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Published in:
Theoretical and Mathematical Physics

DOI:
10.1007/BF01102213

Citation for published version (APA):
QUANTUM GROUPS IN THE HIGGS PHASE

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In the Higgs phase we may be left with a residual finite symmetry group $H$ of the condensate. The topological interactions between the magnetic and electric excitations in these so-called discrete $H$ gauge theories are completely described by the Hopf algebra or quantum group $D(H)$. In 2+1 dimensional space time we may add a Chern-Simons term to such a model. This deforms the underlying Hopf algebra $D(H)$ into the quasi-Hopf algebra $D^\omega(H)$ by means of a 3-cocycle $\omega$ on $H$. Consequently, the finite number of physically inequivalent discrete $H$ gauge theories obtained in this way are labelled by the elements of the cohomology group $H^3(H, U(1))$. We briefly review the above results in these notes. Special attention is given to the Coulomb screening mechanism operational in the Higgs phase. This mechanism screens the Coulomb interactions, but not the Aharonov-Bohm interactions.

1. INTRODUCTION

By means of the Higgs mechanism a continuous gauge group $G$ (for convenience assumed to be simply connected) of some gauge theory can be spontaneously broken down to a finite residual symmetry group $H$. It has been known for some time that such theories support magnetic excitations labelled by $\pi_1(G/H) \simeq H$ [1, 2]. These magnetic excitations are stringlike in 3 spatial dimensions and pointlike in the arena in which we will discuss matters, namely the plane. Quite recently it has been realized that besides these magnetic excitations, the Higgs phase with non-trivial residual symmetry group $H$ also supports charges labelled by the unitary irreducible representations (UIR’s) of $H$ [3]. Since the electromagnetic fields are massive in the Higgs phase, these charges do not carry Coulomb fields. They are nevertheless still able to take part in long range interactions through Aharonov-Bohm (AB) scattering with the magnetic excitations [4].

The physical mechanism behind this screening of Coulomb charges and non-screening of AB charges in the Higgs phase was uncovered in [5].

The large distance physics of these spontaneously broken models is described by a so-called discrete $H$ gauge theory. As was shown in [6], the underlying symmetry algebra is the Hopf algebra (also called quantum group) $D(H)$. In 2+1 dimensional space time, we may add a Chern-Simons (CS) term to the action [7]. This deforms the underlying Hopf algebra $D(H)$ into the quasi-Hopf algebra $D^\omega(H)$ by a 3-cocycle $\omega$ on $H$. As a result, there exists a finite number of distinct discrete $H$ gauge theories labelled by the elements of the cohomology group $H^3(H, U(1))$. These elements are determined by the CS parameter.

The quasi-Hopf algebra $D^\omega(H)$ was originally [8] constructed as the symmetry algebra of orbifold models [9] and the related discrete topological gauge theories studied in [10]. The connection between these models and discrete $H$ gauge theories arising in the Higgs phase, which in some sense can be viewed as a regularized version of [10], certainly calls for further exploration.

Since discrete gauge theories may have emerged after some symmetry breaking phase transition in the early universe, our considerations find a context in cosmology [3]. There are also applications in condensed matter systems, such as nematic crystals [11], and type II Landau-Ginzburg superconductors.

These notes intend to review the results mentioned above. The outline is as follows. In Section 2 we discuss some basic features of discrete $H$ gauge theories with a CS term. The example $H \simeq Z_N$ arising from the symmetry breaking scheme $SU(2) \to U(1) \to Z_N$ will be dealt with in some detail. We emphasize the difference between the CS screening mechanism in unbroken CS electrodynamics and the Higgs screening mechanism entering the scene when $U(1)$ is spontaneously broken down to $Z_N$. In both mechanisms Coulomb interactions are screened, while AB interactions survive. The peculiarities of non-abelian discrete $H$ gauge theories are briefly reviewed in Section 2.2., whereas the symmetry algebra $D^\omega(H)$ behind them is discussed in Section 3. In Section 3.1. we will apply this machinery to the $Z_N$ gauge
2. DISCRETE H GAUGE THEORIES

As mentioned before, discrete H gauge theories [3, 6, 12] naturally arise whenever the continuous symmetry group G of some gauge theory is spontaneously broken down to a finite group H by the Higgs mechanism. We will illustrate this scheme starting from a G \cong SU(2) gauge theory in (2+1)-dimensional Minkovski space

\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu}^a F^{a\mu \nu} + \frac{\mu}{4} e^{\kappa \sigma \rho} F_{\kappa \sigma} A^a_{\rho} + \frac{1}{2} \varepsilon^{\alpha \beta \gamma} A^a_{\alpha} A^b_{\beta} A^c_{\gamma} \]

+ \left( \partial_\mu \Phi \right) \left( \partial^\mu \Phi \right) - V(\Phi) + \mathcal{L}_{\text{matter}}. \quad (1)

\[ \Phi \]

Greek indices run from 0 to 2, whereas latin indices label the three (hermitian) generators of SU(2). In our convention the metric r has signature (+, -, -). The covariant derivative takes the form \( D_\mu \Phi = (\partial_\mu + i e A^2_{\mu}) \Phi \), with the generators \( T^a \) of SU(2) in the representation of the Higgs field \( \Phi \). In \( \mathcal{L}_{\text{matter}} \), we have introduced additional matter fields minimally coupled to the vector-potential, so that all conceivable charge sectors can be discussed. We have included a CS term in (1) as well [7]. The completely anti-symmetric three dimensional Levi-Civita tensor \( \varepsilon \) appearing in this term is normalized such that \( \varepsilon^{012} = 1 \). The demand that the Lagrangian (1) should give rise to a gauge invariant quantum theory leads to a quantization condition for the topological mass \( \mu \) [7]

\[ \mu = pe^2/4\pi \quad \text{with} \quad p \in \mathbb{Z}. \quad (2) \]

By an appropriate choice of the representation of the Higgs field \( \Phi \) and the potential \( V(\Phi) \), the gauge symmetry SU(2) can be spontaneously broken down to any finite subgroup H [13]. If we are dealing with energies well below the symmetry breaking scale, we are in a Higgs phase with a residual finite gauge symmetry group H. The effective theory we are left with has been called a discrete H gauge theory [3, 6, 12]. It is the purpose of this section to identify the complete spectrum of charges and magnetic fluxes of such a theory, together with the topological interactions between them. We will do so in the simplest example first, namely the discrete gauge theory that emerges if we break SU(2) down to \( H \cong \mathbb{Z}_N \). This will be the content of Section 2.1.. The characteristic features of discrete gauge theories with a non-abelian residual gauge group H will be briefly reviewed in Section 2.2.. Throughout these notes we work with units, such that \( \hbar = c = 1 \).

