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Published in:
Nuclear Physics B

DOI:
10.1016/0550-3213(94)00546-Q

Citation for published version (APA):
Fermion production despite fermion number conservation

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Received 30 June 1994; accepted 24 November 1994

Abstract

Lattice proposals for a nonperturbative formulation of the Standard Model easily lead to a global U(1) symmetry corresponding to exactly conserved fermion number. The absence of an anomaly in the fermion current would then appear to inhibit anomalous processes, such as electroweak baryogenesis in the early universe. One way to circumvent this problem is to formulate the theory such that this U(1) symmetry is explicitly broken. However we argue that in the framework of spectral flow, fermion creation and annihilation still in fact occurs, despite the exact fermion number conservation. The crucial observation is that fermions are excitations relative to the vacuum, at the surface of the Dirac sea. The exact global U(1) symmetry prohibits a state from changing its fermion number during time evolution, however nothing prevents the fermionic ground state from doing so. We illustrate our reasoning with a model in two dimensions which has axial-vector couplings, first using a sharp momentum cutoff, then using the lattice regulator with staggered fermions. The difference in fermion number between the time evolved state and the ground state is indeed in agreement with the anomaly. Both the sharp momentum cutoff and the lattice regulator break gauge invariance. In the case of the lattice model a mass counterterm for the gauge field is sufficient to restore gauge invariance in the perturbative regime. A study of the vacuum energy shows however that the perturbative counterterm is insufficient in a nonperturbative setting and that further quartic counterterms are needed. For reference we also study a closely related model with vector couplings, the Schwinger model, and we examine the emergence of the $\theta$-vacuum structure of both theories.
1. Introduction

A tractable nonperturbative formalism for the analysis of chiral gauge theories is an outstanding problem in modern field theory. Among the difficulties it has been argued that lattice formulations (for reviews see Refs. [1,2]) or indeed any regularization, cannot yield the correct physics if it implies an exact global symmetry which should be broken by anomalies [2-4]. For example, such models would be unable to describe anomalous fermion number violation in the Standard Model which plays an important role in current explanations of the observed baryon asymmetry in the universe.

A lattice regularized version of the Standard Model is of course supposed to describe these effects correctly. However typical forms of the lattice fermion action, \[ S_f = -\sum_{xy} \bar{\psi}_x D_{xy} \psi_y \] have an exact global U(1) invariance \( \psi \rightarrow \exp(i\omega) \psi, \bar{\psi} \rightarrow \exp(-i\omega) \bar{\psi} \) and since the lattice fermion measure is generally also invariant, one expects this global symmetry to be accompanied by exact fermion number conservation, suggesting that such models cannot display the anomalous fermion production of the Standard Model. We shall call this the electroweak U(1) problem.

Some recent discussions have focused on the question of heavy fermion decoupling [1,4-6], but the U(1) problem is more general. We also remark that its resolution should not depend on alternative definitions of the fermion number current, constructed such that their divergence has the right anomaly (see e.g. Refs. [6,7]). Once the action and measure in the path integral are defined nonperturbatively, the stage is set, and we should just have to see what physics emerges.

One way out would be to construct lattice actions such that they have no unwanted exact global symmetries [3,4,8-10] and it is reasonable to assume that the violation of the global symmetry turns in the scaling region into the desired anomaly [11].

Here we want to reconsider this standard lore and study the effects of the unwanted global U(1) symmetry in general terms. We examine the spectral flow to see whether creation or annihilation of fermions, i.e. fermionic particles, takes place in a simple model in 1+1 dimensions which has the exact global U(1) symmetry corresponding to fermion number conservation. We study the spectrum of the fermionic Hamiltonian in an external gauge field which changes slowly in time, the idea being that we consider time slices of an instanton-like configuration, which starts as a vacuum gauge field, goes through a sphaleron configuration and ends up as a different vacuum gauge field related to the first by a topologically non-trivial ("large") gauge transformation. If the change with time is very slow, we can make use of the adiabatic theorem [12] and deduce the evolution of the states by continuity, following the eigenstates of the time dependent Hamiltonian. We can then check explicitly whether an initial fermionic vacuum state has evolved into a state with particles produced by the gauge field in accordance with the anomaly. Particle annihilation proceeds of course by the inverse process. For a recent discussion of anomalies and spectral flow see Ref. [13]; an earlier spectral flow

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analysis in lattice gauge theory was given in Ref. [14] and more recently in the context of domain wall fermions [15] in Ref. [16].

Based on the spectral flow analysis we argue that fermion production occurs despite the exact fermion number conservation. The crucial observation is that fermions are excitations relative to the vacuum, at the surface of the Dirac sea, which are defined as the ground states in external fields that are "pure gauge". The exact global U(1) symmetry prohibits a state from changing its fermion number during time evolution, however nothing prevents the fermionic ground state, the state of lowest energy which depends on the adiabatic external gauge field, from changing its fermion number. We show that the difference in fermion number between the time evolved state and the ground state after the sphaleron transition is indeed in agreement with the anomaly.

Our arguments do not really depend on lattice regularization. Still, we would like to indicate how they might fit into a possible lattice formulation of chiral gauge theories, taking into account the experience gained from investigations over the past years showing the failure of many promising proposals [1]. Recall an often followed practice in the continuum: the fermions are integrated out first and from the resulting determinant a well defined effective action is constructed for the bosonic fields. This can be done in a variety of ways; in general it requires the addition of counterterms to get a finite answer upon removing the fermionic regulator, assuming the fermions are in an anomaly free representation of the gauge group. Subsequently, the integration over the bosonic fields is to be performed. One first removes the fermion regulator and then the boson regulator. We see no reason in principle why such a practice cannot be followed on the lattice. It means dealing with lattice fermions in smooth external gauge fields, a relatively simple situation for which any lattice fermion method should work. Counterterms are usually needed to restore (chiral) gauge invariance, and with a gauge invariant effective action the introduction of ghost fields [17] may be avoided as in non-chiral lattice gauge theory (the problem noted in Ref. [18] need not apply). Taking the fermion lattice distance all the way to zero first ("the desperate's method" [2,19]), may not be necessary if the violation of gauge invariance can be controlled so that gauge symmetry restoration may be invoked [20]. Obviously, much work still remains to be done in this direction, and our present study of the subtleties of anomalous fermion production seems a necessary step, which is also of interest in its own right. A preliminary account of this work has already been presented in Ref. [21].

The outline of the paper is as follows: In Section 2 we introduce the two-dimensional massless models, the Schwinger model and its equivalent axial version (axial QED₂), which we shall use in this paper to illustrate our reasoning. In Section 3 we first discuss the spectral flow of these models with a sharp momentum cutoff on the number of modes. We argue that fermion creation and annihilation in axial QED₂ is possible in spite of the exact U(1) symmetry. The sharp momentum cutoff is unsatisfactory because it is both non-local and gauge variant. Local lattice versions of the two models are introduced in Section 4, using staggered fermions. The lattice axial QED₂ model lacks gauge invariance, which can be restored for smooth external gauge fields with a mass counterterm for the gauge field. We then study the spectral flow in both models to see how the properties of the well understood vector model compare with those of the axial model. Fermion creation and annihilation takes place in the same way as in the
toy models with the sharp momentum cutoff. There are furthermore interesting aspects
to the way the counterterm restores gauge invariance to the ground state energy in the
axial model, and to the way the Dirac sea acquires a gauge invariant bottom in the
vector model. We summarize our conclusions in Section 5. For clarity of presentation
we have delegated many details of the staggered fermion formalism and the calculation
of the energy spectrum to Appendices A–C.

2. Axial and vector QED$_2$ in the continuum

The massless axial QED$_2$ model is given in the continuum by the following (real
time) action,

$$S = - \int d^2 x \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + S_f,$$

$$S_f = - \int d^2 x \bar{\psi} \gamma^\mu (\partial_\mu + iA_\mu \gamma_5) \psi,$$

where $\gamma^1 = \gamma_1 = \sigma_1$, $\gamma^0 = -\gamma_0 = -i\sigma_2$, $\gamma_5 = \sigma_3$. We take $A_\mu$ to be an external gauge
field; only the fermion fields are quantized. The model is invariant under local axial
gauge transformations: $\psi(x) \rightarrow \exp(i\omega(x)\gamma_5)\psi(x)$, $\bar{\psi}(x) \rightarrow \bar{\psi}(x) \exp(i\omega(x)\gamma_5)$,
$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \omega(x)$, with gauge current $j_5^\mu = i\bar{\psi} \gamma^\mu \gamma_5 \psi$. The action is fur-
thermore invariant under the global U(1) symmetry $\psi(x) \rightarrow \exp(i\alpha)\psi(x)$, $\bar{\psi}(x) \rightarrow \bar{\psi}(x) \exp(-i\alpha)$, with a corresponding vector current $j^\mu = i\bar{\psi} \gamma^\mu \psi$. According to stan-
dard lore the gauge current, upon quantization, has to be divergence free, $\partial_\mu j_5^\mu = 0$,
while the divergence of the vector current becomes anomalous

$$\partial_\mu j^\mu = -2q, \quad q = \frac{1}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu} \equiv \partial_\mu C^\mu,$$

where $C^\mu(x) = (1/2\pi) \varepsilon^{\mu\nu} A_\nu(x)$ is the Chern–Simons current and $q(x)$ is the topo-
logical charge density ($\varepsilon_{01} = +1$); the topological charge is defined by $\int d^2 x \, q(x)$. Fermion and Chern–Simons numbers are given by the spatial integrals over the time
components of the corresponding currents, $Q = \int dx^1 j^0(x)$, $C = \int dx^1 C^0(x)$. For later
purposes we also introduce here the axial charge $Q_5 = \int dx^1 j_5^0(x)$ which is exactly
conserved in this model. From Eq. (2.3) one can deduce the equation

$$Q(t) - Q(0) = -2(C(t) - C(0)) = -2 \int^t_0 \int dx \, q, \quad x \equiv x^1, \quad t \equiv x^0,$$

which relates a change in the Chern–Simons number to a change in fermion number. The
aim of this paper is to investigate whether, in those models which have the exact global
U(1) symmetry and where $Q$ is thus conserved, a change in $C$ due to a sphaleron
transition can still give rise to a change in the number of fermions in the spirit of
Eq. (2.4).
By charge conjugation of the right-handed fermion fields,

\[ \psi_R = (\psi'_R C)^T, \quad \bar{\psi}_R = -(C^\dagger \psi'_R)^T, \quad \psi_L = \psi'_L, \quad \bar{\psi}_L = \psi'_L, \]  

