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DOI

[10.1016/0370-2693\(94\)90070-1](https://doi.org/10.1016/0370-2693(94)90070-1)

Publication date

1994

Published in

Physics Letters B

[Link to publication](#)

Citation for published version (APA):

Gaite, J. (1994). Singularity Theory for W-Algebra Potentials. *Physics Letters B*, 325(1-2), 51-57. [https://doi.org/10.1016/0370-2693\(94\)90070-1](https://doi.org/10.1016/0370-2693(94)90070-1)

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Singularity theory for W -algebra potentials^{*}

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Received 4 January 1994

Editor: P.V. Landshoff

Abstract

The Landau potentials of W_3 -algebra models are analyzed with algebraic-geometric methods. The number of ground states and the number of independent perturbations of every potential coincide and can be computed. This number agrees with the structure of ground states obtained in a previous paper, namely, as the phase structure of the IRF models of Jimbo et al. The singularities associated to these potentials are identified.

1. Introduction

After the remarkable work of Zamolodchikov (see also Ludwig-Cardy) [1] showing how to construct Landau-Ginzburg potentials for the minimal models of 2d CFT, there have been many interesting extensions and generalizations. In particular, we saw in a previous paper [2] that Zamolodchikov's method with some qualifications works perfectly for the natural generalization of the Virasoro minimal models to higher discrete symmetry, namely, the minimal models of W -algebras. We found the corresponding lagrangians and analyzed the phase structure that they entail. The chief result was the construction of a perturbation that produces the desired state diagrams, namely, dominant-weight diagrams of the corresponding Lie algebra, thus reproducing the phase structure of the IRF models of Jimbo et al. [3]. In the case of $W_{(3)}$, they are triangular diagrams with a particular structure [2].

However, in absence of further analysis we cannot assert that the state diagram mentioned above is the maximum possible unfolding of ground states. Neither do we know that the algebra of perturbations cor-

responds to the relevant fields of a definite model. In fact, for all we know, it might even not be finite. In comparison, these problems do not arise for the potentials of the minimal Virasoro models, since they coincide with the ADE singularities, the resolution of which is standard in the literature [4].

This paper is devoted to establish the unfolding of extremal points and dimension of perturbation algebras of $W_{(3)}$ -potentials and to prove therefore that they are well-defined and just right to describe the desired phase structure. We should point out that both features, namely, number of extremal points and dimension of the perturbation algebra, are intimately related and actually coincide, as will be explained below. We will work out in detail the low- p cases, explicitly exhibiting the perturbation algebras. Finally, we find that these potentials correspond to a simple series of two-variable singularities, of which the first two are already known.

2. Ground states and algebra of perturbations. General methods

Let us recall the Landau potential for $W_{(3)}^p$

^{*} Research supported in part by CICYT.

$$\begin{aligned}
 V &\simeq \sigma^{p-4} \bar{\sigma}^{p-1} + (\sigma \bar{\sigma})^{p-2} + \text{c.c.} \\
 &= (\sigma \bar{\sigma})^{p-4} (\sigma^3 + \bar{\sigma}^3) + (\sigma \bar{\sigma})^{p-2}. \quad (1)
 \end{aligned}$$

From now on, we consider the complexified potential, letting σ and $\bar{\sigma}$ take arbitrary complex values and we rename them as x and y , respectively^{#1}. Since numerical coefficients were disregarded in obtaining (1) and they not play any role in this paper either, we will choose them as to simplify the calculations. Thus we write the potential as

$$V = \frac{(xy)^{p-2}}{p-2} - (xy)^{p-4} (x^3 + y^3). \quad (2)$$

It gives two algebraic equations for the equilibrium (extremal) points,

$$\begin{aligned}
 V_x &\equiv \partial_x V \\
 &= y (xy)^{-5+p} \\
 &\quad \times ((-p+1)x^3 + x^2 y^2 + (-p+4)y^3) \\
 &= 0, \quad (3)
 \end{aligned}$$

$$\begin{aligned}
 V_y &\equiv \partial_y V \\
 &= x (xy)^{-5+p} \\
 &\quad \times ((-p+4)x^3 + x^2 y^2 + (-p+1)y^3) \\
 &= 0. \quad (4)
 \end{aligned}$$

