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DOI

[10.1103/PhysRevB.49.4593](https://doi.org/10.1103/PhysRevB.49.4593)

Publication date

1994

Published in

Physical Review. B, Condensed Matter

[Link to publication](#)

Citation for published version (APA):

Cohen, M. H., & Pruisken, A. M. M. (1994). Mesoscopic Block Models for Macroscopic Conductances. *Physical Review. B, Condensed Matter*, 49, 4593.
<https://doi.org/10.1103/PhysRevB.49.4593>

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Mesoscopic block models for macroscopic conductances

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(Received 17 December 1992)

The existence of mesoscopic conductance fluctuations invalidates the traditional concept of impurity average conductances even for metallic systems which are much larger than the inelastic-scattering length. We use the methodology of classical conductance networks in order to incorporate the effect of mesoscopic conductance fluctuations in a self-consistent extended scheme for computing macroscopic conductances. Scaling ideas are reconsidered with a main emphasis on the two-dimensional disordered electron gas and, in particular, the integral quantum Hall regime. We find that the singularity structures of the theories of “macroscopic” and “impurity averaged” conductances are quite generally the same. Significant numerical differences may occur, however, in the regime of strong quantum interference effects. Specific experimental work on the integral quantum Hall regime is being proposed for the purpose of resolving some of the outstanding questions raised in this paper.

I. INTRODUCTION

Following the work of Abrahams *et al.*,¹ it has become conventional to focus in the theory of transport in isotropic disordered materials on the scaling properties of the conductance tensor $\mathbf{G}$ with length. The scaling length, i.e., the maximum dimension over which the elastic scattering associated with translational disorder acts unimpeded, is usually fixed by inelastic-scattering processes² and identified as the inelastic diffusion length L_{in} . The scaling variable X is the ratio of this length to the localization length ξ_l when the Fermi energy is in the region of localized states, or to the correlation length ξ_e for amplitude fluctuations when it is in the region of extended states.³

The starting point for all theoretical analyses done to date is the Kubo-Greenwood formula or some equivalent representation of the conductance, averaged over an ensemble of realizations of the disorder in the material. It is implicitly assumed that this “averaged” conductance $\bar{G}$ is identical to the “measured” or “macroscopic” conductance G_M except when the sample size L is comparable to L_{in} . It is well recognized that in the latter case G_M fluctuates with L , with the Fermi level, and from sample to sample.⁴

It is the purpose of the present paper to call attention to another regime in which G_M is sharply distributed ($L \gg L_{in}$) but can differ significantly from $\bar{G}$ because of conductance fluctuations. The main idea is that inelastic-scattering events not only provide an effective sample size for scaling, but that the breaking of phase coherence also implies that the local conductance parameters which one can assign to each area of linear dimension L_{in} become statistically independent. These local or mesoscopic conductances can be regarded as static since the elastic collision time τ_e is much shorter than the inelastic relaxation time at low enough temperatures. It is therefore appropriate to consider the macroscopic conductance process in terms of a semiclassical random conductance network which consists of “blocks” of meso-

scopic size L_{in} , each block representing a different realization of an impurity ensemble and having a statistically independent local conductance value.

A direct consequence of our approach is that the internal electric field in a macroscopic sample no longer appears as a constant as assumed somewhat unrealistically in the standard applications of the Kubo formalism. Instead, it “fluctuates” along with the local conductances in order for the continuity equations to be satisfied. For conductance distributions which are not too broad it is known that the macroscopic or measured transport parameters can be obtained with remarkable accuracy⁵ from the effective-medium theory (EMT).^{6–8} This paper then is aimed toward developing a scaling theory for macroscopic conductances which incorporates the effect of conductance fluctuations, or rather conductance “distributions,” within the techniques of the EMT.

It is self-evident that our model of “mesoscopic” blocks presupposes a disorder-dominated electronic system which is adequately described in terms of Landau quasiparticles. The main difference with the more conventional philosophy is that the Coulomb interactions, besides being a source of inelastic scattering at finite temperatures, also act as the source of the internally generated macroscopic electric field entering the constitutive relation.⁹ The existence of conductance fluctuations now forces the internal electric field to be spatially inhomogeneous, the effect of which is accounted for approximately by our use of the EMT.

Notice that a complete microscopic theory with a first-principles treatment of the internally generated electric field does not exist even in the semiclassical limit of weak localization. Nevertheless, there are very good reasons to believe that the concepts of macroscopic and averaged conductances differ significantly, especially for problems dominated by strong quantum interference effects. For instance, in Sec. II of this paper we shall discuss the standard example of localization of spinless electrons in two dimensions.

One of the principal statements made by the renormal-

ization theory of localization is that the entire scenario in two-dimensions (2D) ranging from weak localization to strong Anderson localization may be understood in terms of an (infinite) sequence of renormalization-group transformations. As pointed out in more detail by one of us in Ref. 3, the conventional wisdom may be formulated by saying that the physical parameters of the theory, to which belong the ensemble-averaged conductance $\bar{\sigma}$, the variance as well as all higher-order moments of the conductance, become functions of the single scaling variable L/ξ , where ξ is the localization length of the levels near the Fermi energy. The distinction between weak and strong localization now translates into a simple statement made about the scaling variable L/ξ (i.e., $L < \xi$ and $L \gg \xi$, respectively) with the understanding, as already noted in Ref. 4, that the probability distribution of σ crosses over from a narrow Gaussian one in weak localization ($L \ll \xi$) to a broad and presumably log-normal one in the strongly localized regime ($L \gg \xi$).