(2.5)

with \( C \) the charge conjugation matrix, or alternatively to only the left-handed fields, one can show that axial QED\(_2\) is equivalent to the massless Schwinger model (vector QED\(_2\)),

\[ S'_f = - \int d^2x \bar{\psi}' \gamma^\mu (\partial_\mu - iA_\mu) \psi'. \]  

(2.6)

For the moment we will label the quantities in the vector model with a prime for clarity. The local axial gauge symmetry of the action (2.2) has turned, after the transformation (2.5), into a local vector gauge symmetry for the \( \psi' \) fields, \( \psi'(x) \to \exp(i\omega(x))\psi'(x) \), \( \bar{\psi}'(x) \to \bar{\psi}'(x) \exp(-i\omega(x)) \), and vice versa the global vector invariance into a global axial invariance \( \psi'(x) \to \exp(i\gamma_5)\psi'(x) \), \( \bar{\psi}'(x) \to \bar{\psi}'(x) \exp(i\gamma_5) \). The corresponding currents and charges transform under (2.5) into each other,

\[ j_5^\mu = i\bar{\psi}' \gamma^\mu \gamma_5 \psi' = -i\bar{\psi}' \gamma^\mu \psi' = -j_5'^\mu \]

\[ j^\mu = i\bar{\psi}' \gamma^\mu \gamma_5 \psi' = -i\bar{\psi}' \gamma^\mu \psi' = -j_5'^\mu \]

(2.7)

We notice that a mass or Yukawa term added to the actions (2.2) and (2.6) would transform under (2.5) into a corresponding Majorana mass or Yukawa term. In this paper we will not include these terms. For a discussion of the spectral flow in QED in presence of a bare mass term see Ref. [14].

In the following we use a finite spatial extent of length \( L \), with periodic boundary conditions for the gauge field and antiperiodic boundary conditions for the fermion fields. We consider gauge fields with vanishing time component, \( A_0 = 0 \), and choose the Coulomb gauge specified by \( A_1(x,t) = A(t) \); then the time dependence is expressed by \( A \)-dependence. In this gauge the Chern–Simons number \( C = -ALA/2\pi \), and going through an instanton-like configuration means that \( C \) changes by one unit. Values \( A = 2\pi k/L \) with integer \( k \) are "pure gauge", since for these values we can write \( A = \Omega t \partial_\tau \Omega^* \), with \( \Omega = \exp(i2\pi kx/L) \) a periodic gauge transformation with winding number \( k \). When \( A = 2\pi k/L \) the Chern–Simons number takes integer values \( C = -k \). An example of such gauge fields is given by the configuration with constant electric field \( f = F_{01} \),

\[ A_0 = 0, \quad A_1(x,t) = A(t) = -f t. \]  

(2.8)

For \( f = 2\pi/LT \) we have \( C(0) = 0, C(T) = 1 \), and the adiabatic limit corresponds to \( T \to \infty \).

The ground state \( |0, A \rangle \) of the fermionic Hamiltonian \( H(A) \) is by definition the state with lowest energy. When \( A \) is pure gauge, these ground states are to be identified with the vacuum. In Hilbert space, gauge transformations on \( A \) induce unitary transformations, and we have to identify states related by gauge transformations, hence also all vacua with differing integer Chern–Simons numbers.
A quantity which we will use in our analysis is the ground state energy $E_0(A)$, $H(A)|0,A\rangle = E_0(A)|0,A\rangle$. It can be obtained from the partition function at inverse temperature $\beta$ in the limit $\beta \to \infty$

$$E_0(A) = -\frac{1}{\beta} \ln \text{Tr} e^{-\beta H(A)}, \quad \beta \to \infty,$$  

(2.9)

or in terms of the euclidean effective action in a space-time volume $L \times \beta$,

$$S_{\text{eff}}(A) = \ln \int D\bar{\psi} D\psi \exp(S_{\text{eff}}^{\text{eucl}}),$$  

(2.10)

$$\rightarrow -\beta E_0(A), \quad \beta \to \infty.$$  

(2.11)

We shall use the euclidean formalism for our lattice models. The quantity $E_0(A)$ can be evaluated as (see for example Ref. [22] and references therein)

$$E_0(A) = \frac{2}{\pi L} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nLA) + \text{const.}$$  

(2.12)

The sum is clearly periodic in $A$ with period $2\pi/L$ and evaluates, in each period, to a quadratic potential.

$$E_0(A) - E_0(0) = \frac{L}{2\pi} A^2, \quad -\frac{\pi}{L} < A < +\frac{\pi}{L}, \quad \text{mod}(2\pi/L).$$  

(2.13)

From (2.13) we can read off the dynamically generated mass of the Schwinger Model, $m^2 = e^2/\pi$ (the factor $e^2$ is due to the $F^2$ term in (2.1)). For a treatment of the Schwinger model on a circle in the Hamiltonian formalism, see Ref. [23].

3. A model with a sharp momentum cutoff

Before giving a full treatment using lattice fermions we first consider a model of fermions regulated by a sharp cut-off in momenta $A$. Such a regulator captures essential features we wish to study in lattice regularization, namely, exact global invariance and restriction to a finite number of modes for the $\psi$ field. It is not gauge invariant, which it shares with the class of chiral lattice models we have in mind, and moreover it is non-local, whereas the lattice models are local. The locality of the lattice models allows for restoration of gauge invariance by a local mass counterterm $\propto \int d^2x A_\mu A^\mu$.

From the above actions (2.2) and (2.6), the axial QED$^2$ Hamiltonian $H$ and the equivalent vector QED$^2$ Hamiltonian $H'$ are given by

$$H = \frac{1}{L} \sum_p [\psi_R^\dagger(p)\psi_R(p)(p + A) + \psi_L^\dagger(p)\psi_L(p)(-p + A)] - AN/2,$$  

(3.1)

$$H' = \frac{1}{L} \sum_p [\psi_R^\dagger(p)\psi_R(p)(p - A) + \psi_L^\dagger(p)\psi_L(p)(-p + A)].$$  

(3.2)
where the summation is over momenta \( p = (n - 1/2)2\pi/L \in [-\Lambda, \Lambda] \) (antiperiodic boundary conditions for the fermions), \( N \) is the total number of modes, and \( L \) and \( R \) denote the left- and right-handed projections \( \gamma_5 \rightarrow \pm 1 \). The Fourier modes \( \psi(p) = \int_0^L dx \exp(-ipx) \psi(x) \) have the commutation relations \( \{\psi(p), \psi^\dagger(q)\} = i\delta_{pq} \). In axial QED, the spatial component of the gauge field couples to fermion number

\[
Q = \int dx \frac{1}{2} [\psi^\dagger(x), \psi(x)]
\]

\[
= \frac{1}{L} \sum_p [\psi_R^\dagger(p)\psi_R(p) + \psi_L^\dagger(p)\psi_L(p)] - \frac{N}{2}.
\]  

(3.3)

Note that upon quantization we have anti-symmetrized the field operators in \( Q \), which is necessary for it to change sign under charge conjugation. Charge conjugation (2.5) on the right handed fields only, relating the axial and vector model, is given by \( \psi_R^\dagger = \psi_R \), \( \psi_L^\dagger = \psi_L \). Denoting this conjugation by \( C_R \), we have \( H' = C_R HC_R^\dagger \), so that \( H' \) has the same spectrum as \( H \).

In these models regularized by a sharp momentum cutoff, there are two conserved charges: both \( Q \) and

\[
Q_5 = \int dx \psi^\dagger(x)\gamma_5\psi(x) = \frac{1}{L} \sum_p [\psi_R^\dagger(p)\psi_R(p) - \psi_L^\dagger(p)\psi_L(p)]
\]  

(3.4)

commute with \( H \). In the vector version the two charges are given by \( Q' = -Q_5 \), \( Q'_5 = -Q \) (cf. Eq. (2.7)).

The spectral flow of the eigenmode energies \( \epsilon \) in the axial model is very simple. Disregarding the mode-independent term \(-AN/2\) we have

\[
\epsilon_x(p, A) = \chi p + A,
\]  

(3.5)

where the chirality \( \chi = \pm 1 \) is the eigenvalue of \( \gamma_5 \) (\( \chi = +1(-1) \) for the \( R(L) \) modes). Each mode's energy increases linearly with \( A \). In the vector version

\[
\epsilon'_x(p, A) = \chi(p - A),
\]  

(3.6)

and the \( R \) (\( L \)) modes decrease (increase) linearly with \( A \) as shown in Fig. 1.