Of course, we are only interested in the real solutions, such that $y = \bar{x}$ and the equations are conjugate of one another. Since they are of degree $2p-5$, one should expect either a maximum of $(2p-5)^2$ solutions or an infinite number, according to Bezout's theorem. In geometrical language, if the two algebraic curves intersect at a number of points larger than the product of their degrees, they must have a branch in common. This latter possibility must be excluded; it implies that the potential is not determinate, namely, there are higher degree terms that cannot be removed by diffeomorphisms [5]. However, it is clearly the general case in (3), (4), where the common factor xy appears in both equations when $p \geq 6$. This problem was already anticipated in [2], as well as its solution, namely, to consider further terms in the potential. We shall come back to it in Section 5, where we will construct these terms.

On the other hand, the number $(2p-5)^2$ of solutions is still larger than the number of extremal points in the state diagram we seek to reproduce, as will be seen more concretely below. Nevertheless, it can still happen that the structure of those equations is such that the maximum number of solutions is never reached. To understand why, it is convenient to resort to the geometrical picture: Although the two algebraic curves must have the $(2p-5)^2$ intersection points (counting multiplicities) in projective space, it can occur that some of them are permanently placed at infinity. Let us see that it is the case. If we homogenize the equations by adding a third variable z and we make $z=0$ to find the points at infinity, we immediately see that some of them are common to both curves. Furthermore, this property is not altered when we introduce perturbations.

The method we will use to establish the actual number of extremal points is half way to solve Eqs. (3), (4). The way one should attempt to solve a pair of equations with two unknown variables is by first eliminating one variable and, therefore, one equation, thus reducing the problem to an ordinary equation in one variable. The systematic way to perform this operation utilizes an algebraic object called resultant. The resultant of two polynomials in one variable is defined as the simplest algebraic object that vanishes whenever they have a common root; namely, the product of differences of roots of each polynomial. Since it is symmetric in each set of roots, it can be expressed in terms of the coefficients of the polynomials. This expression is obtained in algebra textbooks but will not be needed. Now, if we regard our two polynomials (3), (4) as polynomials in one variable, y say, with coefficients that are polynomials in the other, x , their resultant is also a polynomial in x . Moreover, for those values of x that make it null the Eqs. (3), (4) will be satisfied. In fact, according to our purpose, we shall not attempt to solve this equation in x but rather to determine the number of solutions; this can be done just by inspection.

We consider now the algebra of perturbations of a potential V . This algebra is given by the quotient of all possible deformations by those that are induced by diffeomorphisms [5],

$$Q = \frac{P[x, y]}{(V_x, V_y)}.$$

^{#1} Do not confuse with the usual notation in which x and y are real variables defined by $\sigma = x + iy$, as in [2].

Alternatively, we can consider, in place of the potential, the curves (3), (4). Each curve has naturally associated to it the ideal of polynomial functions that vanish on it, namely (V_x) or (V_y) . The algebras of polynomial functions on the curves are $Q_x = P[x, y]/(V_x)$ and $Q_y = P[x, y]/(V_y)$, respectively. The algebra of polynomials on the intersection is Q . Since we are assuming that this intersection is just a finite set of points, Q is finite and contains as many elements as there are intersection points (counting multiplicities) [6].

In conclusion, we have a suitable method to obtain the number of extremal points or elements of the perturbation algebra. Nevertheless, to actually determine the ones or the others, one is to undertake detailed computations: Respectively, to completely solve Eqs. (3), (4) to find the extremal points or to use these equations to eliminate all perturbations that they generate and see which ones are left.

3. The $W_{(3)}^4$ potential and its relation to the D_4 singularity

The potential for the first model in the series, $W_{(3)}^4$, is well known. It was described in [7] in the context of symmetric catastrophe theory. There were found the six relevant perturbations; with the addition of the identity, a total number of seven independent perturbations. Correspondingly, it was shown that the unfolding consists of seven points in a triangular array, which is the first instance of the series later analyzed in [2]. We now intend to reobtain these results within our present notation and objectives.

