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19-vertex version of the fully frustrated XY model

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We investigate a 19-vertex version of the two-dimensional fully frustrated XY (FFXY) model. We construct Yang-Baxter equations for this model and show that there is no solution. Therefore we have chosen a numerical approach based on the transfer matrix. The results show that a coupled XY Ising model is in the same universality class as the FFXY model. We find that the phase coupling over an Ising wall is irrelevant at criticality. This leads to a correction of earlier determinations of the dimension $x_{h,Is}^*$ of the Ising disorder operator. We find $x_{h,Is}^* = 0.123(5)$ and a conformal anomaly $c = 1.55(5)$. These results are consistent with the hypothesis that the FFXY model behaves as a superposition of an Ising model and an XY model. However, the dimensions associated with the energy, $x_t = 0.77(3)$, and with the XY magnetization $x_{h,XY} \approx 0.17$, refute this hypothesis.

I. INTRODUCTION

The fully frustrated XY (FFXY) model is very interesting from a theoretical point of view, as well as experimentally. Because of the direct relation between the FFXY model and Josephson junctions arrays (arrays of weakly coupled superconducting islands) in a perpendicular magnetic field theoretical predictions can be tested experimentally.^{1,2}

In recent years a considerable number of theoretical papers has appeared³⁻¹⁵ concerning the universality class to which the FFXY model belongs. So far all numerical results are based on applications of the Monte Carlo technique and are therefore subject to statistical errors. By mapping a FFXY model onto a 19-vertex model we are able to obtain numerically exact results with the transfer-matrix technique. One might expect that the improved accuracy would be helpful in the analysis, and perhaps shed more light on the problem of the nature of the transition.

In this Introduction we present a survey of the most recent results concerning FFXY models. First we summarize different models which are believed to be in the same universality class as the FFXY model. Then we explain briefly the quantities determined in these models. The results are summarized in Table I.

A FFXY model can be defined on a square lattice by the reduced Hamiltonian

$$-\beta\mathcal{H} = K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j - A_{ij}), \quad (1)$$

where $\beta = 1/k_B T$ and the sum is over all nearest neighbor pairs. At every site of the lattice is a spin, which is able to rotate in a two-dimensional plane. Thus the model is a planar model. The XY spin at site i is specified by the angle θ_i . The coupling strength of the interaction is K . The bond variables A_{ij} are restricted by the relation $\sum_p A_{ij} = 2\pi f$. The sum is a directed sum over each plaquette and f is the frustration. For the FFXY model $f = \frac{1}{2}$. Once chosen, the bond variables are fixed. We need not insist on the precise form [Eq. (1)] of the Hamiltonian. More generally, FFXY models are characterized by a global rotational invariance, by an interaction periodic in neighbor angle variables, and by the full frustration in each plaquette.

The relation with Josephson junction arrays is straightforward. The angle θ_i represents the phase of the superconducting order parameter in the corresponding island. The coupling K is related to the critical current i_c . The bond variables A_{ij} are due to the magnetic field perpendicular to the surface of Josephson junctions arrays. The relation with the magnetic field B is given by $f = BS/\Phi_0$, where S is the area of a unit cell and $\Phi_0 = h/2e$ is the magnetic flux quantum.

The ground state of this model [Eq. (1)] shows two types of ordering. The frustration $f = \frac{1}{2}$ in each plaquette is equally distributed over all the surrounding bonds. This leads to plaquettes with a positive or negative chirality. Thus the ground state has, apart from an arbitrary uniform rotation of all the spins, a choice between two checkerboardlike configurations of different

chirality (see Ref. 3). The set of ground states thus has a continuous rotational $U(1)$ symmetry and a discrete Z_2 Ising symmetry. Therefore one might expect XY -like as well as Ising-like critical behavior. We consider three possible scenarios for the phase transitions.⁴

(i) A Kosterlitz-Thouless (KT) transition and an Ising transition occur at different temperatures.

(ii) A KT transition and an Ising transition occur at the same temperature. The coupling between the XY and Ising degrees of freedom is irrelevant. Then we may describe the FFX Y model as a superposition of an Ising model and an XY model. This model has a conformal anomaly $c = \frac{3}{2}$ (which is the sum of the Ising value $c = \frac{1}{2}$ and the XY value $c = 1$).

(iii) The XY transition and the Ising transition occur at the same temperature, and the coupling between both types of degrees of freedom is relevant. This may mean a new universality class, with exponents that are different from the well-known Ising and KT ones.

Which possibility is realized is the subject of this investigation.

Another representation of the FFX Y model uses the Coulomb gas. It is plausible that models with a somewhat different periodic interaction are still in the same universality class. This applies also to the Coulomb gas model, which is derived exactly from a FFX Y model with a Villain type of interaction.⁵ To obtain the Coulomb gas the spin wave degrees of freedom in a FFX Y model of the Villain type are integrated out, and the vortices remain. The ground state consists of a checkerboard configuration of vortices $m = \pm \frac{1}{2}$. Considering this ground state as a neutral background there are two types of excitations. Inverting one single vortex creates a charge ± 1 . The corner of an Ising wall between two areas of different checkerboard configuration has a charge $\pm \frac{1}{4}$. By adding an extra Ising coupling J between nearest neighbor vortices m the Ising wall energy can be tuned independently. The Hamiltonian of this Coulomb gas representation of the FFX Y model is given by

$$-\beta\mathcal{H} = 4J \sum_{\langle rr' \rangle} m_r m_{r'} + K \sum_{rr'} m_r m_{r'} V(r - r'), \quad (2)$$

where $V(r - r')$ is the Coulomb potential. The phase diagram of this model was studied by Thijssen and Knops.⁷ It has three types of critical behavior. For large Ising wall energies J the Ising degrees of freedom are more or less frozen out and there is a line of KT transitions. If the Coulomb interaction K is very small, there is a line of Ising transitions. At a certain point, the branch point, these two lines meet and merge together into one single critical line. It is believed⁵ that along this line the XY transition is triggered by the Ising transition. At the Ising transition the charges $\pm \frac{1}{4}$ at the corners of the Ising walls become free, thus inducing the XY transition.

Another related model is a coupled XY -Ising model. This model is obtained from the FFX Y model by considering a Ginzburg-Landau description of the excitations from the ground state.¹⁰ This Hamiltonian is sufficiently general to vary the Ising wall energy between two domains of different ordering of the chirality. Each site car-

ries an XY spin θ_i and an Ising spin s_i . The Hamiltonian is given by

$$-\beta\mathcal{H} = \sum_{\langle ij \rangle} \left\{ \frac{1}{2} K (1 + s_i s_j) \cos(\theta_i - \theta_j) + J s_i s_j \right\}. \quad (3)$$

The Ising spins s_i represent one of the two checkerboard configurations and J is the coupling between nearest neighbor Ising spins. According to Eq. (3) the XY coupling between two nearest neighbor spins is absent whenever the Ising spins at these sites are opposite. The phase diagram of this model was studied by Granato *et al.*⁸ Their result is in agreement with that found by Thijssen and Knops⁷ using the Coulomb gas representation. In the case of Eq. (3) the Ising and XY -transition merge because there is no phase coupling over an Ising wall. Thus also for the coupled XY -Ising model the XY transition is triggered by the Ising transition. However, in the case of the FFX Y model of Eq. (1), the phases between different Ising domains are coupled, at least in the ground state (see Ref. 3). Therefore the model of Eq. (3) misses a “phase-coupling” term of the form $(1 - s_i s_j) \cos[2(\theta_i - \theta_j)]$. The question whether this term is relevant will be addressed later.

