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Convergence of modified approximants associated with orthogonal rational functions

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Abstract

Let \{\xi_k\} be a sequence in the unit disk \(D = \{ z \in \mathbb{C} : |z| < 1 \}\) consisting of a finite number of points cyclically repeated, and let \(\mathcal{S}_z\) be the linear space generated by the functions \(B_k(z) = \prod_{i=0}^k (1 - \xi_i z)\). Let \(\{\phi_n(z)\}\) be orthogonal rational functions obtained from the sequence \(\{B_n(z)\}\) (orthogonalization with respect to a given functional on \(\mathcal{S}_z\)), and let \(\{\psi_n(z)\}\) be the corresponding functions of the second kind (with superstar transforms \(\phi^*_n(z)\) and \(\psi^*_n(z)\) respectively). Interpolation and convergence properties of the modified approximants \(R_n(z, u, v) = (u_n \phi_n(z) - v_n \psi_n(z)) / (u_n \phi_n(z) + v_n \psi_n(z))\) that satisfy \(|u_n| = |v_n|\) are discussed.

Keywords: Orthogonal rational functions; Rational interpolation

1. Preliminaries

We shall use the notation \(T = \{ z \in \mathbb{C} : |z| = 1 \}\), \(D = \{ z \in \mathbb{C} : |z| < 1 \}\) for the unit circle and the unit disk. The kernel \(D(t, z)\) is defined by

\[
D(t, z) = \frac{t + z}{t - z}.
\]

(1.1)

Let \(\mu\) be a finite Borel measure on \([-\pi, \pi]\). The integral transform \(\Omega_\mu\) is defined as the Carathéodory function

\[
\Omega_\mu(z) = \int_T D(t, z) \, d\mu(t).
\]

(1.2)

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(We use the simplified notation above for $\int_{-\pi}^{\pi} D(e^{i\theta}, z) \, d\mu(\theta)$, and analogously in similar cases.)

The real part of a Carathéodory function is a positive harmonic function in $D$, and vice versa. (Recall the Riesz–Herglotz representation theorem. Note that the real part of the kernel $D(t, z)$ is the Poisson kernel.)

The substar conjugate $f^*_*$ of a function $f$ is defined as

$$f^*_*(z) = \overline{f(1/z)}.$$  (1.3)

When $f$ is a rational function or a series expansion, this may also be written as

$$f^*_*(z) = \overline{f(1/z)},$$  (1.4)

where the bar denotes conjugation of the coefficients. The inner product $\langle , \rangle_\mu$ is defined on $C(T) \times C(T)$ by

$$\langle f, g \rangle_\mu = \int_T f(t) \overline{g(t)} \, d\mu(t) = \int_T f(t) g^*_*(t) \, d\mu(t).$$  (1.5)

Let $\{\alpha_n; n = 1, 2, \ldots\}$ be an arbitrary sequence of (not necessarily distinct) points (interpolation points) in $D$. We define the Blaschke factor $\zeta_n(z)$ as the function

$$\zeta_n(z) = \frac{\alpha_n - z}{|\alpha_n| (1 - \overline{\alpha_n} z)}, \quad n = 1, 2, \ldots.$$  (1.6)

(Here $|\alpha_n| = -1$ if $\alpha_n = 0$.) We also define

$$\pi_0(z) = 1, \quad \pi_n(z) = \prod_{k=1}^{n} (1 - \overline{\alpha_k} z), \quad n = 1, 2, \ldots,$$  (1.7)

$$\omega_0(z) = 1, \quad \omega_n(z) = \prod_{k=1}^{n} (z - \alpha_k), \quad n = 1, 2, \ldots.$$  (1.8)

The Blaschke products $B_n(z)$ are defined by

$$B_0(z) = 1, \quad B_n(z) = \prod_{k=1}^{n} \zeta_k(z) = \eta_n \frac{\omega_n(z)}{\pi_n(z)}, \quad n = 1, 2, \ldots,$$  (1.9)

where

$$\eta_n = (-1)^n \prod_{k=1}^{n} \frac{\alpha_k}{|\alpha_k|}. $$  (1.10)
We shall also make use of the functions $B_{n,k}(z)$ defined by

$$B_{n,0}(z) = 1, \quad B_{n,k}(z) = B_{n}(z)/B_{k}(z) = \prod_{j=k+1}^{n} \zeta_j(z) \quad \text{for } 0 \leq k < n, \quad n = 1, 2, \ldots.$$  \hspace{1cm} (1.11)

(The product means the constant 1 when $k = n$.)

