_{\alpha=1}^{n+1} \frac{1}{u_j - e_\alpha} \{\dot{\beta}_\alpha, \dot{x}_\alpha\}. \hfill (5.17)
\]

Separation of variables is then the process of changing to the new variables \(v_j, u_j, c,\) and \(f^3\). In fact we have that
\[
\frac{1}{4} \{\dot{\beta}_\alpha, \dot{x}_\alpha\} = \frac{\dot{x}_\alpha^2}{c} \left[ -i f^3 - \sum_{j=1}^{n} \frac{1}{e_\alpha - u_j} D_j v_j \right], \quad D_j = -\frac{\prod_{i=1}^{j-1} (u_j - e_\alpha)}{\prod_{i \neq j} (u_j - u_i)}. \hfill (5.18)
\]

The separation equations can be obtained by substituting \(u = u_j\) into the \(q\)-det \(L(u)\), making the choice
\[
v_j = i \left( \frac{\partial}{\partial u_j} + \frac{1}{4} \sum_{\alpha=1}^{n+1} \frac{1}{u_j - e_\alpha} \right) \hfill (5.19)
\]
and looking for the solutions of the spectral problem \(\mathcal{H}_u \Psi = h_u \Psi\). The separation equations are then
\[
\frac{d^2}{du_j^2} \psi_j + \frac{1}{2} \left( \sum_{\alpha=1}^{n} \frac{1}{u_j - e_\alpha} \right) \frac{d}{du_j} \psi_j = \sum_{\alpha=1}^{n+1} \frac{h'_\alpha}{u_j - e_\alpha} \psi_j, \hfill (5.20)
\]
where
\[
\Psi = \prod_{j=1}^{n+1} \psi_j(u_j), \quad h'_\alpha = h_\alpha + \frac{1}{8} \sum_{\beta \neq \alpha} \frac{1}{e_\alpha - e_\beta}.
\]

These are the separation equations for the Helmholtz equation in generic ellipsoidal coordinates. Note that for this choice of \(v_j\) we have taken \((s_\alpha, s_\alpha) = -\frac{3}{16}\). The \(L\) operator is given in direct analogy with the classical case
\[
L(u) = \sum_{\alpha=1}^{n+1} \frac{s_\alpha}{u - e_\alpha}. \hfill (5.21)
\]
The treatment of generic ellipsoidal coordinates on $E_{nc}$ follows similar lines, with the $L$ operator given in this case by

$$L(u) = \sum_{\alpha=1}^{n} \frac{s_{\alpha}}{u - e_{\alpha}} + \frac{1}{4} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}.$$  (5.22)

The limiting procedures described in the classical case work also in the quantum case. All coordinate systems that were obtained in the classical case appear again. In the case of cyclidic coordinates we can adopt the same strategy as in Sec. III: we impose the conditions $f=0$ and proceed as before. The natural setting in this case is again to use hyperspherical coordinates. The total momentum has components as in Eq. (5.10). If $\tilde{p}_{\alpha}, \tilde{x}_{\alpha}$ are now vectors in hyperspherical coordinates then we can derive the standard quantum $r$ matrix algebra as above with the same formulas valid. The constraints are now $2(f^2 + i f^2) = \tilde{\mathbf{x}} \cdot \tilde{\mathbf{x}} - 0, \tilde{\mathbf{r}}^2 = n(n-4)$. Coordinates $u_{\gamma}$ and their conjugate operators $v_{\gamma}$ can be chosen as before. In particular if we make the choice

$$v_{\gamma} = \frac{\partial}{\partial u_{\gamma}} + \frac{1}{4} \sum_{\alpha=1}^{n+2} \frac{1}{u_{\gamma} - e_{\alpha}}$$  (5.23)

and look for solutions of $H \psi = 0$ of the form $\psi = e^{(n-2)/2} \Pi_{\gamma=1}^{n+2} \psi_{\gamma}(u_{\gamma})$, then we obtain the separation equations which coincide with the the equations given by Bocher,$^{12}$ viz.,

$$\frac{d^2}{du_{\gamma}^2} \psi_{\gamma} + \frac{1}{2} \sum_{\alpha=1}^{n+2} \frac{1}{u_{\gamma} - e_{\alpha}} \frac{d}{du_{\gamma}} \psi_{\gamma}$$

$$= \left( -\frac{1}{16} (n^2-4)u_{\gamma}^2 - \frac{1}{16} (2n-n^2)(\Sigma_{\alpha=1}^{n+2} e_{\alpha})u_{\gamma}^{n-1} + \Sigma_{\beta=0}^{n-2} \lambda_{\gamma} u_{\gamma}^\beta \right) \Pi_{\gamma=1}^{n+2} (u_{\gamma} - e_{\gamma})$$  for suitable $\gamma$. Note that the solutions $\Psi$ are not strictly separable in this case but are what is termed $R$ separable, i.e., separable to within a nonseparable factor $e^{(n-2)/2}$.$^{11}$

