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ON ADOPTING KRIPKE SEMANTICS IN SET THEORY

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Abstract. Several philosophers have argued that the logic of set theory should be intuitionistic on the grounds that the open-endedness of the set concept demands the adoption of a nonclassical semantics. This paper examines to what extent adopting such a semantics has revisionary consequences for the logic of our set-theoretic reasoning. It is shown that in the context of the axioms of standard set theory, an intuitionistic semantics sanctions a classical logic. A Kripke semantics in the context of a weaker axiomatization is then considered. It is argued that this semantics vindicates an intuitionistic logic only insofar as certain constraints are put on its interpretation. Wider morals are drawn about the restrictions that this places on the shape of arguments for an intuitionistic revision of the logic of set theory.

1. Introduction: Philosophical preliminaries. One of the central features of the concept of set is its *open-endedness*: whenever we succeed in defining a totality of sets, we can define a larger totality by an application of the *set of* operation. That the set concept is open-ended is the main lesson that should be drawn from the set-theoretic paradoxes. Russell's paradox, for instance, tells us that the supposition that we have succeeded in defining the set of all sets that are not members of themselves leads to a contradiction. The contradiction is removed by assuming that there can be no such set, and the reason why there can be no such set is that the concept of set is open-ended.

The ensuing philosophical discussion has focused on the possibility of articulating a conception of set that captures and indeed motivates the set concept's open-endedness. The iterative conception of set constitutes an attempt to articulate such a conception. On the iterative conception, sets are formed in stages. In the beginning, we have individuals, or *Urelemente*, as they are sometimes called. At stage zero, the set containing all previous elements is formed. If there are no individuals, this set is the empty set. At any finite stage, any sets formed at earlier stages are collected into a set. After the finite stages come the infinite ones corresponding to the ordinals ω, $\omega + 1$, $\omega + 2$, $\omega + \omega$, etc. At limit stages, we form sets corresponding to arbitrary collections of items formed at earlier stages. The resulting picture of the set-theoretic universe as a cumulative hierarchy divided into levels[1] provides a heuristic motivation for (at least some of) the axioms of standard set theory –

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[1] The picture is captured by the following definition by transfinite recursion. Let α be an ordinal. Then, the levels of the hierarchy, V, are organized as follows:

$$V_0 = \{x : x \text{ is an individual}\};$$

$$V_{\alpha+1} = V_\alpha \cup \mathcal{P}(V_\alpha);$$

$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha \text{ if } \lambda \text{ is a limit ordinal.}$$

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by which I mean, as usual, ZFC, i.e. first-order Zermelo-Fraenkel set theory (ZF) with the axiom of choice (AC).[2]

Thus, philosophical debate has traditionally centered around the consequences that the open-endedness of the set concept has for the choice of the *axioms* of set theory. More recently, however, it has been suggested that this open-endedness also has consequences for the *logic* of set theory.[3] In particular, several philosophers have drawn on the set concept's open-endedness in formulating arguments for an *intuitionistic* revision of the logic of our set-theoretic reasoning.

The most notable example are perhaps Michael Dummett's arguments from indefinite extensibility (Dummett, 1991, 1993). According to Dummett, the concepts defining all the fundamental mathematical theories are indefinitely extensible, which in the case of the set concept is open-endedness. He argues that the indefinite extensibility of these concepts makes it impossible for us to have a definite conception of what elements belong to the domain of such theories. But mathematical truth does not outrun our conception of the objects that fall under a mathematical concept. So, he concludes, quantified statements ranging over *all* the objects falling under an indefinitely extensible concept must be understood as embodying a claim to cite an instance or an effective operation. And the logic governing such statements is not classical but intuitionistic.[4]

A revisionary argument along similar lines has been offered by Lear (1977). He argues that the open-endedness of the set concept gives rise to a tension that can be solved only by taking the extension of 'set' and the range of the set-theoretic quantifiers as depending on our conceptual development. He then claims that a Kripke semantics captures our understanding of set theory once the extension of set is taken as changing over time. This, he contends, vindicates the adoption of intuitionistic logic in set theory.

A related view has been expressed by Tait (1998). According to him, the open-endedness of the set concept, exemplified by the iterative conception, precludes any mathematical characterization of the universe of all sets. It is not possible to speak of the universe of all sets as a well-defined extension. So when we affirm the truth of a proposition about the universe of all sets, we do so on the basis of operations for constructing sets that we have already accepted. For this reason, the logic that applies to quantified statements, when these are interpreted as ranging over all sets, is not classical but intuitionistic.[5]

Although these arguments differ in some or several respects, they all share the idea that the open-endedness of the set concept determines the need for a nonclassical semantics in set theory. My aim in this paper is to examine to what extent adopting a nonclassical

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2 For a statement and discussion of this and other well-known axioms mentioned in this paper, see Jech (2003).

3 Of course, some philosophers have claimed that a proper understanding of the set-theoretic paradoxes already demands a revision of the logic of set theory rather than of its principles. See, e.g., Weir (1998).

4 Velleman (1993) accepts Dummett's arguments from indefinite extensibility but denies that the concept of natural number is indefinitely extensible. His position is therefore one that retains classical logic in the domain of arithmetic but endorses an intuitionistic reading of the quantifiers in the domains of analysis and set theory.