Within the EMT, the macroscopic conductance is defined as an implicit functional of the probability distribution. Application of the EMT to the case at hand next indicates that the macroscopic and averaged conductances start to differ significantly when the variance and $\bar{\sigma}^2$ become comparable. This corresponds, roughly speaking, to the crossover point where $L \simeq \xi$ and $\bar{\sigma} \simeq e^2/h$, i.e., the onset of strong quantum interference effects in two dimensions.

This discussion of ordinary Anderson localization motivates our investigation in Sec. III of possible semiclassical effects in the quantum Hall regime¹⁰ for which detailed experimental data are available taken from dirty heterostructures.^{11–13} Following the predictions of the renormalization theory for spin-polarized free electrons (Fig. 1),¹⁴ we there deal with the more interesting critical zero-temperature singularities of a metal-insulator transition which describe the delocalization of the levels near the center of the Landau band. This transition is purely topological in origin,¹⁵ and by all standards a strong

quantum interference phenomenon.³

An extension of the renormalization theory to include the fluctuations in the dissipative conductance σ_{xx} and Hall conductance σ_{xy} has recently been developed by one of us,³ and is summarized briefly in Sec. III A. The physical parameters of this theory, i.e., the averaged conductance $\bar{\sigma}_{\alpha\beta}$, the variance-covariance matrix, and the higher-order moments, can now be expressed in terms of the two scaling variables L/ξ_{cr} and L/ξ_l (Fig. 1). The former (L/ξ_{cr}) formally describes the crossover between the weak quantum interference singularities at $\bar{\sigma}_{xx} \gg e^2/h$ and the strong quantum interference critical singularities of the integral quantum Hall regime with $\bar{\sigma}_{xx}^* \lesssim e^2/h$ (Fig. 1). The latter (L/ξ_l) appears as the only relevant scaling variable in the strong-magnetic-field experiments (with ξ_l = localization length of the levels near E_F), and it describes the crossover between the “metallic” critical fixed point with a diverging ξ_l at $\bar{\sigma}_{xy}$ = half-integer and the “quantum Hall” fixed point at $\bar{\sigma}_{xy}$ = integer (Fig. 1).

In the Appendix we derive the EMT formulas for the macroscopic conductances σ_{xx}^M and σ_{xy}^M which are now given as two coupled integral equations involving the probability density for both σ_{xx} and σ_{xy} . The general structure of these equations, together with results of Ref. 3, allows us to make several very interesting statements about the scaling behavior of the macroscopic conductances (Sec. III B). For instance, since $\bar{\sigma}_{\alpha\beta}$ and $\sigma_{\alpha\beta}^M$ are related in a regular (i.e., nonsingular) fashion, the renormalization-group flow diagram for $\sigma_{xx}^M, \sigma_{xy}^M$ is essentially the same as the original one, expressed in terms of $\bar{\sigma}_{xx}, \bar{\sigma}_{xy}$ (Fig. 1). The main difference is in the absolute value for the fixed point σ_{xx}^* at σ_{xy} = half-integer (Fig. 1) which now depends explicitly on the details of the distribution of the mesoscopic conductances. This dependence, however, turns out to be marginal even for broad but realistic distributions with a variance comparable to or larger than the mean, squared. The remarkable “overall” stability of the σ_{xx}, σ_{xy} renormalization diagram, quite unobvious at the outset, is due to the fact that the mesoscopic fluctuations in σ_{xx} and σ_{xy} have an opposite effect on the numerical value of the macroscopic σ_{xx}^M (Sec. III C).

It is gratifying to conclude from our analysis that the inclusion of a highly nontrivial physical phenomenon does not alter the main predictions of the renormalization theory, and only sharpens the universality aspects. On the other hand, our block theory can be used to explain at least part of the numerical discrepancy between theory and experiment, as will be discussed in Sec. III C.

In Sec. III D, we discuss an extension of our block model which incorporates the statistics of inelastic-scattering times. We then investigate whether this more realistic description possibly enforces limitations on observing scaling. We conclude with a summary (Sec. IV).

II. THE ANDERSON PROBLEM IN 2D ($B = 0$)

Information about the probability distribution for the conductance of 2D disordered films may be obtained

FIG. 1. Renormalization-group flow diagram of “averaged” conductances $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{xy}$. The arrows indicate the scaling toward the infrared ($L \rightarrow \infty$). The ξ_{cr} and ξ_l are the characteristic length scales in the problem describing the crossover between weak localization ($\bar{\sigma}_{xx} \rightarrow \infty$) and criticality at $\bar{\sigma}_{xx}^*$, and between criticality and the quantum Hall fixed points at $\bar{\sigma}_{xy}$ = integer, respectively.