Finally we mention a related model to the FFX Y model, studied by den Nijs.¹⁶ For a certain choice of the parameters, this model is dual to the coupled XY -Ising model of Eq. (3). It is a coupled 19-vertex-Ising model. It is studied in the context of a competition between surface roughening and reconstruction transitions in (110) facets of fcc crystals. As the two transitions merge the reconstruction transition induces the roughening transition. The exponents are Ising like with respect to the reconstruction degrees of freedom and KT like with respect to the roughening degrees of freedom.

In this paper we contribute another model to the discussion. We have studied a 19-vertex version of the FFX Y model. A derivation will be given in Sec. II.

First we summarize the different quantities determined so far in these models in earlier papers^{8,9,12,14-16} as well as in this paper. The way they are determined will be explained in Sec. V, where we present our results. The conformal anomaly c characterizes the universality class. We distinguish two magnetic Ising-like dimensions: one belonging to a spin-spin correlator $x_{h,1s}$ and one belonging to a disorder-disorder correlator $x_{h,1s}^*$. For the pure Ising model these dimensions are related by duality and are thus equal to each other. Here we are careful to make a distinction since the coupling between Ising and XY -like degrees of freedom might spoil this duality relation. We consider the thermal dimension x_t as the leading dimension belonging to the temperature direction K . The last quantity to be calculated concerns the XY -like degrees of freedom. The scaling dimension corresponding to the XY magnetization can be obtained in two ways: either from the free energy increase due to a seam with a phase shift ϕ or directly from the action of the operator $\exp(i\theta)$. When the XY model renormalizes to a free Gaussian model both methods give an independent estimate of the renormalized Gaussian coupling g . The results for g of these two methods will be denoted with

a superscript (1) and (2), respectively.

In Table I we summarize the most recent results. We present the results in a chronological order, indicating the model which is studied, the method which is used, and the size of the largest considered system.

Only the results of den Nijs¹⁶ are in agreement with the hypothesis that the FFXXY model can be described as a superposition of an Ising model and an XY model. However, this author has transfer-matrix results only up to size 7. This is rather small to obtain conclusive results.

The results of Ramirez-Santiago and José¹⁴ are based on extensive Monte Carlo simulations. The values $x_{h,Is}$ and x_t in Table I are calculated, using standard scaling relations, from the exponents $\nu = 0.875$ and $2\beta = 0.1936$ obtained by them. One should however be careful in this interpretation since the values $2\gamma = 1.82$ and $2\gamma' = 1.025$ quoted in the same paper do not obey the scaling relations $2\beta + \gamma = d\nu = 2\beta + \gamma'$, where d is the dimensionality.

In the rest of the paper we discuss our results. Our approach to the problem is based on an exact mapping of a FFXXY model onto an alternating 19-vertex model. This mapping is described in Sec. II. In Sec. III we define Yang-Baxter equations for a somewhat larger class of alternating 19-vertex models and check for the presence of nontrivial solutions. None are found. However, for systems with linear sizes up to $L = 15$, the partition function can be obtained numerically using a transfer-matrix technique. This calculation is presented in Sec. IV and the results are given in Sec. V. Section VI contains a discussion and Sec. VII the conclusions.

II. MAPPING A FFXXY MODEL ONTO AN ALTERNATING 19-VERTEX MODEL

We start with the FFXXY model of Eq. (1). The bond variables A_{ij} can be distributed in several ways over the square lattice. We use the Landau gauge where all vertical bonds have $A_{ij} = 0$ and the rows of horizontal bonds alternate between $A_{ij} = 0$ and $A_{ij} = \pi$. Thus there are ferromagnetic vertical couplings and rows of horizontal couplings which are alternating ferromagnetic and antiferromagnetic. The partition function of this model is given by

$$Z = \int_{-\pi}^{\pi} d\{\theta\} \prod_{\langle ij \rangle} \exp[K \cos(\theta_i - \theta_j - A_{ij})]. \quad (4)$$

This expression is, however, difficult to handle using transfer matrices. For this reason we choose a slightly different FFXXY model with the partition sum

$$Z = \int_{-\pi}^{\pi} d\{\theta\} \prod_{\langle ij \rangle} [1 + K \cos(\theta_i - \theta_j - A_{ij})]. \quad (5)$$

For K small enough, this partition function still has the characteristic properties of Eq. (4): the U(1) symmetry and the frustration.

This model is equivalent with an alternating 19-vertex model. This is shown by writing the partition sum as

$$Z = \int_{-\pi}^{\pi} d\{\theta\} \prod_{\langle ij \rangle} \left\{ 1 + \frac{K}{2} \exp[i(\theta_i - \theta_j - A_{ij})] + \frac{K}{2} \exp[-i(\theta_i - \theta_j - A_{ij})] \right\}. \quad (6)$$

Expanding this product we can represent each term by a directed graph on the square lattice. The three terms between square brackets in Eq. (6) are represented graphically by no arrow, an arrow pointing from site i to site j , and an arrow pointing from site j to site i , respectively. After integration over the θ variables, only those terms survive in which all vertices have an equal number of incoming and outgoing arrows. These vertices are precisely those of the 19-vertex model. In order to account for the frustration a minus sign is assigned to each horizontal arrow across the antiferromagnetic couplings and associated with the vertex weight to its right. For a row with ferromagnetic couplings the weights are denoted as $W(\alpha, \beta, \gamma, \delta)$; for a row with antiferromagnetic couplings the weights are denoted as $\tilde{W}$. W and $\tilde{W}$ are given in Fig. 1. The partition function of this alternating 19-vertex model is given by

$$Z = \sum_{\text{arrows}} \prod_{i=1}^{19} W_i^{n_i} \tilde{W}_i^{\tilde{n}_i}, \quad (7)$$

where n_i and $\tilde{n}_i$ are the number of vertices in an arrow

TABLE I. A survey of recent results for various representations of the FFXXY model. The entries on the lowest line are the result of the present work. The results are obtained by Monte Carlo simulations (MC), Monte Carlo transfer matrix calculations (MCT), or transfer matrix calculations (T). The values of $x_{h,Is}$ and x_t from Ref. 14 are calculated, using scaling relations, from the exponents $\nu = 0.875$ and $2\beta = 0.1936$ obtained in this reference. The values quoted for the scaling dimension $x_{h,Is}^*$ are not corrected for the free boundary conditions. The corrected values are 0.138(10), 0.128(15), and 0.123(5), respectively.

