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Nonlinear Hydrodynamics and Ripplon Interactions on Superfluid Helium

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*We discuss the elementary processes which determine the lifetime of ripplons at low temperatures, considering interaction of ripplons with ripplons and phonons. The interaction Hamiltonian is constructed from the nonlinear hydrodynamic equations of motion using the approach of Khalatnikov and Pokrovsky [Physica A **166**, 288 (1990)]. The rate of spontaneous ripplon decay is calculated. Interaction of ripplons with thermally excited ripplons and phonons is considered. In the limit of low temperature the ripplon subsystem decouples from the phonon subsystem, which is promising for the observation of surface second sound.*

1. INTRODUCTION

The quantum hydrodynamic description of the low-energy excitations of superfluid liquid helium is well established (see, e.g., the book by Khalatnikov¹). The basic idea is that the macroscopic equations of hydrodynamics provide a complete description of these elementary excitations, whose wavelength is much larger than microscopic length scales such as the distance between the atoms of the liquid. The interactions between the excitations also reside within this phenomenological framework.

This picture also holds for the excitations of the free surface. The leading temperature dependence of the surface tension, for example, is a direct consequence of the thermodynamics of an ideal gas of capillary waves (ripplons).² This gas can be considered as a normal component of superfluid helium and can, in principle, support entropy waves, or surface second sound.³ The formalism of quantum hydrodynamics for the case of a compressible liquid with a free surface, describing both ripplons and phonons, was first developed by Saam.⁴

The purpose of this paper is to discuss the elementary processes which determine the lifetime of ripples at low temperatures, considering interaction of ripples with ripples and phonons. The interaction between ripples is important for the problem of the establishment of thermal equilibrium within the ripple subsystem and between the ripple and phonon subsystems of liquid helium. It is also essential for the discussion of the phenomenon of surface second sound. Knowledge of ripple lifetimes is also germane to problems involving electron or ion crystals, or physisorbed atoms (H, ^3He), at the free surface of liquid helium.

We work in the limit of long-wavelength excitations and adopt the equations of hydrodynamics as defining the elementary excitations and their interactions. We derive the Hamiltonian for the interaction of ripples with ripples and phonons from the hydrodynamic equations of motion of a compressible liquid using the approach of Khalatnikov and Pokrovsky.⁵

Elementary processes involving ripples were first considered by Saam.^{6,4} The damping of capillary waves by thermal ripples was calculated by Gould and Wong.⁷ Recently, Jones *et al.*⁸ treated damping of both sub-thermal and thermal (i.e., quantum) capillary waves, verifying a disturbing discrepancy between theory and experiments. In a crucial experiment, Roche *et al.*⁹ showed convincingly that the lifetime of capillary waves was much longer than predicted by theory and was not limited by interaction with ripples, but with phonons. Using the Lagrangian formulation of fluid mechanics, Roche *et al.*¹⁰ rederived the quantum hydrodynamics of Saam⁴ and obtained the cubic terms in the Hamiltonian responsible for the dominant interactions of ripples and phonons. Although our approach is rather different from that of Roche *et al.*, our results agree with theirs where comparison is possible.

Very recently, the problems of the fate of phonons incident on the surface of liquid helium and the damping of surface second sound have been considered by Baddar and Edwards,¹¹ who made use of the interaction Hamiltonian due to Roche *et al.* We will be able to compare our results with theirs at several points of contact.

Our primary interest here is the interaction of *thermal* ripples with ripples and phonons. There are three kinematically-allowed processes which involve ripples. The first of these is the "three-ripple process" ($r \rightleftharpoons r + r$). The vertex for this process which we obtain using the method of Khalatnikov and Pokrovsky (KP) is consistent with an elegant result of fluid dynamics.^{12,13} It also agrees with the Hamiltonian derived by Roche *et al.*¹⁰ The second process is the annihilation of a phonon at the free surface with the creation of a ripple pair ($P \rightleftharpoons r + r$). We call this the Faraday process by virtue of the strong analogy to the Faraday instability of the

free surface on a vertically oscillated liquid. The last process is the Cherenkov process ($P \Rightarrow P + r$), which is responsible for the damping of sub-thermal ripples. In the case of thermal ripples, as we shall see, the Cherenkov process is actually the least important one, and the three-ripple process is dominant.

We begin, in Sec. 2, by reviewing the paper by KP,⁵ which deals with the (classical) Hamiltonian description of the hydrodynamics of a compressible liquid with a free surface, making minor corrections. We start with the well-known hydrodynamic equations of motion including free-surface boundary conditions. These equations are linearized and solved to give the normal modes of the liquid; this was first done by Saam.⁴ The harmonic Hamiltonian, which dictates the linear evolution, is exhibited. Finally, the anharmonic terms in the Hamiltonian which describe resonant nonlinear processes are obtained from the nonlinear part of the equations of motion using the interpretation of the Hamiltonian as the generator of the motion.

In Sec. 3 we specialize to the case of a weakly-compressible deep liquid, which is appropriate when discussing thermal ripples on liquid helium at dilution refrigerator temperatures. The role of the Gibbs surface mass density is briefly discussed. It is interesting that the leading processes are of a universal character—the equilibrium values of the liquid density, speed of sound, and surface tension, required to determine the linear behaviour, also determine the nonlinear interactions.

In Sec. 4 we calculate the width (inverse lifetime) of ripples due to the three-ripple process. First we consider the spontaneous decay of a ripple which occurs even at absolute zero. Next we treat finite temperatures and calculate the width of a ripple in a background of thermally-excited ripples. We discuss the simple scaling of this width with ripple momentum and temperature. Specialization to the case of sub-thermal ripples is made.

Section 5 deals with the Faraday process. This is the dominant energy transfer process between the ripple and phonon subsystems of liquid helium, as pointed out by Kagan *et al.*¹⁴ The ripple width due to this process is calculated and simple scaling with momentum and temperature is found.

In Sec. 6 we calculate the ripple width due to the Cherenkov process. Numerical results are given for ripples as a function of their momentum (including thermal ripples). In the limiting case of sub-thermal ripples, we reproduce the analytical result of Roche *et al.*⁹ for the damping of capillary waves by phonon scattering.

In our numerical results, we use the following ground-state properties of liquid helium: density $\rho = 145 \text{ kg m}^{-3}$, speed of sound $c = 238 \text{ m s}^{-1}$, and surface tension $\alpha = 378 \text{ } \mu\text{J m}^{-2}$.

A preliminary report of this work appeared as Ref. 15.

2. HYDRODYNAMICS

2.1. Equations of Motion

We have in mind superfluid liquid helium at zero temperature, but our treatment is appropriate to any simple compressible liquid for which entropy and dissipation are negligible. Since sound and capillary-gravity waves are the only fluid motions of interest here, we assume irrotational flow. Use cartesian coordinates (x, y, z) ; we will write $\mathbf{R} = (\mathbf{r}, z)$ to denote a point in space, with $\mathbf{r} = (x, y)$ the transverse coordinate. In equilibrium the liquid occupies the volume between a hard floor at $z=0$ and the free surface at $z=h$.