Consider now the flow of the state \( |\Psi, A\rangle \), which starts out as the vacuum state at \( A = 0 \), i.e. the Dirac sea with all negative energy modes occupied. Each mode is doubly degenerate (\( L \) and \( R \)). In the axial model the \( \epsilon_p \) flow upwards with \( A \) and at \( A = 2\pi/L \) they have taken the place of their predecessor, except for the two modes starting out as \( \epsilon = -\pi/L \) and ending up at \( \epsilon = +\pi/L \). The quantum numbers \( Q \) and \( Q_5 \) are conserved, both are zero for all states \( |\Psi, A\rangle \), as for the initial vacuum state \( |0, 0\rangle \). The energy \( E_{\Psi} \) of the state \( |\Psi, A\rangle \) also happens to be constant in this regularization: each mode shifts upwards by \( A \) and the \( N/2 \) modes contribute \( AN/2 \), which is compensated by the mode-independent term \(-AN/2\). However, the ground state \( |0, A\rangle \) looses two occupied modes half way at \( A = \pi/L \), when \( \epsilon = -\pi/L + A \) crosses zero. The final vacuum energy \( E_0 \) at \( A = 2\pi/L \) differs from the initial one, due to the lack of gauge invariance of the sharp momentum cutoff.
The important point is that the final ground state has two occupied states (L and R) less than the initial ground state, so the final vacuum quantum numbers are \( Q = -2 \) and \( Q_5 = 0 \). Hence,

\[
\Delta Q = (Q_\psi - Q_\psi)_{\text{final}} - (Q_\psi - Q_\psi)_{\text{initial}} = 2.
\]

just as expected from the anomaly in the vector current, \( \Delta Q = -2 \Delta C \). In the vector model the L modes move upwards and the R modes move downwards, with \( (Q_5, \psi)_{\text{initial}} = (Q_5, \psi)_{\text{final}} = 0 \). Midway, the ground state loses an L mode and gains an R mode, such that \( (Q_5, \psi)_{\text{initial}} = 0, (Q_5, \psi)_{\text{final}} = +2 \) and \( \Delta Q' = 0, \Delta Q' = -2 = 2 \Delta C \). In the axial model we have the creation of two particles (L and R), whereas in the vector model we have the creation of a particle (L) and an antiparticle (R) (an R-hole).

In both models the final energy difference of \( |\Psi, A\rangle \) with \( |0, A\rangle \) is two units \( \pi/L \), which corresponds to the creation of the two particles (using the generic term “particle” also for the antiparticles). It is amusing that this can be interpreted in classical terms, as the work which is provided by the external field in creating two fermions [13]. The time component of the Lorentz force \( dp_\mu/dt = \pm F_\mu \, dx^\nu/dt \) on a particle of charge \( \pm 1 \) can be integrated for the gauge field of constant field strength (2.8). Using \( dx^1/dt = \pm 1 \) for the massless particles, and taking into account that they appear midway at \( t = T/2 \), this gives a final energy per particle \( p^0(T) = \int_{T/2}^T dt \, dp^0/dt = \pi/L \).

Obviously, the ground state energy in these cutoff models is very different from the continuum form (2.13). It is given by \( E_0(A) - E_0(0) = 0 \) for \( 0 \leq A \leq \pi/L \) and \( E_0(A) - E_0(0) = -2(A - \pi/L) \) for \( \pi/L \leq A \leq 2\pi/L \). A “counterterm” piece-wise linear in \( A \) would be required to rectify the situation. This is at variance with the expectation that a simple quadratic mass counterterm \( \propto \int d^2 x \, A_\mu A^\mu \) should be sufficient
to restore gauge invariance. The sharp momentum cutoff is also unsatisfactory because it is non-local. The local lattice models to which we shall turn in the next section are well behaved in this respect.

4. Level shifting in the lattice models with staggered fermions

Our lattice axial and vector QED$_2$ models are formulated in terms of staggered fermion fields. In order to keep the technicalities to a minimum, we have collected the details of this formulation to the appendices and here we just record the essentials. We use the Euclidean lattice formulation with the partition functions given by

$$Z = \int (\prod_x d\bar{\chi}_x d\chi_x) \exp S_f,$$

with

$$S_f = -\sum_{x,\mu} \eta_{\mu x} \frac{1}{2} (\bar{\chi}_x U_{\mu x} \chi_{x+\hat{\mu}} - \bar{\chi}_{x+\hat{\mu}} U_{\mu x}^* \chi_x),$$

for vector QED$_2$ and

$$S_f = -\sum_x \left\{ \sum_{\mu} \cos A_{\mu x} \eta_{\mu x} \frac{1}{2} (\bar{\chi}_x U_{\mu x} \chi_{x+\hat{\mu}} - \bar{\chi}_{x+\hat{\mu}} U_{\mu x}^* \chi_x) + \sum_{\mu,\nu} \sin A_{\mu x} \eta_{\mu x} \frac{1}{2} (\bar{\chi}_x \chi_{x+\hat{\nu}} + \bar{\chi}_{x+\hat{\nu}} \chi_x) \right\} + \sum_{x,\mu} \gamma_2 A_{\mu x}^2,$$

for axial QED$_2$. We use lattice units, in which the lattice distance $a = 1$. $U_{\mu x} = \exp(-iA_{\mu x})$ is the lattice gauge field, the $\eta_{\mu x}$ are the usual staggered fermion sign factors representing the gamma matrices, $\eta_{1x} = 1$, $\eta_{2x} = (-1)^{x_1}$ $(\gamma_2 = i\gamma^0, \epsilon_{12} = +1)$. The last term in (4.3) is a counterterm. The vector model is gauge invariant; the axial model is not, however invariance is restored in the scaling region by the addition of this counterterm. The derivation of the continuum models (2.6) and (2.2) from the staggered fermion actions (4.2) and (4.3) is sketched in Appendix A, where we give also the perturbative result for the coefficient $\tau$ of the counterterm. A nonperturbative extension of the counterterm is discussed below.

The exact U(1) phase invariance of the actions and measure in the path integrals above is obvious. For the vector model this is just the global limit of its local gauge invariance. For the axial model we have an exact global U(1) symmetry leading to fermion number conservation. To derive the divergence relation of the current associated with this symmetry we replace $\chi_x \rightarrow \exp(i\omega_x)\chi_x$ and $\bar{\chi}_x \rightarrow \exp(-i\omega_x)\bar{\chi}_x$ in the path integral (4.1) and note that it is invariant under this transformation. After collecting terms linear in $\omega_x$ we find from the fact that their coefficients must vanish, the exact current divergence relation on the lattice,

$$\sum_\mu \langle j_{\mu x} - j_{\mu x+\hat{\mu}} \rangle = 0,$$
where the brackets denote the average over the fermion fields. For the vector theory the current is given by

\[ j_{\mu x} = i\eta_{\mu x} \frac{1}{2} (\bar{X}_x U_{\mu x} X_{x+\hat{\mu}} + \bar{X}_{x+\hat{\mu}} U_{\mu x}^* X_x), \quad (4.5) \]

while for the axial theory

\[ j_{\mu x} = i \cos A_{\mu x} \eta_{\mu x} \frac{1}{2} (\bar{X}_x X_{x+\hat{\mu}} + \bar{X}_{x+\hat{\mu}} X_x) \]
\[ + i \sin A_{\rho x} \epsilon_{\mu \nu} \eta_{\mu x} \frac{1}{2} (\bar{X}_x X_{x+\hat{\mu}} - \bar{X}_{x+\hat{\mu}} X_x). \quad (4.6) \]

The current in the vector theory (4.5) is of course gauge invariant, but this is lacking in the axial theory (4.6). Presumably the combination

\[ j_{\rho x} + 2i C_{\mu x} = j_{\mu x} + (i/\pi) \epsilon_{\mu \nu} A_\nu, \]

which has the correct euclidean anomaly, will become gauge invariant in the scaling region and can be compared with the continuum current in (2.3).

The above staggered fermion models actually describe axial or vector QED$_2$ with two flavors. The corresponding SU(2)$_L \times$SU(2)$_R$ flavor symmetry is reduced on the lattice to the exact global U(1) symmetry

\[ X_x \rightarrow \exp (i\epsilon_x) X_x, \quad -X_x \rightarrow \exp (-i\epsilon_x) X_x, \]
\[ \epsilon_x = (-1)^{x_1+x_2}, \]

which corresponds to a flavor non-singlet axial transformation in the scaling region [30]. The charge corresponding to this symmetry is given in Appendix B. As long as the gauge field is external, the two flavors are an unessential extension of the previous models.

Other currents may be constructed which reduce in the scaling region to the expected continuum currents. These are somewhat arbitrary because they do not correspond to symmetries of the action, although natural choices exist [7]. It has been shown in Ref. [24] that the natural choice for the axial current in the gauge invariant vector QED$_2$, for example, has the expected anomalous divergence equation; see Ref. [25] for U(4) $\times$ U(4) currents in staggered fermion QCD.

The gauge potential $A_{\mu x}$ is taken to be independent of euclidean ("imaginary") time $x_2$. It depends parametrically on the real time $t$. From the euclidean path integrals above we can derive the transfer operator $T$ in the standard way (cf. Appendices B and C), $Z = \text{Tr} T^{\beta/2}$, where $\beta/2$ is the number of pairs of time slices and $\beta$ the inverse temperature. The Hamiltonian $H$ is defined in terms of $T$ by $T = \exp (-2H)$, such that $Z = \text{Tr} \exp (-\beta H)$.

For the simple case $A_1 = A(t)$ (i.e. independent of $x_1$ and $x_2$), $A_2 = 0$, we can obtain the spectrum of the transfer operator explicitly, and from this the energy levels of the associated Hamiltonian as a function of $t$—the spectral flow. From the calculations outlined in Appendices A–C we find the following formulas for the energies. For vector QED$_2$:

\[ \epsilon_X(p, A) = \epsilon_{0x}(p - A), \]
\[ \epsilon_{0x}(p) = \ln (\chi \sin p + \sqrt{\sin^2 p + 1}), \quad \text{or} \quad \sinh \epsilon_{0x} = \chi \sin p, \quad (4.7) \]
\[ p = n \frac{2\pi}{L} - \frac{\pi}{L}, \quad n = -\frac{L}{4} + 1, \ldots, \frac{L}{4}. \]
Fig. 2. Spectral flow for the axial and vector models with staggered fermions. The energies $\epsilon_x(p, A)$ are shown as function of $LA/2\pi$ for $L = 32$. Ground state $|0, A\rangle$ and $|\psi, A\rangle$ are represented as in Fig. 1.