Model	Method	$L_{\max}$	Ref.	c	$x_{h,Is}$	$x_{h,Is}^*$	x_t	$g^{(1)}$	$g^{(2)}$
FFXY	MCT	12	9	1.66(4)		0.20(1)	≈ 1	4.7	
XY-Ising	MC	30	8	1.5-2.0	0.14-0.21		0.7-0.85		
FFXY	MC	40	12		0.155(15)		0.82(4)		
19-vertex-Ising	T	7	16	1.50(5)	0.129(5)		1.0(1)		4
FFXY	MCT	14	15	1.61(3)		0.190(15)	0.75(6)		
FFXY	MC	96	14		0.11		0.86		5.2
19-vertex	T	15		1.55(5)	0.14(1)	0.185(5)	0.77(3)	3.6(2)	≈ 3

configuration with weight W_i and $\tilde{W}_i$, respectively. The sum is over all possible arrow configurations.

III. YANG-BAXTER EQUATIONS FOR A FFXY MODEL

As a consequence of the existence of classes of commuting transfer matrices many statistical mechanics models are solvable (i.e., the free energy can be calculated exactly). To satisfy the Yang-Baxter (YB) equations, or the star-triangle equations, is a sufficient condition for transfer matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ to commute (see Ref. 17).

In order to check for the solvability of the FFXY model we consider the YB equations for the alternating 19-

vertex model. By commuting the transfer matrices the frustration may not be changed. Therefore we have to write the commutation relation as $\mathbf{T}_1 \tilde{\mathbf{T}}_2 = \mathbf{T}_2 \tilde{\mathbf{T}}_1$, where $\mathbf{T}$ adds a vertex with ferromagnetic couplings and $\tilde{\mathbf{T}}$ a vertex with antiferromagnetic couplings. The YB equations belonging to this commutation relation are represented in Fig. 2. Summed over the internal indices γ, μ_2 , and ν_2 , the left hand side (LHS) has to be equal to the right hand side for each choice for the external indices $\alpha, \beta, \mu, \mu_1, \nu$, and ν_1 . The indices can take the values $+, 0, -$. The open circles denote the antiferromagnetic couplings.

Solving this set of equations we have used a generalization of the weights deduced in the previous section. These are

$$\begin{aligned}
 W(0, 0, 0, 0) &= a, \\
 W(0, 0, +, -) &= W(0, 0, -, +) \\
 &= W(+, -, 0, 0) = W(-, +, 0, 0) = b, \\
 W(+, 0, 0, -) &= W(-, 0, 0, +) \\
 &= W(0, +, -, 0) = W(0, -, +, 0) = c, \\
 W(0, +, 0, -) &= W(0, -, 0, +) \\
 &= W(+, 0, -, 0) = W(-, 0, +, 0) = d, \\
 W(+, -, -, +) &= W(-, +, +, -) = e, \\
 W(+, +, -, -) &= W(-, -, +, +) = f, \\
 W(+, -, +, -) &= W(-, +, -, +) = g.
 \end{aligned} \tag{8}$$

Due to symmetries and the ice rule (same number of arrows in as out) the 729 equations reduce to 23 different equations. Using formula manipulation software it appears that these remaining equations are too determined to have a nontrivial solution. Too many weights have to be equal to 0 to satisfy the equations.

A generalization of this approach is to allow different weights for W_3 on the LHS and the RHS of the YB equations. The requirement of commuting transfer matrices thus leads to a second set of YB equations. These two sets are represented in Fig. 3. Again the set equations are too determined and no nontrivial solution is found. The weights [Eq. (8)] do not include nontrivial solutions to the YB equations and thus no solvable FFXY model was found.

FIG. 1. Vertex weights W and $\tilde{W}$ of the present 19-vertex model. These weights apply to vertices with ferromagnetic and antiferromagnetic horizontal couplings, respectively. The arguments v_i of W denote the arrows surrounding each vertex, counterclockwise, and starting on the left.

FIG. 2. A graphical representation of the Yang-Baxter equations for the 19-vertex version of the FFXY model. After summation over the internal indices, the weight of the left hand side has to be equal to the right hand side, for each choice of the external indices. The open circles denote the antiferromagnetic couplings. There are three sets of vertex weights W .

FIG. 3. A more general set of Yang-Baxter equations for the 19-vertex version of the FFXY model. The open circles denote the antiferromagnetic couplings. There are four sets of vertex weights W .

IV. TRANSFER-MATRIX CALCULATIONS FOR A FFXY MODEL

Transfer-matrix calculations are very powerful in determining accurately the free energy for models with a cylindrical geometry. For our calculations we have used two transfer matrices. One transfer matrix T_h works in the horizontal direction; each multiplication with T_h adds one column. Since the horizontal couplings alternate between subsequent rows, T_h can only be used for cylinders with an even number of sites on the circumference. The other transfer matrix T_v works in the vertical direction. Each multiplication with T_v adds two rows. T_v can be used for cylinders with even and odd finite size.

The construction of these transfer matrices uses the resemblance with a one-dimensional spin-1 quantum Hamiltonian.¹⁸ The outgoing arrows, zeros, and incoming arrows of the last added row or column are identified with $s_z = 1, 0, -1$, respectively. A sparse-matrix decomposition greatly improves the efficiency of the calculations; for details we refer to Ref. 19.

Because the total sum of arrows between two neighboring columns or between two neighboring rows is invariant, the transfer matrices T_h and T_v both decompose into diagonal blocks. Block S contains all the states with total sum of the arrows S . The number of blocks is $2L + 1$ where L is the circumference of the cylinder.

The free energy per site in the limit of an infinitely long cylinder with finite size L is given by

$$f_L = \zeta \frac{1}{L} \ln \lambda_L^{(0)}, \quad (9)$$

where the geometric factor ζ is the inverse number of layers added by the transfer matrix ($\zeta = 1$ or $\frac{1}{2}$), and $\lambda_L^{(0)}$ is the largest eigenvalue of the transfer matrix. We expect to find this eigenvalue in block 0.

Transfer-matrix calculations can also be used to determine the characteristic lengths of a number of correlation functions, e.g., the energy-energy and the magnetization-magnetization correlation functions. Correlation lengths ξ_i are associated with λ_i :

$$\xi_{L,i}^{-1} = \zeta \ln(\lambda_L^{(0)} / \lambda_L^{(i)}), \quad (10)$$

where $\lambda_L^{(i)}$ is an eigenvalue of T . The right hand side of

Eq. (10) can be seen as a free energy gap.

It is also possible to determine free energy gaps by changing the boundary conditions for the cylinder. In most cases a cylinder with periodic boundary conditions (PBC's) has a minimal free energy. By determining the free energy of a cylinder with other boundary conditions (OBC's) an energy gap can be determined. The corresponding correlation length is given by

$$\xi_L^{-1} = L[f_L(\text{PBC}) - f_L(\text{OBC})]. \quad (11)$$

Boundary conditions are changed by adapting the transfer matrix.

By means of the direct iteration-Hessenberg algorithm (see Ref. 20) we are able to calculate the two largest eigenvalues in each block. By imposing the symmetry of the corresponding eigenvector it is also possible to calculate further eigenvalues. Thus we have calculated three eigenvalues in block 0 and the leading eigenvalue in block 1 for system sizes up to $L = 15$. We have also calculated free energies with four different boundary conditions. Most of the calculations are done using T_v . We will explicitly indicate where we have used T_h .