2.1. A \( \mathbb{Z}_N \) gauge theory. One of the interesting features of CS terms is the fact that they endow the electromagnetic fields with a mass proportional to \( \mu \) [7]. Thus charges are screened in the presence of a CS term. As such, CS terms provide a welcome alternative to the Higgs mechanism. The nature of these two screening mechanism is quite different though. We will contrast the two of them in the U(1) phase that arises when the gauge group SU(2) of (1) is spontaneously broken down to U(1) at some high energy scale. If the U(1) phase remains unbroken, the Coulomb screening is due to the CS mechanism, whereas the Higgs mechanism becomes effective if U(1) is spontaneously broken down to a cyclic group \( Z_N \) at some lower energy scale.

Suppose that the Higgs potential \( V(\Phi) \) in (1) is such that the symmetry group SU(2) is spontaneously broken down to U(1) (see for instance [15] for more details). The U(1) regime is governed by the following effective Lagrangian

\[ \mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu \nu}^{\psi} F^{\psi \mu \nu} + \frac{\mu}{4} e^{\kappa \sigma \rho} F_{\kappa \sigma} A^{\psi}_{\rho} + (D_\mu \psi) (D^\mu \psi) - V(|\psi|) + \mathcal{L}_{\text{matter}}, \quad (3) \]

where \( \psi \) denotes an additional Higgs field that we absorbed in \( \mathcal{L}_{\text{matter}} \) in (1). We assume that this Higgs field \( \psi \) carries a global U(1) charge \( Ne/2 \), i.e., \( D_\mu \psi = (\partial_\mu + i e A^{\psi}_{\mu}) \psi \), so that we obtain a \( \mathbb{Z}_N \) gauge theory if this field condenses at a lower energy scale [3, 6, 12]. The global U(1) charges are quantized in units of \( e/2 \) as a consequence of the embedding in SU(2). In this strictly abelian model, we have omitted the massive modes associated with the broken generators, and the massive neutral Higgs particles. Note that this phase also contains instantons labelled by \( \pi_2(SU(2)/U(1)) \cong \mathbb{Z} \). In 3 euclidean dimensions these instantons are monopoles carrying magnetic charge \( g = 4\pi k/e \) with \( k \in \mathbb{Z} \), while in this (2+1)-dimensional Minkowski setting they describe quantum tunneling events between states with magnetic flux.
difference $|\Delta \phi| = 4\pi/e$. This $U(1)$ gauge theory is spontaneously broken down to $Z_N$ by endowing the Higgs field $\psi$ with a non-vanishing vacuum expectation value $<\psi> = v$ through the following choice of the potential:

$$V(|\psi|) = \frac{\lambda}{4}(|\psi|^2 - v^2)^2, \quad \lambda, v > 0. \quad (4)$$

Before we turn to the subtleties of this spontaneous symmetry breakdown however, we first consider the unbroken case. Thus we set $v = 0$ for the moment. Variation of (3) w.r.t. the $U(1)$ vector-potential $A_\sigma$ then yields the following field equation:

$$\partial_\nu F^{\sigma\nu} + \frac{\mu}{2} \epsilon^{\sigma\tau\rho} F_{\tau\rho} = j^\sigma + j_H^\sigma, \quad (5)$$

where $j_H^\sigma = iNe(\psi^* D_\sigma \psi - (D_\sigma \psi)^*) \psi$ denotes the Higgs current, and the current $j^\sigma$ consists of contributions of the matter fields contained in $\mathcal{L}_{\text{matter}}$. Integrating the zeroth component of (5) over the plane leads to Gauss's law

$$Q = q + \mu \phi + q_H = 0, \quad (6)$$

with $Q = \int d^2x \mathbf{E}$ the Coulomb charge, $q = \int d^2x j^0$ and $q_H = \int d^2x j_H^0$ global $U(1)$ charges, and $\phi \equiv \int d^2x \epsilon^{ij} \partial_i A^j$ the total magnetic flux. The Coulomb charge $Q$ in (6) vanishes because the Coulomb fields carry a mass $\mu$ in the presence of a CS term, and therefore vanish exponentially. The screening mechanism operating in unbroken CS electrodynamics attaches fluxes $\phi = -q/\mu$ and $\phi = -q_H/\mu$ of characteristic size $1/|\mu|$ to the global $U(1)$ point charges $q$ and $q_H$ respectively [7]. This leads to an identification of charge and flux at distances $\gg 1/|\mu|$. The spectrum of this theory at such distances, where only AB interactions remain between the excitations [4], is depicted in Fig. 1. Note that the AB fields (which are pure gauge) around the fluxes are still solutions of the field equations. It is not the vector-potential $A_\rho$, which is massive in CS electrodynamics, but rather the electromagnetic fields [7]. The interaction part $-(j^\rho + j_H^\rho - \frac{\mu}{4} \epsilon^{\rho\sigma\tau} F_{\sigma\tau}) A_\rho$ of the Lagrangian (3) gives rise to the AB phase [4]

$$\mathcal{R}^2 |q_1 > |q_2 >= e^{i(q_1 \phi_2 + q_2 \phi_1 + \mu \phi_1 \phi_2)} |q_1 > |q_2 >= e^{(-i \mu \phi_1 \phi_2)} |q_1 > |q_2 > \quad (7)$$

if we take a charge $q_1$ counterclockwise around $q_2$ once. This process is effectuated by the square of the braid operator $\mathcal{R}$, which interchanges the two particles in a counterclockwise direction. In the last equality sign, we used the aforementioned identification of charge and flux. For the statistical parameter [14] (generated by $\mathcal{R}$) of the excitation with charge $q = -pe$ (with $p \in Z$ defined in (2)) and flux $\phi = 4\pi/e$, we find $\exp i \theta = \exp(-i \phi^2) = \exp(-2\pi up) \approx 1$. So this excitation is a boson, just as the vacuum in which it is tunnelled by an instanton. In fact, this can be seen as an alternative way to derive the quantization of the CS parameter; in the presence of instantons $\mu$ has to satisfy (2), otherwise they would tunnel states with different quantum-statistical properties into each other. Note that the instantons are charged in the sense that they tunnel between excitations not only with flux difference $\Delta \phi = -4\pi/e$, but also with charge difference $\Delta q = pe$ [15]. Using (7), it is also easily verified that the excitations connected by instantons are indistinguishable by AB scattering processes with the other excitations of the spectrum.