The velocity of the fluid is given by the vector field $\mathbf{v}(\mathbf{R})$. For irrotational flow $\nabla \times \mathbf{v} = 0$ and the velocity is the gradient of the velocity potential $\varphi(\mathbf{R})$, i.e., $\mathbf{v} = \nabla\varphi$. The first equation of motion of the fluid is

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2} \nabla\varphi \cdot \nabla\varphi + \mu(\rho) + gz = 0 \quad (1)$$

where $\mu = \partial\varepsilon/\partial\rho$ is the chemical potential function, or Gibbs function, per unit mass (ε is the elastic energy density), and g is the acceleration due to gravity. If desired, it is possible to express gradients in chemical potential in terms of pressure gradients by using the following expression for the pressure P :

$$P = \frac{\partial(\rho\varepsilon)}{\partial\rho} = \rho\mu - \varepsilon \quad (2)$$

so that $\nabla P = \rho \nabla\mu$. Let ρ_0 denote the equilibrium density (we will write this ρ when there is no danger of confusion). For $g=0$, without loss of generality we can take $\varepsilon(\rho_0)=0$, and, furthermore, since ρ_0 is the equilibrium density, $\mu(\rho_0)=0$. With $g>0$, there is a slight z -dependence to ρ_0 , so that $\mu(\rho_0) + gz = 0$; the variation of ρ_0 with z is otherwise unimportant. Ultimately we consider $g=0$ but for now we will keep the general expressions.

The second equation of motion is the so-called equation of continuity,

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho \nabla\varphi) = 0 \quad (3)$$

which reflects conservation of matter.

The free surface of the liquid is given by $z = h + \zeta(\mathbf{r}, t)$. At the free surface we have two boundary conditions, one kinematical and one dynamical. The kinematical boundary condition is that there is no flux of the fluid through the free surface,

$$\frac{\partial \zeta}{\partial t} + \left(\nabla \zeta \cdot \nabla \varphi - \frac{\partial \varphi}{\partial z} \right) \Big|_{z=h+\zeta} = 0 \quad (4)$$

Note that $\nabla \zeta$ is a vector in the xy plane. The dynamical boundary condition is that the liquid pressure at the surface is given by the Laplace expression,

$$P|_{z=h+\zeta} = -\nabla \cdot \left(\frac{\alpha \nabla \zeta}{\sqrt{1 + \nabla \zeta \cdot \nabla \zeta}} \right) \quad (5)$$

where α is the surface tension. We consider here the simplest possible surface: α is independent of the curvature of the surface, and there is no surface mass. Using (1) and (2) the dynamical boundary condition (5) can be written

$$\left(\frac{\partial \varphi}{\partial t} + \frac{1}{2} \nabla \varphi \cdot \nabla \varphi + \frac{\varepsilon(\rho)}{\rho} \right) \Big|_{z=h+\zeta} + g\zeta - \frac{1}{\rho} \nabla \cdot \left(\frac{\alpha \nabla \zeta}{\sqrt{1 + \nabla \zeta \cdot \nabla \zeta}} \right) = 0 \quad (6)$$

At the bottom of the liquid ($z=0$) we have a hard-floor boundary condition: $\partial \varphi / \partial z|_{z=0} = 0$.

2.2. Normal Modes

To construct the normal modes of the liquid (sound waves in the bulk and capillary-gravity waves at the surface) we begin by linearizing the equations of motion (1), (3), (4), and (6). We will use $q(\mathbf{R}, t)$ to denote the fluctuation in density, so that $\rho = \rho_0 + q$. Equation (1) becomes

$$\frac{\partial \varphi}{\partial t} + \frac{c^2}{\rho_0} q = 0 \quad (7)$$

where c is the speed of sound in the liquid, given by $c^2 = \rho \partial \mu / \partial \rho = \partial P / \partial \rho$. Following KP, we allow c to vary with density, although this is not important for processes involving capillary waves. The continuity equation (3) linearizes to

$$\frac{\partial q}{\partial t} + \rho_0 \Delta \varphi = 0 \quad (8)$$

where Δ is the (three-dimensional) Laplacian. In the kinematical boundary condition (4) only the z component of velocity survives linearization:

$$\left. \frac{\partial \zeta}{\partial t} - \frac{\partial \varphi}{\partial z} \right|_{z=h} = 0 \quad (9)$$

The dynamical boundary condition (6) reduces to

$$\left. \frac{\partial \varphi}{\partial t} \right|_{z=h} + g\zeta - \frac{\alpha}{\rho_0} \Delta \zeta = 0 \quad (10)$$

Here Δ turns out to be the two-dimensional Laplacian, since ζ is not a function of z .

In the usual way, we solve the linearized equations of motion by seeking solutions of the form $\Psi(\mathbf{R}, t) = \Psi(\mathbf{R}) e^{-i\omega t}$, where

$$\Psi(\mathbf{R}) = \begin{bmatrix} \varphi(\mathbf{R}) \\ \varrho(\mathbf{R}) \\ \zeta(\mathbf{r}) \end{bmatrix} = \begin{bmatrix} \varphi(z) \\ \varrho(z) \\ \bar{\zeta} \end{bmatrix} e^{i\mathbf{p} \cdot \mathbf{r}} = \begin{bmatrix} \bar{\varphi} \cos(qz) \\ \bar{\varrho} \cos(qz) \\ \bar{\zeta} \end{bmatrix} e^{i\mathbf{p} \cdot \mathbf{r}} \quad (11)$$

Here $\bar{\varphi}$, $\bar{\varrho}$, and $\bar{\zeta}$ are constants, to be determined. The chosen form automatically satisfies the hard-floor boundary condition and corresponds to excitations of definite transverse (i.e., in the xy plane) wavevector $\mathbf{p}$. Equations (7) and (8) give the following relations for the oscillation frequency ω :

$$\omega = ck, \quad k \equiv \sqrt{p^2 + q^2} \quad (12)$$

while the surface boundary conditions (9) and (10) require

$$\omega^2 = -\Lambda(p) q \tan(qh) \quad (13)$$

with an effective wavenumber-dependent gravitational acceleration

$$\Lambda(p) \equiv g + \frac{\alpha}{\rho_0} p^2 \quad (14)$$

The natural, or normal, modes of oscillation of the liquid have frequency ω real. This occurs if q is real (bulk wave, phonon, denoted P) or if q is imaginary (surface wave, ripplon, denoted r). For given p the condition (13) is satisfied by an infinite sequence of real $q > 0$ and a unique imaginary $q = i\kappa$, where κ is the ripplon attenuation depth. For any given p the surface wave is the lowest frequency mode. For the typical situation in

which the compressibility of the liquid is small we have $\kappa \simeq p$ and the usual dispersion relation for capillary-gravity waves on an incompressible fluid: $\omega^2 = p\Lambda(p) \tanh(ph)$.

The linearized equations also give us the following relations between the coefficients $\bar{q}$, $\bar{\varphi}$, and $\bar{\zeta}$:

$$\frac{\bar{q}}{\rho_0} = \frac{ik}{c} \bar{\varphi} \quad (15)$$

$$\bar{\zeta} = \frac{q}{i\omega} \sin(qh) \bar{\varphi} = \frac{i\omega}{\Lambda} \cos(qh) \bar{\varphi} \quad (16)$$

The normal modes are enumerated by mode index λ or, equivalently, by the three-component wavevector $\mathbf{k}$ with xy components $\mathbf{p}$ and z component q . We denote by $\Psi_\lambda(\mathbf{R})$ the normal-mode field pattern for mode λ .

The above orthonormal modes solving the linearized equations for a compressible liquid with a free surface were obtained by Saam,⁴ who showed that they form a complete set.