5 Saul Kripke is often referred to as the first person to suggest that the open-endedness of the set concept might be made sense of by construing the set-theoretic quantifiers intuitionistically rather than classically. See Benacerraf & Putnam (1983, p. 29), Lear (1977, p. 94, fn. 15), and Parsons (1974, p. 47, fn. 20), who also attributes the view to Lawrence Poszgay. More recently, Kripke has been credited with the idea that statements about all sets have a schematic character and should be expressed using free variables. See Parsons (2006, p. 218, fn. 34).

semantics has revisionary consequences for the logic of set theory. I begin by showing that in the context of the axioms of standard set theory, an intuitionistic semantics sanctions a classical logic. I then turn to a weaker axiomatization and consider what happens when we adopt a Kripke semantics in the context of such axiomatization. The focus on this semantics is motivated by 2 considerations. First, as already noticed, Lear's argument proceeds from the set-concept's open-endedness to an intuitionistic revision of the logic of set theory via the adoption of a Kripke semantics. Thus, exploring whether such a semantics vindicates an intuitionistic logic will help us to assess one of the above-mentioned revisionary arguments. Second, and most importantly, Kripke semantics seems to be the standard semantics for intuitionistic set theory.[6] So when discussing the need for a nonclassical semantics in set theory, Kripke semantics stands as the natural candidate to consider. The results are mixed. Although a Kripke semantics does sanction an intuitionistic logic, it does so only insofar as we put certain constraints on its interpretation. And these constraints are ones that are incompatible with Lear's argument. I end by drawing wider conclusions about the restrictions that this places on the shape of revisionary arguments in set theory.

2. Nonclassical semantics and standard set theory. When addressing the need for a nonclassical semantics in set theory, it is very natural to begin by considering what happens when we adopt such a semantics in the context of the axioms of current standard set theory. And, as I mentioned earlier, it is commonplace to take this set theory to be ZFC. For example, when mathematicians say that what they prove can be formalized in set theory, what they usually mean is that it can be formalized in ZFC. Moreover, many set theorists seem to think that if something has been proved with the axiom of constructibility[7] ($V{=}L$) or with some large cardinal axiom, it has been proved from an assumption. Thus, it does not come as much of a surprise that Lear, when discussing the virtues of a Kripke semantics in accounting for the strength of our set-theoretic assertions, takes the axiomatic basis of his theory to be that of ZFC. Indeed, sometimes he seems to suggest that our current state of information is such that we *know* the axioms of ZFC.[8] What the open-endedness of the set concept implies, he seems to think, is only that our conception of the cumulative hierarchy could expand in such a way that, for instance, we recognize the existence of an inaccessible cardinal[9] or of sets that are not constructible.

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[6] A search through the relevant literature reveals that a Kripke semantics has been used in the context of various intuitionistic set theories to, e.g., prove completeness theorems (Powell, 1976), compare the strength of closely related theories (Goodman, 1985), and provide independence results (Lubarski, 2005).

[7] Let α be an ordinal. Then, the levels of the constructible hierarchy, L, are organized as follows:

$$L_0 = \{x : x \text{ is an individual}\};$$

$$L_{\alpha+1} = \mathrm{def}(L_\alpha), \text{ i.e., the sets of all sets that are first-order definable over } L_\alpha;$$

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \text{ if } \lambda \text{ is a limit ordinal.}$$

The hypothesis of constructibility is the assertion that all sets are constructible.

[8] See Lear (1977, pp. 97–98).

[9] A cardinal κ is *inaccessible* if and only if (i) $\kappa > \omega$; (ii) if α is a cardinal less than κ, then $2^\alpha < \kappa$; and (iii) if $(\alpha_i)_{i\in\iota}$ is a family of cardinals less than κ, indexed by a cardinal $\iota < \kappa$, then $\bigcup_{i\in\iota} \alpha_i < \kappa$. Clause (ii) guarantees that the cardinality of the powerset of a cardinal smaller than an inaccessible is also smaller than it. Clause (iii) guarantees that the union of fewer than κ cardinals each of cardinality less than κ is still less than κ. This means that an inaccessible

Natural as this suggestion might seem, far from vindicating an intuitionistic logic, it sanctions a classical logic. The reason is that the law of excluded middle (LEM) follows intuitionistically from the axioms of ZFC, thus making the logic of set theory classical even when we adopt a Kripke semantics – indeed, *any* intuitionistic semantics – for it. For let Z_0 be the theory in which the axioms of empty set, extensionality, separation, and unordered pairs hold. Then we have the following:

Theorem 1. AC *intuitionistically implies* LEM *in the presence of the axioms of* Z_0.

We give a proof of the theorem that makes use of a weaker assumption than AC, *viz.* that each doubleton has a choice function.

Proof. Consider the sets $0 = \emptyset$, $1 = \{0\}$, and $2 = \{0, 1\}$, whose existence is guaranteed by the axioms of empty set and unordered pairs, and use the axiom of separation to define the sets $a = \{x \in 2 | x = 0 \vee \varphi\}$ and $b = \{x \in 2 | x = 1 \vee \varphi\}$. Now suppose given a choice function f on the set $\{a, b\}$, which exists by the axiom of unordered pairs. We then have $f(a) \in a$ and $f(b) \in b$, i.e., $(f(a) = 0 \vee \varphi) \wedge (f(b) = 1 \vee \varphi)$. Hence, $f(a) = 0 \wedge (f(b) = 1 \vee \varphi)$, from which it follows that $f(a) \neq f(b) \vee \varphi$. On the other hand, by the axiom of extensionality, $\varphi \rightarrow a = b \rightarrow f(a) = f(b)$, which implies that $f(a) \neq f(b) \rightarrow \neg\varphi$. But $f(a) = f(b) \vee f(a) \neq f(b)$ even intuitionistically. So we obtain LEM. □

Essentially, this is a particular case of a category-theoretic result of Diaconescu, which states that AC implies that the logic of a topos must be classical.[10]

Changing the semantics of standard set theory, therefore, does not suffice to change its logic. Although Lear's argument proceeds by arguing for the adoption of a Kripke semantics in the context of the ZFC axioms, however, it need not do so. Indeed, none of the revisionary arguments under consideration need to assume the truth of the axioms of standard set theory. What these arguments do need is that the set concept exhibits the kind of open-endedness which is captured by the iterative conception. And it is at least controversial whether the iterative conception justifies AC. Indeed, the received view is probably that the iterative conception does *not* justify AC.[11] In what follows, therefore, I will focus on an axiomatization which does not include AC.