In the broken case ($v \neq 0$) the situation is more involved. The Higgs field $\psi$ condenses at energy scales lower than $M_H = \sqrt{v} \sqrt{2\lambda}$. The $Z_N$ Higgs phase arising at these energy scales is described by the following simplification in (3):

$$|D_\kappa \psi|^2 \rightarrow \frac{M_A^2}{2} \tilde{A}_\kappa \tilde{A}_\kappa, \quad (8)$$

$$\tilde{A}_\kappa \equiv A_\kappa + \frac{2}{N} \partial_\kappa \text{Im} \log <\psi>, \quad (9)$$

with $M_A = \frac{N^2}{2} v \sqrt{2}$ and $<\psi>$ the vacuum expectation value of $\psi$. Consequently, the gauge invariant combination $\tilde{A}$ acquires the mass [16]

$$M_{\pm} = \sqrt{M_A^2 + \frac{1}{2} \mu^2 \pm \frac{1}{2} \mu^2 \sqrt{\frac{4M_A^2}{\mu^2} + 1}}, \quad (10)$$

where $+$ and $-$ stand for two different components of the photon.
The fact that $\tilde{A}$ is massive does not immediately imply that $A$ should also fall off exponentially. It can instead remain pure gauge. This is the case around topological defects of the Higgs condensate corresponding to magnetic vortices [1]. To meet the requirement that the Higgs condensate is single-valued outside the cores of these vortices, their magnetic flux $\phi$ is quantized as $\phi = 4\pi m/N_e$ with $m \in \pi_1(U(1)/Z_N) \simeq Z$ [1]. It can be shown that the magnetic field $(F^1_2)$ related to the vortices fall off with $M_-$, and not with $M_+$ [17]. The shape of this magnetic field depends on the parameters. For $\mu = 0$ we are dealing with the Abrikosov-Nielsen-Olesen vortex with maximal magnetic field at its centre, falling off with mass $M_- = M_A$ at large distances [1]. For increasing $|\mu|$ the magnetic field at the centre of the vortex decreases until it vanishes and becomes a minimum in the CS limit $e, |\mu| \rightarrow \infty$, with fixed ratio $e^2/\mu$ [17]. (In our case, where the topological mass $\mu$ is quantized as (2) this limit simply means $e \rightarrow \infty$ leaving the CS parameter $p$ fixed.) In this CS limit, which boils down to neglecting the Maxwell term in (3), the magnetic field is localized in a ring-shaped region around $1/M_H$ [17, 19]. In the presence of a CS term the vortices are endowed with further peculiar properties. As we see from Eq. (6), a bare flux would imply a non-vanishing Coulomb charge $Q$, which is inconsistent with the massivity of the electromagnetic fields. It is the Higgs condensate that brings salvation. As follows from (6) and (8), at distances $\geq 1/M_H$ it conspires with the vector-potential to become a charge density $j_{scr}^0$

$$q_H \longrightarrow q_{scr} \equiv \int d^2 x j_{scr}^0 \equiv -\int d^2 x M_A^2 \tilde{A}_0 = -\mu \phi,$$

establishing the exponential decay of the Coulomb fields induced by the flux $\phi$ of the vortex [5]. This screening charge density $j_{scr}^0$ is localized in a ring outside the core of the vortex (see [17] and references given there). Remarkably enough, the screening charge $q_{scr}$ does not couple to the AB interaction [5]. The associated current $j_{scr}^\kappa \equiv -M_A^2 \tilde{A}^\kappa$ would only interact with the AB field (produced by some remote vortex) if there was a term in the Lagrangian of the form $-j_{scr}^\kappa A^\kappa = M_A^2 \tilde{A}^\kappa A_\kappa$ [4]. Instead we only encounter the term $\frac{1}{2} M_A^2 \tilde{A}^\kappa \tilde{A}_\kappa$ in (8). In other words, $q_{scr}$ couples to $\tilde{A}$ rather than to $A$, and thus does not feel AB fields related to remote vortices, which have non-vanishing $A$-component, but strictly vanishing $\tilde{A}$ at large distances from their cores. This implies that taking a vortex with flux $\phi_1$ counterclockwise around another vortex with flux $\phi_2$ (a process denoted by $R^2$) generates the AB phase $\exp(i\mu \phi_1 \phi_2)$, entirely due to the coupling $\frac{\mu}{4} \epsilon^\rho_\kappa_\sigma F_\rho_\sigma A_\rho$ in (3). Here we assume that the vortices never overlap. Identical vortices with flux $\phi$ then behave as anyons [14] with statistical parameter $e \phi^2/2\pi s$. A result which is in complete accordance with the spin-statistics connection $\exp i\theta = \exp(2\pi i s)$, where $s = \mu \phi^2/4\pi$ denotes the spin that can be calculated for the classical vortex solution [17].

An important issue is whether vortices will actually form or not, that is, whether the superconductor we are de-
scribing here is type II or I respectively. In ordinary superconductors ($\mu = 0$) an evaluation of the free energy yields that we are dealing with a type II superconductor if $M_H/M_A = 2\sqrt{\lambda}/Ne \geq 1$, and a type I superconductor otherwise. A perturbation analysis for small $\mu \neq 0$ shows that the type II region is extended [18]. In the following, we will always assume that our parameters are adjusted such that we are in the type II region.

