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Trigonometric R Matrices Related to ‘Dilute’ Birman–Wenzl–Murakami Algebra

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Abstract. Explicit expressions for three series of R matrices which are related to a ‘dilute’ generalisation of the Birman–Wenzl–Murakami algebra are presented. Of those, one series is equivalent to the quantum R matrices of the $D_{n+1}^{(2)}$ generalised Toda systems, whereas the remaining two series appear to be new.

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A ‘dilute’ generalisation of the Birman–Wenzl–Murakami (BWM) algebra [1, 2] has recently been introduced [3, 4]. It appears [3] as a generalised braid-monoid algebra [5] related to certain exactly solvable lattice models of two-dimensional statistical mechanics. Alternatively, it can be regarded as a particularly simple case of a two-colour braid-monoid algebra [4] where one colour is trivially represented (in the sense that the corresponding representation of the subalgebra generated by the elements of this colour is one-dimensional). In [3], it was shown that representations of this algebra can be ‘Baxterised’ [6], i.e., one can find a general expression for a local Yang–Baxter operator $X_j(u)$ [5] (u denoting the spectral parameter) in terms of the generators of the dilute BWM algebra for which the Yang–Baxter relations

$$\begin{aligned} X_j(u)X_{j+1}(u+v)X_j(v) &= X_{j+1}(v)X_j(u+v)X_{j+1}(u), \\ X_j(u)X_k(v) &= X_k(v)X_j(u), \quad \text{for } |j-k| > 1, \end{aligned} \tag{1}$$

follow algebraically. This implies that every suitable representation of the dilute BWM algebra defines a solvable lattice model. As an example, one series of R matrices of this kind has been given in [3] which were shown to be equivalent to the $D_{n+1}^{(2)}$ vertex models [7, 8].

In this Letter, we present the explicit form of three such series of R matrices (where the series mentioned above is included for completeness). In this case, the Yang–Baxter operator $X_j(u)$ acts on a tensor space $V \otimes V \otimes \cdots \otimes V$ (where $V \cong \mathbb{C}^{d+1}$ with some integer d) as

$$X_j(u) = I \otimes I \otimes \cdots \otimes I \otimes \check{R}(u) \otimes I \otimes \cdots \otimes I \otimes I, \tag{2}$$

where $\check{R}(u) = PR(u)$ (P is the permutation map on $V \otimes V$, i.e., $P: v \otimes w \mapsto w \otimes v$) acts on the j and $j + 1$ factors in the tensor product and $R(u)$ is the corresponding R matrix.

The representations of the dilute BWM algebra considered in what follows can be regarded as the ‘dilutisation’ of well-known representations of the BWM algebra itself. These are the representations which describe the $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ vertex models [7–10] (and the $A_n^{(2)}$ models as well, see, e.g., [3]), we will refer to the corresponding series of (dilute) models as the (B), (C), and (D) series for short. The corresponding representations of the dilute algebra are obtained by adding a single state (which carries the second (‘trivial’) colour) to the local space $\tilde{V} \cong \mathbb{C}^d$ (with $d = 2n + 1$ for the (B) and $d = 2n$ for the (C) and (D) series, respectively) yielding $V = \tilde{V} \oplus \mathbb{C} \cong \mathbb{C}^{d+1}$.

For all three series, the $(d + 1)^2 \times (d + 1)^2$ matrix $\check{R}(u)$ is given by the following general expression [3]

$$\begin{aligned} \check{R}(u) = & p^{(1,1)} + \\ & + \zeta^{-1} \eta^{-1} (z - z^{-1}) (\tau^{-1} z b^{+(1,1)} - \tau z^{-1} b^{-(1,1)}) + \\ & + \eta^{-1} (\tau z^{-1} - \tau^{-1} z) (p^{(1,2)} + p^{(2,1)}) - \\ & - \kappa_1 \zeta^{-1} \eta^{-1} (z - z^{-1}) (\tau z^{-1} - \tau^{-1} z) (b^{(1,2)} + b^{(2,1)}) + \\ & + \kappa_2 \eta^{-1} (z - z^{-1}) (e^{(1,2)} + e^{(2,1)}) + \\ & + (1 - \zeta^{-1} \eta^{-1} (z - z^{-1}) (\tau z^{-1} - \tau^{-1} z)) p^{(2,2)}, \end{aligned} \quad (3)$$

where

$$z = \exp(iu), \quad \zeta = (\sigma - \sigma^{-1}), \quad \eta = (\tau - \tau^{-1})$$

and where one can choose arbitrary signs $\kappa_1^2 = \kappa_2^2 = 1$ (cf. [3]). Here, the relationship between σ and τ is given by $\tau^2 = \sigma^{2n}$, $-\sigma^{2n+1}$, σ^{2n-1} for the (B), (C), and (D) series, respectively.