Before moving on to assess the revisionary consequences of the adoption of a Kripke semantics in set theory, a further remark concerning the choice of the axioms is in order. For, even dropping AC and adopting an intuitionistic semantics in the context of *some* axiomatizations of ZF still sanction a classical logic. The reason is that, quite surprisingly but very straightforwardly, the usual formulation of the axiom of foundation – which states that every nonempty set u contains an $\in$-*minimal* element, i.e., an element disjoint from u[12]

cardinal cannot be reached 'from below', that is by iterated applications of the powerset and union operations through which the cumulative hierarchy is built.

10 See Diaconescu (1975). Goodman & Myhill (1978) prove that AC implies LEM within CST, a constructive set theory introduced by Myhill (1975). For an extensive discussion of Diaconescu's theorem, as well as of the relation between AC and other choice principles on the one hand and classical logic and constructive logics on the other, see DeVidi (2004).

11 The *locus classicus* is Boolos (1971). See also Boolos (1989) and Paseau (2007) for discussion.

12 Formally: $\forall x(x \neq \emptyset \rightarrow \exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y)))$.

– intuitionistically implies LEM in the presence of the other axioms of ZF. More precisely, we have the following.

Theorem 2 (Myhill). *The axiom of foundation intuitionistically implies* LEM *in the presence of the axioms of* Z_0.

Proof. Consider again the sets 0, 1, and 2, defined as above, and use the axiom of separation to define the set $a = \{x \in 2 | (x = 0 \wedge \varphi) \vee x = 1\}$. Clearly, a is nonempty, since $1 \in a$. Hence, by the axiom of foundation, it has an $\in$-minimal element, i.e., $\exists y((y \in 2 \wedge ((y = 0 \wedge \varphi) \vee y = 1)) \wedge \neg\exists z(z \in a \wedge z \in y))$. If the $\in$-minimal element of a is 0, φ must hold. If, on the other hand, the $\in$-minimal element is 1, $\neg\exists z(z \in a \wedge z \in 1)$, and so $\neg(0 \in a \wedge 0 \in 1)$. Since $0 \in 1$ and $\neg(\varphi \wedge \psi) \wedge \varphi \to \psi$ is intuitionistically valid, we can conclude $0 \notin a$. But, by definition of a, this is equivalent to $\neg\varphi$. Hence, LEM must hold. □

This is the reason why in intuitionistic ZF, usually called ZFI,[13] the version of foundation as a principle of transfinite induction on $\in$[14] is preferred to the standard version of foundation. For, although classically equivalent, the 2 principles are not intuitionistically equivalent, and LEM cannot be derived in ZFI.[15] In this connection, it is worth noticing that Lear himself mentions Powell (1976) as the first person, to his knowledge, to make use of a Kripke semantics in set theory. And indeed Powell does use such a semantics to provide a completeness theorem for ZF. He then goes on to conclude, rightly, that there seems to be no general method for verifying LEM within that framework. But this conclusion only follows because he uses an axiomatization of ZF in which the standard version of foundation is replaced by $\in$-induction. In the remainder of this paper, I will assume, unless otherwise stated, that we are working in the context of the axiomatization of ZF that Powell is considering. For simplicity, I will refer to it as ZFI. This will enable us to properly assess the revisionary consequences of the adoption of a Kripke semantics in set theory. It is to this issue that we now turn.

3. Must a Kripke model for set theory be linear?

First, however, we need to introduce some key definitions and terminology. A *Kripke model* is a structure indexed by a set T of times t, called the *nodes* of the model, partially ordered by the relation $\leq$. For each $t \in T$, $\mathcal{D}(t)$ is the domain of sets at t, with the restriction that if $t \leq t'$, then $\mathcal{D}(t) \subseteq \mathcal{D}(t')$. Moreover, to each $t \in T$ and each $\mathcal{D}(t)$ is associated a set of sentences which hold true at t. When a sentence ϕ holds true at t, t is said to *force* ϕ, symbolized by '$t \Vdash \phi$'. The relation $\Vdash$ is monotonic – i.e., if $t \Vdash \phi$ and $t' \geq t$, then $t' \Vdash \phi$ – and recursively defined as follows:

(1) $t \Vdash \phi(c_1, \ldots, c_k) \Leftrightarrow \phi(c_1, \ldots, c_k)$ holds at t, if ϕ is atomic.
(2) $t \Vdash \phi \wedge \psi \Leftrightarrow t \Vdash \phi$ and $t \Vdash \psi$.

[13] ZFI was introduced by Myhill (1973) as the result of joint work with Harvey Friedman. For a survey of ZFI and other intuitionistic set theories, see Troelstra & van Dalen (1988, pp. 619–635).

[14] That is: $\forall x(\forall y \in x\, \varphi(y) \to \varphi(x)) \to \forall x \varphi(x)$.

[15] More precisely, $\in$-induction is a contrapositive of the schema: $\exists x\varphi(x) \to \exists x(\varphi(x) \wedge \forall y \in x \neg\varphi(y))$. In the presence of the other axioms of ZF, all other instances of this schema follow from the standard version of foundation.

(3) $t \Vdash \phi \vee \psi \iff t \Vdash \phi$ or $t \Vdash \psi$.
(4) $t \Vdash \neg\phi \iff$ for all $t' \geq t, t' \nVdash \phi$.
(5) $t \Vdash \phi \rightarrow \psi \iff$ for all $t' \geq t$, if $t' \Vdash \phi$, then $t' \Vdash \psi$.
(6) $t \Vdash \exists x\phi x \iff$ there is an $s \in \mathcal{D}(t)$ such that $t \Vdash \phi(c_s)$.
(7) $t \Vdash \forall x\phi x \iff$ for all $t' \geq t$ and all $s \in \mathcal{D}(t'), t' \Vdash \phi(c_s)$.