Let us now turn to the fate of the global $U(1)$ matter charges $q$ in this $Z_N$ Higgs phase. We first consider the situation where the CS term is absent, that is $\mu = 0$. Substituting (9) and (6) in (5) yields the Klein–Gordon equation for $\tilde{A}^\sigma$, which indicates that the Coulomb fields generated by the charge $q$ fall off with mass $M_+ = M_A$. It follows from Gauss’s law (6) with $\mu = 0$, that this is achieved by surrounding the charge $q$ by the screening charge density $J_{scr}^0 = -M^2_A \tilde{A}_0$

\[ q_H \rightarrow q_{scr} \equiv -\int d^2x M^2_A \tilde{A}_0 = -q, \]  

with support in a ring-shaped region localized at distances $\sim 1/M_H$. At larger distances, the contribution of the screening charge $q_{scr}$ to the Coulomb fields completely cancels the contribution of $q$. As we saw before, the screening charge does not couple to the AB interaction, and the AB phase $\exp(\imath q \phi)$ generated if $q$ encircles a vortex $\phi$ in counterclockwise direction will not be canceled by the screening cloud $q_{scr}$ around $q$ [5]. Thus the Coulomb interactions are exponentially damped by the Higgs mechanism, while the AB interactions are not. If we turn on the CS term we have in principle two competing screening mechanisms. Only the Higgs mechanism can be effective at distances $\gg 1/M_H$ though. This must be clear already from the fact that the fluxes $-q/\mu$, attached to the point charges $q$ by the CS mechanism, in general do not satisfy the flux quantization condition $\phi = 4\pi m/Ne$ with $m \in Z$, that arises at these distances. It is illuminating to illustrate this in the situation where we have adjusted our parameters such that we are in the CS limit; thus $e$ is large with fixed CS parameter $p$ (therefore the topological mass $|\mu|$ given as (2) is large as well). As we have argued before, if $U(1)$ is not spontaneously broken, i.e., $v$ in (4) is sent to 0, then the point charges $q$ will be surrounded by fluxes localized within a region of radius $1/M_+ = 1/\mu$. The photon component $M_- = 0$ decouples, and the spectrum at distances $\gg 1/\mu$ boils down to Fig. 1. Now suppose that we turn on the Higgs mechanism with a small, but non-vanishing value $v$, i.e., $0 < v \ll 1$. To make sure we are in the type II region, we make the additional assumption $\lambda \geq (Ne/2)^2$. So $\mu \gg M_H > M_A$, and in first-order approximation in $(2M_A/\mu)^2$ we find from (10) that $M_+ \approx \mu$ and $M_- \approx M_A^2/\mu$. From continuity reasons, we expect that the charge $q$ will still be surrounded by a flux $-q/\mu$ with support extending to distances $\sim 1/\mu$, so that the Coulomb fields of the point charges $q$ still fall off with the $M_+$ component of the photon. At distances $\gg 1/M_H$ we have to satisfy the flux quantization condition, which states that the total flux must be a multiple of $4\pi/Ne$. To achieve this, we have the vortex solution at our disposal. We expect that in this range of parameters, the complete solution of the field equations for a point charge $q$ will be a superposition of that for unbroken CS electrodynamics at short distances $\ll 1/M_H$, supplemented with a vortex solution with a flux $\phi$ with support at distances $1/M_H$, such that $-\frac{2}{\mu} + \phi \in 4\pi m/Ne$. Recall that in the CS limit the vortex solution is located in a ring-shaped region with radius $\sim 1/M_H$ falling off with the $M_-$ component at large distances. Moreover, it becomes trivial at small distances $\ll 1/M_H$ [17, 19]. This indicates that we find the spectrum of unbroken CS electrodynamics (Fig. 1) at distances $1/\mu \ll \phi < 1/M_H$, while the spectrum of the $Z_N$ Higgs phase (depicted in Fig. 2 for $N = 4$, $p = 1$, and explained in the next paragraphs) emerges at distances $\gg 1/M_H$. It would be interesting to verify this analysis by means of a numerical evaluation.

The main conclusion from all this is that in the broken case the identification of charge and flux is lost at distances $\gg 1/M_H$; charge and flux become independent quantum numbers in the $Z_N$ Higgs phase. In the $Z_N$ phase the spectrum will not reside on a line as in Fig. 1, but rather on the lattice spanned by the charge $q = e/2$ and the flux $\phi = 4\pi/Ne$. We will denote the excitations on this lattice as $|m, n>, \text{ where } m \text{ stands for the number of flux units } 4\pi/Ne, \text{ and } n \text{ for the number of charge units } e/2$. The AB phases generated between these excitations follow from the coupling $-(j^\rho - \frac{e}{4} \epsilon^{\rho\sigma\kappa} F_{\kappa\sigma}) A_\rho$ as

\[ \mathcal{R} |m_1, n_1> |m_2, n_2> = e^{i\hat{Q}^2} e^{i\Phi} |m_2, n_2> |m_1, n_1> = e^{i\hat{Q}^2} ((m_2 + \frac{4\phi}{\lambda}) m_1) |m_2, n_2> |m_1, n_1>, \]  

with the charge $\hat{Q}$ given by

\[ \hat{Q} \equiv q + \frac{\mu}{2} \phi. \]  

There is a large redundancy in the spectrum as we have sketched it so far. We have not taken care of the modulo $N$ calculus yet. The proper labelling of the magnetic flux sectors in the full theory is by $\pi_1(SU(2)/Z_N) \simeq Z_N$ and not
Fig. 2. The spectrum of a Higgs phase with residual gauge group $Z_4$. We depict the flux $\phi$ against the global $U(1)$ charge $q$, the Noether charge $\bar{Q}$ and the screening charge $-q_{scr} = q + \mu \phi$ respectively. The CS parameter $\mu$ is set to its minimal non-trivial value $\mu = e^2/4\pi$, i.e., $p = 1$. The identification of the encircled excitation with an excitation inside the dashed box is indicated with an arrow. As in Fig. 1 for unbroken CS electrodynamics, this arrow visualizes the effect of a charged instanton.