V. RESULTS

From the theory of conformal invariance²¹ it follows that for the free energy per site of a cylinder with finite size L , f_L , the leading finite size correction of the free energy per site of an infinite system, f_∞ , is determined by the conformal anomaly c .²² For cylinders with even finite size L ,

$$f_L \approx f_\infty + \frac{\pi c}{6L^2}, \quad L \text{ even}. \quad (12)$$

For cylinders with odd finite size L the relation is different. Because the ground state does not fit on an odd cylinder, there is in effect an Ising wall with an interface energy σ , which vanishes at the critical temperature. Therefore for cylinders with odd finite size the relation is

$$f_L \approx f_\infty + \frac{\sigma}{L} + \frac{\pi \tilde{c}}{6L^2}, \quad L \text{ odd}, \quad (13)$$

where $\tilde{c} = c_0 - 12x_{h,Is}^*$, and $x_{h,Is}^*$ is the scaling dimension belonging to the disorder operator that generates the Ising wall. We will discuss the value for c_0 in relation with c below. The amplitude $\tilde{c}$ is not the conformal anomaly, because the ground-state energy of odd systems is not determined by the identity operator, as is the case for even systems.²¹ It also follows from the theory of conformal invariance that the asymptotic behavior of the correlation length ξ for large L is related to the anomalous dimension x pertaining to ξ as^{21,23}

$$\frac{L}{\xi_L} \simeq 2\pi x. \quad (14)$$

First we focus on the determination of the conformal anomaly as function of K . We calculated the free energy for cylinders with even finite sizes $L = 2$ up to $L = 14$ for several values of K . Using Eq. (12), the conformal

anomaly is estimated as $c_{L,L+2} = 3(f_L - f_{L+2})L^2(L+2)^2/2\pi(L+1)$. This quantity converges to c as $L \rightarrow \infty$. Results in terms of $c_{L,L+2}$ can be found in Fig. 4. These results indicate that the Ising and XY transitions occur at the same temperature K_c . For $K > K_c$ ($K_c \approx 1.2$; see Fig. 4) the conformal anomaly estimates go to $c = 1$ for increasing sizes. This is the value for the low T phase of the XY model, also called the algebraic phase. Below K_c the model is not critical; thus we find convergence to $c = 0$. For $K > 1.4$ the results for the conformal anomaly deviate from $c = 1$. Presumably for these K values Eq. (5) ceases to be a good FFX model, at least for the finite sizes we have studied. Too many weights have become negative [the weights of Eq. (4) are always positive].

The location of K_c is determined in several ways. At K_c the Ising wall energy σ vanishes. We calculated σ for several values for $K > K_c$, using system sizes $L = 3$ up to $L = 13$ and Eq. (13). The results are in Fig. 5: σ is decreasing almost linearly when K approaches K_c . This gives a determination of K_c : $K_c = 1.192(2)$.

Another way to determine K_c is to study the scaled gap $x(L)$ between the free energies of cylinders with odd and even size. Since we can interpret the Ising wall, induced for odd systems, as a change in the boundary conditions, we can define the scaled gap, using Eqs. (11) and (14), as

$$x(L) = \frac{L}{2\pi} \left\{ Lf_L - \frac{1}{2}[(L-1)f_{L-1} + (L+1)f_{L+1}] \right\}. \quad (15)$$

The intersections of the $x(L)$ curves converge for increasing system sizes to K_c . We have determined the K values of these intersections using sizes $L = 2$ up to $L = 15$. Extrapolation to $L \rightarrow \infty$ yields $K_c = 1.191(1)$.

To determine the conformal anomaly at K_c we fitted the data for the free energy for even finite sizes $L = 2$ up to $L = 14$ to Eq. (12). To reduce the effect of corrections

FIG. 4. The conformal anomaly, estimated as $c_{L,L+2}$, as a function of K . Data are shown for even sizes $L = 2$ up to $L = 14$. The sharper the peak in $c_{L,L+2}$, the larger is L .

FIG. 5. The Ising wall energy σ as function of K . We have used data for odd sizes $L = 3$ up to $L = 13$ to extrapolate σ to $L \rightarrow \infty$. At K_c , σ vanishes.

to scaling we used an iterated fit procedure as described in Ref. 19. The result is $c = 1.55(5)$. This value is remarkably close to the value $c = 1 + \frac{1}{2}$ to be expected for a decoupled XY -Ising system. Due to large corrections to scaling we were unable to obtain a better precision than that quoted by Thijssen and Knops⁹ who used a Monte Carlo transfer-matrix method.

At the critical temperature our model has negative Boltzmann weights [see Eq. (5)] and therefore may be not unitary. Therefore the question rises whether we indeed found the conformal anomaly by using Eq. (12). In a nonunitary theory it is possible that negative dimensions occur. If this is the case, we only found an upper bound for c . However, for $K < 1.0$, our model is unitary. The continuation of the leading eigenvalue for $K < 1.0$ is still the leading eigenvalue at K_c . There is no intersection with another eigenvalue. Therefore there is no negative dimension and we indeed found the right c .

Next we change the boundary conditions by applying a seam along the cylinder, along which we add an extra phase shift ϕ . This is done by changing the weights [in Eq. (5)] of the bonds which cross this seam from $1 + \cos(\theta_i - \theta_j - A_{ij})$ to $1 + \cos(\theta_i - \theta_j - A_{ij} - \phi)$. In the ordered phase this extra phase ϕ will be distributed in differences $\theta_i - \theta_j$ across the cylinder. The scaled gap $x_\phi(L)$ belonging to this seam is defined as [see Eqs. (11) and (14)]

$$x_\phi(L) = \frac{L^2}{2\pi} [f_L(0) - f_L(\phi)], \quad L \text{ even}. \quad (16)$$

For odd sizes the definition is different, because in the geometry we use, the Ising wall leads to a phase mismatch of $\frac{\pi}{2}$ in the ground state.³ Thus we define

$$x_\phi(L) = \frac{L^2}{2\pi} \left[f_L\left(\frac{\pi}{2}\right) - f_L\left(\frac{\pi}{2} - \phi\right) \right], \quad L \text{ odd}. \quad (17)$$

At the critical temperature the scaled gap converges to the critical dimension x_ϕ for increasing L . The dimension

x_ϕ is an XY-like dimension. For the ordinary XY model this dimension is given by (see Ref. 24)

$$x_\phi = \frac{g}{2} \left(\frac{\phi}{2\pi} \right)^2, \quad (18)$$

where g is the renormalized coupling, when the XY model is renormalized to the Gaussian model:

$$-\beta\mathcal{H} = \frac{g}{4\pi} \int d^2r [\nabla\theta(r)]^2. \quad (19)$$

At the KT transition $g = 4$, and in the algebraic phase, or the low temperature region, $g > 4$.