The potential

$$V = \frac{x^2 y^2}{2} - \frac{x^3 + y^3}{3} \quad (5)$$

produces the equations

$$V_x = -x^2 + x y^2 = 0, \quad (6)$$

$$V_y = x^2 y - y^2 = 0. \quad (7)$$

Given their simplicity, we can actually solve them. We thus have (0,0) with multiplicity 4 and 3 other $(x = \text{cubic root of } 1, y = x^2)$, seven solutions altogether. It is indeed fewer than the expected number for two third-degree equations, which is nine. A more convenient

way to realize this fact is to compute the resultant^{#2},

$$\text{Resultant}[V_x, V_y, y] = x^4 - x^7. \quad (8)$$

From the geometrical point of view, the cause is clear: The curves (6), (7) comprise two paraboles and two intersection points are placed at infinity.

In this case, it is also straightforward to calculate the perturbation algebra Q . It is clear that it contains all monomials up to the second degree, $1, x, y, x^2, y^2, x y$. At the third degree, we have to exclude right away two monomials, $x y^2, x^2 y$, because of Eqs. (6), (7). Furthermore, we can easily obtain $x^3 = y^3$, thus only leaving $x^3 + y^3$. Now, it is not difficult to see that every remaining monomial can be expressed in terms of the ones just selected. For example, $x^4 = x y^3 = x^2 y = y^2$.

We must remark that, strictly speaking, $x^3 + y^3$ cannot be considered a perturbation for if it is removed from the potential, the remaining quartic term is the degenerate double-cusp catastrophe, an indeterminate potential. On the other hand, the quartic term itself can be safely removed: the cubic term constitutes by itself a well defined and known potential, the D_4 singularity of the ADE classification. We can further appreciate this point by observing that the unperturbed potential has a quadruple extremal point at the origin and three distinct others. The latter are obviously irrelevant for the singularity, although not for the potential as a whole, since they are precisely the ground states.

4. The $W_{(3)}^5$ potential. Arnold's N_{16} singularity

The next case, $W_{(3)}^5$, has the potential

$$V = \frac{x^3 y^3}{3} - x y (x^3 + y^3) \quad (9)$$

which has not been thoroughly analyzed before. We know, however, that a definite symmetric perturbation will produce the desired triangular array of 19 extremal points, of which 6 are minima [2]. Proceeding as above, we consider the equations

$$V_x = -4 x^3 y + x^2 y^3 - y^4 = 0, \quad (10)$$

$$V_y = -x^4 + x^3 y^2 - 4 x y^3 = 0. \quad (11)$$

^{#2} We use throughout the Mathematica object Resultant.

Despite their unfriendly aspect, they are solvable. However, we must not bother to actually solve them, since we can get similar information from the resultant,

$$\text{Resultant}[V_x, V_y, y] = 3375 x^{16} - 27 x^{19}. \quad (12)$$

We see that we have indeed a maximum of 19 solutions, 16 of which are degenerate at (0,0), while the other 3 are again cubic roots.

We are now to calculate the perturbation algebra Q . Right away, we can count in the 15 monomials up to the fourth degree. At the fifth degree, we must exclude two, according to (10), (11). Furthermore, less obviously, $x^4 y = y^4 x$. Thus we have left $x^5, y^5, x^4 y + y^4 x$. At sixth degree, there is an independent one yet, $x^6 + y^6$, whereas the remaining and any of higher degree are reducible to those already mentioned. The total number is 19, of course.

We have to face now the problem analogous to that of the previous potential; namely, we cannot consider $x^4 y + y^4 x$ as a perturbation, for the same reason. Moreover, we have three other terms, x^5, y^5 , and $x^6 + y^6$, that do not fit the expected perturbations for this potential. We are thus well advised to resort to singularity theory to further analyze this potential. In analogy with the previous case, we may expect the term $x^4 y + y^4 x$ to furnish the appropriate singularity. Let us see whether it has the correct properties. First of all, we know from (12) that its critical point must be 16-fold degenerate. It is easy to convince oneself that removing the 6th degree term from the potential implies reducing the resultant to just x^{16} . Besides, the candidate singularity has already been identified and named N_{16} ; it has codimension 12 and 3 modular parameters, associated to the monomials $x^2 y^3, y^2 x^3$ and $x^3 y^3$ [4].