Paseau (2001, 2003) claims that in set theory, adoption of a Kripke semantics does not vindicate an intuitionistic logic, since, he argues, the relevant Kripke model validates the weak law of excluded middle (WLEM), i.e., $\neg\phi \vee \neg\neg\phi$, which is not intuitionistically valid. Paseau's argument is based on the following model-theoretic result:

Proposition 3. *A directed Kripke model*[16] *validates* WLEM.

Proof. If ϕ is forced at some later node t_i, the directedness of the model ensures that for all t subsequent to the present time t_0, there is a later time at which ϕ is forced, i.e., for all $t \geq t_0$, there is a $t' \geq t$ such that $t' \Vdash \phi$. This means that for all $t \geq t_0$, $t \nVdash \neg\phi$, which, in turn, means that $t_0 \Vdash \neg\neg\phi$. If, on the other hand, ϕ is never forced, then for all $t \geq t_0$, $t \nVdash \phi$, which means that $t_0 \Vdash \neg\phi$.[17] □

Paseau aims to show that in set theory, adoption of a Kripke semantics does not vindicate an intuitionistic logic by showing that the relevant Kripke model satisfies a property stronger than directedness, *viz.* linearity,[18] thereby validating WLEM. In this connection, it is worth pointing out that the logics to which directed and linear Kripke models, respectively, give rise – although both intermediate between intuitionistic and classical logic – are different. For while directed Kripke models sanction LQ, the logic one obtains by adding WLEM to any standard system for intuitionistic logic, linear Kripke models sanction LC, the (stronger) logic one obtains by adding the linearity axiom $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$ to it.[19] In fact, we have the following result.

Proposition 4. *A linear Kripke model validates* $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$.

Proof. If ϕ is never forced at any time subsequent to the present time t_0 – i.e., for all $t \geq t_0$, $t \nVdash \phi$ – it is vacuously the case that $t_0 \Vdash \phi \rightarrow \psi$, similarly for the case where ψ is never forced. So suppose there are $t_1, t_2 \geq t_0$ such that $t_1 \Vdash \phi$ and $t_2 \Vdash \psi$, and take each to be

16 A Kripke model is *directed* if and only if for all t, t' there is a t'' such that $t \leq t''$ and $t' \leq t''$.

17 Notice that this proof is nonconstructive, since it assumes that either ϕ is forced at some time t or it is not. On these grounds, Hazen (1982) argues that the result does not undermine the adequacy of directed Kripke models for intuitionistic logic, since an intuitionist might have an argument for disallowing the use of classical reasoning over the nodes of the model. As Paseau (2003, p. 390) points out, however, this requires a different, more general argument to the effect that the correct logic for reasoning about time or modality should be nonclassical, whereas the revisionary arguments we are concerned with do not call into question classical reasoning for domains other than set theory or mathematics in general.

18 A Kripke model is *linear* if and only if for all $t, t', t \leq t'$ or $t' \leq t$.

19 Details of the proof and model theory of LQ are discussed in Akama (1991). LC was first introduced by Gödel (1932) in order to show that intuitionistic logic does not admit a characteristic finite matrix and was later axiomatized by Dummett (1959), which is why it is also known as *Gödel–Dummett logic*. For a survey of the literature on intermediate logics up to 1970, see Hosoi & Ono (1973).

the earliest such. Then, the linearity of the model ensures that $t_1 \leq t_2$ or $t_2 \leq t_1$. If $t_1 \leq t_2$, $t_0 \Vdash \psi \to \phi$, while if $t_2 \leq t_1$, $t_0 \Vdash \phi \to \psi$. □

Paseau (2001, pp. 373–374) considers 2 possible interpretations of the nodes of a Kripke model for set theory: a *temporal* interpretation, according to which the nodes of the model are temporal knowledge states, and a *modal* interpretation, according to which the nodes are possible knowledge states.[20] He argues that on both interpretations the linearity condition is secured and the Kripke model validates a logic stronger than the intuitionistic one. In what follows, I will restrict attention to the modal interpretation, since I think that Paseau's argument for the linearity condition is successful on the temporal interpretation.[21] Moreover, the modal interpretation is the standard interpretation of the nodes of a Kripke model and arguably the one that Lear is endorsing.[22] Hence, focusing on this interpretation will enable us to evaluate Lear's argument while assessing the viability of Kripke semantics as a semantics for intuitionistic set theory.

On the assumption that a possible knowledge state consists in a model of set theory, Paseau's reasoning to obtain the linearity condition is as follows (where V_κ is a segment of the hierarchy of inaccessible height):

> The axiom of choice implies that all cardinals are comparable. Combined with Zermelo's proof that the natural models for set theory are the V_κ, this shows that the models of set theory are linearly ordered by inclusion – they inherit the cardinal ordering. An immediate consequence of this objective linear order is that the possible knowledge states are themselves linearly ordered. (Paseau, 2001, p. 374)

Now the Comparability of Cardinals, although not provable in ZF, is provable in ZFC. Indeed, it is equivalent to AC over the axioms of ZF. Paseau's reasoning does not need AC, however, since the levels of the hierarchy, rather than being indexed by cardinals, are indexed by ordinals, and the ordinals can be shown to be comparable without AC.[23] Although Paseau's argument does not require AC, however, it does seem to require classical reasoning about sets in the metatheory. Suspicion on this matter is raised by the fact that

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[20] Paseau (2003, pp. 386–391) also considers a 'many models' interpretation, as opposed to the temporal and modal versions of the 'many nodes' interpretation. According to the 'many models' interpretation, the class of valuations with respect to which the notion of semantic consequence is defined is constituted by different possible Kripke models. The arguments that I offer below concerning the modal version of the 'many nodes' interpretation straightforwardly carry over to the 'many models' interpretation.