First we treat the results for the odd systems. In Fig. 6 the scaled gap is plotted as function of ϕ for odd sizes $L = 3$ up to $L = 9$. From this figure it can be seen that in our geometry an Ising wall indeed creates a phase mismatch of $\frac{\pi}{2}$ in the ground state. However, the surprising result is that the scaled gap converges for increasing L to $x_\phi = 0$ for all ϕ , though for $K > K_c$ we do observe a nonzero x_ϕ . Thus there is no phase coupling over an Ising wall at the critical temperature. This has two important consequences.

(i) The phase coupling term which is neglected in the coupled XY-Ising representation of the FFXXY model is irrelevant. Therefore this model is indeed in the same universality class as the FFXXY model.

(ii) Systems with L odd have an Ising wall along the cylinder. At the critical temperature there is no phase coupling over this wall, so that the XY degrees of freedom do not have periodic boundary conditions but rather free boundary conditions. This must have an effect on the value of the conformal anomaly c . It is well known²² that the finite size correction [see Eq. (12)] for a Gaussian (or XY) model is reduced from $\pi c/6L^2$ for periodic boundary conditions to $\pi c/24L^2$ for free boundary conditions. The situation here is more complicated: The free boundary conditions are to be taken along a fluctuating Ising wall. There might also be an effect due to the coupling between the XY and Ising degrees of freedom [although this effect seems to be weak as far as the conformal anomaly is concerned since $c = 1.55(5)$; see above]. Neglecting these complications we estimate c_o , defined in Eq. (13), as $c_o \approx c - \frac{3}{4}$.

We have to take this second point into account when

FIG. 6. The scaled gap $x_\phi(L)$ as function of ϕ . Data are shown for odd sizes $L = 3$ up to $L = 9$.

we calculate the scaling dimension belonging to an Ising wall. From a finite size analysis using Eq. (13), with $c_o \approx c - \frac{3}{4}$ [$c = 1.55(5)$], follows $x_{h,Is}^* = 0.12(1)$. Just as our value for c this result is consistent with the hypothesis that the FFXXY model is a superposition of an Ising model and an XY model. For the Ising model the magnetic dimension is $\frac{1}{8}$. Also the results of Thijssen and Knops⁹ and of Granato and Nightingale¹⁵ are in agreement with this value if we correct their analysis of the free energy for odd systems.

An analogous reasoning is used by den Nijs¹⁶ when he interprets his results. For a coupled 19-vertex-Ising model he considered strips with different boundary conditions to determine several free energy gaps. One of these gaps $\eta(\pi/2, 1)$ could only be interpreted as a KT free-energy gap when the Ising spins are treated as subject to an open boundary condition.

The value for the magnetic Ising-like dimension can be determined more directly by extrapolating the scaled gap between the free energies of cylinders with odd and even sizes [Eq. (15)] rather than by taking the difference between two extrapolated values. At K_c this scaled gap converges to $x_{h,Is}^* + \frac{1}{16}$ as $L \rightarrow \infty$. The correction $\frac{1}{16}$ follows from the different values for the finite size amplitude for L even and odd [see Eqs. (12), (13), and (15)]. Thus we obtain $x_{h,Is}^* = 0.123(5)$.

Now we return to the results for even L in the case of a seam with a phase shift ϕ . The results for the scaled gap $x_\phi(L)$ as function of ϕ for $L = 4$ up to $L = 10$ are shown in Fig. 7. For ϕ small we do indeed observe a quadratic behavior, as expected from Eq. (18). This can explicitly be seen from the results of a finite size analysis for $\phi = \frac{\pi}{10}$ and $\phi = \frac{\pi}{5}$. The results are, respectively, $x_\phi = 0.0045(3)$ and $x_\phi = 0.018(1)$. When we calculate g from these results using Eq. (18) we find $g = 3.6(2)^{(1)}$. This is an unexpected low value for g , because for $g \leq 4$ even single XY vortices are free according to the KT theory. (From the ideas of Korshunov⁵ one would rather expect $g > 4$ at K_c for the FFXXY model, because of the liberation of fractional vortices that reside at the corners of the Ising walls.)

For larger values of ϕ there is no quadratic behavior anymore. This could be because it becomes cheaper in terms of the free energy to have two Ising walls instead of distributing the extra phase shift across the cylinder. Then the scaled gap converges to $2x_{h,Is}^* = \frac{1}{4}$ for approx-

FIG. 7. The scaled gap $x_\phi(L)$ as function of ϕ . Data are shown for even sizes $L = 4$ up to $L = 10$.

imately $\phi > \frac{4}{5}\pi$. Thus we expect to observe a plateau in Fig. 7. However, our results do not indicate a plateau for large values of ϕ . Another possibility is that the same behavior occurs as is observed²⁵ in the Izergin-Korepin model,²⁶ which is a 19-vertex representation of an $O(n)$ model. The behavior of x_ϕ as function of ϕ is described as two parabolas. Extrapolated to $n = 2$ these parabolas are given by $\frac{g}{2}(\frac{\phi}{2\pi})^2$ for $\phi < \frac{\pi}{2}$ and by $\frac{g}{16} - \frac{g}{2}(\frac{\pi-\phi}{2\pi})^2$ for $\phi > \frac{\pi}{2}$. This behavior is not understood. A direct fit of our results to this formula would yield $g = 3.5(1)^{(1)}$.

The other way to determine the effective Gaussian coupling is from the correlation between two spins θ_0 and θ_r at a distance r . For $r \gg 1$ this XY magnetic correlation function behaves as

$$\langle \cos(\theta_0 - \theta_r) \rangle \sim r^{-2x_{h,XY}}. \quad (20)$$

The scaling dimension $x_{h,XY}$ can be obtained as

$$x_{h,XY}(L) = \frac{L}{2\pi} (\ln \lambda_L^{(0,0)} - \ln \lambda_L^{(0,1)}), \quad (21)$$

where $\lambda_L^{(0,0)}$ and $\lambda_L^{(0,1)}$ are the leading eigenvalues in sector 0 and sector 1 of the transfer matrix respectively (see, e.g., Ref. 19). In Appendix A it is shown that also for the FFX Y model this scaling dimension is related to the coupling g , as defined by Eq. (18), by

$$x_{h,XY} = \frac{1}{2g}. \quad (22)$$

For a KT transition $x_{h,XY} = \frac{1}{8}$ at K_c and $x_{h,XY} < \frac{1}{8}$ above K_c .

To compare the results of the FFX Y model with the XY model we have also mapped an XY model onto a 19-vertex model, in the same way as described in Sec. II. This yields a 19-vertex model with only positive weights W (see Fig. 1).

First we present the results of the XY model. The results for $x_{h,XY}$ as function of $\tilde{K}$, the coupling in the XY model, are shown in Fig. 8. Data are used for $L = 2$ up to $L = 8$. The critical coupling is determined as $\tilde{K}_c = 1.0866(2)$. It follows from this figure that our approximation of an XY model is valid for at least $\tilde{K} < 2$. As expected we observe that at $\tilde{K}_c$, $x_{h,XY} \approx \frac{1}{8}$ and above

FIG. 8. The scaled gap $x_{h,XY}(L)$ for the ferromagnetic XY model as function of $\tilde{K}$. Data are shown for $L = 2$ up to $L = 8$. At $x_{h,XY}(L) = 0.25$ the leftmost curve is for $L = 2$ and the rightmost one for $L = 8$.