We define the spaces $\mathcal{L}_n$ and $\mathcal{L}_n^*$ by

$$\mathcal{L}_n = \text{Span}\{B_k: k = 0, 1, \ldots, n\}, \quad (1.12)$$

$$\mathcal{L}_n^* = \{f^*: f \in \mathcal{L}_n\}, \quad (1.13)$$

and set $\mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n$, $\mathcal{L}^* = \bigcup_{n=0}^{\infty} \mathcal{L}_n^*$. We may then write

$$\mathcal{L}_n = \left\{ \frac{p_n(z)}{\pi_n(z)} : p_n \in \Pi_n \right\}, \quad (1.14)$$

$$\mathcal{L}_n^* = \left\{ \frac{q_n(z)}{\omega_n(z)} : q_n \in \Pi_n \right\}, \quad (1.15)$$

where $\Pi_n$ denotes the space of all polynomials of degree at most $n$.

For $f_n \in \mathcal{L}_n$ we define its superstar conjugate $f_n^*$ by

$$f_n^*(z) = B_n(z)f_n(z).$$ \hspace{1cm} (1.16)

Note that this transformation depends on $n$. It must be clear from the context what $n$ is. Also note that when $f_n \in \mathcal{L}_n$, then $f_n^* \in \mathcal{L}_n^*$.

The theory of the function spaces described above is connected with the Nevanlinna–Pick interpolation problem with interpolation points $\{z_n\}$ (cf. [16, 17]). These function spaces were introduced by Djrbashian in 1969 (see [11]), and independently in [1, 2, 10]. The theory has recently been further developed in [3, 5, 6, 8] (cf. also [14]). For connections between Nevanlinna–Pick interpolation and system theory, see [9].

We shall in this paper mainly be concerned with a special case, which we shall call the cyclic case. In this case the sequence $\{z_n\}$ consists of a finite number $p$ of points cyclically repeated. Thus $z_{p+k} = z_k$ for $k = 1, \ldots, p, \quad q = 0, 1, 2, \ldots$. For more details on the cyclic case see [4, 7, 12].

When all the interpolation points coalesce at the origin, the space $\mathcal{L}$ reduces to the space of polynomials, and the orthogonal rational functions in $\mathcal{L}$ (see Section 2) are orthogonal polynomials, Szegő polynomials. For a survey of this special situation, see e.g. [13].

2. Orthogonal rational functions

Let the sequence $\{\varphi_n: n = 0, 1, 2, \ldots\}$ be obtained by orthonormalization of the sequence $\{B_n: n = 0, 1, 2, \ldots\}$ with respect to $\langle , \rangle_\mu$. These functions are uniquely determined by the requirement that the leading coefficient $\kappa_n$ in

$$\varphi_n(z) = \sum_{k=0}^{n} \kappa_k B_k(z)$$ \hspace{1cm} (2.1)
is positive. We then have $\kappa_n = \varphi_n^* (\varphi_n)$. The following orthogonality properties are valid:

$$\langle f, \varphi_n \rangle_{\mu} = 0 \quad \text{for } f \in \mathcal{L}_{n-1}, \quad (2.2)$$

$$\langle g, \varphi_n^* \rangle_{\mu} = 0 \quad \text{for } g \in \mathcal{L}_{n-1} \quad (2.3)$$

(see [3, 5]). We define the functions $\varphi_n(z, u, v)$ by

$$\varphi_n(z, u, v) = u \varphi_n(z) + v \varphi_n^*(z), \quad u, v \in \mathbb{C}, \quad (u, v) \neq (0, 0). \quad (2.4)$$

We note that $\varphi_n(z, u, v)$ belongs to $\mathcal{L}_n$ (as a function of $z$). We call these functions paraorthogonal when $|u| = |v|$.

We define the functions $\psi_n$ of the second kind by

$$\psi_0(z) = 1, \quad \psi_n(z) = \int D(t, z) [\varphi_n(t) - \varphi_n(z)] d\mu(t), \quad n = 1, 2, \ldots \quad (2.5)$$

For the functions $\psi_n$ and $\psi_n^*$ various equivalent expressions can be given. Let us recall the following result (see [3, 5]).