from the usual perturbation expansion in the weak quantum interference regime ($\bar{\sigma}_{xx} \rightarrow \infty$). A standard result obtained from the Kubo-Greenwood formula for σ , averaged over the impurity ensemble, may be written as

$$\bar{\sigma} = \sigma^0 - \beta_1 \ln \frac{L}{L_0} + O((\sigma^0)^{-1}), \quad (1)$$

which is the well-known logarithmic correction to the Drude result $\sigma^0 = ne^2\tau/m^*$. Here L_0 denotes the mean free path for elastic scattering, whereas L stands for an effective sample size determined by inelastic-scattering processes (i.e., $L = L_{in} \propto T^{-p/2}$). In analogy to what is known in asymptotically free field theories, we can interpret the Anderson localization problem in terms of a “dynamically” generated correlation length ξ by writing³

$$\bar{\sigma} = -\beta_1 \ln \frac{L}{\xi}, \quad \xi = L_0 e^{\sigma^0/\beta_1}, \quad (2)$$

where ξ now stands for the localization length of the electronic levels near E_F . On the other hand, the moments of the conductance have been obtained as a power series in $(\bar{\sigma})^{-1}$, e.g.,^{3,4}

$$\begin{aligned} \Delta\sigma^2 &\equiv (\sigma - \bar{\sigma})^2 = c_0 + c_1 \bar{\sigma}^{-1} + O(\bar{\sigma}^{-2}), \\ (\sigma - \bar{\sigma})^3 &= d_1 \sigma^{-1} + O(\bar{\sigma}^{-2}). \end{aligned} \quad (3)$$

Equations (2) and (3) imply that the probability density $P[\sigma; (L/\xi)]$ depends on the single scaling variable L/ξ only, and reduces to a narrow Gaussian distribution in weak localization. Note that the constant c_0 in Eq. (3) stands for what is known as the “universal conductance fluctuations.” The dependence of $\Delta\sigma^2$ on L/ξ only appears in the higher-order corrections terms, and is indeed much weaker than in $\bar{\sigma}$ [Eq. (2)] itself.

The appearance of a finite localization length ξ in Eq. (2) gives rise to the belief that the system becomes truly insulating at sufficiently large distances:¹

$$\bar{\sigma} \propto e^{-L/\xi}, \quad L \gg \xi. \quad (4)$$

In this regime the probability distribution $P(\sigma)$ presumably becomes log-normal, in analogy with the exact results in one dimension.¹⁶ The statement for $D \neq 1$ relies on a plausibility argument based on a random distribution for the quantity L/ξ as well as the application of the central limit theorem. The log-normal distribution is defined by

$$\begin{aligned} P\left[\sigma; \frac{L}{\xi}\right] &= (2\pi v)^{-1/2} \sigma^{-1} e^{-(\ln \sigma / \sigma_0)^2 / 2v}, \\ \sigma_0 &= \bar{\sigma} e^{-v/2}, \\ v &= \ln \left[1 + \left(\frac{\Delta\sigma}{\sigma} \right)^2 \right], \\ \sigma_0 &\propto e^{-L/\xi}, \quad L \gg \xi. \end{aligned} \quad (5a)$$

On the other hand, Eqs. (2) and (3) imply

$$\begin{aligned} P\left[\sigma; \frac{L}{\xi}\right] &= (2\pi c_0)^{-1/2} e^{-(\sigma - \bar{\sigma})^2 / 2c_0}, \\ \bar{\sigma} &= -\beta_1 \ln \frac{L}{\xi}, \quad L \ll \xi. \end{aligned} \quad (5b)$$

From now on we shall assume that the full probability $P[\sigma; (L/\xi)]$ interpolates smoothly between the two limiting forms of Eqs. (5a) and (5b).

Following the discussion of inelastic scattering given in Sec. I we next substitute L_{in} for L such that $P[\sigma; (L_{in}/\xi)]$ is the probability density of σ for a finite mesoscopic block of size L_{in} . A macroscopic sample consists of many such blocks and the macroscopic conductance σ^M may be obtained from the EMT result (Appendix)

$$\int d\sigma P\left[\sigma; \frac{L_{in}}{\xi}\right] \frac{\sigma - \sigma^M}{\sigma + \sigma^M} = 0. \quad (6)$$

A. Narrow distribution

When P is narrow, Eq. (6) reduces to

$$\frac{\sigma^M}{\bar{\sigma}} = 1 - \frac{1}{2} \frac{\Delta\sigma^2}{\bar{\sigma}}, \quad (7)$$

which is applicable to the weak-localization regime. From Eqs. (3) and (7) we have

$$\sigma^M = \bar{\sigma} - \frac{1}{2} \frac{c_0}{\bar{\sigma}} \quad (\bar{\sigma} \gg 1) \quad (8)$$

which shows that the EMT correction to $\bar{\sigma}$ is one order in σ_0^{-1} higher than the usual weak-localization logarithmic correction [Eq. (1)], and hence negligible. Equations (7) and (8) already indicate, however, that the correction becomes substantial in the crossover regime $\bar{\sigma} \simeq 1$.