Following closely the notation of [8] (with $\sigma = k$ and $\tau^2 = \xi$), we find the following explicit expressions for the matrices

$$p^{(a,b)} = P^{(a)} \otimes P^{(b)}, \quad b^{\pm(a,b)} \quad \text{and} \quad e^{(a,b)} \quad (a, b \in \{1, 2\}):$$

$$P^{(1)} = \sum_{\alpha} E_{\alpha, \alpha}, \quad (4a)$$

$$P^{(2)} = E_{d+1, d+1}, \quad (4b)$$

$$\begin{aligned} b^{+(1,1)} = & \sum_{\alpha} \sigma^{-1} (1 + (\sigma - 1) \delta_{\alpha, \alpha'}) E_{\alpha, \alpha} \otimes E_{\alpha, \alpha} + \\ & + \sum_{\alpha \neq \beta} (1 + (\sigma - 1) \delta_{\alpha, \beta'}) E_{\alpha, \beta} \otimes E_{\beta, \alpha} - \\ & - (\sigma - \sigma^{-1}) \sum_{\alpha < \beta} E_{\alpha, \alpha} \otimes E_{\beta, \beta} + \\ & + (\sigma - \sigma^{-1}) \sum_{\alpha > \beta} \varepsilon_{\alpha} \varepsilon_{\beta} \sigma^{\bar{\alpha} - \bar{\beta}} E_{\alpha', \beta} \otimes E_{\alpha, \beta'}, \end{aligned} \quad (5a)$$

$$\begin{aligned}
b^{-(1,1)} = & \sum_{\alpha} \sigma(1 + (\sigma^{-1} - 1)\delta_{\alpha,\alpha'}) E_{\alpha,\alpha} \otimes E_{\alpha,\alpha} + \\
& + \sum_{\alpha \neq \beta} (1 + (\sigma^{-1} - 1)\delta_{\alpha,\beta'}) E_{\alpha,\beta} \otimes E_{\beta,\alpha} + \\
& - (\sigma - \sigma^{-1}) \sum_{\alpha > \beta} E_{\alpha,\alpha} \otimes E_{\beta,\beta} - \\
& - (\sigma - \sigma^{-1}) \sum_{\alpha < \beta} \varepsilon_{\alpha} \varepsilon_{\beta} \sigma^{\bar{\alpha} - \bar{\beta}} E_{\alpha',\beta} \otimes E_{\alpha,\beta'},
\end{aligned} \tag{5b}$$

$$b^{(1,2)} = \sum_{\alpha} E_{d+1,\alpha} \otimes E_{\alpha,d+1}, \tag{5c}$$

$$b^{(2,1)} = \sum_{\alpha} E_{\alpha,d+1} \otimes E_{d+1,\alpha}, \tag{5d}$$

$$e^{(1,2)} = - \sum_{\alpha} \varepsilon_{\alpha} \sigma^{(d+1)/2 - \bar{\alpha}} E_{d+1,\alpha} \otimes E_{d+1,\alpha'}, \tag{6a}$$

$$e^{(2,1)} = - \sum_{\alpha} \varepsilon_{\alpha} \sigma^{\bar{\alpha} - (d+1)/2} E_{\alpha',d+1} \otimes E_{\alpha,d+1}, \tag{6b}$$

where in all expressions the summation variables are restricted to values $1 \leq \alpha, \beta \leq d$. Here, $E_{k,l}$ denote $(d+1) \times (d+1)$ matrices with elements $(E_{k,l})_{i,j} = \delta_{i,k} \delta_{j,l}$. Furthermore, ‘charge conjugated’ states are defined by

$$\alpha' = d+1 - \alpha \quad (1 \leq \alpha \leq d) \quad \text{and} \quad (d+1)' = (d+1).$$

We use $\varepsilon_{\alpha} = 1$ for the (B) and (D) series whereas

$$\varepsilon_{\alpha} = \begin{cases} 1, & \alpha < \alpha' \\ -1, & \alpha > \alpha' \end{cases} \tag{7}$$

for the (C) series. Finally,

$$\bar{\alpha} = \begin{cases} \alpha \pm \frac{1}{2}, & \alpha < \alpha', \\ \alpha, & \alpha = \alpha', \\ \alpha \mp \frac{1}{2}, & \alpha > \alpha', \end{cases} \tag{8}$$

where the upper sign applies for series (B) and (D) and the lower for the (C) series, respectively.