2.3. Harmonic Hamiltonian

As pointed out explicitly by KP, the harmonic Hamiltonian H_0 induces a natural inner product on the vector space defined by the normal modes. The harmonic Hamiltonian $H_0(\Psi) = (\Psi, \Psi)$, where the inner product

$$\begin{aligned} (\Psi_1, \Psi_2) = & \int dx dy \int_0^h dz \left(\frac{\rho}{2} \nabla \varphi_1^* \cdot \nabla \varphi_2 + \frac{c^2}{2\rho} \varrho_1^* \varrho_2 \right) \\ & + \int dx dy \left(\frac{\alpha}{2} \nabla \zeta_1^* \cdot \nabla \zeta_2 + \frac{\rho g}{2} \zeta_1^* \zeta_2 \right) \end{aligned} \quad (17)$$

For discussion of nonlinear processes it is best to work with the "dynamically normalized" eigenvectors Ψ_λ satisfying

$$(\Psi_\lambda, \Psi_\nu) = \frac{\omega_\lambda \delta_{\lambda, \nu}}{2} \quad (18)$$

Then, with an arbitrary field $\Psi(\mathbf{R})$ represented by a superposition

$$\Psi(\mathbf{R}) = \sum_{\lambda} a_{\lambda} \Psi_{\lambda} + \text{c.c.} \quad (19)$$

where the a_λ are (time-dependent) complex coefficients, we find that the energy in the harmonic approximation has the familiar diagonal form

$$H_0(\Psi) = \sum_{\lambda} \omega_{\lambda} a_{\lambda}^* a_{\lambda} \quad (20)$$

Using (17) and the relations (15) and (16), our chosen normalization condition (18) can be written

$$|\bar{\varphi}_{\lambda}|^{-2} = \frac{S\rho}{\omega_{\lambda}} \left[k_{\lambda}^2 h + \frac{p_{\lambda}^2 - q_{\lambda}^2}{2q_{\lambda}} \sin(2q_{\lambda}h) \right] \quad (21)$$

where S is the area of the fluid on the xy plane. Let us consider the form of $\bar{\varphi}_{\lambda}$ in the case of "deep water": the liquid is assumed to be only weakly compressible ($\Lambda(p_{\lambda}) \ll p_{\lambda} c^2$) and very deep ($p_{\lambda} h \gg 1$). For deep-water phonons we can take

$$\bar{\varphi}_{\lambda} = i\sigma_{\lambda} \left(\frac{\omega_{\lambda}}{V\rho k_{\lambda}^2} \right)^{1/2} \quad (22)$$

where $V = Sh$ is the volume of the liquid and the sign $\sigma_{\lambda} = \pm 1$ can be chosen as we like; to make later expressions simple we take it to be the sign of $\sin(q_{\lambda}h)$. For deep-water ripplons we can take

$$\bar{\varphi}_{\lambda} = -i \left(\frac{\omega_{\lambda}}{2S\rho p_{\lambda}} \right)^{1/2} 2e^{-p_{\lambda}h} \quad (23)$$

The factor $-i$ is chosen so that the amplitudes a_{λ} (annihilation operators in the quantized picture) will correspond to the definition of Saam.⁴

The expressions of Saam for the the surface displacement and velocity fields, in our notation and for the "deep water" case, can be found in the appendix.

Another form of the inner product which is useful is obtained from (17) using Green's formula to rewrite the z integral and integrating over x and y to display the transverse momentum conservation:

$$\begin{aligned} (\Psi_{\lambda}, \Psi_{\nu}) = & \frac{S\rho}{2} \left\{ \int_0^h \left[-\Delta \varphi_{\lambda}^*(z) \varphi_{\nu}(z) + \frac{c^2}{\rho^2} \varrho_{\lambda}^*(z) \varrho_{\nu}(z) \right] dz \right. \\ & \left. + \Lambda(p_{\lambda}) \zeta_{\lambda}^* \zeta_{\nu} + \frac{\partial \varphi_{\lambda}^*}{\partial z} \varphi_{\nu} \right|_{z=h} \Bigg\} \delta_{\mathbf{p}_{\lambda}, \mathbf{p}_{\nu}} \end{aligned} \quad (24)$$

2.4. Resonant Nonlinear Processes

To move beyond the harmonic approximation, KP adopt the mode amplitudes a_λ and a_λ^* as canonical variables. Poisson brackets provide the symplectic structure: $\{a_\lambda, a_\nu^*\} = -i\delta_{\lambda,\nu}$ and $\{a_\lambda, a_\nu\} = \{a_\lambda^*, a_\nu^*\} = 0$. The evolution of the mode amplitudes is given by Hamilton's equation of motion, $\dot{a}_\lambda = \{a_\lambda, H\}$. In the harmonic approximation $\dot{a}_\lambda = \{a_\lambda, H_0\} = -i\omega_\lambda a_\lambda$. Of interest here is the three-wave process $X_1 \rightleftharpoons X_2 + X_3$, with conservation of momentum parallel to the surface and conservation of energy:

$$\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3 \quad (25)$$

$$\omega_1 = \omega_2 + \omega_3 \quad (26)$$

Each wave can be a sound wave (P) in the bulk or a capillary-gravity wave (r) on the surface. Kinematically, the following reactions are allowed: $r \rightleftharpoons r + r$, $P \rightleftharpoons r + r$, and $P \rightleftharpoons P + r$. The term in the Hamiltonian, to be determined, which is responsible for the three-wave process is

$$\delta H = V_{1,23} a_1^* a_2 a_3 + \text{c.c.} \quad (27)$$

The approach taken by KP, and which we follow here, is to obtain the vertex $V_{1,23}$ using the interpretation of the Hamiltonian as the generator of the motion. Our discussion is inspired by Phillips.¹⁶

The system of nonlinear equations of motion (1), (3), (4), and (6) can be written $\mathcal{L}\Psi + \mathcal{N}(\Psi) = 0$, where $\mathcal{L}$ is a linear operator on the field Ψ , and $\mathcal{N}$ is a nonlinear function of Ψ . To find the cubic anharmonic vertex it is sufficient to substitute a linear solution of the form (19); corrections to (19) will enter only at higher order. We have

$$\sum_{\lambda} (\dot{a}_\lambda + i\omega_\lambda a_\lambda) \Psi_\lambda = -\Xi_{23} a_2 a_3 + \dots \quad (28)$$

where the left-hand side, $\mathcal{L}\Psi$, describes an ensemble of harmonic oscillators and the right-hand side, $-\mathcal{N}(\Psi)$, is a weak driving force. Imagine that we have only excited modes 2 and 3. From the nonlinear part $\mathcal{N}$, we extract the "coefficient" Ξ_{23} of $a_2 a_3$ as indicated. Taking the inner product with Ψ_1 of both sides of (28) we can obtain the equation of motion for the amplitude of any desired mode 1:

$$\dot{a}_1 + i\omega_1 a_1 = -\frac{(\Psi_1, \Xi_{23})}{(\Psi_1, \Psi_1)} a_2 a_3 + \dots \quad (29)$$

Since Ξ_{23} will vary in space as $\exp[i(\mathbf{p}_2 + \mathbf{p}_3) \cdot \mathbf{r}]$, mode 1 will be excited only if momentum conservation (25) is satisfied. Furthermore, there will be

no appreciable effect unless the time dependence $e^{-i(\omega_2 + \omega_3)t}$ is resonant with mode 1, in accordance with our idea of energy conservation (26) in elementary processes. Nonresonant terms in the rhs of (28) and (29), denoted by ellipsis, are not of interest at the moment.