by $\pi_1(U(1)/Z_N) \simeq Z$. The apparent difference can be understood if the role of the instantons in the model is taken into account. As mentioned earlier, they connect vortices with a flux difference $|\Delta \phi| = 4\pi/e$, thus establishing the desired $Z_N$ calculus. We will argue that this magnetic $Z_N$ calculus becomes twisted in the presence of a CS term, while the $Z_N$ calculus for the charges is unaffected. To that end consider the process in which an arbitrary composite $(m_3, n_3)$ encircles a composite $(m_1 + m_2, n_1 + n_2)$. The sums $m_1 + m_2$ and $n_1 + n_2$ do not necessarily lay between 0 and $N - 1$. Using the notation $(m_1 + m_2)$ for $(m_1 + m_2)$ modulo $N$, chosen between 0 and $N - 1$, we can rewrite the AB phase for this process, generated by $\phi$, as

$$e^{i(Q_3 \phi_3 + Q_1 \phi_1 + Q_2 \phi_2 + Q_3 \phi_3)} = e^{i(Q_3 \phi_12 + Q_1 \phi_12)},$$

with the definitions

$$Q_{12} \equiv q_{12} + \frac{\mu}{2} \phi_{12},$$

$$\phi_{12} \equiv \frac{4\pi}{Ne}(m_1 + m_2).$$

Equations (16) and (17) express the way charges and fluxes 'add', i.e., they specify the fusion rules for the excitations in our model. In terms of the quantum states $|m, n>$ these read

$$|m_1, n_1 > \ast |m_2, n_2 > = |[m_1 + m_2], [n_1 + n_2 + 2p]/N (m_1 + m_2) - [m_1 + m_2]) >.$$  

Hence the spectrum can be confined to an $N$ by $N$ charge/flux lattice. The modulo $N$ calculus for the fluxes is twisted by the CS parameter $p$ though. Phrased in more physical terms: the instantons $\Delta \phi = -4\pi/e$ carry a charge $\Delta q = pe$, in complete accordance with our findings in unbroken CS electrodynamics.

It is clarifying to summarize the foregoing discussion in pictures, as is done in Fig. 2 for a $Z_4$ gauge theory. From these pictures it is immediate that for odd $N$ the fusion algebra $Z_N \times Z_N$ of a discrete $Z_N$ gauge theory without a CS term is altered into $Z_{kN} \times Z_{N/k}$ in the presence of a CS term [9]. Here we defined $k \equiv N/(p, N)$ with $(p, N)$ the greatest common divisor of the CS parameter $p$ and $N$. So in particular for $p = 1$, the complete spectrum is generated by a single excitation. For even $N$ we find a similar result, except that the formula for $k$ has to be replaced by $k \equiv N/(2p, N)$.

The symmetry algebra behind this spectrum is the Hopf algebra $D^N(Z_N)$ [6]. It consists of $Z_N$ gauge transformations and projection operators signalling the flux of a state. We will denote the elements of this algebra by $m_1$.
which performs a gauge transformation with parameter $l$, and subsequently projects the state on the flux $m$

$$m^{l}_{m, n} |m_1, n_1> = \delta_{m, m_1} e^{\frac{2\pi i m}{N} (n_1 + p m_1)} |m_1, n_1> .$$

(19)

Note that it is the charge $\tilde{Q}$ (14), which appeared in the braid process (13), that gives the response of a state to a gauge transformation [6]. A braid process has the same effect as a gauge transformation. We leave a more detailed description of the symmetry algebra for Section 3.

2.2. **Non-abelian discrete gauge theories.** Of course we may also be left with a non-abelian finite symmetry group $H$ of the Higgs condensate [13]. The discussion of such non-abelian discrete gauge theories is slightly more involved. As in the abelian case, there are three types of excitations in these theories, namely purely magnetic vortices, AB charges, and of course the dyons that are composites of these last two excitations. If we assume that $G$ is simply connected, the magnetic vortices are labelled [2] by the elements of $\pi_1(G/H) \cong H$. They are stringlike in 3 spatial dimensions, while they are particle like in the plain. A residual gauge transformation $g \in H$ acts on the magnetic charges $h \in H$ through conjugation

$$g: |h> \rightarrow |ghg^{-1}> .$$

(20)

So the gauge invariant labelling of the magnetic charges is by means of the conjugacy classes $AC$ of $H$. We have to keep in mind, however, that physical properties such as braiding [2, 3, 6, 12]

$$R |h> |k> = |hkh^{-1}> |h>,$$

(21)

depend on the specific element of $AC$ by which the magnetic flux is represented. As before, the effect of braiding and gauge transformations is similar.

The free electric charges are labelled by the UIR's of the gauge group $H$ [3]. In the presence of a magnetic flux $h \in AC$ the gauge group $H$ is broken down to the centralizer $hN$ of $h$ in $H$ through [3, 11], and the electric charges we can put on the magnetic flux $h$ are labelled by the UIR's of $hN$ [6]. If we now use the fact that the centralizers of the different $h \in AC$ are isomorphic, then we can summarize the superselection sectors of a discrete $H$ gauge theory as

$$(AC, \Gamma),$$

with $\Gamma$ a UIR of the centralizer $hN$ associated with the conjugacy class $AC$. As we shall see in the next section, these superselection sectors exactly coincide with the irreducible representations of the Hopf algebra $D(H)$, and it turns out that the topological properties of the excitations in discrete $H$ theories can be completely described in terms of this algebra and its representation theory [6]. If we add a CS term to the action of these theories, then the underlying Hopf algebra $D(H)$ is deformed into the quasi-Hopf algebra $D^\omega (H)$ by means of a 3-cocycle $\omega$ on $H$.

3. **THE QUASI-HOPF ALGEBRA $D^\omega (W)$**

The quasi-triangular quasi-Hopf algebra $D^\omega (H)$ was first discussed by Dijkgraaf, Pasquier and Roche [8] and we will adopt their notation. For a general introduction into the notion of quasi-triangular quasi-Hopf algebras we refer to the work of Drinfeld [20].