Therefore, we now realize that the term $x^3 y^3$ in the potential is just a modular deformation that needs not be present^{#3}. Then, Eqs. (10), (11) become homogeneous of 4th degree, and indicate that two monomials of this degree do not belong to the perturbation algebra; let them be x^4 and y^4 . The remaining 13 monomials up to degree 4th account for the codimension 12. Besides, there are two other monomials in the

perturbation algebra, $x^2 y^3 = x^5/4$ and $y^2 x^3 = y^5/4$, corresponding to modular deformations.

It is pertinent to point out that there is another form of the N_{16} singularity more suited to computations, namely $x^5 + y^5$. However, it does not have D_3 symmetry (it has D_5 instead) and is thus not convenient for our purposes.

5. $W_{(3)}^6$ and the general $W_{(3)}^p$ case. The series W^l of singularities

The potential of $W_{(3)}^6$,

$$V = x^4 y^4 - x^2 y^2 (x^3 + y^3) \quad (13)$$

is incomplete, as was mentioned in Section 2. It is easy to see that essentially we have only two possible additions with the necessary symmetry, $x^6 + y^6$ or $x y (x^6 + y^6)$. The second one yields a resultant of degree 49th, showing that the number of extremal points is the maximum allowed by the degree of the potential. The first one yields

$$\begin{aligned} \text{Resultant}[V_x, V_y, y] = & -1024 x^{25} + 23168 x^{28} \\ & -25120 x^{31} + 8944 x^{34} - 1024 x^{37}, \end{aligned} \quad (14)$$

with 37 extrema. They account for the 10 minima, 18 saddle points and 9 maxima of the corresponding triangular array, as was depicted in [2, Fig. 1]. With some effort one could equally obtain the 37 elements of Q but it is not worthwhile.

Let us now consider the type of singularity of this potential. The multiplicity at the origin is 25, according to (14). This is precisely the multiplicity of $V = x^6 + y^6$, as can be seen at once from the equations $V_x = 0$ and $V_y = 0$ ^{#4}. As far as we know, this singularity has never been studied in the literature. It has codimension 18 and 6 modular parameters. One of these modular parameters corresponds to $x^4 y^4$, of course.

We should attempt to generalize the previous results to arbitrary p . The first problem is to find out the correct way to complete the potential. To this end, we can appeal to the results in [2]. Let us recall that we obtained there the algebraic curve that considered as a level curve of some potential gives the necessary extremal-point structure. That curve was the product

^{#3} We must remark again though that it is essential for the potential to have absolute minima and, in particular, the minima that we expect.

^{#4} More generally, the multiplicity of $x^n + y^n$ is $(n-1)^2$.

$$\prod_{i=1}^{(l+1)/2} (I_0^2 + I_1 + w_i I_0) \\ = I_0^{l+1} + I_0^{l-1} I_1 + \dots = 0, \quad (15)$$

for l odd, or

$$\prod_{i=1}^{l/2} (I_0^2 + I_1 + w_i I_0) (I_0 - c) \\ = I_0^{l+1} + I_0^{l-1} I_1 + \dots = 0, \quad (16)$$

for l even. The point was that the two first terms coincide with those of the potential (2) if $l = p - 3$. However, it must be noticed that there are further terms that should be regarded as belonging to the unperturbed potential, since they remain when all perturbation parameters, w_i and c , are made null. Nevertheless, there is an excessive number of monomials in x and y ; e.g., the lowest degree term in (15), $I_1^{(l+1)/2}$, or in (16), $I_0 I_1^{l/2}$, produce when expanded a string of monomials with only two essential ones, $x^{3(l+1)/2} + y^{3(l+1)/2}$ or $(x y) (x^{3l/2} + y^{3l/2})$, respectively. Keeping only the essential terms, we are thus led to propose a potential