**Theorem 2.1.** For $n = 1, 2, \ldots$ the following formulas are valid:

$$\psi_n(z) = \int D(t, z) \left[ \frac{B_k(z)}{B_k(t)} \varphi_n(t) - \varphi_n(z) \right] d\mu(t), \quad k = 0, 1, \ldots, n - 1, \quad (2.6)$$

$$\psi_n^*(z) = - \int D(t, z) \left[ \frac{B_{n,k}(z)}{B_{n,k}(t)} \varphi_n^*(t) - \varphi_n^*(z) \right], \quad k = 0, 1, \ldots, n - 1. \quad (2.7)$$

We shall next prove a result valid in the cyclic situation.

**Theorem 2.2.** In the cyclic case with $p$ points the following formulas are valid for $n = p + 1, p + 2, \ldots$:

$$\psi_n(z) = \int D(t, z) \left[ \frac{B_{n,qp}(z)}{B_{n,qp}(t)} \varphi_n(t) - \varphi_n(z) \right] d\mu(t) \quad \text{where } qp < n, \quad (2.8)$$

$$\psi_n^*(z) = - \int D(t, z) \left[ \frac{B_{qp}(z)}{B_{qp}(t)} \varphi_n^*(t) - \varphi_n^*(z) \right] d\mu(t) \quad \text{where } qp < n. \quad (2.9)$$

**Proof.** We may write

$$B_{n,qp}(z) = \prod_{j=n-qp+1}^{n} \zeta_j(z) = \prod_{j=1}^{qp} \zeta_j(z) = B_{qp}(z).$$

The results now follow by using $k = qp$ in (2.6) and (2.7).
We define the functions $\psi_n(z, u, v)$ of the second kind by
\begin{equation}
\psi_n(z, u, v) = u\psi_n(z) - v\psi_n^*(z), \quad u, v \in \mathbb{C}, \quad (u, v) \neq (0, 0).
\end{equation}

**Theorem 2.3.** In the cyclic case with $p$ points the following formulas are valid for $n = p + 1, p + 2, \ldots$:
\begin{align}
\psi_n(z, u, v) &= \int_T D(t, z) \left[ \frac{B_{q_p}(z)}{B_{q_p}(t)} \varphi_n(t, u, v) - \varphi_n(z, u, v) \right] d\mu(t) \quad \text{where } q_p < n, \\
\psi_n(z, u, v) &= \int_T D(t, z) \left[ \frac{B_{n,q_p}(z)}{B_{n,q_p}(t)} \varphi_n(t, u, v) - \varphi_n(z, u, v) \right] d\mu(t) \quad \text{where } q_p < n.
\end{align}

**Proof.** Follows by combining (2.7) and (2.8) (resp. (2.6) and (2.9)) for the situation $k = q_p$. \qed

3. Interpolation by rational approximants

We shall in this section study interpolation properties of the rational functions
\begin{equation}
R_n(z, u, v) = \frac{\psi_n(z, u, v)}{\varphi_n(z, u, v)}
\end{equation}
given by (2.4) and (2.10) to the function $-\Omega_n(z)$ defined in (1.2).

Let us recall the following result (see [8]).

**Theorem 3.1.** The function $\Omega_n(z)$ has in $D$ the following Newton series expansion:
\begin{equation}
\Omega_n(z) = \varphi_n + 2 \sum_{m=1}^{\infty} \mu_m z^{\omega_{m-1}}(z),
\end{equation}
where the general moments $\mu_m$ are given by
\begin{equation}
\mu_m = \int_T \frac{d\mu(t)}{\omega_m(t)}, \quad m = 0, 1, 2, \ldots.
\end{equation}

In the following we shall use the notation $q(n), r(n)$ as defined below:
\begin{equation}
n = q(n)p + r(n), \quad r(n) \in \{1, \ldots, p\}.
\end{equation}

**Theorem 3.2.** The rational function $R_n(z, u, v)$ interpolates the function $-\Omega_n(z)$ in the sense that for $n > p$:
\begin{equation}
\psi_n(z, u, v) + \varphi_n(z, u, v)\Omega_n(z) = f_n(z)z^{\omega_{n-1}}(z),
\end{equation}
where $f_n(z)$ is analytic in $D$. 
Proof. One can easily establish the identity
\[
1 + 2 \sum_{m=1}^{n-1} \frac{z \omega_{m-1}(z)}{\omega_m(t)} = \frac{t + z}{t - z} \left[ 1 - \frac{z \omega_{n-1}(z)}{t \omega_{n-1}(t)} \right] - \frac{z \omega_{n-1}(z)}{t \omega_{n-1}(t)}.
\] (3.6)