B. Log-normal distribution

For a log-normal distribution [Eq. (5a)] the solution for the macroscopic σ^M becomes simply

$$\sigma^M = \sigma_0 = \frac{\bar{\sigma}}{\left[1 + \frac{\Delta\sigma^2}{\sigma^2} \right]^{1/2}} = \bar{\sigma} e^{-(1/2)v}. \quad (9)$$

To see this, introduce a change of variables

$$u = \ln \frac{\sigma}{\sigma_0}, \quad u_M = \ln \frac{\sigma^M}{\sigma_0} \quad (10)$$

such that the EMT condition becomes

$$f(u_M) \equiv \int_{-\infty}^{\infty} \frac{du}{(2\pi v)^{1/2}} e^{-u^2/2v} \frac{e^{u_M}}{e^u + e^{u_M}} = \frac{1}{2}. \quad (11)$$

The function $f(x)$ is monotonically increasing, whereas

$$f(x) + f(-x) = 1. \quad (12)$$

It then follows that $u_M = 0$, and hence Eq. (9) is the solution. Equation (9) is valid as long as $\Delta\sigma/\bar{\sigma}$ and hence v is

not too large. For completeness we give a result valid for large v :

$$\frac{\sigma^M}{\bar{\sigma}} \gtrsim \frac{a^s}{e^s} \frac{\bar{\sigma}/\Delta\sigma}{\left[3\pi \ln \frac{\Delta\sigma^2}{\bar{\sigma}^2}\right]^{1/2}}, \quad \frac{\Delta\sigma^2}{\bar{\sigma}^2} \gg 1, \quad (13)$$

which follows from an argument by Kirkpatrick for estimating a highest lower bound.¹⁷ Here, the quantities $a \simeq 1$ and s are the prefactor and exponent in 2D percolation theory.

An interpolation between the weak quantum-interference result [Eq. (8)] and the strong-localization result [Eqs. (9) and (13)] requires a knowledge of $\Delta\sigma^2/\bar{\sigma}^2$ as a function of $\bar{\sigma}$. For purposes of illustration we have sketched the quantity $\sigma^M/\bar{\sigma}$ with varying $\bar{\sigma}$ for different values of the quantity v by taking $\bar{\sigma} \simeq 1$ as the crossover point in Fig. 2.

The most important feature displayed in Fig. 2 is that there is no longer a unique, one-to-one relation between σ^M and $\bar{\sigma}$. The variance v , which naturally relates to the strength of the disorder, parametrizes the various branches in the strong quantum-interference regime. These branches, however, collapse into a single “universal” curve in the weak quantum interference regime ($\sigma \gg 1$). It is understood that along each branch, σ^M depends on the scaling variable L/ξ only.

In summary, we can say that our block model for macroscopic conductances in effect reduces the value of the ensemble averaged conductance. In this way it only enforces the process of localization. For strong disorder and, in particular, in the regime of strong quantum interference, the effect on σ^M can be dramatic.

III. THE INTEGRAL QUANTUM HALL REGIME

A. Introduction

In order to set up a theory of macroscopic conductances for the integral quantum Hall regime, we first recapitulate some of the basic predictions of the renormalization theory^{14,15} which are formulated in terms of ensemble-averaged quantities alone. As usual we limit ourselves to short-range disorder, since it facilitates in many ways the search for universality.³

(i) The symmetry of the $\bar{\sigma}_{xx}, \bar{\sigma}_{xy}$ flow diagram (Fig. 1) about the unstable fixed point at $\bar{\sigma}_{xy} = \text{half-integer}$. This symmetry is a direct consequence of the topological nature of the Hall conductance, and should be regarded as a

FIG. 2. Sketch of $\sigma^M/\bar{\sigma}$ for different v [Eq. (9)].

universal feature of the disordered electron gas.

(ii) The physical parameters of the theory, i.e., the averaged conductances, the conductance fluctuations, and the higher-order moments, can all be expressed in terms of the scaling variables L/ξ_{cr} discussed Sec. I. In particular,³

$$\bar{\sigma}_{\alpha\beta}(L, B) = F_{\alpha\beta} \left[\pm \left(\frac{L}{\xi_l} \right)^{1/\nu}; \left(\frac{L}{\xi_{cr}} \right)^{-\nu_\sigma} \right], \quad (14)$$

where $F_{\alpha\beta}(X; Y)$ is regular (i.e., power series expandable) in small X and Y . For an isolated Landau band, ξ_l diverges near the band center (B^*) like $\xi_l \propto |B - B^*|^{-\nu}$. It is understood that varying B (keeping the electron density fixed) is equivalent to varying the Fermi level.

Equation (14) simultaneously describes the weak-localization regime ($L \gg \xi_{cr}$) and the strong quantum interference effects of the quantum Hall regime ($L \lesssim \xi_{cr}$). Our main interest is in the latter, and we may write³⁻¹⁴

$$\bar{\sigma}_{\alpha\beta}(L, B) = f_{\alpha\beta} \left[\pm \left(\frac{L}{\xi_l} \right)^{1/\nu} \right], \quad (15)$$

where $f_{\alpha\beta}(X) \equiv F_{\alpha\beta}(X; Y=0)$. Provided the conditions for this asymptotic result [Eq. (15)] are met in practice, we next distinguish between the “metallic” regime $L < \xi_l$ which describes the critical singularities of delocalization near the center of the Landau band, and the quantum Hall effect which occurs away from the band center where $L \gg \xi_l$. We expect (Fig. 3).

$$f_{xx} \left[\pm \left(\frac{L}{\xi_l} \right)^{1/\nu} \right] \propto e^{-L/\xi_l}, \quad (16)$$

$$f_{xy} \left[\pm \left(\frac{L}{\xi_l} \right)^{1/\nu} \right] = \begin{cases} n+1 - Ae^{-L/\xi_l} \\ n + Ae^{-L/\xi_l} \end{cases} \quad (L \gg \xi_l).$$

Notice that there is a formal similarity with the previously discussed $B=0$ example in that a single scaling variable (L/ξ) describes the entire crossover between a (quasi) metallic phase and an exponentially localized phase. The physical scenario, however, is very different for each of these cases and in what follows we shall limit ourselves to a discussion of the “metallic” phase alone.

