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Bisimulation is two-way simulation

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Abstract

We give here a simple proof of the fact that on transition systems bisimulation is the equivalence relation generated by simulation via functions. The proof entirely rests on simple rules of the calculus of relations.

Keywords: Theory of computation; Concurrency; Transition systems; Equivalence

Simulation is a standard notion of graph homomorphism that has been used in the study of flow diagram programs (see, e.g. [6,7,11]). Bisimulation is an equivalence on transition systems introduced by Park [9] in connection with Milner's work on concurrency [8]. In [3] we have shown that bisimulation is the equivalence relation generated by simulation via functions by using a translation between flowchart schemes and process graphs. We give here a simple proof of this fact using transition systems and simple rules of the calculus of relations, only.

To this end we show that (1) bisimulation via functions coincides with simulation via functions and (2) for two bisimilar processes one may find a common refinement which is bisimilar via functions with each of them.

In a categorical formulation this result also appears in [10]. A similar decomposition of bisimulation in terms of simpler relations was given by Castellani in [5]. She has shown that two nondeterministic processes (i.e., certain acyclic graphs that occur by unfolding transition systems) are bisimilar iff they may be reduced to a common one. The reductions used by Castellani are given by certain abstract homomorphisms, which roughly correspond to the relation on the unfolded processes induced by our simulation via functions defined on the level of (possible cyclic) transition systems. (See also the comments in the final part of the present paper.) This result was further extended in [1] to the logical equivalence of transition systems.

The result of this paper is used in [4] as a key point in order to extend the axiomatisation of process graphs modulo isomorphism to the case of bisimilar ones.

Remarks on notation. We use " $\circ$ " to denote relational composition of two binary relations (i.e., $R \circ T = \{(x,z) \mid \exists y\colon x\,R\,y \text{ and } y\,T\,z\}$) and Id_A to denote the identity relation on a set A (i.e., $\mathrm{Id}_A = \{(x,x) \mid x \in A\}$). Moreover, if R is a binary relation and T a ternary one, then $R \circ T = \{(x,z,w) \mid \exists y\colon (x,y) \in R \text{ and } (y,z,w) \in T\}$ and $T \circ R = \{(x,y,w) \mid \exists z\colon (x,y,z) \in T \text{ and } (z,w) \in R\}$.

Definitions. Let A be a set of atomic actions. A *transition system over A with multiple entries and multiple*

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exits $P : I \xrightarrow{S} O$ consists of three sets of vertices: I, S and O with $S \cap (I \cup O) = \emptyset$ (I specifies the start vertices, S the internal ones and O the end ones) and a transition relation $T \subseteq (I \cup S) \times A \times (O \cup S)$. Two transition systems $P : I \xrightarrow{S} O$ and $P' : I \xrightarrow{S'} O$ are *similar* via a function $\phi : S \to S'$ – written $P \to_\phi P'$ – if the corresponding transition relations T and T' fulfill

- $(\mathrm{Id}_I \cup \phi) \circ T' = T \circ (\mathrm{Id}_O \cup \phi)$.

P and P' are *bisimilar* via a relation $\rho \subseteq S \times S'$ – written $P \leftrightarrow_\rho P'$ – if the corresponding transition relations T and T' fulfill

- $(\mathrm{Id}_I \cup \rho) \circ T' \subseteq T \circ (\mathrm{Id}_O \cup \rho)$,
- $(\mathrm{Id}_I \cup \rho^{-1}) \circ T \subseteq T' \circ (\mathrm{Id}_O \cup \rho^{-1})$.

Note that simulation $\to$ is a transitive, but not a symmetric relation, whilst bisimulation $\leftrightarrow$ is an equivalence relation. We denote by $\leftarrow$ the converse of $\to$.

Lemma 1. *For two transition systems* $P : I \xrightarrow{S} O$ *and* $P' : I \xrightarrow{S'} O$ *(with the corresponding transition relations* T *and* T'*) and a function* $\phi : S \to S'$ *the following equivalence holds*:

(i) $(\mathrm{Id}_I \cup \phi^{-1}) \circ T \subseteq T' \circ (\mathrm{Id}_O \cup \phi^{-1})$

$\iff$

(ii) $T \circ (\mathrm{Id}_O \cup \phi) \subseteq (\mathrm{Id}_I \cup \phi) \circ T'$.

Proof. Since ϕ is a function, $\phi \circ \phi^{-1} \supseteq \mathrm{Id}_S$ (i.e., ϕ is a total relation) and $\phi^{-1} \circ \phi \subseteq \mathrm{Id}_{S'}$ (i.e., ϕ is a univocal relation).

($\Rightarrow$) Compose (i) on the left with $\mathrm{Id}_I \cup \phi$ and on the right with $\mathrm{Id}_O \cup \phi$. Then

$$(\mathrm{Id}_I \cup \phi \circ \phi^{-1}) \circ T \circ (\mathrm{Id}_O \cup \phi) \subseteq (\mathrm{Id}_I \cup \phi) \circ T' \circ (\mathrm{Id}_O \cup \phi^{-1} \circ \phi)$$

and this implies (ii) by the above properties of ϕ.