The symmetry algebra $D^\omega (H)$ behind discrete $H$ gauge theories is spanned by the elements $\{ g_{x} \}_{g, x \in H}$, denoting a residual gauge transformation $x \in H$ followed by a projection on the flux state $|g>$. In terms of these basis elements the multiplication, the co-multiplication $\Delta$, the associator $\varphi$, and the $R$-matrix $R$ read

$$g_{x} h_{y} = \delta_{g, xh_{x}} g_{x} \vartheta_{y}(x, y)$$

(22)

$$\Delta(g_{x}) = \sum_{\{ h, k | h = g \}} h_{x} \otimes k_{x} \gamma_{x}(h, k)$$

(23)

$$\varphi = \sum_{g, h, k} \omega^{-1}(g, h, k) g_{e} \otimes h_{e} \otimes k_{e}$$

(24)

$$R = \sum_{g, h} g_{e} \otimes h_{g}$$

(25)
where \( \theta, \gamma, \) and \( \omega \) are phases that equal 1 whenever one of their variables is the unit \( e \) of \( H \). The algebra morphism \( \Delta \) from \( D^\omega(H) \) to \( D^\omega(H) \otimes D^\omega(H) \) enables us to construct the tensor product of representations \((\Pi_1, V_1)\) and \((\Pi_2, V_2)\) of \( D^\omega(H) \), and therefore to extend the action of the symmetry algebra from 1-particle states to 2-particle states. The associator \( \alpha \) establishes the isomorphism \( \Pi_1 \otimes \Pi_2 \otimes \Pi_3(\alpha) \) between the representation-spaces \((V_1 \otimes V_2) \otimes V_3 \) and \( V_1 \otimes (V_2 \otimes V_3) \), constructed by \((\Delta \otimes id)\Delta \) and \((id \otimes \Delta )\Delta \) respectively. The \( R \)-matrix describes the braiding properties of the particles, as will become clear later on.

For a consistent implementation \([8, 20]\) of \( \alpha \) on tensor products of four representations \( \omega \) has to satisfy the 3-cocycle condition

\[
\delta \omega(g, h, k, l) = \omega(g, h, k, l) \omega(g, h, k, l) \omega(g, h, k, l) = 1,
\]

where \( \delta \) denotes the coboundary operator (interested readers are referred to \([6]\) for more details on the cohomology structure appearing here). So \( \omega \) is an element of the cohomology group \( H^3(H, U(1)) \). This element is determined by the CS parameter \( p \) \([6]\). To proceed, the phases \( \theta \) and \( \gamma \) are completely prescribed by \( \omega \) \([8]\). This implies that \( \theta \) is a conjugated 2-cocycle

\[
\tilde{\delta} \theta(x, y, z) = \frac{\theta_{x^{-1}y}}{\theta_{x(x, y)}} = 1,
\]

with \( \tilde{\delta} \) the 'conjugated' coboundary operator. Equation (27) expresses associativity of the multiplication (22). It is also easily verified that the co-multiplication is quasi-coassociative.

The irreducible representations of \( D^\omega(H) \), which label the excitations of a discrete \( H \) gauge theory, can be found by inducing the unitary irreducible representations (UIR's) of the centralizer subgroups. Let \( \{ AC \} \) be the set of conjugacy classes of \( H \) and introduce a fixed but arbitrary ordering \( AC \) = \( \{ A_{g_1}, A_{g_2}, \ldots, A_{g_k} \} \). Let \( AN \) be the centralizer of \( A_{g_1} \) and \( \{ A_{x_1}, A_{x_2}, \ldots, A_{x_k} \} \) be a set of representatives of the equivalence classes of \( H/AN \), such that \( A_{g_i} = A_{x_i} A_{g_1} A_{x_i}^{-1} \). Choose for convenience \( A_{x_1} = e \). Now consider the complex vector space \( V^A \) spanned by the basis \( \{ A_{y_j}, \sigma_{v_j} >, j \in \{ 1, \ldots, k \} \} \), where \( \sigma_{v_j} \) denotes a basis element of the UIR \( \Pi^T \) of \( AN \). This vector space carries an irreducible representation \( \Pi^A \) of \( D^\omega(H) \) given by

\[
\Pi^A(\frac{g}{x}) A_{g_1} \sigma_{v_j} > = \delta_{g_2, x} \varepsilon g_2, x^{-1} \varepsilon g(x) | x A_{g_1} x^{-1}, \sigma T(\frac{A_{x}^{-1}x A_{x_i}}{A_{x}^{-1}x A_{x_i}}) \varepsilon_{v_j} >,
\]

with \( A_{x_k} \) defined through \( A_{g_k} = A_{y_j} A_{x_k}^{-1} \). The new ingredient here is the phase \( \varepsilon g \) that is related to \( \theta_{g} \) by

\[
\theta_{g}(x, y) = \frac{\varepsilon g(x, y)}{\varepsilon g(x, y)} = 1,
\]

in order to make (28) a representation. Note that (29) is equivalent to the statement that \( \theta_{g} \) is a 'conjugated' 2-coboundary, a property that is not automatically assured by (27). In passing, we also mention that it can be shown (e.g. \([6]\)) that the representation theory of \( D^\omega(H) \) indeed only depends on the cohomology class of \( \omega \) and not on the representative we choose in such a class.

We turn to the fusion rules of \( D^\omega(H) \). Let \( \Pi^A \) and \( \Pi^B \) once again denote irreducible representations of \( D^\omega(H) \). The tensor product representation \( \Pi^A \otimes \Pi^B \) defined by means of the co-multiplication (23) need not be irreducible. In general, it gives rise to a decomposition

\[
\Pi^A \otimes \Pi^B = \sum_{C, \gamma} N^{A B C}_{C, \gamma} \Pi^C,
\]

with \( N^{A B C}_{C, \gamma} \) the multiplicity of the irreducible representations \( \Pi^C \). In other words, the tensor product representation \( \Pi^A \otimes \Pi^B \) is completely reducible. Relation (30) is called a fusion rule of \( D^\omega(H) \). Phrased physically: it determines which excitations \( (C, \gamma) \) can be formed in non-elastic scattering processes among an excitation \( (A, \alpha) \) and an excitation \( (B, \beta) \). The fusion algebra spanned by the elements \( \Pi^A \) with multiplication rule (30) is commutative and associative. It can therefore be diagonalized by a single matrix, the so-called modular \( S \) matrix. For \( D^\omega(H) \), this matrix takes the form \([9]\)

\[
S_{A B}^{A B} = \frac{1}{|H|} \sum_{A_{g_1} \in A, B_{g_2} \in B} \alpha_{g_1}^{A_{g_1}^{-1}B_{g_2} A_{g_2}} \beta_{g_2}^{B_{g_2}^{-1}A_{g_1} B_{g_1}} \sigma(A_{g_1}, B_{g_2}),
\]

\[
364
\]
with \( \alpha(g) \equiv \text{tr} \, \alpha \Gamma(g) \) and the phase \( \sigma(g|h) \equiv \varepsilon_g(h)\varepsilon_h(g) \). We denote the order of the group \( H \) by \(|H|\). The modular \( S \) matrix contains all information about the fusion algebra. In particular, the multiplicities \( N_{\alpha\beta}\gamma \) can be obtained from the modular \( S \) matrix by means of Verlinde's formula \([21]\)

\[
N_{\alpha\beta}\gamma = \sum_{D,\delta} S_{\alpha\delta}^{AD} S_{\beta\delta}^{BD} (S^*)^{CD} \frac{S_{\delta\delta}^{ED}}{S_{\delta\delta}^{ED}}.
\]  