We next quote a scaling result for the variance-covariance matrix of the conductances:³

FIG. 3. Sketches of $f_{\alpha\beta}(X)$ describing the crossover between “metallic” behavior at $X \lesssim 1$ and the quantization of the Hall conductance for $X \gg 1$.

$$\overline{(\sigma_{\alpha\beta} - \bar{\sigma}_{\alpha\beta})^2} = g_{\alpha\beta} \left[\pm \left[\frac{L}{\xi_l} \right]^{1/\nu} \right], \quad (17)$$

$$\overline{(\sigma_{xx} - \bar{\sigma}_{xx})(\sigma_{xy} - \bar{\sigma}_{xy})} = h \left[\pm \left[\frac{L}{\xi_l} \right]^{1/\nu} \right],$$

where $g_{\alpha\beta}(X) = g_{\alpha\beta}(-X)$ and $h(X) = -h(-X)$ are, just like $f_{\alpha\beta}(X)$, regular (power series expandable) in X . An important observation is that near criticality ($L \ll \xi_l$) the variance [Eq. (17)] is not a small quantity. In fact, the average $\bar{\sigma}_{xx}$ and the root-mean-square fluctuations are all of the same order (e^2/h), indicating that “macroscopic” and “averaged” conductances are presumably very different concepts in this problem.

When the system crosses over to the quantum Hall state ($L \gg \xi_l$), we expect the fluctuations [Eq. (4)] to vanish exponentially in the variable (L/ξ_l). This statement is consistent with the formulation of Ref. 3, in which the averaged conductances, the fluctuations, and higher-order moments appear as response functions to external perturbations or, more precisely, to changes in boundary conditions.

B. Macroscopic conductances

Equation (17), together with a similar statement about the higher moments of the conductance tensor, implies that we have a probability density for the conductances which is of the general form $P[\sigma_{xx}, \sigma_{xy}; (L/\xi_l)]$, with σ_{xx} and σ_{xy} independent random variables:

$$\int_0^\infty d\sigma_{xx} \int_{-\infty}^\infty d\sigma_{xy} P \left[\sigma_{xx}, \sigma_{xy}; \frac{L}{\xi_l} \right] = 1. \quad (18)$$

Substitution of L_{in} for L now leads to a block model with both σ_{xx} and σ_{xy} as local, random variables. In the Appendix we derive the EMT formulas for a random σ_{xx}, σ_{xy} conductance network, and the result is

$$\int_0^\infty d\sigma_{xx} \int_{-\infty}^\infty d\sigma_{xy} P \left[\sigma_{xx}, \sigma_{xy}; \frac{L_{in}}{\xi_l} \right] F_{\alpha\beta} = 0, \quad (19)$$

$$F_{xx} = \frac{\sigma_{xx}^2 - (\sigma_{xx}^M)^2 + (\sigma_{xy} - \sigma_{xy}^M)^2}{(\sigma_{xx} + \sigma_{xx}^M)^2 + (\sigma_{xy} - \sigma_{xy}^M)^2}, \quad (19a)$$

$$F_{xy} = \frac{\sigma_{xy} - \sigma_{xy}^M}{(\sigma_{xx} + \sigma_{xx}^M)^2 + (\sigma_{xy} - \sigma_{xy}^M)^2}. \quad (19b)$$

We do not have detailed knowledge of the probability density P except for some very general symmetry and analyticity arguments which are sufficient to determine the singularity structure of the theory.

(i) Along the lines $\bar{\sigma}_{xy} = \text{half-integer}$, integer [i.e., $(L_{in}/\xi_l) = 0$ and $\pm\infty$, respectively], the probability P is an even function of $(\sigma_{xy} - \bar{\sigma}_{xy})$. Equation (19) then implies

$$\frac{L_{in}}{\xi_l} = 0 \rightarrow \bar{\sigma}_{xy} = \sigma_{xy}^M = \text{half-integer}, \quad (20a)$$

$$\frac{L_{in}}{\xi_l} = \pm\infty \rightarrow \bar{\sigma}_{xy} = \sigma_{xy}^M = \text{integer}. \quad (20b)$$

(ii) The fact that P is power series expandable in the quantities $(L_{in}/\xi_l)^{1/\nu}$ implies that σ_{xx}^M and σ_{xy}^M are also regular functions of this scaling variable, i.e.,

$$\sigma_{xx}^M = f_{xx}^M \left[\left[\frac{L_{in}}{\xi_l} \right]^{1/\nu} \right], \quad (21a)$$

$$\sigma_{xy}^M = f_{xy}^M \left[\left[\frac{L_{in}}{\xi_l} \right]^{1/\nu} \right]. \quad (21b)$$

(iii) The argument can be extended to include $L_{in}/\xi_{cr} \neq 0$. This then leads to the conclusion that the scaling diagram in the variables $\sigma_{xx}^M, \sigma_{xy}^M$ is essentially the same as that of Fig. 1 which involves the averaged parameters. In particular, the symmetry about $\bar{\sigma}_{xy} = \text{half-integer}$ also is rigorously retained in the scaling diagram for macroscopic conductances.