Hence, after integrating (3.6) with measure \( \mu \), we get
\[
\mu_0 + 2 \sum_{m=1}^{n-1} \mu_m z \omega_{m-1}(z) = \int_T \left\{ D(t, z) \left[ 1 - \frac{z \omega_{n-1}(z)}{t \omega_{n-1}(t)} \right] - \frac{z \omega_{n-1}(z)}{t \omega_{n-1}(t)} \right\} d\mu(t).
\] (3.7)

By combining (2.11) and (3.7) we then obtain (since \( q(n)p < n \))
\[
\psi_n(z, u, v) + \phi_n(z, u, v) \left[ \mu_0 + 2 \sum_{m=1}^{n-1} \mu_m z \omega_{m-1}(z) \right]
= \int_T D(t, z) \left[ \frac{B_{q(n)p}(z)}{B_{q(n)p}(t)} \phi_n(t, u, v) - \frac{z \omega_{n-1}(z)}{t \omega_{n-1}(t)} \phi_n(z, u, v) \right] d\mu(t)
- \phi_n(z, u, v) z \omega_{n-1}(z) \int_T \frac{1}{t \omega_{n-1}(t)} d\mu(t)
\] (3.8)

and hence
\[
\psi_n(z, u, v) + \phi_n(z, u, v) \left[ \mu_0 + 2 \sum_{m=1}^{n-1} \mu_m z \omega_{m-1}(z) \right]
= -\mu'_n \phi_n(z, u, v) z \omega_{n-1}(z) + \omega_{q(n)p}(z) \sigma_n(z),
\] (3.9)

where
\[
\mu'_n = \int_T \frac{1}{t \omega_{n-1}(t)} d\mu(t)
\] (3.10)

and
\[
\sigma_n(z) = \int_T D(t, z) \left[ \frac{\pi_{q(n)p}(t)}{\pi_{q(n)p}(z) \omega_{q(n)p}(t)} \phi_n(t, u, v) - \frac{z \prod_{k=q(n)+1}^{n-1} (z - \alpha_k)}{t \omega_{n-1}(t)} \phi_n(z, u, v) \right] d\mu(t).
\] (3.11)

(If \( q(n)p = n - 1 \), the product means the constant 1.)

We are going to prove that \( \sigma_n(\alpha_k) = 0 \) for \( q(n)p + 1 \leq k \leq n - 1 \). Let \( q(n)p + 1 \leq k \leq n - 1 \), if \( n(q) < n - 1 \). Then
\[
\sigma_n(\alpha_k) = \frac{1}{\pi_{q(n)p}(\alpha_k)} \int_T D(t, \alpha_k) \frac{\pi_{q(n)p}(t)}{\omega_{q(n)p}(t)} \phi_n(t, u, v) d\mu(t).
\] (3.12)
We note that

\[ D(t, \alpha_k) \left[ \frac{\pi_{q(n)}p(t)}{\omega_{q(n)}p(t)} \right] = c \frac{1 + \tilde{\alpha}_k t \omega_{q(n)}p(t)}{1 - \tilde{\alpha}_k t \pi_{q(n)}p(t)} = c \zeta_n(t) L(t), \]

where \( L(t) \in \mathcal{L}_{n-1} \) and \( c \) is a constant, while also

\[ D(t, \alpha_k) \frac{\omega_{q(n)}p(t)}{\pi_{q(n)}p(t)} \in \mathcal{L}_{n-1}. \]

Because we may note that

\[ \frac{(1 + \tilde{\alpha}_k t)\omega_{q(n)}p(t)}{(1 - \tilde{\alpha}_k t)\pi_{q(n)}p(t)} = \frac{(t - \alpha_k)\pi_{q(n)}p(t)}{(1 - \tilde{\alpha}_k t)\pi_{q(n)}p(t)}, \]

where \( s_{q(n)}p(t) \) is a polynomial of degree \( q(n)p \), that \( (1 - \tilde{\alpha}_k t)\pi_{q(n)}p(t) \) is a factor in \( \pi_n(t) \), and that \( t - \alpha_k \) is a factor in \( \omega_{q(n)}p(t) \), thus

\[ \left[ \frac{\pi_{q(n)}p(t)}{\omega_{q(n)}p(t)} \right] \in \mathcal{L}_{n-1} \cap \zeta_n \mathcal{L}_{n-1}, \]

and hence

\[ \sigma_n(\alpha_k) = \frac{1}{\pi_{q(n)}p(\alpha_k)} \left\langle \varphi_n(t, u, v), \left[ \frac{\pi_{q(n)}p(t)}{\omega_{q(n)}p(t)} \right] \right\rangle_\mu = 0. \quad (3.13) \]

Analogously we find \( \sigma_n(0) = 0 \).