To find the anharmonic vertex, we notice that $\dot{a}_1 = \{a_1, H_0 + \delta H\} = -i\omega_1 a_1 - iV_{1,23}a_2a_3$. Comparing this equation with (29), we see that $V_{1,23}$ is given by

$$V_{1,23} = -i \frac{(\Psi_1, \Xi_{23})}{(\Psi_1, \Psi_1)} \quad (30)$$

It is instructive to separate out the fast time dependence from the mode amplitudes and write

$$a_\lambda = \bar{a}_\lambda e^{-i\omega_\lambda t}, \quad \dot{\bar{a}}_\lambda e^{-i\omega_\lambda t} = \dot{a}_\lambda + i\omega_\lambda a_\lambda \quad (31)$$

In the limit of small amplitudes (harmonic approximation) we have $\dot{\bar{a}}_\lambda = 0$. The time evolution of $\bar{a}_\lambda$ is due to nonlinearity. For a resonant process

$$\dot{\bar{a}}_1 = -iV_{1,23}\bar{a}_2\bar{a}_3 \quad (32)$$

Considering the nonlinear behaviour of the hydrodynamic equations, we obtain a set of second-order equations for the motion for $\bar{a}_1$. The first equation, from (1), is $\dot{\bar{a}}_1\varphi_1(\mathbf{R}) = -\bar{a}_2\bar{a}_3A$, where the coefficient A , which arises from kinetic energy and departures from Hooke's law, is given by

$$A = \nabla\varphi_2 \cdot \nabla\varphi_3 + \frac{\partial^2\mu}{\partial\rho^2}\varrho_2\varrho_3 \quad (33)$$

The continuity equation (3) gives $\dot{\bar{a}}_1\varrho_1(\mathbf{R}) = -\bar{a}_2\bar{a}_3B$, where the transport of density fluctuations leads to

$$B = \nabla \cdot (\varrho_2 \cdot \nabla\varphi_3 + \varrho_3 \cdot \nabla\varphi_2) \quad (34)$$

The kinematic boundary condition (4) gives $\dot{\bar{a}}_1\zeta_1(\mathbf{r}) = -\bar{a}_2\bar{a}_3C$, where nonlinearity due to tipping and displacement of the surface is described by

$$C = \left(\nabla\zeta_2 \cdot \nabla\varphi_3 + \nabla\zeta_3 \cdot \nabla\varphi_2 - \zeta_2 \frac{\partial^2\varphi_3}{\partial z^2} - \zeta_3 \frac{\partial^2\varphi_2}{\partial z^2} \right) \Big|_{z=h} \quad (35)$$

The dynamical boundary condition (6) is the most involved. Nonlinearity arises from the kinetic energy term and from the variation of the energy per unit mass, ε/ρ , with density. The Laplace pressure term contributes second-order nonlinearity only through the variation of density in the prefactor.

One last complication is that the time derivative of the velocity potential should be evaluated at the free surface. To account for this we make use of the fact that to second order in small quantities

$$\left. \frac{\partial \varphi(t)}{\partial t} \right|_{z=h+\zeta(t)} = \left. \frac{\partial \varphi(t)}{\partial t} \right|_{z=h} + \zeta(t) \left. \frac{\partial}{\partial t} \frac{\partial \varphi(t)}{\partial z} \right|_{z=h} \quad (36)$$

Hence the second-order equation of motion arising from the dynamical boundary condition is $\ddot{a}_1 \varphi_1(\mathbf{r}, h) = -\bar{a}_2 \bar{a}_3 D$, with

$$D = \left(\nabla \varphi_2 \cdot \nabla \varphi_3 + \frac{c^2}{\rho^2} \varrho_2 \varrho_3 - \frac{\alpha}{\rho^2} (\varrho_2 \Delta \zeta_3 + \varrho_3 \Delta \zeta_2) \right) \Big|_{z=h} - (\omega_2^2 + \omega_3^2) \zeta_2 \zeta_3 \quad (37)$$

The above equations of motion can be consolidated as

$$\ddot{\mathbf{a}} \begin{pmatrix} \varphi_1(\mathbf{R}) \\ \varrho_1(\mathbf{R}) \\ \zeta_1(\mathbf{r}) \\ \varphi_1(\mathbf{r}, h) \end{pmatrix} = -\bar{a}_2 \bar{a}_3 \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \quad (38)$$

We see that the quantities A , B , C , and D can be considered the components of the vector Ξ_{23} in the representation associated with the form (24) of the inner product. It is now straightforward to apply (30) to obtain the anharmonic vertex. The result is

$$\begin{aligned} V_{1,23} = & -\frac{iS\rho}{\omega_1} \left\{ -\int_0^h \Delta \varphi_1^* \left(\nabla \varphi_2 \cdot \nabla \varphi_3 + \frac{\partial^2 \mu}{\partial \rho^2} \varrho_2 \varrho_3 \right) dz \right. \\ & + \frac{c^2}{\rho^2} \int_0^h \varrho_1^* \nabla \cdot (\varrho_2 \nabla \varphi_3 + \varrho_3 \nabla \varphi_2) dz \\ & + \Lambda(p_1) \zeta_1^* \left(\nabla \zeta_2 \cdot \nabla \varphi_3 + \nabla \zeta_3 \cdot \nabla \varphi_2 - \zeta_2 \frac{\partial^2 \varphi_3}{\partial z^2} - \zeta_3 \frac{\partial^2 \varphi_2}{\partial z^2} \right) \Big|_{z=h} \\ & + \frac{\partial \varphi_1^*(h)}{\partial z} \left[\left(\nabla \varphi_2 \cdot \nabla \varphi_3 + \frac{c^2}{\rho^2} \varrho_2 \varrho_3 - \frac{\alpha}{\rho^2} (\varrho_2 \Delta \zeta_3 + \varrho_3 \Delta \zeta_2) \right) \Big|_{z=h} \right. \\ & \left. \left. - (\omega_2^2 + \omega_3^2) \zeta_2 \zeta_3 \right] \right\} \quad (39) \end{aligned}$$

Now we use (11) and the relations between the amplitudes, (15) and (16), to write the anharmonic vertex as

$$\begin{aligned}
 V_{1,23} = & -\frac{iS\rho}{\omega_1} \bar{\varphi}_1^* \bar{\varphi}_2 \bar{\varphi}_3 \left\{ 2k_1^2 q_2 q_3 \int_0^h \cos(q_1 z) \sin(q_2 z) \sin(q_3 z) dz \right. \\
 & - 2k_1^2 (\mathbf{p}_2 \cdot \mathbf{p}_3 + Gk_2 k_3) \int_0^h \cos(q_1 z) \cos(q_2 z) \cos(q_3 z) dz \\
 & + q_1 \left[\mathbf{p}_2 \cdot \mathbf{p}_3 + k_2 k_3 \left(\frac{g}{\Lambda_2} + \frac{g}{\Lambda_3} - 1 \right) \right] \sin(q_1 h) \cos(q_2 h) \cos(q_3 h) \\
 & + \frac{\omega_1 q_2}{\omega_2} (\mathbf{p}_2 \cdot \mathbf{p}_3 - q_3^2) \cos(q_1 h) \sin(q_2 h) \cos(q_3 h) \\
 & + \frac{\omega_1 q_3}{\omega_3} (\mathbf{p}_2 \cdot \mathbf{p}_3 - q_2^2) \cos(q_1 h) \cos(q_2 h) \sin(q_3 h) \\
 & \left. - q_1 q_2 q_3 \left(1 + \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_2} \right) \sin(q_1 h) \sin(q_2 h) \sin(q_3 h) \right\} \quad (40)
 \end{aligned}$$

This expression remedies several errors and omissions in the result of Ref. 5. The Grüneisen constant $G \equiv (p/c) \partial c / \partial p$; the value $G = 2.84$ has been measured for helium.¹⁷ Factors of ω/Λ have been made implicit using (13).

3. APPROXIMATIONS

We will apply (40) to kinematically-allowed processes involving thermal ripples on bulk liquid helium ($h \rightarrow \infty$) in the interesting temperature range between 1 mK and several hundred mK. For this purpose it can easily be verified that we may set $g = 0$; the energy of a ripplon at the cross-over between capillary and gravity wave behavior is $\hbar(\rho g^3/\alpha)^{1/4} \simeq 1$ nK.