[21] The argument is straightforward: "assuming that the temporal evolution of set theory unfolds linearly, this temporal Kripke model must accordingly be linearly ordered" (Paseau, 2003, p. 389). To be sure, the argument rests on the physical assumption that time is linear, but a discussion of this assumption lies beyond the scope of this paper.

[22] The evidence that Paseau (2001, p. 373, fn. 7) offers for a temporal interpretation of Lear's text, alongside a modal interpretation, is unconvincing. For instance, when Lear (1977, p. 93, my italics) says that "the points t are points *in* time in the development of set theory," he might simply be taken as saying that to every possible knowledge state is associated a time.

[23] To stress: it is true that the models will all be segments of the hierarchy of inaccessible height, which might suggest that AC is needed to prove that they are linearly ordered. The cardinality of the models, however, does not prevent us from indexing them by means of ordinals in the usual way, which makes it possible to prove that they are linearly ordered without AC.

the argument needs the Comparability of Ordinals, which is yet another case of a statement which is problematic in intuitionistic set theory. In fact, we have the following:[24]

Theorem 5. *The Comparability of Ordinals intuitionistically implies* LEM *in the presence of the axioms of* Z_0.

Proof. Consider the sets 0, 1, and 2, defined as before, and use the axiom of separation to define the set $\alpha = \{x \in 2 | x = 0 \wedge \varphi\}$. Clearly, α is an ordinal, since $\alpha \subseteq 1$. By the Comparability of Ordinals and the usual set-theoretic definition of ordering, we then have $\alpha \in 1 \vee \alpha = 1$. If $\alpha \in 1$, then $\alpha = 0$, which implies $\neg\varphi$ by definition of α. If $\alpha = 1$, then φ. Hence, LEM must hold. □

Moreover, the argument makes use of Zermelo's categoricity theorem, whose proof requires distinctively classical resources. If one takes models of set theory to be sets, this might also be a source of potential concern. However, Paseau (2003, p. 390) claims that the logic in which the consequences of the adoption of the semantics are to be assessed should be classical if we want to produce a dialectically suasive argument against the classical mathematician. But be that as it may: even if we grant that the logic of the metatheory should be classical, the use of Zermelo's categoricity theorem makes Paseau's argument problematic for other reasons.

Paseau presents Zermelo's result as a "proof that the natural models of set theory are the V_κ" – where a natural model is one which is "faithful to our conception of the set-theoretic universe" (Paseau, 2003, p. 381). And the idea, presumably, is that only a model of this kind can constitute a possible knowledge state. But even if we restrict attention to natural models, philosophical argument is needed to establish that these models are exhausted by those to which Zermelo's result applies. For what Zermelo (1930) proved is that *second-order* ZF (ZF_2) is only satisfiable in the V_κ for κ an inaccessible ordinal[25] – where ZF_2 is, as usual, the theory one obtains if one replaces ZF's axiom schema of replacement with the corresponding second-order axiom.[26] Hence, Paseau's argument rests on the assumption that only models of second-order set theory reflect our understanding of the hierarchy of sets and can thereby constitute possible knowledge states.

Is this assumption legitimate? The issue turns on the status of second-order set theory and on whether appeal to it is neutral in the present context. We can begin by considering the current status of the debate in set theory. Take the dispute over the truth-value of the continuum hypothesis (CH), for instance.[27] In the context of this dispute, various attempts have been made to find plausible principles that settle CH one way or the other.

24 See Aczel & Rathjen (2000/2001, section 9-2).

25 Zermelo's original result concerns a theory without infinity, which can also be satisfied in V_ω.

26 Second-order separation is a consequence of second-order replacement, which, in the presence of the standard version of foundation, also implies the second-order version of $\in$-induction. The presence of foundation obviously raises issues concerning the underlying logic of the theory whose models we are considering. For the sake of simplicity, and in keeping with Paseau's discussion, I will set these problems aside for present purposes and focus on models of ZF_2. What I will say carries over to other second-order set theories.

27 CH states that there is no cardinal κ such that $\aleph_0 < \kappa < 2^{\aleph_0}$. If we assume AC, then $\aleph_0 < \aleph_1 \leq 2^{\aleph_0}$ and CH is equivalent to $2^{\aleph_0} = \aleph_1$. If we do not assume AC, however, CH is strictly weaker than the equation $2^{\aleph_0} = \aleph_1$, which is then equivalent to the conjunction of CH and the claim that $2^{\aleph_0}$ is not an aleph.

For example, the axiom of determinacy, whose adoption has been suggested to avoid the allegedly paradoxical consequences of AC, implies CH.[28] Another, more famous example of a principle which implies CH is $V = L$. On the other hand, other set-theoretic principles have been shown to imply that $2^{\aleph_0} = \aleph_2$, e.g., Martin's maximum[29] or an axiom proposed by Woodin (1999). But if set theory is ZF_2, it is hard to make sense of this search for new set-theoretic axioms or intuitive considerations which might settle CH. For ZF_2 decides CH, in the sense that there is a sentence, call it CH*, in the language of pure second-order logic which is a logical truth if and only if CH is true and another sentence, call it NCH*, which is a logical truth if and only if CH is false.[30] To be sure, ZF_2 decides CH only in a semantic sense, not in a proof-theoretic one. That is, neither $\mathsf{ZF}_2 \vdash \mathsf{CH}$ nor $\mathsf{ZF}_2 \vdash \neg\mathsf{CH}$.[31] Nonetheless, the search for new axioms would take a very different form if set theory was ZF_2. What set theorists would do is try to find plausible *proof-theoretic* principles that settle CH, not *set-theoretical* ones. In other words, they would try to establish the truth-value of CH by expanding not the axiomatic basis of set theory but its proof-theoretic resources. This shows that the set theorists' conception of the hierarchy is not second order, which already casts doubts on Paseau's contention that only the second-order models are faithful to our conception of the hierarchy. At any rate, it is an unwelcome consequence of Paseau's argument that it rules out the current shape of the dispute over the status of CH as wrongheaded.