C. Numerical example

In order to provide some quantitative insight into the effect of conductance fluctuations, we have investigated numerically the value of $(\sigma_{xx}^*)^M$ of the unstable fixed point at $\bar{\sigma}_{xy} = (\sigma_{xy}^*)^M = \text{half-integer}$. We have assumed the following form for P :

$$P(\sigma_{xx}, \sigma_{xy}; 0) = p_1(\sigma_{xx}) p_2(\sigma_{xy}), \quad (22a)$$

$$p_1(x) = (2\pi v)^{-1/2} x^{-1} e^{-(\ln x / \sigma_0)^2 / 2v}, \quad (22b)$$

$$p_2(y) = (2\pi w)^{-1/2} e^{-[y - n - (1/2)]^2 / 2w}, \quad (22c)$$

with

$$\sigma_0 = \bar{\sigma}_{xx}^* e^{-(1/2)v}, \quad n + \frac{1}{2} \equiv \bar{\sigma}_{xy}^*. \quad (22d)$$

The relative conductance fluctuations are denoted as

$$\hat{v} \equiv \frac{(\sigma_{xx} - \bar{\sigma}_{xx}^*)^2}{(\bar{\sigma}_{xx}^*)^2} = e^v - 1, \quad (23a)$$

$$\hat{w} \equiv \frac{(\sigma_{xy} - \bar{\sigma}_{xy}^*)^2}{(\bar{\sigma}_{xx}^*)^2} = \frac{w}{(\bar{\sigma}_{xx}^*)^2}. \quad (23b)$$

Note that in the limit of very sharp distributions p_1 and p_2 , the EMT conditions [Eq. (19)] imply that the averaged and macroscopic parameters are the same. We are interested in the ratio R^* of these quantities which can be expanded as

$$R^* \equiv \frac{(\sigma_{xx}^*)^M}{\bar{\sigma}_{xx}^*} = 1 - \frac{1}{2}\hat{v} + \frac{1}{2}\hat{w} \quad (\hat{v}, \hat{w} \ll 1). \quad (24)$$

This result already indicates that the fluctuations in σ_{xx} and σ_{xy} have an opposite effect on the measured value $(\sigma_{xx}^*)^M$. In Fig. 4 we have plotted the various curves with constant R^* in the $\hat{v}, \hat{w}$ plane. We observed that the line $\hat{v} = \hat{w}$ with $R^* \cong 1$ extends far beyond the regime where the asymptotic result of Eq. (24) is valid. Although the EMT is of limited use for broad distributions, Fig. 4 nevertheless indicates a remarkably small difference between the averaged and macroscopic conductances σ_{xx}^* , even for the case of large (relative) conductance fluctuations.

FIG. 4. Lines of constant R^* in the $\hat{v}, \hat{w}$ plane [Eq. (23)].

The results of this section as well as Sec. III D provide a clear demonstration of the lack of universality in the numerical value of $(\sigma_{xx}^*)^M$ and in the scaling functions $f_{\alpha\beta}^M(X)$ [Eqs. (15) and (21)]. At this stage, however, we do not have enough information to carry on with precise comparisons between our block model for macroscopic conductances and experiment. For instance, an experimental value of $\sigma_{xx}^* \approx 0.2(e^2/h)$ has been extracted from the low-temperature data taken from the InP-In GaAs heterostructure.¹³ On the other hand, we expect the fluctuations in σ_{xx} and σ_{xy} at criticality to be of the order unity in e^2/h .³ Hence the statement $(\sigma_{xx}^*)^M \approx \hat{\sigma}_{xx}^*$ would now imply that $\hat{v}$ and $\hat{w}$ are in the range $\hat{v} \approx \hat{w} \lesssim 25$ [Eq. (23)], which is rather large and beyond the range of validity of the EMT. Nevertheless, this type of estimate *does* leave room for an alternative possibility in which $\bar{\sigma}_{xx}^*$ and $(\sigma_{xx}^*)^M$ are in fact significantly different due to a difference in $\hat{v}$ and $\hat{w}$ or due to a complicated structure in the actual form for P [Eq. (22)].

The only serious estimate for $\bar{\sigma}_{xx}^*$ quoted in the literature is from Hikami's series analysis,¹⁸ which seems to give $\bar{\sigma}_{xx}^* \approx 0.5(e^2/h)$. If correct, the numerical difference between averaged and experimental conductances by a factor 2 or 3 can only partly be explained by a block model of mesoscopic conductances, and is most likely due to different aspects of the quasiparticle gas. It is important to stress, however, that the experimental data otherwise leave very little room for speculation, since the behavior at both high and low temperatures precisely follows the predictions of the free-electron gas with short-range disorder.¹¹⁻¹³

In summary we can say that our block model for macroscopic conductances does not modify or complicate the basic predictions of the renormalization theory which were originally made in terms of the averaged parameters alone. In fact, the analysis of the present section only sharpens the universal aspects of the renormalization theory, while simultaneously providing a self-consistent and possibly more realistic framework for analyzing the experiments.

D. Statistics of L_{in}

Our formulation of a block model for computing macroscopic conductances was based on the assumption that L_{in} can be substituted for the scaling length which appears in the renormalization of the elastic diffusion process. It is more appropriate, however, to consider L_{in} as a statistical quantity, and to replace the fixed-sized blocks by a distribution of blocks with random sizes. In this section we shall digress on the subject of statistics in L_{in} and see the extent to which the results of the EMT are modified.