Here the $\epsilon_{0x}(p)$ are the free staggered fermion energies and $\chi = \pm 1$ is the chirality of the mode, the eigenvalue of $\gamma_5$. Each mode is doubly degenerate because of the two flavors. The eigenvalues of the Hamiltonian,

$$E = L \ln 2 + \sum_{\text{occupied modes}} \epsilon_x(p, A),$$  

(4.8)

consist of the sum of the occupied mode energies plus an overall constant $L \ln 2$, which could be avoided by redefining the fermion measure in the partition function. For axial QED$_2$:

$$\epsilon_x(p, A) = \epsilon_{0x}(p) + \ln \sqrt{\frac{1 + \sin A}{1 - \sin A}},$$  

(4.9)

where we assume $\cos A > 0$ (below we shall furthermore restrict ourselves to $|A| < \pi/4$). In this case

$$E = L \ln 2 - L \ln(1 + \sin A) + \sum_{\text{occupied modes}} \epsilon_x(p, A),$$  

(4.10)

where $E$ does not include the counterterm. The term $-L \ln(1 + \sin A)$ is the analogue of the “charge term” $-AN/2$ in (3.1), generalized to two flavors (in lattice units $L = Na = N$).

We have plotted in Fig. 2 the energies $\epsilon_x(p, A)$ as a function of $AL/2\pi$ in the interval between 0 and 1, for a lattice of size $L = 32$. Each energy eigenvalue in (4.7) and (4.9) appears twice because of the two flavors. For momenta $p$ and gauge fields $A$ which are

...
small compared to the cutoff $\pi/2$, (4.7) and (4.9) reduce to the linearly spaced modes in formulas (3.6) and (3.5). The spectra close to the surface of the Dirac sea, for small $A$, are as in the continuum. For the modes near the maximal negative or positive energies we clearly see regularization effects. But most importantly, the interpretation of anomalous fermion creation and annihilation for the spectra in Fig. 2 is just as in the previous section: a given quantum state respects the global $U(1)$ invariance of the action, while the vacuum adjusts, producing or absorbing the charge in concordance with the anomaly.

There are furthermore some interesting details which are worth mentioning. In the vector case the two spectra describing the vacuum at $A = 0$ and $A = 2\pi/L$ are identical, in accordance with gauge invariance. As in the sharp cut-off model, the $L$-modes move upwards, whereas the $R$-modes move downwards. This however does not hold for the upper- and lower-most modes. For example, the maximal negative energy mode appears to start out right handed $A = 0$ (going down) while ending up left handed (going up). This effective chirality flip occurs because the staggered fermion analogue of $\gamma_5 (p - A)$ is $\gamma_5 \sin(p - A)$ the slope of which, $(\partial/\partial A) \gamma_5 \sin(p - A) = -\gamma_5 \cos(p - A)$, changes sign at $A = \pi/2 + \pi/L$ (at the bottom of the Dirac sea). From a more general point of view we expect in coupled staggered fermion theories to be able to recover $\gamma_5$ only in the scaling region, i.e. for sufficiently small $p$. Far from the scaling regime the labels “$L$” and “$R$” may lose their usual meaning. In the present situation, however, the perfectly smooth (constant) $A$ allows for a more detailed identification of $\gamma_5$ also outside the scaling region, which leads to the chirality flip in the minimum energy modes at $A = \pi/L$.

The chirality of an eigenmode may be viewed as the eigenvalue of an axial charge $\tilde{Q}_5$, as defined in (B.27), which commutes with the transfer operator. This $\tilde{Q}_5$ is the analogue of $\int dx \psi^\dagger \gamma_5 \psi - 2C$ in continuum considerations and it is not gauge invariant. On the other hand the natural gauge invariant but not conserved $Q_5$ constructed from the current $j_\mu = -ie_{\mu\nu} j_\nu$ gives only sensible chiralities in the scaling region (cf. (4.5), (B.26)).

The behavior under large gauge transformations is also interesting. Although the spectrum is invariant, e.g. comparing $A = 0$ and $A = 2\pi/L$, the eigenmodes shift into each other, $\psi_{L,R}(p) \rightarrow \psi_{L,R}(p - 2\pi/L)$, except for the last modes with minimum momentum $p_{\text{min}} = -\pi/2 + \pi/L$ which turn into the $p_{\text{max}} = \pi/2 - \pi/L$ modes while flipping chirality and flavor. This follows from the transformation properties (B.22) of the eigenmodes. The Dirac sea vacua have all positive momentum $L$-modes and negative momentum $R$-modes filled, such that $|0\rangle_{L} \equiv |0, A = 2\pi l/L\rangle$ transforms under large gauge transformations $\Omega_k(x) = \exp(i2\pi x k/L)$ as

$$\Omega_k |0\rangle_{L} = |0\rangle_{k+1},$$

assuming a suitable choice of phases. The action of $\tilde{Q}_5$ on these states is given by

$$\tilde{Q}_5 |0\rangle_{k} = 2k |0\rangle_{k},$$

which also expresses the non-gauge invariance of this operator.
The vector QED$_2$ ground state energy $E_0(A)$ is periodic in $A$ with period $2\pi/L$, which expresses invariance under large gauge transformations. In the continuum limit it reduces to the quadratic form (2.13). The subtle regularization effects in the spectrum away from the scaling region concord with gauge invariance and the $A$ dependence of $E_0$ is a surface effect of the Dirac sea (cf. Eq. (A.16)). In physical units, the relative discretization errors in the $A$ dependence are of order $a^2 A^2$. In the infinite volume limit the ground state energy density $E_0$ defined by

$$E_0(A) = \lim_{L \to \infty} \frac{E_0(A)}{L}$$

becomes independent of $A$, $E_0(A) = E_0(0)$, since the variation in $E_0(A)$ is of order $1/L$ (cf. (2.13)).

For the axial model Fig. 2 shows that the breaking of gauge invariance has the effect that the spectra describing the vacua at $A = 2\pi/L$ and $A = 0$ differ from each other, especially for high momentum modes which are far away from the surface of the Dirac sea. The regularization dependence of the axial spectrum is also a subtle effect, which is just right so that the quadratic counterterm can restore gauge invariance to the ground state energy in the scaling region. In Fig. 3 we have plotted the dimensionless function $L(E_0(A) - E_0(0))$, without the counterterm, with the counterterm $-\tau L^2 A^2/2$, and with a more refined subtraction in terms of the energy density of the ground state, $-L^2 E_0(A)$. The latter subtraction is equivalent to the perturbative counterterm in the scaling region since $E_0(A) - E_0(0) = \tau A^2/2 + O(A^4)$ (cf. Appendix A), but leads clearly to better overall behavior for the renormalized ground state energy. (The size of the scaling region will be discussed shortly.)

Fig. 4 shows a close-up. We clearly see how the counterterm restores gauge invariance in the scaling region, which for a lattice of 32 sites appears to be given by $|LA|/2\pi \lesssim 1$. The cusps at $LA/2\pi = \pm 1/2$ correspond to the energy barrier between the vacua $E_0(\pi/L) - E_0(0) = \pi/L$ (for two flavors). In the region $-1 \leq LA/2\pi \leq 1$ containing

![Graph showing the scaled energy difference $L\Delta E_0 = L(E_0(A) - E_0(0))$ for the axial model as a function of $AL/2\pi$, on a lattice with $N = 32$. Shown are the unrenormalized energy (dashed); with perturbative counterterm $-\tau L^2 A^2/2$ added (dotted); with $L^2 (E_0(A) - E_0(0))$ subtracted (solid).]
the first three minima, the vector and axial ground state energies are very close.

Notice that the abscissas in Figs. 3 and 4 are limited to $|A| < \pi/4$; the reason is that at $A = \pm \pi/4$ (up to corrections of order $1/L$) even the lowest/highest lying mode has been pulled out/pushed into the Dirac sea: $\epsilon_\chi(p, A) = 0$ for $p = \mp \pi/2, A = \pm \pi/4$. When the Dirac sea evaporates ($A \approx \pi/4$) or the Dirac sky condenses ($A \approx -\pi/4$), the physics clearly ceases to make sense, and we have to restrict the allowed values of $A$ in the axial model to $|A| < \pi/4$.

We also see from Figs. 3 and 4 that the quadratic counterterm is much too strong in the large field region, outside the scaling region. With dynamical gauge fields it would be bad to have the minima of the classical energy lying outside the scaling region and away from $A = 0$. To bring the minimum of the energy into the scaling region we must add suitable higher order terms to the action, e.g. $-\int d^2x \tau' a^2(A_\mu A_\mu)^2$, with $\tau' > 0$.

In the static case the best choice is perhaps to subtract $L\epsilon_0(A)$ from the energy.

We have checked numerically using a variety of lattice sizes that for given $LA$, the relative discretization errors are again $O(a^2A^2)$. However, in contrast to the vector case, convergence is not periodic in $LA$. We can estimate the size of the scaling region to a given accuracy on a spatial lattice of $N$ sites. To this end we restore the lattice distance $a$, $L = Na$, and write $E_0(A)/L = a^{-2}f_N(aaA), \epsilon_0(A) = a^{-2}f_\infty(aaA)$, where $E_0$ does not include the counterterm. Figs. 3 and 4 show that $LAE_0 \equiv L(E_0(A) - E_0(0)) = N^2(f_N(aaA) - f_N(0))$ behaves like $L^2(\epsilon_0(A) - \epsilon_0(0)) = N^2(f_\infty(aaA) - f_\infty(0))$, modulated with "oscillations" which in the scaling region turn into the physical $LAE_0$ (i.e. the quadratic $(LA)^2/\pi$ mod $2\pi$ potential). Since $f_\infty(aaA) - f_\infty(0) = \frac{1}{2}\tau(aaA)^2 + \tau'(aaA)^4 + \ldots$, we expect that upon subtracting the counterterm, $LAE_0 - \frac{1}{2}\tau(LA)^2$ equals the physical $LAE_0$ up to terms of order $N^2(aaA)^4 = N^{-2}(LA)^4$. It follows that to given accuracy, $LA$ can be at most of order $\sqrt{N}$. Hence the scaling region for $LA$ grows as $\sqrt{N}$. 