We have now seen that the second term on the right-hand side of (3.9) in addition to having the factor \( \omega_{q(n)}p(z) \) also has the extra factor \( z \) and the extra factors \( (z - \alpha_k) \) for \( q(n)p + 1 \leq k \leq n-1 \) (since \( \sigma_n(0) \) and \( \sigma_n(\alpha_k) = 0 \) for the values of \( k \) indicated).

It follows that the second term on the right of (3.9) is of the form \( A_n(z)z\omega_{n-1}(z) \). Thus

\[ \psi_n(z, u, v) + \varphi_n(z, u, v) \left[ \mu_0 + 2 \sum_{m=1}^{n-1} \mu_m z \omega_{m-1}(z) \right] = g_n(z)z \omega_{n-1}(z), \quad g_n(z) \text{ analytic.} \quad (3.14) \]

Since

\[ \Omega_n(z) + \left[ \mu_0 + 2 \sum_{m=1}^{n-1} \mu_m z \omega_{m-1}(z) \right] = h_n(z)z \omega_{n-1}(z), \quad h_n(z) \text{ analytic}, \quad (3.15) \]

we conclude that (3.5) holds. \( \square \)

4. Convergence of rational approximants

We recall that we call the function \( \varphi_n(z, u, v) \) paraorthogonal when \( |u| = |v| \). Paraorthogonal functions give rise to quadrature formulas. Let us recall the following result (see [3, 6]).

**Theorem 4.1.** The zeros of \( \varphi_n(z, u, v) \) for \( |u| = |v| \) are all simple and lie on \( T \). Let the zeros be denoted by \( \xi_k^{(n)}(u, v) \), \( k = 1, \ldots, n \). Then there exist positive constants \( \lambda_k^{(n)}(u, v) \) such that the quadrature
formula
\[ \int L(t) \, d\mu(t) = \sum_{k=1}^{n} \xi_k^{(n)}(u, v) L(\zeta_k^{(n)}(u, v)) \] (4.1)
is valid for \( L \in \mathcal{L}_{n-1} + \mathcal{L}_{(n-1)_a} \).

We shall in the rest of this section again consider only the cyclic case with \( p \) points, and use the same notation as in Section 3 and Theorem 4.1.

**Theorem 4.2.** Let \(|u| = |v|\), and assume \( n > p \). Then \( R_n(z, u, v) \) has the partial fraction decomposition
\[ R_n(z, u, v) = -\sum_{m=1}^{n} \xi_m^{(n)}(u, v) D(\zeta_m^{(n)}(u, v), z). \] (4.2)