But there are further worries about taking set theory to be second-order set theory in the present context. For notice that the fact that ZF_2 decides CH has not led anyone to try to settle its truth-value by reflecting on the acceptability of CH* as a logical truth. This suggests that we do not have a grasp of the range of the second-order quantifier such that reflection on the acceptability of CH* as a logical truth is an easier task than reflection on the acceptability of new set-theoretic principles on the basis of the set concept, which, in turn, suggests that the concept of well-defined property might itself be open-ended. The fact that the concept of well-defined property is open-ended is familiar to advocates of intuitionistic logic. It has been emphasized, for example, by Dummett (1963, p. 198). The reason is that, once we have specified a language and recognized its well-defined properties, we can specify, by reference to the expressions of that language, a further well-defined property that is not expressible in it. But now recall that the point of the arguments we are concerned with is that the open-endedness of the set concept demands the adoption of a nonclassical semantics in set theory. What we are assessing are the revisionary consequences, if any, of the adoption of such a semantics. But to argue that a Kripke semantics does not sanction an intuitionistic logic by assuming that we can grasp the concept of well-defined property in such a comprehensive way that we can quantify over all properties would not be a legitimate assumption if this open-endedness were partly responsible for that of the set concept. And that this is the case can be seen by reflecting on the relation between our notion of *all properties* and our conception of the hierarchy of sets.

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28 However, the axiom of determinacy also implies that $2^{\aleph_0}$ is not an aleph. It is therefore incompatible with AC and entails that $2^{\aleph_0} \neq \aleph_1$, which does not contradict CH in the absence of AC.

29 See Foreman *et al.* (1988).

30 As far as I know, this was first observed by Kreisel (1967).

31 The proof of this result is due to Weston (1977).

Recall that on the iterative conception, each level $V_{\alpha+1}$ of the hierarchy is said to contain *all subsets* of sets of the immediately preceding level V_α. Paseau claims that the natural models of set theory may only differ in height, i.e., in how many levels they have. It is widely believed, however, that they may also differ in width, i.e., in how rich their levels are: the constructible hierarchy is the thinnest possible, while the widest is that in which the powerset of a given set includes all its subsets as classically conceived. The thought is that, in the case of infinite sets, our notion of *all subsets* can expand – just like our conception of *all ordinals* – and our conception of the width of the hierarchy varies accordingly.[32] To put it another way, not only is the concept of set open-ended in height, but it is also open-ended in width. But now notice that the powerset axiom gets its strength only in conjunction with the axiom of separation, which specifies what subsets of a given set there are,[33] which is to say that the width of models of set theory is capable of expansion just because our conception of what counts as *all properties* can expand. Now what Zermelo's categoricity theorem tells us is that each level of the hierarchy includes all the sets it is forced to have by the second-order axiom of separation, that is, it includes $\{x \in V : Xx\}$ for all properties X. So Zermelo's categoricity theorem effectively requires us to give up the open-endedness in width of the set concept. To sum up, the belief that the natural models of set theory – models answering to our conception of the set-theoretic hierarchy – may differ in width cannot be overturned by appealing to Zermelo's categoricity theorem. For it is a second-order result and, as such, depends on the assumption that we have a comprehensive grasp of the range of the second-order quantifier. And this assumption is, in a sense, just what is at stake.

While the fact that the natural models of set theory may differ in width invalidates Paseau's argument, it does not show that the relevant Kripke model is not linear. In fact, someone might argue that our conception of the hierarchy may expand in width only provided that it expands in height. For our conception of the hierarchy expands in width as our notion of *arbitrary subset* expands and, on this view, it is only when our conception of the hierarchy expands in height that subsets may be definable which were not definable before and hence that our notion of an *arbitrary subset* can expand. It is not difficult to show, however, that this is not the case, since we can conceive of models of set theory which are wider than others while having the same height as them and *vice versa*.

Example 6. *Let $\mathfrak{A}'$ be a model of*

1. ZF
2. $\neg(V = L)$
3. $\exists\kappa(\kappa$ *is an inaccessible cardinal)*

and let $\mathfrak{A}$ be $\mathfrak{A}'$ up to the first inaccessible. Moreover, let $\mathfrak{L}$ be the constructible hierarchy in $\mathfrak{A}$ and $\mathfrak{L}'$ be the constructible hierarchy in $\mathfrak{A}'$. Now let $\mathfrak{L}$ be $\mathcal{D}(t_0)$ for some time t_0, $\mathfrak{A}$ be $\mathcal{D}(t_i)$ for some time $t_i \geq t_0$, and $\mathfrak{L}'$ be $\mathcal{D}(t_j)$ for some time $t_j \geq t_0$. Clearly, $\mathcal{D}(t_i) \nsubseteq \mathcal{D}(t_j)$, since $\mathfrak{A}$ is wider than $\mathfrak{L}'$. On the other hand, $\mathcal{D}(t_j) \nsubseteq \mathcal{D}(t_i)$, since $\mathfrak{L}'$ is higher than $\mathfrak{A}$. But this, together with the restriction imposed on the Kripke model that if $t \leq t'$, then $\mathcal{D}(t) \subseteq \mathcal{D}(t')$, implies that $\neg(t_i \leq t_j \vee t_j \leq t_i)$, that is, $\leq$ is not linear.