Fig. 4. As in Fig. 3, on a smaller scale (unrenormalized energy (dashed); with perturbative counterterm $-rL^2A^2/2$ added (dotted); with $L^2(\epsilon_0(A) - \epsilon_0(0))$ subtracted (solid)).
In Fig. 5 we display this scaling region for a lattice of size $N = 256$. The nonperturbatively renormalized ground state energy $L\Delta E_0 - L^2 \Delta E_0$ of the axial model (cf. Fig. 4) is plotted superimposed on the gauge invariant (periodic) potential of the vector model. One can clearly see that several axial vacua near $A = 0$ are degenerate and indistinguishable from the gauge invariant vector case, with violations of gauge invariance setting in at $|C| \gtrsim 5$.

In the continuum limit $N \to \infty$ we recover the equivalence with the vector model. The scaling region then contains an arbitrarily large number of vacua $|0\rangle_k$ which transform into each other under large gauge transformations as in (4.11), with the conserved fermion charge acting as

$$Q|0\rangle_k = -2k|0\rangle_k,$$

(4.14)

similar to (4.12). Clearly, the conserved $Q$ is not gauge invariant; it is the analogue of $-\tilde{Q}_5$ in the vector model.

5. Discussion

We have shown using simple models in two dimensions, from spectral flow in the external field approximation that fermion creation and annihilation is possible even when fermion number is exactly conserved. The transition rate is determined by the usual anomaly formula, as if the fermion current had an anomalous divergence. We see no reason why the same resolution of the electroweak $U(1)$ problem would not apply in four dimensions.
Furthermore, our study of the ground state energy of the axial model reveals that a standard perturbative mass counterterm for the gauge field, which was needed for restoration of gauge invariance, is insufficient nonperturbatively. Additional terms are needed ("oversubtraction") to get the global minimum of the ground state energy in the desired scaling region. In the vector model, the elegant way in which the staggered fermion method is able to give a bottom to the Dirac sea [14] in a gauge invariant way (thereby circumventing the "infinite hotel" explanation of the anomaly [13]) is also worth noting.

To complete our arguments, they have to be extended to the full theory with dynamical gauge fields. We also have to recover zero modes, fermion number violation in correlation functions, etc. Since we used continuous real time in the spectral flow arguments this may not be a problem. However, with discrete time e.g. in the euclidean formulation exact zero modes are non-generic and the consequences of this will have to be understood (for analogous questions in QCD see Refs. [25,26]). These questions lie outside the scope of our present inquiry. Nonetheless we shall briefly outline some expectations for the full theory with dynamical gauge fields.

Consider first the vector theory as defined by the lattice regulated euclidean path integral in the compact formulation, in which the dynamical gauge field variables are phase factors $U_{\mu,x} = \exp(-iA_{\mu,x})$ with $A_{\mu,x} \in (-\pi, \pi]$, in lattice units. The path integral is then well defined without gauge fixing and its Hilbert space interpretation leads naturally to temporal gauge quantization, in which the integration over the time components $U_{2,x}$ of the gauge field provides the projection onto the gauge invariant subspace, the implementation of "Gauss's law". In the previous sections we used the Coulomb gauge however, so let us fix to this gauge: $U_{1,x} = U_x = \exp(-iA_x)$ independent of $x$. In this case the integration over the time components $U_{2,x}$ in the path integral can be split into a summation over large gauge transformations $\Omega_k = \exp(i2\pi k x/L)$ plus fluctuations. The summation $P_0 = \sum_k \Omega_k$ over the large gauge transformations leads again to a projection onto the gauge invariant subspace whereas the integration over fluctuations leads to the Coulomb interaction. This interpretation of the integration over $U_{2,x}$ is appropriate in the scaling region.

To compare with analysis in the continuum and also with the axial theory to be discussed below, let us assume that in the scaling region there is a $Q_5$ which commutes with the Hamiltonian (but not with $P_0$). This is not entirely obvious, since with fluctuating gauge fields the spin-flavor interpretation of staggered fermions emerges only in the scaling region and even though $A$ is constant, the local charge densities in the Coulomb interaction can only be expected to commute with such a $Q_5$ in the scaling region. In the scaling region the Hilbert space before projection with $P_0$ may then be split into sectors characterized by the eigenvalue of $Q_5$, $Q_5|\Psi\rangle_k = 2k|\Psi\rangle_k$, similar to the external gauge field case. Under large gauge transformations these sectors transform into one another, $\Omega_k|\Psi\rangle_k = |\Psi\rangle_{k+t}$, and consequently physical states $P_0|\Psi\rangle_k = \sum_l |\Psi\rangle_l$ have no well defined value of $Q_5$. In particular $Q_5$ does not annihilate the ground state $|0\rangle$, which is like a $\theta$-vacuum superposition with $\theta = 0$, $|0\rangle = \sum_k |0\rangle_k$.

The axial case is more difficult since we have not given a complete nonperturbative construction of the model with quantized gauge fields. The difficulty is the lack of gauge
invariance at the fermion regulator scale. However, as mentioned in Section 1, we have in mind an implementation which is gauge invariant on the scale of the lattice distance for the gauge field, which is much larger than the lattice distance for the fermions. Thus we may expect to regain invariance under gauge transformations which are smooth on the fermion regulator scale, in particular large gauge transformations, as we have seen in the scaling region of the external field model.

This leads us to the conclusion that also in the axial case with dynamical gauge fields $|0\rangle = \sum_k |0\rangle_k$ as well. Here however it is fermion number $Q$, which is analogous to $\tilde{Q}_5$ in the vector case, that is not well defined in the vacuum. Since the corresponding global $U(1)$ symmetry is an exact symmetry of both the action and the fermion measure in the path integral, this suggests that the symmetry is broken spontaneously via the $\theta$-vacuum: $Q|0\rangle = -\sum_k 2k|0\rangle_k \neq 0$. This picture though needs to be reconciled with the expectation that spontaneous symmetry breaking occurs only in the infinite volume limit and that spontaneous breaking of a continuous symmetry in two dimensions is not possible. This will have to await the detailed construction of the model with dynamical gauge fields.

Despite the simple nature of the models studied here, we hope the above picture is a valid description of the physics, also in similar four-dimensional models where it mimics old ideas on the spontaneous breaking of $U(1)_A$ in continuum treatments of QCD. Our conserved vector current $j^\mu$ in the axial QED$_2$ model is similar to the analogous axial current in QCD, $j^\mu_5 = i\bar{\psi} \gamma^\mu \gamma_5 \psi - 2nfC^\mu$ ($n_f$ is the number of flavors), which is conserved but not gauge invariant. The $U(1)$ symmetry generated by the conserved charge $\tilde{Q}_5 = \int d^3x j_5^\mu$ is in this scenario supposed to be spontaneously broken due to instanton-$\theta$-vacuum effects. (There appears to be no universal agreement on this point [27, 28]); the $U(1)_A$ problem in QCD can also be solved without invoking a gauge variant conserved $\tilde{Q}_5$ [29]; for lattice implementations see Ref. [25].)

To someone working exclusively in the continuum formulation, where models are constructed from formal unregulated expressions proceeding to well defined regulated expressions "along the way", the gist of above remarks may appear so familiar that the question may arise, what's new? We repeat however, that with a nonperturbative regulator such as the lattice, models are well defined from the start and we have to determine the fate of the exact global $U(1)$ symmetry and fermion number conservation. We have presented a resolution, at least in the external field approximation.

Acknowledgements

We would like to thank Jeroen C. Vink for interesting discussions. This research was supported by the Stichting voor Fundamenteel Onderzoek der Materie (FOM) and by the DOE under contract DE-FG03-90ER40546.

Appendix A. Staggered fermion interpretation and ground state energy

Various routes to the staggered fermion actions (4.2) and (4.3) from the continuum actions (2.6), (2.2) have been described in Refs. [2, 7, 10, 31]. Here we shall briefly
review the continuum interpretation of the staggered fermion actions.

The interpretation is done in momentum space on a lattice of \( N_1 N_2 \) points; for simplicity we use lattice units \( a = 1 \). The momenta in the Brillouin zone are parameterized as \( p_\mu + \pi A_\mu \), with \(-\pi/2 < p_\mu < +\pi/2\) and \( \pi A = \{ \pi_0 = (0, 0), \pi_1 = (\pi, 0), \pi_2 = (\pi, \pi), \pi_3 = (0, \pi) \} \), \( A = 1, \ldots, 4 \). The regions of the Brillouin zone labeled by \( A \) provide the spin-flavor components of the fermion field according to \( \psi^A = \sum_x \exp[-i(p + \pi A) x] \chi^A \). For the simple case of constant \( A_\mu \), the actions (4.2), (4.3) can then be rewritten in the form \([30,31]\)

\[
S_f = -\frac{4}{N_1 N_2} \sum_{p,\mu} \bar{\psi}_p i \Gamma_\mu \sin(p_\mu - A_\mu) \psi_p \tag{A.1}
\]

for the vector case, and

\[
S_f = -\frac{4}{N_1 N_2} \sum_{p,\mu} \bar{\psi}_p \left( i \Gamma_\mu \sin p_\mu \cos A_\mu + \sum_\nu \epsilon_{\mu\nu} \Gamma_\nu \cos p_\nu \sin A_\mu \right) \psi_p \tag{A.2}
\]

for the axial case. Here the tilde on \( \sum_p \) indicates the restricted momentum range given by \( p_\mu = (n_\mu - 1/2)(2\pi/N_\mu), n_\mu = -N_\mu/4 + 1, \ldots, N_\mu/4 \). The \( 4 \times 4 \) real symmetric matrices \( (\Gamma_\mu)_{AB} \) come out \([30,31]\) naturally as tensor products of Pauli matrices \( \sigma_k \) and \( \tau_k \). They commute with \( 4 \times 4 \) real symmetric flavor matrices \( \Xi_\mu \) which correspond to shift symmetries of the staggered fermion action. The \( \Gamma_\mu \) and \( \Xi_\mu \) are given by

\[
\Gamma_1 = \sigma_3, \quad \Gamma_2 = \sigma_1 \tau_3, \quad \Xi_1 = \sigma_3 \tau_1, \quad \Xi_2 = \tau_3. \tag{A.3}
\]

By a suitable unitary transformation the \( \Gamma \)'s and \( \Xi \)'s can be brought into block diagonal form, \( \Gamma_\mu, \Xi_\nu \to \gamma_\mu, \xi_\nu \), with \( \gamma_{1,2,5} = \sigma_{1,2,3}, \xi_{1,2,5} = \tau_{1,2,3} \). The classical continuum limit corresponds to small \( p \) and \( A \) where the sines and cosines take their limiting forms. In the general case of non-constant gauge fields the classical continuum limit involves small momenta and values of \( A_\mu x \) as well.