(32)

The elastic scattering processes between the excitations are governed by the braid operator \( \mathcal{R} \), associated with the \( R \) matrix \((25)\)

\[
\mathcal{R}_{\alpha\beta} = \sigma \circ (\Pi^A \otimes \Pi^B)(R),
\]

(33)

where \( \sigma \) effectuates a permutation. To be explicit, the braid operation \( \mathcal{R} \) on the state \( |A_{y_i}, \alpha_{v_j}\rangle \leq |B_{y_k}, \beta_{v_l}\rangle \in V^A \otimes V^B \) reads

\[
\mathcal{R}_{\alpha\beta} |A_{y_i}, \alpha_{v_j}\rangle = |B_{y_k}, \beta_{v_l}\rangle = \varepsilon_{H_{gm}}(A_{y_i}) |B_{y_m}, \beta_{\Gamma(B_{x_{m^{-1}}, m^{-1}} A_{y_i} B_{x_{k}})}\rangle \beta_{v_l} > |A_{y_i}, \alpha_{v_j}\rangle,
\]

(34)

where \( B_{x_{m}} \) is defined through \( B_{y_{m}} \equiv A_{y_{i}} B_{y_{k}} A_{y_{i}}^{-1} \). Note that \((34)\) incorporates the braid properties found in \((13)\) and \((21)\). For a non-trivial 3-cocycle \( \omega \), the braid operator \( \mathcal{R} \) does not satisfy the ordinary-, but rather the quasi Yang-Baxter equation

\[
\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_1 = \mathcal{R}_2 \mathcal{R}_1 \mathcal{R}_2.
\]

(35)

\( \mathcal{R}_1 \) acts on the three-particle states in the space \((V_1 \otimes V_2) \otimes V_3 = \mathcal{R} \otimes 1 \) and \( \mathcal{R}_2 \) as \( \Phi^{-1} \cdot (1 \otimes \mathcal{R}) \cdot \Phi \) with \( \Phi \equiv \Pi_1 \otimes \Pi_2 \otimes \Pi_3(\varphi) \), \( \Phi^{-1} \equiv \Pi_2 \otimes \Pi_1 \otimes \Pi_3(\varphi^{-1}) \) and \( \varphi \) the associator \((24)\).

The cross sections of elastic two-particle Aharonov-Bohm scattering are completely determined by the monodromy matrix \( \mathcal{R}^2 \)

\[
\frac{d\sigma}{d\varphi} = \frac{1}{2\pi k \sin^2(\varphi/2)} \frac{1}{2} [1 - \text{Re} \langle \psi_{in} | \mathcal{R}^2 | \psi_{in} \rangle],
\]

(36)

With \( |\psi_{in} \rangle \) the incoming two-particle state and \( k \) the relative momentum (recall that we are working with natural units \( h = e = 1 \)) \([4, 6, 23]\).

3.1. \( Z_N \) gauge theories revisited. We briefly illustrate these rather mathematical considerations by approaching the \( Z_N \) gauge theories (Section 2.1.) from the Hopf algebra point of view.

Every element \( m \in Z_N \) constitutes a conjugacy class and has the full group \( Z_N \) as its centralizer. The \( N \) different UIR's \( \Gamma \) of \( Z_N \) are all 1-dimensional and are given by \( \Gamma(m) = e^{i\frac{2\pi}{N} im}. \) Thus the representations of \( D^\omega(Z_N) \) in turn can be labelled as \( \Pi_n(m) \equiv (m, n) \), where \( m \) denotes an element of \( Z_N \) and \( n \) the \( Z_N \) representation \( \Gamma^\omega. \) With \( m \) the number of flux units \( 4\pi Ne \), and \( n \) the charge in units \( e/2 \), this is exactly the spectrum we found for a \( Z_N \) gauge theory.

It is wellknown \([22]\) that all \( H^{even}(Z_N, U(1)) \simeq 1 \), while \( H^{odd}(Z_N, U(1)) \simeq Z_N \). An explicit realization of \( \omega \in H^2(Z_N, U(1)) \) is given by

\[
\omega(m_1, m_2, m_3) = e^{i \frac{2\pi}{N} m_1 (m_2 + m_3 - [m_2 + m_3])},
\]

(37)

with \( p \in [0, \ldots, N - 1] \). It is easily inferred \([6]\) that \( \theta_{m_1}(m_2, m_3) = \gamma_{m_1}(m_2, m_3) = \omega(m_1, m_2, m_3). \) Consequently relation \((29)\) is solved by

\[
\varepsilon_{m_2}(m_2) = e^{i \frac{2\pi}{N} m_1 m_2}.
\]

(38)

With the help of \((38)\) we now obtain \((19)\) from \((28)\), \((18)\) from \((32)\), and \((13)\) from \((34)\).

Note that in this abelian example we find \( \mathcal{R}_2 = \mathcal{R}_2 \), because of the symmetry of \( \omega \) in the last two entries: \( \omega(m_1, m_2, m_3) = \omega(m_1, m_3, m_2) \). This implies that the quasi Yang-Baxter equation \((35)\) projects down to the ordinary Yang-Baxter equation \( \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_1 = \mathcal{R}_2 \mathcal{R}_1 \mathcal{R}_2 \). To proceed, the Aharonov-Bohm crosssections \((36)\) take the form

\[
\frac{d\sigma}{d\varphi}((m_1, n_1), (m_2, n_2)) = \frac{\sin^2 \frac{\pi}{N}(n_1 m_2 + n_2 m_1 + \frac{2\pi}{N} m_1 m_2)}{2\pi k \sin^2(\varphi/2)}.
\]

(39)

where the factor \( \frac{1}{\sin^2 \varphi/2} \) has to be replaced by \( \frac{1}{\sin^2 \varphi/2} + \frac{1}{\cos^2 \varphi/2} \) if the two particles are indistinguishable \([6]\).
3.2. Cheshire charges and Alice fluxes. In a non-abelian setting intriguing phenomena such as Alice fluxes and Cheshire charges arise [3, 6, 12]. We briefly discuss these phenomena in a $D_2$ gauge theory.