($\Leftarrow$) Similar. □

Corollary. *Simulation via a function coincides with bisimulation via a function.* □

In order to pass from bisimilar transition systems to chains of similar ones we need a characterization of bisimulation in terms of bisimulation via functions. The result given here is based on a construction of a common refinement of two bisimilar transition systems.

Lemma 2 (Interpolation property). *Suppose two transition systems* $P' : I \xrightarrow{S'} O$ *and* $P'' : I \xrightarrow{S''} O$ *and a relation* $\rho \subseteq S' \times S''$ *are given such that* $P' \leftrightarrow_\rho P''$. *Then there exist a transition system* $P : I \xrightarrow{S} O$ *and two functions* $\phi : S \to S'$ *and* $\psi : S \to S''$ *such that* $P \leftrightarrow_\phi P'$ *and* $P \leftrightarrow_\psi P''$.

Proof. Take a decomposition of ρ as $\phi^{-1} \circ \psi$ for two functions $\phi : S \to S'$ and $\psi : S \to S''$. Bisimulation $P' \leftrightarrow_\rho P''$ means

$$\text{(i)}\quad (\mathrm{Id}_I \cup \phi^{-1} \circ \psi) \circ T'' \subseteq T' \circ (\mathrm{Id}_O \cup \phi^{-1} \circ \psi)$$

and

$$\text{(ii)}\quad (\mathrm{Id}_I \cup \psi^{-1} \circ \phi) \circ T' \subseteq T'' \circ (\mathrm{Id}_O \cup \psi^{-1} \circ \phi)$$

Take the transition system $P : I \xrightarrow{S} O$ given by the transition relation

$$T = [(\mathrm{Id}_I \cup \phi) \circ T' \circ (\mathrm{Id}_O \cup \phi^{-1})] \cap [(\mathrm{Id}_I \cup \psi) \circ T'' \circ (\mathrm{Id}_O \cup \psi^{-1})].$$

Then $P \leftrightarrow_\phi P'$. Indeed, one inclusion is straightforward:

$$(\mathrm{Id}_I \cup \phi^{-1}) \circ T \subseteq (\mathrm{Id}_I \cup \phi^{-1} \circ \phi) \circ T' \circ (\mathrm{Id}_O \cup \phi^{-1}) \subseteq T' \circ (\mathrm{Id}_O \cup \phi^{-1})$$

where in the last step we have used the fact that ϕ is a univocal relation, hence $\phi^{-1} \circ \phi \subseteq \mathrm{Id}_{S'}$.

For the second one we use (ii). By a left composition of (ii) with $\mathrm{Id}_I \cup \psi$ we get

$$(\mathrm{Id}_I \cup \psi \circ \psi^{-1} \circ \phi) \circ T' \subseteq (\mathrm{Id}_I \cup \psi) \circ T'' \circ (\mathrm{Id}_O \cup \psi^{-1} \circ \phi)$$

By using $\mathrm{Id}_S \subseteq \psi \circ \psi^{-1}$ (i.e., ψ is a total relation) it follows that the left-hand side of this inclusion contains $(\mathrm{Id}_I \cup \phi) \circ T'$. On the other hand, the right-hand side of this inclusion is equal to the second part of the definition of T composed on right with $\mathrm{Id}_O \cup \phi$, hence is included in $T \circ (\mathrm{Id}_O \cup \phi)$. Consequently,

$$(\mathrm{Id}_I \cup \phi) \circ T' \subseteq T \circ (\mathrm{Id}_O \cup \phi)$$

In a similar way one may prove that $P \leftrightarrow_\psi P''$. □

Theorem. *Bisimulation is the equivalence relation generated by simulation via functions. More precisely,*

$$\leftrightarrow \quad = \quad \leftarrow \circ \rightarrow.$$

Proof. From the lemmas above. □

Simulation is an useful tool to speak about minimization. Simulation via surjective functions models identification of the states with the same behaviour and the *converse* of the simulation via injective functions models the deletion of the nonaccessible parts. We may give an alternative proof of the Theorem by replacing Lemma 2 with another one using minimization (as in [11] or as in Theorem 2.7.13 in [2] regarding normal forms), but the characterization is less direct, i.e., instead of the present characterization

$$\leftrightarrow \quad = \quad \leftarrow \circ \rightarrow,$$

we get

$$\leftrightarrow \quad = \quad \rightarrow_{\mathrm{sur}} \circ {}_{\mathrm{inj}}\!\leftarrow \circ \rightarrow_{\mathrm{inj}} \circ {}_{\mathrm{sur}}\!\leftarrow,$$

where $\rightarrow_{\mathrm{sur}}$ and $\rightarrow_{\mathrm{inj}}$ are the restrictions of simulation to simulation via surjective and injective functions, respectively. The latter characterization says that two transition systems are bisimilar iff by the identification of the states with the same behaviour and the deletion of the nonaccessible ones both systems may be reduced to the same minimal one.

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