$\tilde{K}_c$, $x_{h,XY} < \frac{1}{8}$. Another characteristic property for the KT transition is that the curves $x_{h,XY}(L)$ are parallel for $\tilde{K} \geq \tilde{K}_c$, because of the marginal thermal exponent.

Let us compare the results for the XY model with the results of the same calculations for the FFX Y model, which are shown in Fig. 9. Data are used for even sizes $2 \leq L \leq 12$. First we remark that the convergence for increasing L is worse than for the other results for the FFX Y model. This poor convergence is due to large corrections to scaling. We observe that for K large enough the curves $x_{h,XY}(L)$ become parallel again, indicating the algebraic phase of the XY model. They converge to a dimension $x_{h,XY} < \frac{1}{8}$, as they should. However, at K_c , $x_{h,XY} \approx 0.17$; thus from Eq. (22) it follows that $g \approx 3^{(2)}$, again significantly below the value $g = 4$ at a KT transition. Moreover, the curves are not parallel at K_c . They intersect, which indicates a relevant thermal dimension.

To be able to combine the results for x_ϕ and $x_{h,XY}$ to results for g we have also determined x_ϕ as function of K for $\phi = \frac{\pi}{10}$. The results are shown in Fig. 10. To compare the results for the FFX Y model with the XY model we have done the same calculations for the XY model. The results are shown in Fig. 11. Comparing, for several values for $K > K_c$, the renormalized coupling calculated from x_ϕ and $x_{h,XY}$, using Eqs. (18) and (22), we find agreement and values $g > 4$. Therefore we may conclude that for $K > K_c$ the FFX Y model behaves as an XY model in its low temperature region. However, at K_c the values for g from x_ϕ and $x_{h,XY}$ [$g = 3.6(2)^{(1)}$ and $g \approx 3^{(2)}$] do not agree with the predictions at the KT transition or in the low temperature phase ($g \geq 4$).

These results indicate that at K_c the FFX Y model does not renormalize to the Gaussian fixed line, which is supposed to govern the strong coupling regime (the renormalization flow will be discussed in Sec. VI). Therefore we propose that for $K \downarrow K_c$ there is a jump in x_ϕ and $x_{h,XY}$ to a nonzero value. For a KT transition the jump is to 0. The values for x_ϕ and $x_{h,XY}$ are related to each other in the same way as for $K > K_c$ (see Appendix A). For $K < K_c$ the dimensions x_ϕ and $x_{h,XY}$ jump to zero. Numerically one cannot observe these two jumps for finite systems; however, the presence of these jumps is in agreement with the apparent divergence of the slopes of the $x_{h,XY}(L)$ curves in Fig. 9.

FIG. 9. The scaled gap $x_{h,XY}(L)$ for the FFX Y model as function of K . The curves apply to even sizes $L = 2$ up to $L = 12$.

FIG. 10. The scaled gap $x_\phi(L)$ for the FFXY model as function of K . These data apply to even sizes $L = 2$ up to $L = 8$.

We have determined the thermal dimension x_t more precisely, by considering an expansion around K_c of the finite size scale function of the free energy and of some scaled gaps.²⁷ The inverse finite size is treated as a relevant scaling field. These expansions are given, respectively, by

$$f(L, t) = f_L + a_1 L^{y_t - d} t + a_2 L^{2y_t - d} t^2 + \dots, \quad (23)$$

$$x(L, t) = x + b_1 L^{y_t} t + b_2 L^{2y_t} t^2 + \dots, \quad (24)$$

where $t = (K - K_c)/K$, $y_t = 2 - x_t$, and $d = 2$ is the lattice dimensionality. We have calculated numerically the first and second derivative with respect to K of the free energy, the Ising scaled gap [Eq. (15)], and the XY magnetization scaled gap [Eq. (22)], at K_c . After iterated fits of these results, using Eqs. (23) and (24), we have determined the thermal dimension. The results indicate that $x_t = 0.77(3)$. This result is also obtained by determining the scaled gap between the two leading eigenvalues in sector 0 of the transfer matrix, which belong to eigenvectors which are symmetric under arrow inversion. Our result is in agreement with the result of Granato and Nightingale.¹⁵ This thermal dimension can neither be identified as an Ising dimension nor as an XY dimension.

We have also determined two other dimensions in the FFXY model: One is Ising like and one is XY like. The Ising-like dimension $x_{h,Is}$ belongs to an order-order operator. The Ising-like order parameter is the chirality. To

FIG. 11. The scaled gap $x_\phi(L)$ for the ferromagnetic XY model as function of $\tilde{K}$. These data apply to even sizes $L = 2$ up to $L = 8$.

calculate the dimension $x_{h,Is}$ we have used the transfer matrix T_h which adds one column with an even number of sites. The chirality can directly be linked to the operator $\sin(\theta_i - \theta_j)$ where site i and site j are connected by a vertical bond. Writing the sine as the sum of two complex phase factors, we note that in analogy with Eqs. (5) and (6) the operator can be represented by an incoming arrow at site i and an outgoing arrow at site j , or vice versa. We are interested in the chirality-chirality correlation function. In order to determine its value over a distance r , we add r columns by means of a factor T_h^r , and then apply another operator $\sin(\theta_{i'} - \theta_{j'})$. Since these operators conserve the total number of arrows, we expect to find the pertinent eigenvalue in the $S = 0$ sector. Furthermore, because the chirality alternates, the eigenvalue has to be negative. The corresponding eigenvector is antisymmetric with respect to arrow inversion. Replacing all outgoing arrows by incoming arrows and vice versa changes the sign of the chirality.

This eigenvalue is obtained by imposing an antisymmetric form of the vector which is multiplied with the transfer matrix. It appears that the leading eigenvalue is indeed negative. The dimension $x_{h,Is}$ is calculated using Eq. (10). The result is $x_{h,Is} = 0.14(1)$. This is rather close to the same dimension in the Ising model, which is equal to $\frac{1}{8}$. However, the latter value is just outside the estimated margin.

Finally we determined a dimension associated with arrow inversion. We use the transfer matrix T_v for L even and L odd. We apply a seam along the cylinder which effectively creates a mirror image of the XY spins. The weight of the bonds which cross this seam is changed into $1 + \cos(\theta_i + \theta_j - A_{ij})$. In the 19-vertex version this means that if two nearest neighbor sites i and j are at different sides of the seam, an outgoing arrow for site i is also an outgoing arrow for site j . The same holds for incoming arrows. Thus across the seam the arrows are inverted. For L even and odd we expect different behavior.

First we consider the case L even. Because across the seam the arrows are inverted, the chirality is also inverted. Thus in effect an Ising wall is created. Because the phase coupling over an Ising wall is irrelevant, there is no other effect of the phase inversion. Therefore we expect that the leading finite size correction of the free energy is given by $\frac{\pi \tilde{c}}{6L^2}$ where $\tilde{c} = c - \frac{3}{4} - 12x_{h,Is}^*$. Thus the free boundary conditions due to the Ising wall are also taken into account. The reason that we still distinguish two parts in the conformal anomaly will be discussed in Sec. VI. We found $\tilde{c} = -0.70(5)$; hence $x_{h,Is}^* = 0.125(8)$. This result is consistent with the result for $x_{h,Is}^*$ described above.