The dihedral group $D_2$ is of order 8 with 5 conjugacy classes $\{e\}, \{e, X_1, X_2\}, \{X_2, X_3\}, \{X_3, X_1\}$. There are four 1 dimensional UIR’s of $D_2$, and one 2-dimensional UIR. We will label the trivial UIR by 1, while the other 1-dimensional UIR’s will be denoted by $J_a$, $a = 1, 2, 3$. The UIR $J_a$ sends all elements into -1 except for $e, \tilde{e}, X_a, \tilde{X}_a$, which are represented by 1. The 2-dimensional UIR labelled by $\phi$ is obtained by sending $X_a/\tilde{X}_a$ into $\pm \sigma_a$ with $\sigma_a$ the Pauli matrices. This is the purely electric part of the spectrum of our $D_2$ gauge theory. Note that $\tilde{e}$ also got the full group $D_2$ as its centralizer. We label these dyonic excitations by overlining the UIR’s of $D_2$. The $X_a$ conjugacy classes have a $Z_4$ centralizer. In the remainder, we will only work with the purely magnetic fluxes, labelled by $\sigma_a^+$.

![Fig. 3. After the $\phi$ charge has encircled the Alice flux $\sigma_1^+$, the flux/anti-flux pair $\sigma_1^+$ and charge/anti-charge $\phi$ carry Cheshire charge $J_a$.](image)

<table>
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<th>$p=2$</th>
<th>$p=3$</th>
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<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>$\phi * \phi$</td>
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<td>$1 + \sum_b J_b$</td>
<td>$1 + \sum_b J_b$</td>
<td>$1 + \sum_b J_b$</td>
</tr>
<tr>
<td>$\sigma_1^+ * \sigma_1^+$</td>
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<td>$1 + J_a + \phi$</td>
<td>$1 + J_a + \phi$</td>
<td>$1 + J_a + \phi$</td>
</tr>
</tbody>
</table>

Table 1. The fusion rules (for different values of the CS parameter $p$) that play a role in the process depicted in Fig. 3.

The fusion rules are periodic in $p$ with period 4.

For the cocycle structure we use the general result $H^3(H, U(1)) \simeq \mathbb{Z}/H$ for subgroups $H$ of $SU(2)$ [22]. Recall that $|H|$ denotes the order of $H$. In our case this leads us to $H^3(D_2) \simeq \mathbb{Z}_8$, so there are in principle 8 different $D_2$ gauge theories. A numerical solution of the cocycle structure shows that there are only 4 different sets of fusion rules though, i.e., in terms of the fusion rules the CS parameter $p$ is periodic with period 4 (see Table 1 and [6]).

Now let us consider the process in which we start from the vacuum $1$ and at some time create a flux/anti-flux pair $\sigma_1^+$ and a charge/anti-charge pair $\phi$. Note that this is possible for all values of the CS parameter $p$, since the vacuum appears in the fusion rules $\phi * \phi$ and $\sigma_1^+ * \sigma_1^+$ (displayed in Table 1) irrespective of the value of $p$. Subsequently, the charge $\phi$ is taken around the flux $\sigma_1^+$, and fused with the other member of the $\phi$ pair again. In terms of quantum states, this process (depicted in Fig. 3) reads

$$1 \rightarrow 1 \otimes 1$$

$$\rightarrow \frac{1}{2} [|X_1 \rangle \otimes [\pi_v_1 \rangle \otimes [\pi_v_2 \rangle - |\phi_v_2 \rangle + |\phi_v_1 \rangle]$$
where $\phi_1 = (1, 0)$ and $\phi_2 = (0, 1)$ are the basis vectors of the 2 dimensional UIR $\phi$, in which the elements $X_a/X_a$ of $D_2$ are represented by $\pm \sigma_a$ with $\sigma_a$ the Pauli matrices. For convenience we only consider $a = 1$. After the $\phi$ charge has encircled the flux, the nature of the charge pair has changed. It is easily verified that the quantum state describing the $\phi$ pair behaves as a $J_1$ charge under $D_2$ gauge transformations, instead of the trivial charge $1$. This is the reason why $\sigma^+_1$ is also called an Alice flux [3]. Going around an Alice flux with a charge has a similar effect as going through the looking glass in Lewis Carroll’s ‘Alice's Adventures in Wonderland’. Actually, the analogy with these adventures can be pushed further. The charge $J_1$ of the $\phi$ pair is not an ordinary charge. It is a property of the pair, it cannot be localized on the constituents of the pair nor anywhere else, just as the elusive smile of the Cheshire cat in ‘Alice’s Adventures in Wonderland’. Therefore the name Cheshire charge has been coined [3]. After amalgamating the members of the $\phi$ pair the Cheshire charge $J_1$ becomes an ordinary localized $J_1$ charge. Similar observations appear for the flux pair. If we recall (see (20)) that the gauge group $\tilde{D_2}$ acts by means of conjugation on the fluxes, we see that the $\sigma^+_1$ pair is also endowed with a Cheshire flux $J_1$ after the charge $\phi$ has encircled the Alice flux $\sigma^+_1$. This Cheshire charge becomes a localized $J_1$ charge upon fusing the members of the flux pair, and, as Table 1 indicates, the two $J_1$ charges we are left with now can be annihilated into the vacuum. So this local process did not alter any global properties, just as it should. Note that the cocycles do not enter this braid process involving the pure charge $\phi$. It only enters braid processes among fluxes, as is clear from the $Z_N$ example.

ACKNOWLEDGEMENTS

M. de W.P. would like to thank the organizers for their kind invitation, and for the enormous hospitality he has enjoyed in Moscow and Alushta.

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