The coefficient \( \tau \) of the counterterm of the axial model can be obtained from the lattice vacuum polarization diagrams along the lines in Ref. [7], which gives

\[
\tau = -\left( \frac{1}{\pi} + I_1 + I_2 \right) \approx 0.3634,
\]

\[
I_2 = -\frac{1}{2} \int_{-\pi/2}^{\pi/2} \frac{d^2 q}{\pi} \frac{\sin^2 q_1}{\sin^2 q_1 + \sin^2 q_2} = -\frac{1}{2},
\]

\[
I_1 = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \frac{d^2 q}{\pi} \frac{\sin^2 q_1 - \sin^2 q_1 \cos^2 q_2}{(\sin^2 q_1 + \sin^2 q_2)^2} \approx -0.1817. \tag{A.4}
\]

This is given in Ref. [7] for a somewhat different but equivalent action. In this sense the action studied in [7] has less gauge symmetry breaking than the "canonical action" (4.3) studied here (notice that the value in Ref. [7] had been normalized to one flavor). Below we shall obtain \( \tau \) again from the energy density of the ground state.
The above representations (A.1) and (A.2) of the actions are useful for calculating the ground state energy \( E_0 = \lim_{N_2 \to \infty} -\frac{1}{N_2} \ln \det D \), where \( D_{xy} \) is the fermion matrix defined by \( S_f = -\bar{\chi}_x D_{xy} \chi_y \); we shall do so for the case \( A_2 = 0, A_1 = A \) considered in this paper. In momentum space \( \det D = \prod_p \det D_p \), where \( D_p = i \Gamma_2 \sin p_2 + \ldots \) can be read off from (A.1) and (A.2). We write \( D_p = \Gamma_2 (i \sin p_2 + \mathcal{H}_p) \), where \( \mathcal{H}_p \) is a hermitian matrix. Since \( \det \Gamma_2 = 1 \), the determinant of \( D \) is given by the product of the eigenvalues of \( (i \sin p_2 + \mathcal{H}_p) \), which come in complex conjugate pairs. Hence, for the evaluation of \( \ln \det D \) we may use \( \ln(zz^*) = \ln z + \ln z^* \), with the principal value for the logarithm \( (\ln z = \ln |z| + i \arg z, -\pi < \arg z < \pi) \). For the vector theory we find

\[
\mathcal{H}_p = \Gamma_5 \sin(p - A),
\]

(A.5)

\[
E_0 = -\sum_p \int_{-\pi/2}^{\pi/2} \frac{dp_2}{\pi} \sum_x \ln(i \sin p_2 + \chi \sin(p - A)),
\]

(A.6)

and for the axial theory

\[
\mathcal{H}_p = \Gamma_5 \cos A \sin p + \sin A \cos p_2,
\]

(A.7)

\[
E_0 = -\sum_p \int_{-\pi/2}^{\pi/2} \frac{dp_2}{\pi} \sum_x \ln(i \sin p_2 + \chi \cos A \sin p + \sin A \cos p_2),
\]

(A.8)

where we have let \( N_2 \to \infty, p = p_1, N = N_1 \), and the chirality \( \chi = \pm 1 \) is the eigenvalue of \( \Gamma_5 \).

As an example we describe the calculation for the axial case. The integrand has period \( \pi \), so we may replace \( \int_{-\pi/2}^{\pi/2} dp_2 / \pi \to \int_{-\pi}^{\pi} dp_2 / 2\pi \). The “fermion doubling” introduced by this replacement is compensated by the factor 1/2. Next we change variables to \( z = \exp ip_2 \) and use

\[
\text{Re} \int_{|z|=1} \frac{dz}{2\pi i z} \ln(z - x) = \ln(|x|) \theta(|x| - 1),
\]

(A.9)

where \( x \) is real and \( \theta \) is a Heaviside step function, \( \theta(x) = 1, x > 0, \theta(x) = 0, x < 0 \). We factorize \( i \sin p_2 + \chi \cos A \sin p + \sin A \cos p_2 = (1 + \sin A)(z - z_+)(z - z_-)/2z \), with

\[
z_{\pm} = \pm \exp(-\epsilon_{\pm}x), \quad \epsilon_{\pm}x = \epsilon_{0x} + \ln \sqrt{(1 + \sin A)/(1 - \sin A)},
\]

(A.10)

\[
\epsilon_{0x} = \ln(\chi s + \sqrt{s^2 + 1}), \quad \sinh \epsilon_{0x} = \chi s, \quad s = \text{sgn}(\cos A) \sin p.
\]

(A.11)

From the “inverse Wick rotation” \( p_2 = ip^0 \) we see that \( z_+ \) and \( \epsilon_+ \) correspond to “normal fermion” excitations, whereas \( p_2 = ip^0 + \pi \) shows that \( z_- \) and \( \epsilon_- \) correspond to “doubler fermion” excitations introduced by doubling the momentum interval. Furthermore, the doublers have effectively opposite chirality, \( \epsilon_{-x} = \epsilon_{+, -x} \). The step function in (A.9)
allows only the negative $\varepsilon_{\pm x}$ to contribute, as expected for the Dirac sea. The axial ground state energy can be evaluated as

$$E_0 = N \ln 2 - N \ln(1 + \sin A) + 2 \sum_p \sum_x \varepsilon_x \theta(-\varepsilon_x), \quad (A.12)$$

where $\varepsilon_x = \varepsilon_{+x}$ depends on $A$ and $p$ according to (A.11).

The corresponding formulas for the vector case can be obtained from the above by letting $A \to 0$ and then $p \to p - A$:

$$\varepsilon_x = \ln \left( \chi \sin(p - A) + \sqrt{\sin^2(p - A) + 1} \right) \sinh \varepsilon_x = \chi \sin(p - A), \quad (A.13)$$

$$E_0 = N \ln 2 + 2 \sum_p \sum_x \varepsilon_x \theta(-\varepsilon_x). \quad (A.14)$$

Using $\varepsilon_-(p, A) = \varepsilon_+(p + \pi, A)$ this can be rewritten as a summation involving $\varepsilon_+$ only, over the whole periodic region $p \in (-\pi, \pi)$. This shows that $E_0(A)$ is periodic in $A$ with period $2\pi/N$. In the infinite volume limit the momentum summation may be replaced by an integration over $p \in (-\pi, \pi)$, and by $p$-translation invariance the energy density becomes independent of $A$, as expected from gauge invariance.

Reintroducing the lattice distance $a$ by $A \to aA$, $E_0 \to aE_0$, $L = Na$, the continuum limit of the finite volume $E_0$ can be found by expansion in $a$, taking $A$ in the first period about $A = 0$. From $\varepsilon_-(p, A) = \varepsilon_+(p - A)$ and the symmetry of the momentum summation, only even powers of $A$ occur. Using $\partial \varepsilon_+/\partial A = -\cos(p - A)/\cosh \varepsilon_+$ and replacing afterwards $\sum_p$ by an integral for $a \to 0$, gives

$$E_0(A) - E_0(0) = LA^2 \int_{-\pi/2}^{0} \frac{dp}{\pi} \left[ \frac{-\cos^2 p \sin p}{(1 + \sin^2 p)^{3/2}} + \frac{-\sin p}{(1 + \sin^2 p)^{1/2}} \right] + \ldots \quad (A.15)$$

$$= LA^2 \int_{-\pi/2}^{0} \frac{dp}{\pi} \frac{\partial}{\partial p} \left( \frac{\cos p}{(1 + \sin^2 p)^{1/2}} \right) + \ldots \quad (A.16)$$

$$= \frac{1}{\pi} LA^2 \left[ 1 + O(a^2 A^2) \right]. \quad (A.17)$$

The two terms in (A.15) correspond to the "ordinary" and "leaf diagram" terms in the vacuum polarization diagram, and the differential identity in (A.16) is essentially a Ward identity [11]. The factor two in comparison to the continuum formula (2.13) is due to the two staggered flavors.

Returning to the axial case, for which $E_0$ is not gauge invariant without the counterterm, we write $E_0 = N f_N(A)$ (in lattice units). In the infinite volume limit the energy density $E_0/N$ approaches a smooth function of $A$. 


\[
\begin{align*}
\quad f_{\infty}(A) &= \ln 2 - \ln(1 + \sin A) + 4 \int_{-\pi/2}^{\pi/2} \frac{dp}{\pi} \epsilon_+(p, A) \theta(-\epsilon_+(p, A)) \\
&= \ln 2 - \ln(1 + \sin A) + 4 \int_{-\pi/4}^{\pi} \frac{d\theta}{\cos \theta} \ln \sqrt{\frac{1 + \sin A}{1 - \sin A}} \frac{1 + \sin \theta}{1 - \sin \theta},
\end{align*}
\]
(A.18)

where in the second line we used the substitution \( \sin p = \tan \theta \) and assumed \( A > 0 \). From \( f_{\infty}(A) = \frac{1}{2} \tau A^2 + O(A^4) \) we get an analytic expression for the coefficient of the counterterm, \( \tau = 1 - 2/\pi = 0.363380 \ldots \)

Appendix B. Transfer operator and energy spectrum in vector QED\(_2\)

In this Appendix we construct the transfer operator for vector QED\(_2\) and rederive from it the energy spectrum. To the path integral a quantum mechanical Hilbert space is associated according to the coherent state formalism [24,33]. A rudimentary set of formulas for this purpose is given by

\[
|a\rangle = \exp(a_i^+ a_i) |\emptyset\rangle, \quad \langle a\rangle = \exp(a_i^+ a_i), \quad 1 = \prod_i da_i^+ da_i \exp(-a_i^+ a_i) |a\rangle \langle a|,
\]

\[
\hat{a}_i |a\rangle = a_i |a\rangle, \quad \langle a| \hat{a}_i^+ = a_i^+ \langle a|, \quad \langle a| \exp(\hat{a}_i^+ M_{ij} \hat{a}_j) |a\rangle = \exp[a_i^+ (\epsilon^M_{ij} a_j).
\]
(B.1)

In these appendices we distinguish operators in Hilbert space by a caret \(^\hat{\cdot}\). The \( \hat{a}_i \) and \( \hat{a}_i^\dagger \), are a generic set of fermionic creation and annihilation operators with the usual anticommutation rules and “empty state” \( |\emptyset\rangle \), the \( a_i \) and \( a_i^\dagger \) are “Grassmann variables”, and \( M \) is some matrix.