For L odd the result of the arrow inversion seam is that the Ising wall is canceled. Therefore we expect that the leading finite size correction of the free energy is given by $\frac{\pi \tilde{c}}{6L^2}$ where $\tilde{c} = c - 12x_{ai,XY}$. The dimension $x_{ai,XY}$ belongs to an arrow inversion seam in the XY model. This dimension can be calculated exactly^{28,29} for the Gaussian model to which the XY model renormalizes. Its value $x_{ai,XY} = \frac{1}{8}$ is independent of the Gaussian coupling g . The original derivation²⁹ is presented in a field theoretic context and uses an analytic continuation procedure. In

Appendix B we present a simple direct calculation. We have also determined it numerically using the transfer matrix of the XY model. We found $x_{\text{ai},XY} \approx 0.12$. Because of the logarithmic corrections at a KT transition, this is a satisfactory result. Thus in the FFXY model we expect that $\tilde{c} = 0$. The result $\tilde{c} = 0.1(2)$ is not very accurate, but consistent with $\tilde{c} = 0$.

VI. DISCUSSION

The results of our transfer-matrix calculations lead to a better understanding of the critical behavior in the FFXY model. We interpret the FFXY model in terms of three degrees of freedom. We distinguish^{30,5} spin waves, vortices, and Ising spins. The spin wave degrees of freedom decouple from the vortices and the Ising spins. This is manifest in a Villain form of the FFXY model,⁵ in which the spin wave degrees of freedom can be integrated out exactly. In a 19-vertex version of the FFXY model it appears numerically that the coupling with the spin waves is irrelevant. As in the XY model, where the conformal anomaly is given by $c = c_{\text{sw}} + c_v = 1 + 0$, the conformal anomaly for the spin waves c_{sw} in the FFXY model does not depend on the conformal anomaly for the vortices and the Ising spins. This is confirmed by our results when applying an arrow inversion seam. These results can only be understood if the coupling with the spin waves is irrelevant. Moreover, the result for $x_{\text{h},\text{Is}}$ is in agreement with the Ising magnetic dimension if we correct the spin wave part of the conformal anomaly for the free boundary conditions as if it is independent.

However, the vortices and Ising spins have a relevant coupling at K_c . This is because the vortices induce a long range interaction between the Ising spins. This appears from our results for g and x_t . These results are not consistent with a preempted KT transition or with an Ising transition. The coupling mechanism is still not understood.

We have found evidence that a coupled XY-Ising model [Eq. (3)] is in the same universality class as the FFXY model. Therefore, in the scaling limit, the 19-vertex version of the FFXY model describes a line in the phase diagram of the coupled XY-Ising model. From our results we propose the following renormalization flow; see Fig. 12. Along the axis $K = 0$ there is an attractive Ising fixed point. A line of Ising transitions renormalizes to this fixed point. The low temperature region of the FFXY model behaves as an XY model and therefore renormalizes to the Gaussian fixed line. The line of KT transitions renormalizes to the point at the Gaussian line where $g = 4$. The area between the Ising and KT transition line renormalizes to $g = 0$. Therefore the branch point B is repulsive in two directions, and thus also along the critical line where the two transitions coincide. It is possible that this line renormalizes to another fixed line; however, it seems more likely that there is an extra fixed point F at this line. Along this line the fixed point F is attractive. Thus the dimensions determined with our transfer-matrix calculations at K_c apply to this fixed point. Because we find

FIG. 12. A sketch of the renormalization flow in the coupled XY-Ising model of Eq. (3), conjectured on the basis of our results. The thick line above is the Gaussian fixed line. The dashed curves represent flow lines. The dotted curves 1 and 2 are temperaturelike trajectories through the parameter space.

a relevant thermal exponent, the fixed point is repulsive in the other direction. This determines the flow lines as sketched in Fig. 12.

The 19-vertex version of the FFXY model describes a line in this phase diagram which cuts the line BE at a certain point; this could be to the left or right from the fixed point F. These two possibilities are sketched as respectively line 1 and line 2 in the phase diagram. If the coupling along line 1 increases above the critical coupling, it follows that g decreases. This can be seen by following the renormalization flow lines which cut line 1. If the coupling along line 2 increases, g increases. From Fig. 9 it follows that g increases as K increases above K_c . Therefore we conclude that the 19-vertex version of the FFXY model is represented by line 2 in the phase diagram. Because the renormalized couplings for $K > K_c$ are rather close to $g = 4$, the new fixed point F lies close to the branch point B.

By extending the coupled XY-Ising model of Eq. (3) to a coupled XY-Ising model with the Hamiltonian

$$-\beta\mathcal{H} = \sum_{\langle ij \rangle} \left\{ \frac{K}{2} (1 + s_i s_j) \cos(\theta_i - \theta_j) + J s_i s_j + A s_i s_j \cos(\theta_i - \theta_j) \right\}, \quad (25)$$

there is a coupling between XY spins over an Ising wall for $A \neq 0$. The XY and Ising transitions do not merge anymore, but they cross. Thus an intermediate phase is possible, in which the Ising spins are disordered and

the XY spins are ordered.¹³ At the decoupling point $A = -K/2$. Then the model is a superposition of an Ising model and an XY model; thus has $c = \frac{3}{2}$. The dimension belonging to the coupling term $s_i s_j \cos(\theta_i - \theta_j)$ is $x_{t,XY} + x_{t,Is} = 2 + 1 = 3$. Thus the coupling term is irrelevant at the decoupling critical point. According to Zamolodchikov's theorem, the conformal anomaly is not increasing along a renormalization flow line. Thus the conformal anomaly of the branch point c_B is larger than or equal to $\frac{3}{2}$, and the conformal anomaly of the fixed point F is smaller than or equal to c_B . Thus on the grounds of Zamolodchikov there is no conclusion about the value of c_F .

VII. CONCLUSIONS

We have shown that an alternating 19-vertex model is, for K small enough ($K < 1.4$), a good representation for the FFX Y model. We have studied the Yang-Baxter equations for this model, which keep the frustration invariant. Unfortunately these equations have only trivial solutions. Thus no exact result is obtained.

The 19-vertex model is suitable for transfer-matrix calculations. This made it possible to obtain numerical results which are the first without the statistical errors due to the Monte Carlo method. We have shown that the coupled XY -Ising model of Eq. (3) is in the same universality class as the FFX Y model. Because there is no phase coupling over an Ising wall at criticality as $L \rightarrow \infty$, we have to take free boundary conditions into account instead of periodic ones when determining the Ising-like magnetic dimension. Thus we found $x_{h,Is}^* = 0.123(5)$. By means of a finite size scaling analysis of the transfer-matrix results we have also obtained $K_c = 1.191(1)$, $c = 1.55(5)$, $x_{h,Is} = 0.14(1)$, $x_{h,XY} \approx 0.17$, $g = 3.6(2)^{(1)}$, $x_t = 0.77(3)$, and $x_{ai,XY} \approx 0.12$.