We conclude this work by pointing out that one can obtain a complete set of constants of the motion associated with an orthogonal separable coordinate system $\{u_1, ..., u_n\}$ for the Schrödinger equation (5.1) directly from Eq. (5.1) itself.$^{16,17}$ It is well known that all orthogonal separable coordinate systems for the Schrödinger equation on a Riemannian space are obtainable via the Stäckel construction, e.g., Refs. 18, 19. Thus if $\{u\}$ is a separable orthogonal coordinate system for Eq. (5.1) there exists an $n \times n$ nonsingular matrix $S = (S_{\alpha\beta}(u_{\alpha}))$ such that $\partial_{\gamma} S_{\alpha\beta} = 0$ if $\gamma \neq \alpha$, and such that the nonzero components of the contravariant metric tensor in the coordinates $\{u\}$ are $g^{\alpha\beta}(u) = T^{\alpha\beta}(u)$, $\alpha = 1, ..., n$, where $T$ is the inverse matrix to $S$

$$\sum_{\beta=1}^{n} T^{\alpha\beta}(u) S_{\beta\gamma}(u_{\beta}) = \delta_{\gamma}^\alpha.$$  (5.24)

The constants of the motion are

$$\mathcal{A}_\beta = \sum_{\alpha=1}^{n} T^{\alpha\beta}(u) \left( \partial_{\gamma}^2 + f_{\alpha}(u_{\alpha}) \partial_{\gamma} u_{\alpha} \right), \quad \mathcal{A}_n = \mathcal{H}.$$  (5.25)

Here, $f_{\alpha}(u_{\alpha}) = \partial_{u_{\alpha}} \ln(\sqrt{g_{\alpha\alpha}})$. (The fact that
\[ \frac{\partial}{\partial u^\beta} \ln(\sqrt{g} g^{\alpha \beta}) = 0, \quad \text{for} \quad \alpha \neq \beta \]

follows for any space of constant curvature by noting that it is equivalent to the statement that the off diagonal elements of the Ricci tensor must vanish for an orthogonal coordinate system:  
\[ R_{\alpha \beta} = 0, \quad \alpha \neq \beta. \]

One can show that \[ [\mathcal{A}_\alpha, \mathcal{A}_\beta] = 0. \]

Furthermore, if \( \Psi = \prod_{\alpha=1}^n \psi_{\alpha}(y_\alpha; h_1, ..., h_n) \) satisfies the separation equations

\[ \frac{\partial^2}{\partial u_\alpha^2} \Psi + f_\alpha(u_\alpha) \frac{\partial}{\partial u_\alpha} \Psi = \sum_{\beta=1}^n S_{\alpha \beta}(u_\alpha) h_\beta \Psi, \quad \alpha = 1, ..., n \]  

then

\[ \mathcal{A}_\beta \Psi = h_\beta \Psi, \quad \beta = 1, ..., n. \]  

Now fix \( \gamma, \) \( 1 < \gamma < n, \) and denote by \( y \) the coordinate choice

\[ y = (u_1, ..., u_{\gamma-1}, \tau, u_{\gamma+1}, ..., u_n), \]

where \( \tau \) is a parameter. We see from Eq. (5.26) that if \( \Psi \) satisfies the separation equations then

\[ \psi_{\gamma}^{-1}(\partial_{u_{\gamma}}^2 \psi_{\gamma} + f_\gamma(u_\gamma) \partial_{u_{\gamma}} \psi_{\gamma}) \big|_{u_{\gamma}=\tau} \Psi(u) = \sum_{\beta=1}^n S_{\gamma \beta}(\tau) \mathcal{A}_\beta \Psi(u). \]  

On the other hand, from Eq. (5.1) we have

\[ \psi_{\gamma}^{-1}(\partial_{u_{\gamma}}^2 \psi_{\gamma} + f_\gamma(u_\gamma) \partial_{u_{\gamma}} \psi_{\gamma}) \big|_{u_{\gamma}=\tau} \Psi(u) = \frac{1}{g^{\gamma \gamma}(y)} (\mathcal{H}^\gamma(u) - \mathcal{H}^\gamma(y)) \Psi(u), \]

where

\[ \mathcal{H}^\gamma(y) = \frac{1}{\sqrt{g(y)}} \sum_{\alpha \neq \gamma} \partial_{u_\alpha}(\sqrt{g(y)} g^{\alpha \gamma}(y) \partial_{u_\alpha}). \]

This suggests the operator identity

\[ \frac{1}{g^{\gamma \gamma}(y)} (\mathcal{H}^\gamma(u) - \mathcal{H}^\gamma(y)) = \sum_{\beta=1}^n S_{\gamma \beta}(\tau) \mathcal{A}_\beta, \]  

i.e., that the left-hand side of Eq. (5.29) is a one-parameter family of constants of the motion and that as \( \tau \) runs over a range of values and \( \gamma = 1, ..., n \) the full space of constants of motion associated with this separable system is spanned.\(^{16,17}\)

Now Eq. (5.29) is equivalent to the conditions

\[ \frac{T^{\alpha \alpha}(u)}{T^{\gamma \gamma}(y)} - \frac{T^{\alpha \alpha}(y)}{T^{\gamma \gamma}(y)} (1 - \delta_\gamma^\alpha) = \sum_{\beta=1}^n S_{\gamma \beta}(\tau) T^{\beta \alpha}(u), \]

and these conditions are easily seen to follow from Eq. (5.24) and from

\[ \sum_{\beta=1, \beta \neq \gamma}^n T^{\beta \gamma}(y) S_{\beta \delta}(u_\beta) + T^{\gamma \gamma}(y) S_{\gamma \delta}(\tau) = \delta_\delta^\gamma. \]

Similarly, for the classical case an expression for the Hamiltonian analogous to Eq. (5.29) generates the constants of the motion.\(^{16,21}\)
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