Our main approximation is to consider that the compressibility of the liquid is small, in the sense that $\Lambda(p)/pc^2$ is a small quantity. Essentially, we treat the liquid helium as "deep water." The characteristic ripplon wavenumber p_c associated with the compressibility of the liquid (i.e., with the fact that the speed of sound is not infinite) is given by $p_c = \rho c^2/\alpha$. Substituting material properties of liquid helium, we obtain $p_c = 23 \text{ nm}^{-1}$; the energy of such a wave, $\hbar p c$, is about 40 K. Effects which are not included in our simple hydrodynamic model, most importantly the Gibbs surface mass, will become important at much lower momentum and energy scales. We will restrict ourselves to wavenumbers below 5 nm^{-1} (ripplon energy below 4 K in the

hydrodynamic model), where the observed ripplon dispersion relation is still reasonably close to the relation given by ordinary hydrodynamics.²¹

We take the bulk limit $h \rightarrow \infty$ in order to simplify the discussion. In principle, the anharmonic vertex given by (40) is applicable to helium films of finite thickness. However, if h is finite then the problem is significantly complicated. First, the phonon spectrum is strongly modified. Consider a film of thickness $h = 10$ nm. Then the lowest phonon mode has frequency $\pi c/2h \simeq 0.3$ K. At lower temperatures, the freezing of phonon modes becomes important. Second, the ripplon velocity field will penetrate to the substrate, requiring the use of the hyperbolic eigenfunctions, unless the ripplon energy is larger than several times 10 mK (so that ph is sensibly larger than unity). A third complication, which should be taken into account when calculating transport properties, is the substrate roughness, which will spoil transverse momentum conservation for phonons and for ripplons whose velocity field penetrates to the substrate.¹⁸ Finally, on a helium film the Van der Waals attraction of the substrate gives rise to a large effective gravity. However, this is less than 10^7 times earth's gravity for a "typical" saturated film, which implies that capillarity dominates over "gravity" if the ripplon energy is above 1 mK. Thus, even on a film we can set $g = 0$ when discussing thermal ripplons.

For the kinematically allowed processes we may assume that excitation 2 or 3 is a ripplon when calculating the vertex $V_{1,2,3}$. Therefore, considering the limit of a weakly compressible liquid we may neglect the $k_2 k_3$ terms in (40) compared to the $\mathbf{p}_2 \cdot \mathbf{p}_3$ terms. Furthermore, it turns out that the terms involving integrals can also be neglected.

The neglected terms are of the same magnitude as terms that would arise if the Gibbs surface mass was taken into account. For example, the variation of surface tension with chemical potential, and hence with density, will give a contribution to (37) whose importance relative to the compression terms is given by $(\rho/\alpha) \partial\alpha/\partial\rho = -\Gamma c^2/\alpha$, where Γ is the Gibbs surface mass density. Taking $\Gamma = \rho\delta$ with the Tolman length $\delta = -0.336$ Å (see, e.g., Edwards and Saam¹⁹), we have $\Gamma c^2/\alpha \simeq -0.7$, of order unity. Hence, to improve upon our present approximations will require the development of a theory of nonlinear processes which includes effects associated with the Gibbs surface mass.

4. THREE-RIPPLON PROCESS

4.1. Anharmonic Vertex

The interaction of three surface waves is described by the nonlinear vertex (40). For three gravity-capillary waves on "deep water" the leading

interaction is, not surprisingly, just that which is present on an incompressible liquid. The integral terms in (40), which represent compression contributions, are smaller than the other terms by a factor $(k/p)^2$. Using the replacements $q \rightarrow ip$, $\cos(qh) \rightarrow (1/2) e^{\rho h}$, and $\sin(qh) \rightarrow (i/2) e^{\rho h}$, the relations (15) and (16), and the normalization (23) we obtain the vertex

$$\begin{aligned}
 V_{1,23} = & -\frac{1}{\omega_1(8S\rho)^{1/2}} \left(\frac{\omega_1 \omega_2 \omega_3}{p_1 p_2 p_3} \right)^{1/2} \\
 & \times \left\{ p_1 p_2 p_3 \left(1 + \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_2} - \frac{\omega_1 p_3}{\omega_2 p_1} - \frac{\omega_1 p_2}{\omega_3 p_1} \right) \right. \\
 & \left. - \omega_1 \mathbf{p}_2 \cdot \mathbf{p}_3 \left(\frac{p_1}{\omega_1} + \frac{p_2}{\omega_2} + \frac{p_3}{\omega_3} \right) \right\} \quad (41)
 \end{aligned}$$

This result agrees with the Hamiltonian of Roche *et al.*¹⁰ We shall now discuss how this vertex can be obtained from a known result in fluid mechanics.

The theory of the resonant interaction of a triad of capillary-gravity waves on an incompressible liquid was developed by McGoldrick¹² and Simmons.¹³ The surface displacement of the liquid is written as a sum of three waves,

$$\zeta(\mathbf{r}, t) = \sum_{j=1}^3 \mathcal{A}_j \cos(\tilde{\mathbf{p}}_j \cdot \mathbf{r} - \tilde{\omega}_j t - \eta_j) \quad (42)$$

where we denote by $\tilde{\omega}_j$ frequencies which may be positive or negative. Resonant interaction, conserving momentum and energy, occurs when $\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2 + \tilde{\mathbf{p}}_3 = 0$ and $\tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3 = 0$. Succinct equations for the slow variation of the amplitudes $\mathcal{A}_j(\mathbf{r}, t)$ and phases $\eta_j(\mathbf{r}, t)$, obtained by Simmons,¹³ are

$$\frac{1}{2} \{ \partial \mathcal{A}_j^2 / \partial t + \nabla \cdot (\mathbf{c}_{gj} \mathcal{A}_j^2) \} = \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 (J / \tilde{c}_{pj}) \sin \eta \quad (43)$$

$$\mathcal{A}_j^2 \{ \partial \eta_j / \partial t + \nabla \cdot (\mathbf{c}_{gj} \eta_j) \} = \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 (J / \tilde{c}_{pj}) \cos \eta \quad (44)$$

where $\tilde{c}_{pj} = \tilde{\omega}_j / p_j$ is the (signed) phase speed, $\mathbf{c}_{gj} = \partial \tilde{\omega}_j / \partial \tilde{\mathbf{p}}_j$ is the group velocity, $\eta = \eta_1 + \eta_2 + \eta_3$ is the "total phase," and $J = K_{12} + K_{23} + K_{31}$ with

$$K_{jk} = -\frac{1}{4} \tilde{\omega}_j \tilde{\omega}_k \left(1 + \frac{\tilde{\mathbf{p}}_j \cdot \tilde{\mathbf{p}}_k}{p_j p_k} \right) \quad (45)$$

To make contact with our own results, let us write the surface displacement in terms of complex amplitudes A_j ,

$$\zeta(\mathbf{r}, t) = \frac{1}{2} \sum_{j=1}^3 A_j \exp[i(\mathbf{p}_j \cdot \mathbf{r} - \omega_j t)] + \text{c.c.} \quad (46)$$

with frequencies ω_j positive. Clearly, the signs of the frequencies $\tilde{\omega}_j$ in (42) are arbitrary, since cosine is an even function. Useful expressions are obtained by the following choice: $\tilde{\omega}_j = \varepsilon_j \omega_j$ and, hence, $\tilde{\mathbf{p}}_j = \varepsilon_j \mathbf{p}_j$, where $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_3 = -1$. In terms of the complex amplitudes $A_j = \mathcal{A}_j e^{i\varepsilon_j \eta_j}$, the evolution equations (43) and (44) may be written for plane waves as

$$(\tilde{\omega}_1/p_1) \dot{A}_1 = iJA_2 A_3 \quad (47)$$

$$(\tilde{\omega}_2/p_2) \dot{A}_2 = iJA_3 A_1 \quad (48)$$

$$(\tilde{\omega}_3/p_3) \dot{A}_3 = iJA_1 A_2 \quad (49)$$

The connection between the complex amplitudes of surface displacements A_j and the mode amplitudes a_j used earlier is determined by the normalization of the eigenmodes, given by (23). The connection is

$$A_\lambda = 2(p_\lambda/2S\rho\omega_\lambda)^{1/2} \bar{a}_\lambda \quad (50)$$

with $\bar{a}_\lambda$ the slowly-varying part of a_λ , given by (31). Comparison of (47) with (32) gives the following expression for the anharmonic vertex:

$$V_{1,23} = -\left(\frac{2}{S\rho} \cdot \frac{p_1 p_2 p_3}{\omega_1 \omega_2 \omega_3}\right)^{1/2} J \quad (51)$$

This vertex is the same as (41) obtained using the method of KP.