It is straightforward to show that the above expressions (4a)–(6b) indeed define a matrix representation of the dilute BWM algebra in the sense of [3]. It follows from the results of [3] that the $\check{R}$ matrix (3) fulfills the quantum Yang–Baxter equation (1)

$$(\check{R}(u) \otimes I)(I \otimes \check{R}(u+v))(\check{R}(v) \otimes I) = (I \otimes \check{R}(v))(\check{R}(u+v) \otimes I)(I \otimes \check{R}(u)). \tag{9}$$

In addition, it satisfies the following relations

$$\check{R}(0) = I \quad (\text{initial condition}), \quad (10a)$$

$$\check{R}(u)\check{R}(-u) = \varrho(u)\varrho(-u)I \quad (\text{inversion relation}), \quad (10b)$$

$$\check{R}(u) = {}^t\check{R}(u) \quad (\text{reflection symmetry}), \quad (10c)$$

$$\check{R}_{lq}^{kp}(u) = 0 \quad \text{unless } \tilde{k} + \tilde{l} = \tilde{p} + \tilde{q} \quad (\text{charge conservation}), \quad (10d)$$

$$R(u) = (S \otimes S) {}^tR(u)(S \otimes S) \quad (\text{CT invariance}), \quad (10e)$$

$$R(u) = (C \otimes I) {}^t_1(PR(\lambda - u)P)(C \otimes I)^{-1} \quad (\text{crossing symmetry}), \quad (10f)$$

where relations (10c) (P invariance) and (10e) (CT invariance) do *not* hold for the (C) series, but the invariance under the combined operation (CPT invariance) persists. Here, the left superscripts t_1 and t denote transposition in the first space and in both spaces, respectively. The function $\varrho(u)$ which enters in the inversion relation (10b) has the form

$$\varrho(u) = \zeta^{-1}\eta^{-1}(\sigma z^{-1} - \sigma^{-1}z)(\tau z^{-1} - \tau^{-1}z). \quad (11)$$

The matrix elements of $\check{R}(u)$ are defined by

$$\check{R}(u) = \sum_{k,l,p,q=1}^{d+1} \check{R}_{lq}^{kp}(u) E_{q,k} \otimes E_{p,l} \quad (12)$$

and $\tilde{k}$ is given by

$$\tilde{k} = \frac{k - k'}{2}. \quad (13)$$

Note that the state $d+1$ is charge conjugated to itself, i.e. $(d+1)' = (d+1)$, and, hence, $\tilde{d}+1 = 0$. Finally, the crossing parameter λ is determined by $\tau = \exp(i\lambda)$ and the matrices S and C in Equations (10e) and (10f) have the elements

$$C_{i,j} = r(i)S_{i,j} = r(i)\delta_{i,j'} \quad (14)$$

with crossing multipliers

$$r(\alpha) = \varepsilon_\alpha \sigma^{\bar{\alpha} - (d+1)/2}, \quad r(d+1) = -\kappa_2. \quad (15)$$

The crossing symmetry (10f) of the $\check{R}$ matrix (3) can be easily verified by looking at the properties of the individual parts (4a)–(6b) under the ‘crossing transformation’

$$\mathcal{O} \mapsto \mathcal{O}^{\text{cr}} = P(C \otimes I) {}^t_1(\mathcal{O}P)(C \otimes I)^{-1}, \quad (16)$$

where $\mathcal{O}$ denotes any of the matrices of Equations (4a)–(6b). For $\kappa_2 = 1$ (see [3]), one observes

$$p^{(1,1)} \mapsto e^{(1,1)} \mapsto p^{(1,1)}, \quad (17a)$$

$$b^{+(1,1)} \mapsto b^{-(1,1)} \mapsto b^{+(1,1)}, \quad (17b)$$

$$p^{(1,2)} \mapsto e^{(1,2)} \mapsto p^{(2,1)} \mapsto e^{(2,1)} \mapsto p^{(1,2)}, \quad (17c)$$

$$b^{(1,2)} \mapsto b^{(2,1)} \mapsto b^{(1,2)}, \quad (17d)$$

$$p^{(2,2)} \mapsto p^{(2,2)}, \quad (17e)$$

where

$$e^{(1,1)} = I \otimes I + \xi^{-1}(b^{+(1,1)} - b^{-(1,1)}).$$

This is exactly what one expects from the diagrammatic interpretation of the corresponding generators of the ‘dilute’ BWM algebra (in which the crossing transformation (16) corresponds to a clockwise rotation of the two-string diagrams by 90 degrees), compare the discussion in [4].

The R matrices defined in this Letter actually possess a charge conservation that is somewhat stricter than shown in Equation (10d). In fact, the charges of both ‘colours’ are conserved separately which implies that only vertices with an even number of states $d + 1$ occur (i.e., the number of states $d + 1$ is conserved modulo two as this state is conjugated to itself).

As shown explicitly in [3], the (B) series is equivalent to the $D_{n+1}^{(2)}$ vertex models, differing from the R matrix of [8] by a local orthogonal transformation only. Essentially, the basis of [8] uses the symmetric and anti-symmetric linear combinations (states $n + 1$ and $n + 2$ in [8], respectively) of the two ‘neutral’ states

$$(d + 1)/2 = n + 1 \quad \text{and} \quad d + 1 = 2n + 2$$

in the present parametrisation. For the other two series, it remains an open question if similar relations to known R matrices exist.

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