**Proof.** Consider the function \( f(t) \) defined by
\[ f(t) = \frac{B_p(z)}{B_p(t)} \frac{\varphi_n(t, u, v) - \varphi_n(z, u, v)}{D(t, z)} \] (4.3)
The function \( \varphi_n(z, u, v) \) can be written as
\[ \varphi_n(z, u, v) = \frac{p_n(z, u, v)}{\pi_n(z)} \] (4.4)
where \( p_n(z, u, v) \in \Pi_n \). It follows that
\[ f(t) = \frac{(t + z)[\omega_p(z) \pi_p(t) p_n(t, u, v) \pi_n(z) - \omega_p(t) \pi_p(z) \pi_n(t) p_n(z, u, v)]}{(t - z) \omega_p(t) \pi_p(z) \pi_n(t)}, \] (4.5)
hence since \( t - z \) is a factor in the numerator:
\[ f(t) = \frac{P_{p+n-1}(z, t)(1 - \bar{\omega}_n t)}{\omega_p(t) \pi_n(t)}, \] (4.6)
where \( P_{p+n-1} \) belongs to \( \Pi_{p+n-1} \) as a function of \( t \). (Note that \( 1 - \bar{\omega}_n t \) is a factor both in \( \pi_p(t) \) and in \( \pi_n(t) \), and also in the numerator.)
It follows that we may write
\[ f(t) = \frac{P_{p+n-1}(z, t)}{\omega_p(t) \pi_{n-1}(t)}, \] (4.7)
hence \( f(t) \in \mathcal{L}_{n-1} + \mathcal{L}_{p_a} \subset \mathcal{L}_{n-1} + \mathcal{L}_{(n-1)_a} \), by partial fraction decomposition. (Note that \( \omega_p(t) \) and \( \pi_{n-1}(t) \) have no common factors.) Since \( f(\xi_m^{(n)}(u, v)) = -D(\xi_m^{(n)}(u, v), z) \varphi_n(z, u, v), \) as
\[ \phi_n(\xi_m^{(n)}(u, v), u, v) \text{ equals zero, application of Theorem 4.1 and formula (2.11) yields} \]
\[ \psi_n(z, u, v) = - \phi_n(z, u, v) \sum_{m=1}^{n} \lambda_m^{(n)}(u, v) D(\xi_m^{(n)}(u, v), z), \]  
\[ (4.8) \]

which is equivalent to (4.2). \[ \square \]

Since (4.1) is valid for \( L = 1 \), the following equality holds:
\[ \sum_{m=1}^{n} \lambda_m^{(n)}(u, v) = \mu_0. \]  
\[ (4.9) \]

**Theorem 4.3.** Let \(|u_n| = |v_n|\) for \( n = 1, 2, \ldots \). Then the sequence \( \{R_n(z, u_n, v_n)\} \) converges locally uniformly on \( D \) to \( - \Omega_\mu(z) \).

**Proof.** It easily follows by (4.2) and (4.9) that the functions \( R_n(z, u, v), |u| = |v| \), are uniformly bounded on every compact subset of \( D \), and thus form a normal family. So there exist subsequences of \( \{R_n(z, u_n, v_n)\} \) converging locally uniformly on \( D \). Let \( v_n(t, u_n, v_n) \) be the measure on \( T \) having masses \( \lambda_m^{(n)}(u_n, v_n) \) at the points \( \xi_m^{(n)}(u_n, v_n) \). By Theorem 4.2 we may then write
\[ R_n(z, u_n, v_n) = - \int_T D(t, z) \, dv_n(t, u_n, v_n). \]  
\[ (4.10) \]

A standard argument shows that a subsequence of \( \{R_n(z, u_n, v_n)\} \) converges locally uniformly on \( D \) to a function \( F(z) \) if and only if the corresponding subsequence of \( \{v_n(t, u_n, v_n)\} \) converges to a measure \( \nu \) such that \( F(z) = - \Omega_\nu(z) \).

Furthermore \( \int_T dv_n(u_n, v_n, t)/\omega_m(t) \) converges to \( \int_T d\nu(t)/\omega_m(t) \) for \( m = 0, 1, 2, \ldots \). On the other hand Theorem 3.2 shows that \( R_n(z, u_n, v_n) + \Omega_\mu(z) = g_n(z)z\omega_{n-1}(z) \), where \( g_n(z) \) is analytic in \( D \). It follows from this and (4.10) that \( \int_T dv_n(t, u_n, v_n)/\omega_m(t) = \int_T d\mu(t)/\omega_m(t) \) for \( m = 0, 1, \ldots, n-1 \).

Consequently \( \int_T d\nu(t)/\omega_m(t) = \int_T d\mu(t)/\omega_m(t) \) for \( m = 0, 1, 2, \ldots \) (cf. [7, 8] where related problems are treated). It is known that the measure giving rise to the moments \( \mu_m = \int_T d\mu(t)/\omega_m(t) \) is unique when \( \sum_{m=1}^{\infty} (1 - |\gamma_m|) = \infty \) (this follows e.g. from the convergence result in [3, Section 21]). This is the case in the cyclic situation. Thus \( \nu = \mu \) and the whole sequence \( \{R_n(z, u_n, v_n)\} \) converges to \( - \Omega_\mu(z) \). \[ \square \]

For convergence properties of the rational approximants \( R_n(z, 0, 1) \) and \( R_n(z, 1, 0) \) see [3]. For a more detailed study of convergence of multipoint Padé approximants, see especially [15].

**References**