4.2. Spontaneous Ripplon Decay

The dominant nonlinear process for ripplons with energy $\hbar\omega \gg k_B T$ is their spontaneous decay into two ripplons. This process was first considered by Saam.⁶ We can calculate the decay assuming that the liquid temperature T is strictly zero. The "reaction" is $r_1 \rightarrow r_2 + r_3$. In the initial state there is one ripplon r_1 , with momentum $\mathbf{p}_1$, on an otherwise quiet liquid. The possible final states are labeled by the momentum $\mathbf{p}_2$ of one of the pair of ripplons; the momentum $\mathbf{p}_3 = \mathbf{p}_1 - \mathbf{p}_2$ is determined by momentum conservation.

The treatment so far has been within the framework of classical hydrodynamics. The correspondence with quantum mechanics is well known. The Poisson bracket $\{f, g\}$ corresponds to the commutator $[f, g]$ divided by $i\hbar$. The field amplitudes a_λ are the annihilation operators times $\hbar^{1/2}$. From now on we simply interpret the a_λ as annihilation operators, multiplying the harmonic Hamiltonian H_0 by $\hbar$ and the anharmonic vertex $V_{1,23}$ by $\hbar^{3/2}$.

The matrix element of the interaction part of the Hamiltonian between initial and final states is just the anharmonic vertex $V_{1,23}$. The inverse lifetime, or width, Γ_0 , of ripplon r_1 due to spontaneous decay is given by Fermi's golden rule:

$$\Gamma_0 = \frac{2\pi}{\hbar^2} \sum_{\mathbf{p}_2} |V_{1,23}|^2 \delta(\omega_2 + \omega_3 - \omega_1) \quad (52)$$

The sum over final states is converted to an integral with the replacement

$$\sum_{\mathbf{p}} \rightarrow \frac{S}{(2\pi)^2} \int d^2p \quad (53)$$

Let θ be the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$. The integration over $\mathbf{p}_2$ is restricted to $\theta > 0$ in order to avoid double counting of final states. The energy conservation delta function may be written as

$$\delta(\omega_2 + \omega_3 - \omega_1) = \left| \frac{p_1 p_2}{p_3} \frac{\partial \omega_3}{\partial p_3} \sin \theta \right|^{-1} \delta(\theta - \theta_0) \quad (54)$$

where the angle θ_0 at which energy conservation is satisfied for a given p_1 and p_2 can be found from the dispersion relation.

It is convenient to use reduced units (with quantities denoted by a hat) such that the initial ripplon r_1 has momentum $(1, 0)$ and unit energy. The relations for momentum and energy conservation are then $\hat{\mathbf{p}}_2 + \hat{\mathbf{p}}_3 = (1, 0)$ and $\hat{\omega}_2 + \hat{\omega}_3 = 1$. Introduce the reduced vertex v by $V_{1,23} = -(\hbar^3 \omega_1 p_1^3 / 8S\rho)^{1/2} v$. In terms of the emission angle θ we have $\hat{\mathbf{p}}_2 = (\hat{p}_2 \cos \theta, \hat{p}_2 \sin \theta)$ and, by momentum conservation, $\hat{\mathbf{p}}_3 = (1 - \hat{p}_2 \cos \theta, -\hat{p}_2 \sin \theta)$. Thus, the reduced vertex is completely specified by the magnitude $\hat{p}_2$ and $\cos \theta$. Considering pure capillary waves ($g=0$) we set $\hat{\omega}_2 = \hat{p}_2^{3/2}$ and $\hat{\omega}_3 = \hat{p}_3^{3/2}$. Then we find that $\cos \theta_0 = (1 + \hat{p}_2^2 - \hat{p}_3^2) / 2\hat{p}_2$.

Using (54) to do the integration over θ we obtain the following expression for the ripplon decay rate:

$$\Gamma_0 = \frac{\hbar p_1^5}{\rho} \int_0^1 dx F(x) \quad (55)$$

Fig. 1. Spectral distribution of ripplons produced by the spontaneous decay process $r_1 \rightarrow r_2 + r_3$, given by $F(x)/\int_0^1 dx' F(x')$. The argument $x = \omega_2/\omega_1$.

where $F(x)$ describes the spectral distribution of ripplons produced by spontaneous ripplon decay. This distribution is plotted in Fig. 1. The function $F(x)$ is given by

$$F(x) = \frac{1}{36\pi} \left(\frac{1}{x} - 1 \right)^{1/3} \frac{v^2}{\sin \theta_0} \quad (56)$$

The argument of F is the ratio ω_2/ω_1 , which determines the reduced vertex v and the emission angle θ_0 . Typically, the ripplon decays into two ripplons of approximately equal energy. This is reminiscent of experimental results in classical fluid mechanics.²⁰ The evaluation of the integral in (55) is done numerically. The final result is that the spontaneous decay rate of a ripplon of wavenumber p is given by

$$\Gamma_0 = 0.0001493 \frac{\hbar p^5}{\rho} \quad (57)$$

This decay rate was also found by Baddar and Edwards.¹¹

It is interesting to calculate the quality factor of a ripplon due to the decay process: $Q = \omega/\Gamma_0 = (42 \text{ nm}^{-1}/p)^{7/2}$. The simple hydrodynamic picture we are using should be applicable as long as the ripplon momentum does not exceed about 5 nm^{-1} (ripplon energy up to about 4 K). In this range of momentum we find $Q > 1700$. Using neutron scattering, Lauter *et al.*²¹ were able to observe well-defined ripplons with momenta as high as $p = 14 \text{ nm}^{-1}$ and energies approaching 8 K (the roton gap). The experimental dispersion relation has been accurately modeled within a generalized

hydrodynamic approach²² and within a density-functional approach.²³ Perhaps one of these approaches could form the basis for treatment of non-linear effects, such as ripplon decay, at momenta higher than 5 nm^{-1} .

4.3. Finite Temperature

Let us now consider the three-riplon process at finite temperature. We will calculate the width of a ripplon which is introduced into the existing gas of thermally excited riplons. There are two channels. Our given ripplon can decay into two riplons or can be absorbed by a thermally excited ripplon. Either channel can be stimulated by the existing riplons. To calculate the width we apply Fermi's golden rule averaged over the thermal distribution of states of the liquid.

For the decay process we let r_1 be our specified ripplon and calculate the rate of $r_1 \rightleftharpoons r_2 + r_3$. To the rhs of Eq. (52) we should append the factor $[1 + N(\omega_2) + N(\omega_3)]$, where $N(\omega) = [\exp(\hbar\omega/k_B T) - 1]^{-1}$ is the Bose distribution function. Here we will calculate the stimulated contribution to the decay, given by

$$\Gamma_1 = \frac{2\pi}{\hbar^2} \sum_{\mathbf{p}_2} |V_{1,23}|^2 \delta(\omega_2 + \omega_3 - \omega_1) [N(\omega_2) + N(\omega_3)] \quad (58)$$

The calculation of Γ_1 proceeds in a manner entirely analogous to the calculation of the spontaneous decay rate. We obtain the following expression for the stimulated rate of decay of a ripplon of wave number p and frequency ω :

$$\Gamma_1 = \frac{\hbar p^5}{\rho} \int_0^1 dx F(x) [N(\omega(1-x)) + N(\omega x)] \quad (59)$$

As pointed out by Baddar and Edwards,¹¹ the ripplon rates are only a function of the zero-temperature (spontaneous) rate and the reduced temperature $k_B T/\hbar\omega$.