32 See, e.g., Lear (1977, p. 99) and Reinhardt (1974, p. 191).
33 See Clark (1993, pp. 242–243).

The possibility of conceiving of, as it were, hybrid models shows that the idea that the models of set theory are nested in the way Paseau supposes them to be is not sustainable, at least if we do not import into set theory the second-order assumption embodied in Zermelo's categoricity theorem. This means that there are possible knowledge states that are not comparable and hence that the order on the index points of the Kripke model need not be linear.

4. Must a Kripke model for set theory be directed? Recall, however, that in order to establish that adoption of a Kripke semantics in set theory does not vindicate an intuitionistic logic, it suffices to show that the relevant Kripke model is directed, and nothing we have said so far tells against this. To be sure, a directed Kripke model vindicates a logic which, albeit stronger than intuitionistic logic, is weaker than the one vindicated by a linear Kripke model. But the directedness of the model would still show that the semantics does not sanction an intuitionistic logic.

Is there any way to argue for the directedness of the Kripke model without assuming a second-order conception of the hierarchy? To answer this question, it is helpful to think of what a nondirected Kripke model for set theory would look like. We have seen that Paseau's argument for the linearity of the Kripke model rested on the assumption that a possible knowledge state has to consist in a model of set theory. With this assumption on board, however, it appears difficult to produce an example of a nondirected Kripke model. For, it seems, for any 2 models $\mathfrak{A}$ and $\mathfrak{B}$, one can always conceive of a model $\mathfrak{C}$ such that $\mathfrak{A} \subseteq \mathfrak{C}$ and $\mathfrak{B} \subseteq \mathfrak{C}$. On the other hand, it is easy to give such an example once we switch from talk of models to talk of sentences.

Example 7. *Let* Th(ZFI) *be the intuitionistic deductive closure of* ZFI *and suppose that* $t_0 \Vdash \text{Th}(\mathsf{ZFI})$. *Suppose, moreover, there are* $t_1, t_2 \geq t_0$ *such that* $t_1 \Vdash \exists\kappa(\kappa$ *is a measurable cardinal)*[34] *and* $t_2 \Vdash \neg\exists\kappa(\kappa$ *is a measurable cardinal), and take each to be the earliest such. Obviously, there can be no* t_k *such that* $t_k \geq t_1$ *and* $t_k \geq t_2$, *since the monotonicity of the forcing relation would guarantee that* $t_k \Vdash \exists\kappa(\kappa$ *is a measurable cardinal) and* $t_k \Vdash \neg\exists\kappa$ *(*κ *is a measurable cardinal), a contradiction.*

However, it is not clear whether this example shows that the relevant Kripke model is not directed and does not validate WLEM. For in order to provide a counterexample to the directedness of the Kripke model, we had to assume that there are possible knowledge states that are not only incomparable but also incompatible. In general, take 2 set-theoretic statements ϕ and ψ, each of which can be known. Then, it seems that their conjunction

[34] A cardinal κ is *measurable* if and only if $\kappa > \omega$ and there is *measure* on κ, i.e., there is a function $\mu : \mathcal{P}(\kappa) \mapsto \{0, 1\}$ such that:

(i) if $\{\chi_i : i \in \imath\}$ is a pairwise disjoint family of subsets of κ and the cardinality of $\imath$ is less than κ, then

$$\bigcup_{i \in \imath} \chi_i = \sum_{i \in \imath} \mu(\chi_i);$$

(ii) $\mu(\kappa) = 1$ and $\mu(\{\alpha\}) = 0$ for $\alpha < \kappa$.

Intuitively, μ is to be thought of as selecting out the 'big' subsets of κ, where a set of μ-measure 0 is 'small' and a set of μ-measure 1 is big. Clause (i) guarantees that the union of fewer than κ small subsets of κ is still small. Clause (ii) guarantees that κ is big and that, for any α less than κ, the singleton $\{\alpha\}$ is small.

can also be known, since coming to know one of them does not make it impossible to come to know the other. The only way to avoid this conclusion is to suppose that there can be incompatible advancements in our set-theoretical knowledge. Hence, a failure of directedness amounts to there being a knowledge state t_0 followed by 2 knowledge states t_1 and t_2 such that $t_1 \Vdash \phi$ and $t_2 \Vdash \neg\phi$.[35] Is it possible for the envisaged situation to obtain when the nodes of the Kripke model are interpreted as *metaphysically* or *mathematically* possible knowledge states? The answer varies depending on the particular revisionary argument we are concerned with.

Consider Tait's view, for instance. According to him, the open-endedness of the set concept precludes any mathematical characterization of the universe of sets, which does not constitute a well-defined extension. Hence, universal generalizations about the universe of all sets can be affirmed only on the basis of operations for constructing sets that we have already accepted. The proposal relies on a constructivist view of the set-theoretic universe according to which the hierarchy is what is obtained by introducing, on the basis of the iterative conception, stronger and stronger operations for constructing sets. And the operations we might be led to introduce might well be incompatible. Tait himself is explicit about this:

> It is not at all clear that, as we try to develop the idea of iteration as far as possible, there is only one direction that this would take us. It is conceivable that there is some development of this idea that leads to the construction, for example, of measurable cardinals; but that there would also be grounds for rejecting the construction. (Tait, 1998, p. 478)

Now take Lear's argument instead. The argument presupposes a Platonist picture of the set-theoretic universe, according to which "there is a determinate universe of sets, and set-theoretic discourse is discourse about these objects" (Lear, 1977, p. 101). But if set-theoretical discourse is discourse about sets *qua* abstract objects and there is a determinate unique universe of sets, then it seems that it may not be possible to come to know that, for instance, there exists a measurable cardinal and to come to know that there does not exist such a cardinal. If we interpret the nodes of the Kripke model as metaphysically or mathematically possible knowledge states, therefore, it seems that Lear's Platonism suffices to establish the directedness of the model.