In the temporal gauge \( U_{2,x} = 1 \), we can rewrite the vector action (4.2) in the form, using \( \bar{x}_x = \chi^+_x \eta_{2x} \),

\[
S = -\sum_{\tau = 0}^{N_2 - 1} \left[ \frac{1}{2} (\chi^+_{\tau} \chi_{\tau+1} - \chi^+_{\tau+1} \chi_{\tau}) + \chi^+_{\tau} \mathcal{H}_x \chi_{\tau} \right],
\]
(B.2)

where we have used \( (x_1, x_2) = (x, \tau) \) and suppressed the summations over \( x \). The identification \( \chi_{x,N_2} = -\chi_{x,0}, \bar{\chi}_{x,N_2} = -\bar{\chi}_{x,0} \) takes into account the antiperiodic boundary conditions. The staggered fermion Dirac Hamiltonian \( \mathcal{H} \) is given by

\[
\mathcal{H}_{x,xy} = \frac{1}{2} (\eta_{2,x,x} \eta_{1,x,x} U_{1,x,x} \delta_{x,x+1} + \eta_{2,y,x} \eta_{1,y,x} U_{1,y,x} \delta_{x,y+1}),
\]
(B.3)

and is manifestly hermitian.

The time derivative terms \( -\chi^+_{\tau} \chi_{\tau+1} + \chi^+_{\tau+1} \chi_{\tau} \) are associated with the normalization factor \( \exp(a^+a) \) and measure factor \( \exp(-a^+a) \) in the coherent state formalism. This leads to the two time-slice construction described in Refs. [24,30,32], where we interpret the \( \chi \) fields at odd time slices as “type a” and at even time slices as “type a\(^+\)”, writing
\[ X^\tau = \sqrt{2a_k}, \quad X^\tau_+ = \sqrt{2b_k}, \quad \tau = 2k + 1, \]
\[ X^\tau = \sqrt{2b_k^+}, \quad X^\tau_+ = \sqrt{2a_k^+}, \quad \tau = 2k, \]

where \( k = 0, \ldots, N_2/2 - 1 \) denotes pairs of time slices. Using these new variables the partition function can be rewritten in the form

\[ Z = \int \prod_{k=0}^{N_2/2-1} da_k^+ da_k db_k^+ db_k \langle a_{k+1}, b_{k+1} | \hat{T} | a_k, b_k \rangle \]
\[ = \text{Tr} \hat{T}^{N_2/2}, \]

with

\[ \langle a_{k+1}, b_{k+1} | \hat{T} | a_k, b_k \rangle = e^{-2N \ln 2} e^{\sigma_1 \sigma_2 + \sigma_2 \sigma_1} e^{-2a_k^+ \sigma_2 b_{k+1}^+ \sigma_2} e^{-2b_k \sigma_2 a_{k+1} \sigma_2}. \]

Recall that \( N = N_1 \) is the total number of spatial sites. For the transfer operator \( \hat{T} \) associated with the matrix element \( \langle a_{k+1}, b_{k+1} | \hat{T} | a_k, b_k \rangle \) we obtain the expression

\[ \hat{T} = e^{-2N \ln 2} e^{-2a^+ \sigma_2 b^+} e^{-2b \sigma_2 a}. \]

The staggered transfer operator can be written as the square of the shift operator in the time direction [34,35], but we do not need this here.

We shall now determine the eigenvalues of \( \hat{T} \) for the case \( U_1 x, r = \exp(-iA) \) needed for the spectral flow (\( A \) may be considered constant in this calculation, as it is independent of the euclidean time \( \tau \)). First we consider the Dirac Hamiltonian \( \hat{\mathcal{H}} \). Similar to the two-dimensional case in (A.1), the spatial Fourier transform brings \( \hat{\mathcal{H}} \) in a more intelligible form

\[ \sum_{x,y} e^{-i(p+\pi_A)x + i(q+\pi_B)y} \hat{\mathcal{H}}_{xy} = (\sigma_2)_{AB} \sin(p - A) N \delta_{p,q}, \]
\[ \sum_{x} e^{-i(p+\pi_A)x} \hat{a}_x = \hat{a}_p, \quad \text{etc.} \]
\[ \hat{b} \hat{\mathcal{H}} \hat{a} = \frac{2}{N} \sum_p \frac{1}{2} \hat{b}_p \sigma_2 \sin(p - A) \hat{a}_p, \quad \text{etc.} \]

Here the indices \( A, B \) take only two values, \( \pi_A = \pi_0, \pi_1 \), and in \( \sigma_2 \) we recognize the upper block of \( \Gamma_5 = \sigma_2 \tau_3 \). We will recover the complete \( \Gamma_5 \) below. The eigenvalues of \( \hat{\mathcal{H}} \) are given by \( \lambda = \pm \sin(p - A) \), where \( \pm 1 \) is the eigenvalue of \( \sigma_2 \). The transfer operator reduces to \( \prod_{\text{modes}} \exp(-2 \ln 2) \exp(-2 \hat{a}^+ \lambda \hat{b}^+) \exp(-2 \hat{b} \lambda \hat{a}) \).

For the moment we continue the calculation for a single mode. We seek linear combinations of \( \hat{a}, \hat{b}^\dagger \) and \( \hat{a}^\dagger, \hat{b} \) which have simple commutation relations with \( \hat{T} \). This leads to a \( 2 \times 2 \) real symmetric eigensystem for the coefficients of the linear combinations. From

\[ \left( \begin{array}{c} \hat{a} \\ \hat{b}^\dagger \end{array} \right) = V \left( \begin{array}{c} \hat{\psi}_+ \\ \hat{\psi}_- \end{array} \right), \]

it is straightforward to check that
\[ \hat{T} \hat{\psi} = \begin{pmatrix} e^{2\omega'} & 0 \\ 0 & e^{-2\omega'} \end{pmatrix} \hat{\psi} \hat{T} = e^{2\omega' \tau_3} \hat{\psi} , \]  
(B.13)

\[ \omega' = \ln(\lambda + \sqrt{\lambda^2 + 1}) , \quad \sinh \omega' = \lambda , \]  
(B.14)

with the orthogonal matrix \( V' \) given by

\[ V' = \frac{1}{\sqrt{2 \cosh \omega'}} \begin{pmatrix} e^{-\omega'/2} & e^{\omega'/2} \\ e^{-\omega'/2} & -e^{\omega'/2} \end{pmatrix} . \]  
(B.15)

We can now retrace our steps and restore the matrix \( \sigma_2 \) to the eigenvalues of \( \mathcal{H} \) in this basis: \( \lambda \rightarrow \sigma_2 \sin(p - A) \), \( \omega' \rightarrow \sigma_2 \omega \), turning \( \hat{\psi}_\pm \) into two component fields on which the \( \sigma \)'s act, hence obtaining four component fields \( \hat{\psi} \) on which both the \( \sigma \)'s and \( \tau \)'s act. Then the 2 \( \times \) 2 orthogonal \( V_t \) gets replaced by a 4 \( \times \) 4 unitary matrix \( V \),

\[ \left( \begin{array}{c} \hat{a} \\ \hat{b}^\dagger \end{array} \right) = V \hat{\psi} , \]  
(B.16)

\[ \hat{T} \hat{\psi} = e^{2\omega' \sigma_2 \tau_3} \hat{\psi} \hat{T} \quad \sinh \omega = \sin(p - A) , \]  
(B.17)

\[ V = \frac{1}{\sqrt{2 \cosh \omega}} \begin{pmatrix} e^{-\omega^2/2} & e^{\omega^2/2} \\ e^{-\omega^2/2} & -e^{\omega^2/2} \end{pmatrix} , \]  
(B.18)

and we have recovered \( \Gamma_5 = \sigma_2 \tau_3 \).

Defining the no-quantum state \( |0\rangle \) by \( \hat{a}|0\rangle = b^\dagger|0\rangle = 0 \), it follows that \( \hat{\psi}|0\rangle = 0 \) and the eigenstates of the transfer operator are obtained by repeated application of \( \hat{\psi}^\dagger \) on \( |0\rangle \),

\[ \hat{T} \hat{\psi}^\dagger |0\rangle = e^{-2\omega' \Gamma_5} \hat{\psi}^\dagger \hat{T} |0\rangle , \]  
(B.19)

\[ \hat{T} |0\rangle = e^{-2N \ln 2} |0\rangle . \]  
(B.20)

The mode eigenvalues are obviously given by \( \chi \omega \), with \( \chi \) the eigenvalue of \( \Gamma_5 \).