For the absorption process it is convenient to take r_2 as our specified ripplon and calculate its width due to $r_2 + r_3 \rightleftharpoons r_1$. The width is given by

$$\Gamma_2 = \frac{2\pi}{\hbar^2} \sum_{\mathbf{p}_1} |V_{1,23}|^2 \delta(\omega_2 + \omega_3 - \omega_1) [N(\omega_3) - N(\omega_1)] \quad (60)$$

The sum over final states includes all directions of $\mathbf{p}_1$. The one-dimensional integral which results is

$$\Gamma_2 = \frac{\hbar p^5}{\rho} \int_1^\infty dx \, 2x^{8/3} F(1/x) [N(\omega(x-1)) - N(\omega x)] \quad (61)$$

The factor 2 arises because the absorbing ripplon can approach from the left or from the right.

The calculated widths are plotted as a function of ripplon momentum p in Fig. 2. The rates Γ_0 , Γ_1 , and Γ_2 , and hence their sum, obey the same scaling with ripplon momentum and reduced temperature. Defining the thermal momentum p_{th} by $\hbar\omega_{th} = \hbar(\alpha/\rho)^{1/2} p_{th}^{3/2} = k_B T$ we find that $\Gamma_0(p, T) = T^{10/3} \gamma_0(p/p_{th})$, $\Gamma_1(p, T) = T^{10/3} \gamma_1(p/p_{th})$, and $\Gamma_2(p, T) = T^{10/3} \gamma_2(p/p_{th})$, where γ_0 , γ_1 , and γ_2 are functions only of the reduced momentum p/p_{th} . This means that we only need to calculate these rates as a function of momentum for one temperature, the results for other temperatures following from the scaling relations.

Fig. 2. Partial contributions to the ripplon width *vs* momentum for $T = 0.1$ K. Denote by r_* the ripplon of interest. Dashed curves: Γ_0 is the rate of spontaneous decay ($r_* \rightarrow r + r$), Γ_1 is the decay rate stimulated by thermal ripples ($r_* \rightleftharpoons r + r$), and Γ_2 is the rate of absorption by thermal ripples ($r_* + r \rightleftharpoons r$). Solid curves: $\Gamma_0 + \Gamma_1 + \Gamma_2$ is the net rate due to the three-ripple process, Γ_3 is the ripplon width due to $r_* + r \rightleftharpoons P$ (Faraday process), and Γ_4 is the width due to $r_* + P \rightleftharpoons P$ (Cherenkov process). Also plotted is the ripplon frequency ω . The thermal ripplon momentum $p_{th} \approx 0.40 \text{ nm}^{-1}$.

For riplons with $\hbar\omega \gg k_B T$ only the decay channel is important. At high enough momentum the stimulation of this process by existing riplons is unimportant and we recover the spontaneous decay calculated previously.

For sub-thermal riplons, $\hbar\omega \ll k_B T$, the width obeys a simple power law. For the decay channel, the ripplon modes are all classically excited, and we can expand the Bose function as $N(\omega) \simeq k_B T/\hbar\omega \gg 1$. Numerical evaluation of the resulting integral yields $\Gamma_1/\Gamma_0 \simeq 5k_B T/\hbar\omega$. For the absorption channel it so happens that the ripplon interacts primarily with riplons whose momentum is not very much higher than that its own. The anharmonic vertex increases rapidly with the momentum of the most energetic ripplon involved in the process, but not quickly enough to outweigh the decrease in the occupation number imbalance $N(\omega_3) - N(\omega_1)$. Hence all modes are classically excited and numerical calculation yields $\Gamma_2/\Gamma_0 \simeq 158k_B T/\hbar\omega$. The absorption channel is the more important because it involves riplons of highest momentum. For both decay and absorption channels, the width depends linearly on the temperature and varies with the ripplon momentum as $p^5/\omega \propto p^{7/2}$. Roche *et al.* found the same damping rate for sub-thermal riplons as we do; Baddar and Edwards appear to have overlooked the factor 2 which appears in the absorption channel.

5. FARADAY PROCESS

5.1. Anharmonic Vertex

The decay of a phonon into two riplons, $P_1 \rightleftharpoons r_2 + r_3$, we call the Faraday process by virtue of its connection to the Faraday instability of the free surface of a liquid subjected to vertical oscillation.²⁴ To find the nonlinear vertex describing this process, we can make use of the fact that for a weakly compressible liquid the two riplons have almost equal and opposite momenta, $\mathbf{p}_3 \simeq -\mathbf{p}_2$, and hence $\omega_3 \simeq \omega_2 \simeq \omega_1/2$. Then, from (40) we find that the anharmonic vertex for this process is given by

$$V_{1,23} = - \left(\frac{\hbar^3 \omega_1}{V\rho} \right)^{1/2} \frac{q_1}{k_1} p_2 \quad (62)$$

The third and last terms in (40) are responsible for this vertex because $\cos(q_1 h) \simeq 0$. The vertex (62) coincides with the vertex used by Reynolds *et al.*²⁵ and Baddar and Edwards.¹¹

The connection with the Faraday instability is clear. The phonon wavelength is much longer than that of the riplons, hence the phonon

is felt at the surface as an oscillating vertical acceleration of the liquid. This is precisely the experimental arrangement for the observation of the Faraday instability.

5.2. Ripplon Width

The union of ripples of essentially opposite momentum to form phonons is the primary mechanism for transfer of thermal energy from the ripplon to the phonon subsystem of liquid helium at low temperature.^{14,25} Hence, we can anticipate that, of processes involving the phonon modes of the liquid, this process gives the dominant contribution to the width of thermal ripples.

Using the finite temperature version of Fermi's golden rule, the rate of absorption of our ripplon (here it is best to take r_3) is

$$\Gamma_3 = \frac{2\pi}{\hbar^2} \sum_{\mathbf{k}_1} |V_{1,23}|^2 \delta(\omega_2 + \omega_3 - \omega_1) [N(\omega_2) - N(\omega_1)] \quad (63)$$

The vertex $V_{1,23}$ is given by (62). The sum over q quantum numbers of phonon states is converted to an integral with the replacement

$$\sum_q \rightarrow \frac{h}{\pi} \int_{q>0} dq \quad (64)$$

The transverse momentum sum transforms as (53), as before. We find the following simple expression for the ripplon width ($p \equiv p_3$, $\omega \equiv \omega_3$):

$$\Gamma_3 = \frac{4\hbar\omega^3 p^2}{3\pi\rho c^3} [N(\omega) - N(2\omega)] \quad (65)$$

For a ripplon with the thermal momentum p_{th} this rate varies as $\omega^3 p^2 \propto T^{13/3}$. For arbitrary momentum a general scaling relation also exists for this process: $\Gamma_3(p, T) = T^{13/3} \gamma_3(p/p_{\text{th}})$.

For sub-thermal ripples we can expand the Bose distribution functions in (65) to obtain

$$\Gamma_3 = \frac{2\hbar\omega^3 p^2}{3\pi\rho c^3} \frac{k_B T}{\hbar\omega} \quad (66)$$

This rate varies with ripplon momentum as $\omega^2 p^2 \propto p^5$, and is smaller than the rate for $r+r \rightarrow r$ by a factor of about $(2\omega/pc)^3 \ll 1$ (ripplon energy being much smaller than the energy of a phonon of the same momentum).