$$V = x^{p-2} y^{p-2} - x^{p-4} y^{p-4} (x^3 + y^3) \\ + x^{p-6} y^{p-6} (x^6 + y^6) + \dots \\ + \left(x^{3(p-2)/2} + y^{3(p-2)/2} \right) \quad (17)$$

for p even, or

$$V = x^{p-2} y^{p-2} - x^{p-4} y^{p-4} (x^3 + y^3) \\ + x^{p-6} y^{p-6} (x^6 + y^6) + \dots \\ + x y \left(x^{3(p-3)/2} + y^{3(p-3)/2} \right) \quad (18)$$

for p odd, which nicely generalizes that of (13) to larger p .

It remains to be proved that this potential (17) or (18) has the properties that we found for those up to $p = 6$; namely, the maximum number of extrema is as the corresponding triangular array and the independent perturbations can be obtained accordingly. Unfortunately, the form of (17) or (18) as a sum of a number of terms proportional to p makes it unsuitable for the algebraic computation of the resultant. Nevertheless, we have performed the calculations up to $p = 11$ and checked that the resultant has the expected degree.

The results for $p = 7$ to 9 are shown in the appendix. We believe that these partial results, beside their own interest, provide some evidence for the general case.

An important result can be directly read off from (17), (18), namely, the series of singularities associated to our potentials:

$$x^{3(l+1)/2} + y^{3(l+1)/2}, \quad l = 1, 3, 5, \dots,$$

or

$$(x y) \left(x^{3l/2} + y^{3l/2} \right), \quad l = 2, 4, 6, \dots.$$

We think that W^l is a convenient name for this series^{#5}; then $W^1 = D_4$ and $W^2 = N_{16}$. The latter form above, when l is even, is equivalent to $(x^{(3l/2)+2} + y^{(3l/2)+2})$. Hence, we see that both belong to a simple series of two-variable singularities, $x^n + y^n$. They have multiplicity $\mu = (n - 1)^2$, moduli $m = (n - 3)(n - 2)/2$ and codimension $c = (n + 3)(n - 2)/2$ such that they fulfill $\mu = c + m + 1$ [4]. Among these large moduli are the other terms in the complete potential (17), (18), as well as those inessential terms in (15), (16) that were discarded.

6. Conclusions

The Landau potentials of W_3 models are analyzed in order to ascertain whether they fit the phase structure of the IRF models of Jimbo et al. Their sufficiency to this purpose has been established in a previous paper [2] by constructing a perturbation that produces the required configuration of extrema. This configuration is now shown to be the maximum possible unfolding. The method used only allows to consider a finite number of cases. The reason is that it is based on an algebraic study of the equations for extrema, which final step is to obtain the resultant with Mathematica. This is done sequentially for increasing p . It has been carried out up to $p = 11$ but only the cases up to $p = 9$ are displayed. The potentials for $p = 4$ and 5 deserve a more detailed study; their algebras of perturbations are also obtained.

^{#5} We follow Arnold in giving names to series of singularities [4]. In fact, he has already used most letters to this effect, including W . Nevertheless, we think that it is the adequate letter here and that no confusion should arise since he places the other letter(s) as subscript whereas we have placed l as superscript.

A necessary step to attain these results is to complete the potentials, which are otherwise indeterminate. This is done in general, yielding a sum of terms of decreasing degree, the last of which depends on the parity of p (17), (18). Precisely these last terms give the singularities of these potentials, while the others can be regarded as modular terms. Of course, this is the mathematical viewpoint, which disregard the physical significance of the various terms. An analysis along a more physical line should require us to resort to the actual fields associated to the monomials according to the identifications realized in [2] and is beyond the scope of the present paper.

7. Acknowledgement

I am grateful to Jean-Bernard Zuber for asking the right questions and for reading this manuscript.