We are interested in the probability distribution $P(l)$, where l is the distance a quasiparticle diffuses between successive inelastic collisions. The $P(l)$ can be expressed as

$$P(l) = \int \bar{P}(l;t) \bar{P}(t) dt, \quad (25)$$

where t is the time interval between successive collisions, $\bar{P}(t)$ the probability distribution of t , and $P(l;t)$ the conditional probability distribution of l given t . $\bar{P}(l;t)$ is given by the Green's function for diffusion in two dimensions:

$$\bar{P}(l;t) = \frac{l}{2Dt} e^{-l^2/4Dt} \quad (26)$$

whereas t obeys Poisson statistics

$$\bar{P}(t) = \frac{1}{\tau_{in}} e^{-t/\tau_{in}}. \quad (27)$$

Equations (25)–(27) now yield

$$P(l) = \frac{l}{L_{in}^2} K_0 \left(\frac{l}{L_{in}} \right), \quad (28)$$

where $K_0(x)$ is the zero-order modified Bessel function, and

$$L_{in}^2 = D\tau_{in}. \quad (29)$$

We next assume that each region of size l is in the scaling regime with a coherence (localization) length ξ_l . Hence the overall probability distribution Π for the conductances becomes

$$\Pi \left[\sigma_{xx}, \sigma_{xy}; \frac{L_{in}}{\xi_l} \right] = \int P \left[\sigma_{xx}, \sigma_{xy}; \frac{l}{\xi_l} \right] P(l) dl, \quad (30)$$

where $P[\sigma; l/\xi_l]$ is the conditional probability distribution of σ given l/ξ_l [Eqs. (18) and (19)].

We see that our ansatz of diffusive motion [Eqs. (26) and (27)] only modifies the conditional probability distribution $P[\sigma; (L/\xi_l)]$ into $\Pi[\sigma; (L/\xi_l)]$ [Eq. (30)]. Since we are working explicitly in the metallic regime ($L_{in} \ll \xi_l$), we can perform an expansion in powers of the scaling variable,

$$P \left[\sigma; \frac{l}{\xi_l} \right] = \sum_{n=0}^{\infty} P_n(\sigma) \left(\frac{l}{\xi_l} \right)^{n/\nu} \quad (31)$$

such that Π [Eq. (30)] becomes

$$\prod \left[\sigma; \frac{L_{\text{in}}}{\xi_l} \right] = \sum_{n=0}^{\infty} c_n P_n(\sigma) \left[\frac{L_{\text{in}}}{\xi_l} \right]^{n/\nu}, \quad (32)$$

$$c_n = \int u^{n+1} K_0(u) du = n^2 c_{n-2}, \quad (32a)$$

$$c_0 = 1, \quad c_1 = \pi. \quad (32b)$$

The recurrence relations [Eqs. (32a) and (32b)] indicate that statistics in L_{in} possibly modify the radius of convergence of the series expansion and further demonstrate the lack of universality of the scaling function $f_{\alpha\beta}^M(X)$ introduced in Sec. III C [Eq. (21)]. These kinds of considerations, however, do not modify the experimental predictions of scaling, which are statements made of $f_{\alpha\beta}^M$ at $X \simeq 0$ alone.

IV. SUMMARY AND DISCUSSION

We have discussed and analyzed the logical consequences of mesoscopic conductance distributions on a length scale (L_{in}) determined by inelastic-scattering processes. The mapping of the problem onto a classical conductance network is motivated in part by the Landauer argument⁶ for internally generated electric fields due to Coulomb interactions. We believe this argument also to be physically correct in the regime of strong quantum interference, when conductance fluctuations generally become significant. Application of the techniques of EMT enables us to formulate a scaling theory for macroscopic or measured conductances.

We have generalized our block model for macroscopic conductances to include the critical metallic phase in dirty Landau-level systems in strong magnetic fields (the integral quantum Hall regime). The results do not really modify the basic prediction of scaling, but only sharpen the concept of universality. For instance, they provide a physical demonstration of the fact that the numerical value of $(\sigma_{xx}^*)^M$ and the details of the function $f_{\alpha\beta}^M(X)$ are model dependent, in contrast to the claims of universality which are sometimes made in the literature.¹⁹

An important conceptual advance which has become more explicit in our recent paper³ and also this work is that criticality in the quantum Hall regime goes hand in hand with broad conductance distributions on mesoscopic length scales. Experiments on dirty mesoscopic samples (i.e., samples size $L \leq L_{\text{in}}$) should provide direct evidence of scaling of the conductance fluctuations in the quantum Hall regime [Eq. (17)]. For instance, by moving the Fermi energy through the center of the Landau band, the finite-size scaling variable L/ξ_l (L fixed) varies from finite values to zero; at the same time, the conductances “fluctuate” [Eq. (17)] with a variance ranging from zero (in the plateau regime) to order unity (in between adjacent quantum Hall plateaus). This type of scaling behavior is new, to our knowledge, and the phenomenon fundamentally different from the fluctuations due to sample inhomogeneities²⁰ which have been observed experimentally in high mobility samples.²¹ To our knowledge finite-size scaling of conductance fluctuations has not yet been investigated experimentally.

As a final remark, we wish to point out that noise ex-

periments at finite temperatures could possibly provide the most significant test of our block model approach. Recall that the applications of the EMT are justified as long as $(L/L_{\text{in}}) \gg 1$. The situation in the metallic regime changes dramatically when L/L_{in} approaches unity. In the range $0 < (L/L_{\text{in}}) < 1$ we expect the noise with frequency $\omega \sim 2\pi/\tau_{\text{in}}$ in the measured transport data to increase linearly in L_{in}/L . The interesting possibility of extracting physical quantities like τ_{in} from the noise spectrum is currently under investigation.