Under a large gauge transformation

\[ \hat{\Omega}_x \hat{a}_x \hat{\Omega}_x = e^{2\pi x/N} \hat{a}_x , \quad \hat{\Omega}^\dagger_x \hat{b}_x \hat{\Omega}_x = e^{2\pi x/N} \hat{b}_x^\dagger , \]  
(B.21)

the \( \hat{\psi}_p(A) \) transform as

\[ \hat{\Omega}_x \hat{\psi}_p(A + 2\pi/N) \hat{\Omega} = \hat{\psi}_{p - 2\pi/N}(A) , \quad p = p_{\min} + 2\pi/N , \ldots , p_{\max} , \]  
(B.22)

where we used \( \sigma_1 = \Gamma_2 \Xi_2 \) and \( p_{\min} = -p_{\max} = -\pi/2 + \pi/N \). We see that the minimum momentum mode flips its chirality \( \chi \) and flavor \( \xi \) (eigenvalue of \( \Xi_5 \)) as it changes into the maximum momentum mode,

\[ \hat{\Omega}_x \hat{\psi}^{\chi,\xi}_{p_{\min}}(A + 2\pi/N) \hat{\Omega} = (-1)^{(\chi + \xi)/2} \hat{\psi}^{\chi,\xi}_{p_{\max}}(A) . \]  
(B.23)

The eigenvalue of \( \Gamma_5 \Xi_5 \) remains the same.
We shall now obtain operator expressions for the charges $Q$, $Q_e$ and $Q_5$. Since fermion number $Q = -i \sum_x j_{x,\tau}$ is time independent we choose $\tau = 0$ = even, and associate with $Q = \frac{1}{2}(\chi^+_r U_{2,\tau} X_{\tau+1} + X^+_{\tau+1} U^*_{2,\tau} X_{\tau}) = a^+ a + b b^+$ the operator
\[
\hat{Q} = \hat{a}^+ \hat{a} - \hat{b}^+ \hat{b} = \hat{a}^+ \hat{a} + \hat{b}^+ \hat{b} - N = \frac{2}{N} \sum_p \frac{1}{2} \hat{\psi}^+_p \hat{\psi}_p - N. \tag{B.24}
\]
This has the expected form (3.3) for two flavors. The other conserved U(1) charge $Q_e = -i \sum_x j^2_{x,\tau} = \sum_x \frac{1}{2}(\epsilon_x \chi^+_r X_{\tau+1} + \epsilon_x X^+_{\tau+1} \chi_{\tau})$ is also time independent (recall $\epsilon_x = (-1)^{x_1+x_2}$). Choosing $\tau = 0$ again leads to
\[
Q_e = -\frac{2}{N} \sum_p \frac{1}{2}(\hat{a}^+_p \sigma_1 \hat{a}_p + \hat{b}^+_p \sigma_1 \hat{b}_p) = -\frac{2}{N} \sum_p \frac{1}{2} \hat{\psi}^+_p V^\dagger \sigma_1 \tau_3 V \hat{\psi}_p 
= \frac{2}{N} \sum_p \frac{1}{2} \hat{\psi}^+_p \Gamma_5 \Xi_5 \hat{\psi}_p, \tag{B.25}
\]
with $\Gamma_5 \Xi_5 = -\sigma_1 \tau_1$ the flavor non-singlet axial generator corresponding to $\epsilon_x$.

Although the chiralities $\chi$ appear naturally in the mode spectrum of the staggered formulation, an axial current $j_{5 \mu}$ or charge operator $Q_5$ is somewhat extraneous to the formulation. A natural candidate is given by $j_{5 \mu} = -i \epsilon_{\mu \nu} j_{\nu}$, $Q_5 = \sum_x j_{1 \, x,\tau}$. However, for a reasonable interpretation in staggered fermion theory, operators usually have to involve some time average as well as space average. This leads us to investigate $Q_5 \equiv \frac{1}{2} \sum_x (j_{1 \, x,\tau} + j_{1 \, x,\tau+1})$ at $\tau = 0$. We find
\[
\hat{Q}_5 = -\frac{2}{N} \sum_p \frac{1}{2}(\hat{a}^+_p \sigma_2 \hat{b}^+_p + \hat{b} \sigma_2 \hat{a}) \cos(p - A) \\
= -\frac{2}{N} \sum_p \frac{1}{2} \hat{\psi}^+_p V^\dagger \tau_1 \sigma_2 V \hat{\psi}_p \cos(p - A) \\
= \frac{2}{N} \sum_p \frac{1}{2} \hat{\psi}^+_p \left[ \frac{\Gamma_5}{\cosh \omega} + \Gamma_1 \Xi_1 \tanh \omega \right] \hat{\psi}_p \cos(p - A), \tag{B.26}
\]
where $\Gamma_1 \Xi_1 = \tau_1$; note that $\omega$ depends on $p - A$ according to Eq. (B.17). In the scaling region $\cos(p - A) \to 1$, $\omega \to 0$ and the rather ugly looking $\hat{Q}_5$ reduces to the continuum form. Away from the scaling region there is a reduction of the "strength of $\Gamma_5$" in $Q_5$ and flavor symmetry breaking.

The operators $\hat{Q}$, $\hat{Q}_e$ and $\hat{Q}_5$ are invariant under the large gauge transformations (B.22), as is obvious from their original gauge invariant expression in terms of the staggered fermion fields. We can define similarly a conserved chiral charge operator $\hat{Q}_5$ which commutes with the transfer operator (B.8) but is not gauge invariant under the large gauge transformations (B.22),
\[
\hat{Q}_5 = \frac{2}{N} \sum_p \frac{1}{2} \hat{\psi}^+_p \Gamma_5 \hat{\psi}_p. \tag{B.27}
\]
The chiralities $\chi = \pm 1$ of the modes $\psi_{R,L} = \frac{1}{2}(1 \pm \Gamma_5)\psi$ are the eigenvalues of this operator $\hat{Q}_5$.

**Appendix C. Transfer operator and energy spectrum in axial QED$_2$**

To calculate the transfer operator and its eigenvalue spectrum in the axial-vector model we follow the same steps as in the previous section, and will therefore only point out the differences in the calculation as compared to the vector case. Using the definitions (B.4), we find for the action (4.3) in terms of the fields $\chi^-$ and $\chi^+_\tau$, the expression

$$S = -\sum_{\tau=0}^{N-1} \left\{ \frac{1}{2} (\alpha \chi^+_{\tau+1} \chi^-_\tau - \beta \chi^+_{\tau+1} \chi^-_\tau) + \chi^+_\tau \tilde{\mathcal{H}}_\tau \chi^-_\tau \right\},$$

(C.1)

$$\alpha = 1 + \sin A, \quad \beta = 1 - \sin A.$$

(C.2)

The matrix $\tilde{\mathcal{H}}_{\tau xy}$ in (C.1) is given by the same expression as in (B.3), except that $U_{1 x,\tau}$ has to be replaced by the real form $\cos A$. With the same assignment of the variables $a_k, \ldots, b_k^+$ to the $\chi^-$ and $\chi^+_\tau$ fields, as in (B.4), and after the rescaling $a_k (a_k^+) \rightarrow a_k/\sqrt{\alpha}$ ($a_k^+ / \sqrt{\alpha}$) and $b_k (b_k^+) \rightarrow b_k/\sqrt{\beta}$ ($b_k^+ / \sqrt{\beta}$) we find that the matrix element of the transfer operator is given by

$$\langle a_{k+1}, b_{k+1} | \hat{T} | a_k, b_k \rangle = e^{-2N \ln 2} e^{N \ln(\alpha \beta)} e^{-2a_k^+ (\tilde{\mathcal{H}}_{2z+1}/\sqrt{\alpha \beta}) b_{k+1}} \times e^{(\beta/\alpha) a_{k+1} a_k + (\alpha/\beta) b_{k+1} b_k} e^{-2b_k (\tilde{\mathcal{H}}_{2z+1}/\sqrt{\alpha \beta}) a_k}.$$

(C.3)

The second exponential factor in (C.3) arises from the Jacobian in the above rescaling of the Grassmann measure: $d(a_k / \sqrt{\alpha}) = \sqrt{\alpha} \ dx_k$, etc. Using the rules (B.1) for transcribing the Grassmann algebra matrix elements to their associated Fock space operators we obtain from (C.3)

$$\hat{T} = e^{-2N \ln 2} e^{N \ln(\alpha \beta)} e^{-2a_k^+ \mathcal{H}_0 b_k^+} e^{\ln(\beta/\alpha) (\bar{a}^+ \bar{a} - \bar{b}^+ \bar{b})} e^{-2b_k \mathcal{H}_0 a_k},$$

(C.4)

$$\mathcal{H}_0_{\tau xy} = \text{sgn} (\cos A) \frac{1}{2} (\eta_{2 x,\tau} \eta_{1 x,\tau} \delta_{x,x+1} + \eta_{2 y,\tau} \eta_{1 y,\tau} \delta_{x,y+1}).$$

(C.5)

For $\text{sgn} \cos A > 0$, which is satisfied because we require $|A| < \pi/4$ (cf. Section 4), the matrix $\mathcal{H}_0$ is just the free staggered fermion Dirac operator. In the factor $\bar{a}^+ \bar{a} - \bar{b}^+ \bar{b}$ we recognize the charge operator $\hat{Q}$ of (B.24). The factor $\exp[\ln(\alpha/\beta) \hat{Q}]$ has simple commutation relations with the fields $\bar{a}, \bar{b}$ and $\bar{a}^+, \bar{b}$ and the analogue of (B.18) in this axial case is given by

$$\hat{T} \hat{\psi} = e^{\ln(\alpha/\beta) + 2\omega_0 \sigma_3} \hat{\psi} \hat{T}, \quad \sinh \omega_0 = \sin p,$$

(C.6)

$$V = \frac{1}{\sqrt{2} \cosh \omega_0} (e^{-\omega_0 \sigma_3 / 2} + i \sigma_2 e^{\omega_0 \sigma_2 / 2}).$$

(C.7)

The interpretation of the operators $\hat{\psi}, \hat{\psi}^+$ follows that of the vector case. The energy spectrum following from (C.6) is the same as in (A.10).
References

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