8. Appendix

We show here that the resultants for the $p = 7, 8$ and 9 potentials precisely account for the number of extrema in the corresponding diagrams. Let us first calculate the number of extrema in the p th triangular diagram, which can be done in full generality by clever use of triangular numbers. This diagram has $p - 2$ minima on each side. Hence, the total number of minima is $(p - 2)(p - 1)/2$. There is exactly one saddle point between each pair of minima, so that saddle points and minima form together a triangular array of side $2p - 5$. Therefore, their total number is $(2p - 5)(2p - 4)/2$. The distribution of maxima is a little more involved; however, they can be split in two triangles of sides $p - 3$ and $p - 4$, the total number being $(p - 3)^2$. Adding both numbers, we obtain $3p(p - 5) + 19$ extrema altogether. In particular, 61, 91 and 127 for $p = 7, 8$ and 9, respectively.

Now we show the results of a computation of the resultants with Mathematica. The potential, equations and resultant are written for each case separately:

– $p = 7$:

$$V = xy \left(x^4 y^4 - x^2 y^2 (x^3 + y^3) + \frac{x^6 + y^6}{3} \right), \quad (19)$$

$$V_x = \frac{7x^6 y}{3} - 6x^5 y^3 + 5x^4 y^5 - 3x^2 y^6 + \frac{y^7}{3} = 0, \quad (20)$$

$$V_y = \frac{x^7}{3} - 3x^6 y^2 + 5x^5 y^4 - 6x^3 y^5 + \frac{7xy^6}{3} = 0, \quad (21)$$

$$\begin{aligned} \text{Resultant}[V_x, V_y, y] \\ = \frac{12230590464}{1594323} x^{49} - \frac{246594502656}{1594323} x^{52} \\ + \frac{249221422080}{1594323} x^{55} - \frac{81199756125}{1594323} x^{58} \\ + \frac{8303765625}{1594323} x^{61}. \end{aligned} \quad (22)$$

– $p = 8$:

$$V = x^6 y^6 - x^4 y^4 (x^3 + y^3) + x^2 y^2 (x^6 + y^6) + \frac{x^9 + y^9}{3}, \quad (23)$$

$$V_x = 3x^8 + 8x^7 y^2 - 7x^6 y^4 + 6x^5 y^6 - 4x^3 y^7 + 2x y^8 = 0, \quad (24)$$

$$V_y = 2x^8 y - 4x^7 y^3 + 6x^6 y^5 - 7x^4 y^6 + 8x^2 y^7 + 3y^8 = 0, \quad (25)$$

$$\begin{aligned} \text{Resultant}[V_x, V_y, y] \\ = 43046721 x^{64} + 2027978856 x^{67} \\ - 75467102058 x^{70} + 2349506627136 x^{73} \\ - 26268032446551 x^{76} + 172417135016664 x^{79} \\ + 42570239554944 x^{82} + 11049268755456 x^{85} \\ + 577322090496 x^{88} + 990677827584 x^{91}. \end{aligned} \quad (26)$$

– $p = 9$:

$$V = x^7 y^7 - x^5 y^5 (x^3 + y^3) + x^3 y^3 (x^6 + y^6) + xy (x^9 + y^9), \quad (27)$$

$$V_x = 10x^9 y + 9x^8 y^3 - 8x^7 y^5 + 7x^6 y^7 - 5x^4 y^8 + 3x^2 y^9 + y^{10} = 0, \quad (28)$$

$$V_y = x^{10} + 3x^9y^2 - 5x^8y^4 + 7x^7y^6 - 8x^5y^7 + 9x^3y^8 + 10xy^9 = 0, \quad (29)$$

Resultant[V_x, V_y, y]

$$\begin{aligned} &= -913517247483640899x^{100} \\ &- 2700744078773949615x^{103} \\ &+ 14078078713498713606x^{106} \\ &- 54656571881643296139x^{109} \\ &+ 62947451766811208022x^{112} \\ &- 62526864228499636560x^{115} \\ &- 8771826922328936340x^{118} \\ &- 1233504732780966240x^{121} \\ &- 38877206300966880x^{124} \\ &- 67659951487955904x^{127}. \end{aligned} \quad (30)$$

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