Can this conclusion be resisted? There seem to be 2 ways of denying that Platonism implies that there can be no incompatible advancements in our set-theoretical knowledge. The first way would be to switch from the standard Platonism Lear seems to be working with to a form of *full-blooded* Platonism[36] or at least to a form of Platonism that allows for the existence of multiple universes of sets. Although Lear (1977, p. 93) briefly considers this option, it is difficult to reconcile this passage with the general tenor of his discussion. At any rate, the reliance on the existence of multiple universes of sets would make the argument dependent on a kind of Platonism which can hardly be taken to be part of

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[35] In this connection, it is worth noticing that the usual semantic proof of the intuitionistic invalidity of WLEM proceeds precisely by constructing a Kripke model in which such a situation obtains. What we are asked to consider is a model with nodes t_0, t_1, and t_2 such that $t_0 < t_1$ and $t_0 < t_2$ and in which $t_1 \Vdash \phi$. Clearly, $t_2 \Vdash \neg\phi$ and so $t_0 \nVdash \neg\neg\phi$. But since $t_1 \Vdash \phi$, also $t_0 \nVdash \neg\phi$. Therefore $t_0 \nVdash \neg\phi \vee \neg\neg\phi$.

[36] Full-blooded Platonism is the view "that all mathematical objects that logically possibly could exist actually do exist" (Balaguer, 1998, p. 191).

'one's original intuitions' or of the 'naively Platonist framework' within which set theorists standardly work.[37] As a result, the force of the argument would be considerably weakened.

Another way of denying that Platonism implies that there can be no incompatible advancements in our set-theoretical knowledge is to interpret the modality at issue differently. Rather than metaphysically or mathematically possible knowledge states, the nodes of the Kripke model could be taken as *epistemically* possible knowledge states. Assuming that we currently know Th(ZFI), the idea is that, for all we know, there may be a later knowledge state in which we come to know that there exists a measurable cardinal and, for all we know, there may also be a later knowledge state in which we come to know that there does not exist a measurable cardinal. If this notion of epistemic possibility can be made precise, the conclusion that the relevant Kripke model is directed can be resisted even on a standard Platonist picture of the universe of sets.

The problem with this option is that the interpretation of the nodes of the Kripke model as epistemically possible knowledge states is incompatible with Lear's argument, which is that classical semantics does not capture the strength of our set-theoretic assertions once the extension of set is taken as changing over time. For example, he writes:

> When one asserts that there is no inaccessible cardinal one asserts now that no future development in set theory will lead us to discover the existence of an inaccessible: for there are none to be discovered. (Lear, 1977, p. 96)

The strength of our set-theoretic assertions, Lear claims, derives not from the specified domain – the extension of set at a certain time – but "from the expression of a current expectation that the statement will continue to hold no matter how our understanding of set theory – and thus the extension of 'set' – may develop over time" (Lear, 1977, p. 95). A classical semantics, he contends, is not capable of capturing such strength, whereas a Kripke semantics – with its account of, for example, negation in terms of what is the case at present or later times – is. But if the nodes of the Kripke model were epistemically possible knowledge states, when asserting that there are no inaccessible cardinals, one would not be asserting that there are none to be discovered but that our current state of information is incompatible with their discovery, which is certainly not the case. So Lear's argument seems to enforce an interpretation of the nodes of the Kripke model that rules out their being epistemically possible knowledge states.

5. Concluding remarks. There is an interpretation of the nodes of the Kripke model, *viz.* the one in terms of epistemically possible knowledge states, which makes Kripke semantics adequate for the adoption of intuitionistic logic in set theory, with the proviso that we are careful in the choice of the axioms. Thus, the use of a Kripke semantics in the development of intuitionistic set theories seems safe, even within a standard Platonist framework.

Alongside this interpretation, there is another one according to which the nodes of the Kripke model are metaphysically or mathematically possible knowledge states. I have argued that within a standard Platonist framework, this modal Kripke model must be directed. However, I have also argued, *contra* Paseau, that this Kripke model need not be linear, since in order to establish its linearity we need the second-order assumption embodied in Zermelo's categoricity theorem, an assumption which is of a piece with giving

[37] See Lear (1977, p. 87).

up one kind of open-endedness of the set concept. And the open-endedness of the set concept is precisely what is supposed to motivate the adoption of a nonclassical semantics.

Hence, once a Kripke semantics is adopted, we have set theories of varying logical strength depending on the interpretation we give to the nodes of the Kripke model and on the background ontology. This has important consequences for revisionary arguments in set theory. In particular, Lear's argument, being incompatible with an interpretation of the nodes of the Kripke model in terms of epistemically possible knowledge states and assuming a Platonist framework, leads at best to the adoption of a logic that is intermediate between the classical and the intuitionistic one. On the other hand, other revisionary arguments such as Tait's seem to fit well with an interpretation of the nodes of the relevant Kripke model which vindicates an intuitionistic logic. Whether any of these arguments is successful – i.e., whether the open-endedness of the set concept determines the need for a nonclassical semantics in the first place – is still an open question.[38]

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[38] Indeed, it is far from obvious that the right semantics for the language of set theory should be an intuitionistic one. Paseau himself (2001, pp. 374–377, 2003, pp. 391–395) criticizes Lear's argument for the adoption of a Kripke semantics. And some philosophers have claimed that the open-endedness of the set concept does demand a revision of the logic and semantics of set theory but along different lines. One alternative is the view that the ordinary set-theoretic quantifiers have some implicit modal force. Specifically, the idea is that '$\forall x$' means something like '$\Box\forall x$' and '$\exists x$' means '$\Diamond\exists x$'. Another alternative, developed by, e.g., Fine (1981) and Parsons (1983), would be to conclude that set theory should be done in a modal language.

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