Due to large corrections to scaling we did not succeed in finding more accurate results than the ones obtained by Monte Carlo simulations. Calculations for the ferromagnetic XY model show that the scaling behavior is bad for large K ($K > 1.3$). However, the frustration in the FFX Y model is making it even worse.

Our results for $x_{h,Is}$, $x_{h,Is}^*$, and x_t are in agreement with the result found by others (see Table I). Our result for c is somewhat smaller than the results quoted in Table I. Probably this is so because our finite size scaling analysis is more accurate. Our value for g clearly deviates from the other results in Table I; it is the first values with $g < 4$. Maybe this is because of the more accurate determination of K_c .

On the one hand our values for the conformal anomaly and the magnetic Ising-like dimensions are rather close to $\frac{3}{2}$ and $\frac{1}{8}$, respectively. Therefore they do not exclude the theory that the FFX Y model behaves as a superposition of an Ising model and an XY model. On the other hand the anomalous dimensions x_t and $x_{h,XY}$ and the renormalized coupling g are not in agreement; they should have been 1, $1/8$, and 4, respectively. Therefore we may conclude that the FFX Y model belongs to a "new" universality class.

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APPENDIX A

The scaling dimensions $x_{h,XY}$ and x_ϕ are related by a Fourier transformation. This can be seen in the following way. The partition function $\exp[-F(\phi)]$ of the 19-vertex version of the FFX Y model on a cylinder with length N_y and circumference N_x and with a seam with a phase shift ϕ satisfies

$$\exp[-F(\phi)] = \sum_h \exp[-\tilde{F}(h)] \exp[ih\phi], \quad (A1)$$

where $\tilde{F}(h)$ is the free energy of the cylinder without a seam but with a fixed height difference h along its length. This relation is due to the fact that the phase ϕ couples to the sum of 19-vertex arrows between two neighboring columns, which is precisely the height difference h . The Fourier inverse of Eq. (A1) is

$$\exp[-\tilde{F}(h)] = \frac{1}{2\pi} \int_0^{2\pi} d\phi \exp[-ih\phi] \exp[-F(\phi)]. \quad (A2)$$

For ϕ small $F(\phi)$ is given by $\frac{1}{2}g(\frac{\phi}{2\pi})^2 N_y/N_x$. For $N_y/N_x \gg 1$ the integral can be approximated as a Gaussian integral:

$$\exp[\tilde{F}(0) - \tilde{F}(h)] = \exp\left[-\frac{\pi}{g} \frac{N_x}{N_y} h^2\right]. \quad (A3)$$

Thus for a height difference $h = 1$ along the cylinder

$$\exp[\tilde{F}(0_{||}) - \tilde{F}(1_{||})] = \exp\left[-\frac{\pi}{g} \frac{N_x}{N_y}\right]. \quad (A4)$$

Using modular invariance, the partition function for a height difference of $h = 1$ across the cylinder is given by

$$\exp[\tilde{F}(0_{\perp}) - \tilde{F}(1_{\perp})] = \exp\left[-\frac{\pi}{g} \frac{N_y}{N_x}\right]. \quad (A5)$$

From this equation and Eq. (21) follows $x_{h,XY} = 1/2g$. Thus x_ϕ and $x_{h,XY}$ are related to each other, also when the model does not renormalize to the Gaussian model.

APPENDIX B

Scaling dimensions can be calculated exactly by determining the finite size dependence of the free energy of a strip with various boundary conditions. We have determined the free energy of the Gaussian model on a strip with periodic, open, and twisted boundary conditions. The twisted boundary imposes a change of sign across the seam; i.e., the interaction between two height vari-

ables h_i and h_j is proportional to $(h_i + h_j)^2$ instead of $(h_i - h_j)^2$. This situation corresponds with a seam with arrow inversion in the 19-vertex model as explained in Sec. V. Thus we have calculated exactly c , c_o , and x_{twi} for the Gaussian model. Here the quantities c and c_o represent only the spin wave part of the similarly named quantities for the FFX model (Sec. V). The scaling dimension x_{twi} corresponds with the scaling dimension $x_{\text{ai},XY}$ in the FFX model.

We start by writing down the free energy F of a strip with length N_y and width N_x for the Gaussian model, $N_y \gg N_x$,

$$F = f_0 N_x N_y + N_y \sum_{q_x} \ln(|S_x| + \sqrt{1 + S_x^2}), \quad (\text{B1})$$

where $S_x = \sin(q_x/2)$. This formula is the result of the integration over the periodic Fourier modes q_y . f_0 belongs to the bulk part of the free energy. The Fourier modes q_x are determined by the boundary conditions. We distinguish three cases

$$q_x = \frac{2n\pi}{N_x}, \quad q_x \in (-\pi, \pi) \text{ periodic}, \quad (\text{B2a})$$

$$q_x = \frac{n\pi}{N_x}, \quad q_x \in (0, \pi) \text{ open}, \quad (\text{B2b})$$

$$q_x = \frac{(2n+1)\pi}{N_x}, \quad q_x \in (-\pi, \pi) \text{ twisted}. \quad (\text{B2c})$$

Using the Poisson summation formula the second term of Eq. (B1) can be written as

$$\frac{N_x N_y}{2\pi} \sum_m \int_{-\pi}^{\pi} \ln(|S_x| + \sqrt{1 + S_x^2}) \exp[iq_x m N_x] dq_x, \quad (\text{B3a})$$

$$\dots \exp[i2q_x m N_x] dq_x, \quad (\text{B3b})$$

$$\dots \exp[iq_x m (N_x + 1)] dq_x. \quad (\text{B3c})$$

The term for $m = 0$ belongs to the bulk part of the free energy. The term of order N_y/N_x is calculated by partially integrating Eqs. (B3a)–(B3c) two times. Higher order terms in $1/N_x$ remain. The partial integration gives a term $\delta(q_x)$ because of the $|S_x|$. Thus the result for order N_y/N_x is

$$\frac{1}{2\pi} \frac{N_y}{N_x} \sum_{m \neq 0} \frac{1}{m^2} = \frac{\pi}{6} \frac{N_y}{N_x}, \quad (\text{B4a})$$

$$\frac{1}{2\pi} \frac{N_y}{N_x} \sum_{m \neq 0} \frac{1}{4m^2} = \frac{\pi}{24} \frac{N_y}{N_x}, \quad (\text{B4b})$$

$$\frac{1}{2\pi} \frac{N_y}{N_x} \left[\sum_{m \text{ even}} \frac{1}{m^2} - \sum_{m \text{ odd}} \frac{1}{m^2} \right] = -\frac{\pi}{12} \frac{N_y}{N_x}. \quad (\text{B4c})$$

It follows from the theory of conformal invariance²¹ that in these cases the finite size amplitude of the order N_y/N_x is given by $\pi c/6$, $\pi c_o/6$, and $\pi \tilde{c}/6$, respectively, where $c_o = c/4$ and $\tilde{c} = (c - 12x_{\text{twi}})$. Thus we find exactly $c = 1$, $c_o = \frac{1}{4}$, and $x_{\text{twi}} = \frac{1}{8}$.

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