ACKNOWLEDGMENTS

This work has been supported in part by the Dutch National Science Foundation (FOM). M.H.C. is indebted to Exxon Research and Engineering Company (Annandale, NJ) and the University of Amsterdam for financial support during his stay in Amsterdam.

APPENDIX

1. Simple derivation of EMT

The following is a simplified derivation of the EMT (Refs. 6–8) based on a particular member of the set of random internal geometries for which the EMT holds. To develop it in full generality requires the use of multiple-scattering theory or its equivalent. The results are the same as long as a local relation between local current density and local electric field holds, and the probability distribution for the local conductance tensor is translation invariant.

Imagine a circular region I of local conductance σ . Outside the disk (region II) the conductance is macroscopic (σ_M):

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix}. \quad (A1)$$

We generally have

$$\mathbf{j} = \sigma \mathbf{E}, \quad (A2)$$

$$\nabla \cdot \mathbf{j} = 0, \quad (A3)$$

$$\mathbf{E} = -\nabla V, \quad (A4)$$

where $\mathbf{j}$ denotes the current density, $\mathbf{E}$ the electric field, and V the potential. Equations (1)–(4) lead to the Laplace equation for V :

$$\nabla^2 V = 0. \quad (A5)$$

The solution of Eq. (A5) in I and II is of the general form

$$V_{\text{II}} = -\mathbf{E}_I \cdot \mathbf{r}, \quad (A6)$$

$$V_{\text{II}} = -\mathbf{E}_M \cdot \mathbf{r} - \frac{a^2}{r^2} \mathbf{E}_{\text{II}} \cdot \mathbf{r}, \quad (A7)$$

where $\mathbf{E}_M$ is the macroscopic field in II, and a is the radius of I. Far from I, $\mathbf{E}_{\text{II}}$ is nonvanishing, whereas $\mathbf{E}_I$ is different from $\mathbf{E}_M$ because σ_M and the local conductance σ are different. The boundary conditions which relate $\mathbf{E}_I$ and $\mathbf{E}_{\text{II}}$ to $\mathbf{E}_M$ are as follows.

(i) V is continuous across the interface between I and II. This requires

$$\mathbf{E}_I = \mathbf{E}_M + \mathbf{E}_{II} . \quad (\text{A8})$$

(ii) The normal component of $\mathbf{j}$ is continuous. This implies

$$\hat{\mathbf{a}} \cdot \boldsymbol{\sigma} \cdot \nabla V_I = \hat{\mathbf{a}} \cdot \boldsymbol{\sigma}_M \cdot \nabla V_{II} , \quad (\text{A9})$$

where $\hat{\mathbf{a}}$ is an arbitrary unit vector normal on the surface of I. Inserting Eqs. (A6)–(A8) into (A9) yields

$$\begin{aligned} \mathbf{E}_{II} &= [\boldsymbol{\sigma} - \boldsymbol{\sigma}_M + 2\sigma_{xx}^M \mathbf{1}]^{-1} \cdot (\boldsymbol{\sigma}_M - \boldsymbol{\sigma}) \cdot \mathbf{E}_M \\ &\equiv \mathbf{F}(\boldsymbol{\sigma}, \boldsymbol{\sigma}_M) \cdot \mathbf{E}_M . \end{aligned} \quad (\text{A10})$$

The statement that $\mathbf{E}_M$ is the macroscopic field is equivalent to saying that $\mathbf{E}_{II}$ must vanish when averaged over $\boldsymbol{\sigma}$. This leads to the effective-medium condition

$$\int \int d\sigma_{xx} d\sigma_{xy} P(\sigma_{xx}, \sigma_{xy}) \mathbf{F}(\boldsymbol{\sigma}, \boldsymbol{\sigma}_M) = 0 , \quad (\text{A11})$$

with $\mathbf{F}$ written out explicitly in Eqs. (19) and (19b) in Sec. III B.

2. Narrow distribution

For a narrow probability distribution $P(\sigma_{xx}, \sigma_{xy})$, the measured σ_{xx}^M and σ_{xy}^M differ little from the mean value $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{xy}$. Write

$$\sigma_{xx}^M = \sigma_{xx} + \delta\sigma_{xx}^M , \quad (\text{A12})$$

$$\sigma_{xy}^M = \bar{\sigma}_{xy} + \delta\sigma_{xy}^M , \quad (\text{A13})$$

then expanding $\mathbf{F}(\boldsymbol{\sigma}, \boldsymbol{\sigma}_M)$ to first order in $\delta\sigma_M$ yields

$$\delta\sigma_{xx}^M = -\frac{1}{2} \frac{(\overline{\delta\sigma_{xx}})^2 - \overline{(\delta\sigma_{xy})^2}}{\bar{\sigma}_{xx}} , \quad (\text{A14})$$

$$\delta\sigma_{xy}^M = \frac{\overline{\delta\sigma_{xx} \delta\sigma_{xy}}}{\bar{\sigma}_{xx}} . \quad (\text{A15})$$

Hence the difference between measured and averaged values is expressed in terms of the elements of the variance-covariance matrix.

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