5.3. Connection with Geophysics

Interestingly, the same physics also occurs on a grand scale in the ocean. Through the nonlinearity of the hydrodynamic equations of motion, colliding gravity waves generate infrasound which is detected as microseisms (continuous earth tremors with period 1–10 seconds). This mechanism for the generation of microseisms was put forward and analyzed by Longuet-Higgins.²⁶

6. CHERENKOV PROCESS

6.1. Anharmonic Vertex

As demonstrated experimentally (and shown theoretically) by Roche *et al.*,⁹ the dominant attenuation mechanism for sub-thermal ripplons is the Cherenkov process $P_1 \rightleftharpoons P_2 + r_3$. As KP pointed out, this process is quasi-elastic: the frequency of the phonon is essentially unchanged although its direction can be changed substantially. To absorb a ripplon of frequency ω_3 , the phonon frequency ω_2 should satisfy $\omega_2 > (cp_3 - \omega_3)/2$ and hence $\omega_2 \gg \omega_3$. The leading contribution to the vertex comes from the last term in (40) because of the large factor ω_2/ω_3 and because $\cos(q_1 h) \simeq \cos(q_2 h) \simeq 0$. Our approximate vertex is thus

$$V_{1,23} = \frac{q_1 q_2 \omega_2}{V k_1} \left(\frac{\hbar^3 S p_3}{2 \rho k_1 k_2 \omega_3} \right)^{1/2} \quad (67)$$

6.2. Ripplon Width

Fermi's golden rule for the width of ripplon r_3 due to the process $P_1 \rightleftharpoons P_2 + r_3$ is

$$\Gamma_4 = \frac{2\pi}{\hbar^2} \sum_{\mathbf{k}_2} \sum_{q_1} |V_{1,23}|^2 \delta(\omega_2 + \omega_3 - \omega_1) [N(\omega_2) - N(\omega_1)] \quad (68)$$

with the anharmonic vertex given by (67). The sum is over the transverse momentum $\mathbf{p}_2$ of the absorbing phonon and the z quantum numbers q_2 and q_1 of the two phonons—the specified momentum of the ripplon, $\mathbf{p}_3$, then determines the transverse momentum $\mathbf{p}_1$ and hence the state of phonon P_1 . The sums are converted to integrals using (53) and (64). The q_1 integral is done immediately to eliminate the energy-conservation delta

function. We arrive at the following expression for the damping rate of the ripplon, Γ_4 , due to this process:

$$\Gamma_4 = \frac{\hbar c p_3}{4\pi^3 \rho \omega_3} \int d^3 k_2 \frac{q_1 q_2^2 k_2}{k_1^2} [N(\omega_2) - N(\omega_1)] \quad (69)$$

where $k_1 = k_2 + k_3$ and $q_1 = \sqrt{k_1^2 - |\mathbf{p}_2 - \mathbf{p}_3|^2}$. The three-dimensional integral (with $q_2 > 0$) was done numerically in order to arrive at the curve displayed in Fig. 2. Like the Faraday process, the Cherenkov process becomes less important, relative to the three-riplon process, at lower temperatures as the compressibility of the liquid plays an ever smaller role.

An important special case, which yields simple expressions, is the damping of sub-thermal ripples or ripples ($\hbar\omega_3 \ll k_B T$). Expanding the difference $N(\omega_2) - N(\omega_1)$ for small $\omega_3 = \omega_1 - \omega_2$, straightforward calculation gives ($p \equiv p_3$)

$$\Gamma'_4 = \frac{\pi^2}{30} \frac{\hbar p}{\rho} \left(\frac{k_B T}{\hbar c} \right)^4 \quad (70)$$

The dominant contribution comes from thermal phonons, as can be appreciated by the similarity between this expression and the Stefan-Boltzmann law. This expression is applicable as long as the mean free path of thermal phonons in liquid helium is much longer than the wavelength of the ripplon under consideration, a condition easily satisfied in practice.

The attenuation of the amplitude of the ripple can be described either by a complex frequency $\omega_c = \omega - i\omega''$ or by a complex wavenumber $p_c = p + ip''$. The connection between these two pictures is easily made by considering the propagation of a wave packet for a time t , during which it moves a distance $x = v_g t$, where v_g is the group velocity. In the complex-frequency picture, the wave packet amplitude attenuates by $\exp(-\omega'' t)$ and in the complex-wavenumber picture the attenuation is $\exp(-p'' x)$. The two pictures should have the same attenuation, so $\omega'' = v_g p''$. On the other hand $\omega = v_p p$, where v_p is the phase velocity. Furthermore, the inverse lifetime Γ (for energy) is equal to $2\omega''$. Hence the wavenumber loss tangent is $p''/p = (v_p/2v_g) \Gamma/\omega = \Gamma/3\omega$. Using this relation we find that (70) is in agreement with the result of Ref. 9.

7. CONCLUSIONS

Following Khalatnikov and Pokrovsky⁵ we have derived the cubic terms in the hydrodynamic Hamiltonian which are responsible for the elementary processes in which ripples interact with each other or with

phonons. For the important case of three-ripplon interaction (one ripplon in, two riplons out), we obtain a vertex which is consistent with well-known equations from fluid mechanics.

We used the obtained anharmonic Hamiltonian to calculate rates of ripplon decay and thermalization of the ripplon subsystem of liquid helium and the interaction of riplons with phonons. It is interesting that the thermalization rate within the ripplon subsystem of liquid helium (due to the three-ripplon process) scales with temperature as $T^{10/3}$, while the thermalization between the ripplon and phonon subsystems (due to the Faraday process) scales more quickly as $T^{13/3}$. Thus, at low temperatures the ripplon subsystem is decoupled from the phonon subsystem on the timescale of its own internal thermalization. Already at 0.1 K the Faraday process is slower than the three-ripplon process. Hence, there is reason to be optimistic that the phenomenon of surface second sound will be observable on pure liquid helium at temperatures attainable using a dilution refrigerator. A firm prediction, however, will require a kinetic treatment of the ripplon gas, as pointed out by Baddar and Edwards.¹¹

APPENDIX—FIELD OPERATORS

This appendix gives the field operators of Saam⁴ for surface displacement and velocity in our notation, in the Schrödinger picture, and for the “deep water” limit. Note that in this appendix $z=0$ corresponds to the equilibrium free surface. For riplons we introduce $\xi_p = (\hbar p / 2S\rho\omega)^{1/2}$, and write

$$\zeta(\mathbf{r}) = \sum_{\mathbf{p}} \xi_p e^{i\mathbf{p} \cdot \mathbf{r}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) \quad (71)$$

$$\mathbf{v}(\mathbf{r}, z) = \sum_{\mathbf{p}} (-i\omega\xi_p) \left(\frac{i\mathbf{p} + \mathbf{e}_z p}{p} \right) e^{i\mathbf{p} \cdot \mathbf{r}} e^{pz} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) \quad (72)$$

For phonons the “deep water” limit means that we can neglect the phase shift of the phonon at the surface. Introducing $\xi_{p,q} = (\hbar c^2 / V\rho\omega)^{1/2} p/\omega$, we have

$$\zeta(\mathbf{r}) = \sum_{\mathbf{p}, q} \xi_{p,q} e^{i\mathbf{p} \cdot \mathbf{r}} (a_{\mathbf{p}, q} + a_{-\mathbf{p}, q}^\dagger) \quad (73)$$

$$\mathbf{v}(\mathbf{r}, z) = \sum_{\mathbf{p}, q} (-i\omega\xi_{p,q}) \left[\frac{i\mathbf{p} \sin(qz) + \mathbf{e}_z p \cos(qz)}{p} \right] e^{i\mathbf{p} \cdot \mathbf{r}} (a_{\mathbf{p}, q} - a_{-\mathbf{p}, q}^\dagger) \quad (74)$$

As is well-known, the fluid motion in the ripplon velocity field describes circles. The phonon velocity field, on the other hand, is directed vertically

near the surface and has an antinode at the surface because the surface presents essentially a zero-pressure boundary condition.

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