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New supersymmetry of the monopole

F. De Jonghe, A.J. Macfarlane¹, K. Peeters, J.W. van Holten

NIKHEF, Postbus 41882, 1009 DB Amsterdam, The Netherlands

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Abstract

The non-relativistic dynamics of a spin- $\frac{1}{2}$ particle in a monopole field possesses a rich supersymmetry structure. One supersymmetry, uncovered by d'Hoker and Vinet, is of the standard type: it squares to the Hamiltonian. In this paper we show the presence of another supersymmetry which squares to the Casimir invariant of the full rotation group. The geometrical origin of this supersymmetry is traced, and its relationship with the constrained dynamics of a spinning particle on a sphere centered at the monopole is described.

1. This paper discusses spin- $\frac{1}{2}$ particles in a magnetic monopole background. The model is described by the Hamiltonian

$$H = \frac{1}{2}(\mathbf{p} - e\mathbf{A})^2 - e\mathbf{B} \cdot \mathbf{S}, \quad (1)$$

where all vectors are 3-dimensional. In particular $\mathbf{S} = \boldsymbol{\sigma}/2$, $\mathbf{B} = g\mathbf{r}/r^3$, and the vector potential $\mathbf{A}$ is to be defined patchwise in the well-known way. It is more than ten years now since it was observed that this model possesses a hidden supersymmetry [1]. This supersymmetry is of the so-called $N = \frac{1}{2}$ type [2–4] in which there is a single hermitian supercharge Q such that $Q^2 = H$. It is unusual in that the phase space of its fermionic sector is of odd dimension: this involves three hermitian Majorana fermions ψ_i , $i = 1, 2, 3$. These satisfy $\{\psi_i, \psi_j\} = \delta_{ij}$ and admit the natural representation $\psi_i = \sigma_i/\sqrt{2}$, in terms of Pauli matrices, which gives rise to the description [5] of the spin of the monopole. In the rest of this paper,

we will work at the level of the quantum theory, i.e. after the transition from Poisson brackets to canonical (anti)commutators. However, we will not use the specific representation of the Grassmann coordinates in terms of Pauli matrices that was mentioned above, since this can obscure the difference between Grassmann even and Grassmann odd operators.

In this paper, we show that the dynamics of a spin- $\frac{1}{2}$ particle in a magnetic monopole field admits a larger hidden supersymmetry structure. Indeed, the theory possesses an additional supercharge $\tilde{Q}$ which obeys

$$\{Q, \tilde{Q}\} = 0, \quad (2)$$

so that $[\tilde{Q}, H] = 0$ is evident. Also we have

$$\begin{aligned} \tilde{Q}^2 &= K, \\ K &= \frac{1}{2}(J^2 - e^2 g^2 + \frac{1}{4}), \end{aligned} \quad (3)$$

where $\mathbf{J}$ is the total angular momentum of the monopole system,

$$\mathbf{J} = \mathbf{r} \times (\mathbf{p} - e\mathbf{A}) + \mathbf{S} - eg\frac{\mathbf{r}}{r}. \quad (4)$$

¹ On leave of absence from DAMTP, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK, and St. John's College, Cambridge.

The operator $\tilde{Q}$ – which coincides up to a constant shift with the operator A in [1] – plays an important role in the determination of the spectrum of the model. To see the equivalence between $\tilde{Q}$ and A , one has to use the representation of the Grassmann coordinates by Pauli matrices, and, in this way, one loses the insight that A is in fact a Grassmann odd operator. In the original analysis, A was introduced only on algebraic grounds, and its geometrical interpretation remained unclear. Here we explain that it corresponds to a new type of supersymmetry of the problem, and our way of obtaining this extra supercharge shows the geometry of the monopole configuration to be considerably richer than anticipated. In particular, since the new supercharge $\tilde{Q}$ squares to essentially J^2 , our analysis shows that the full supersymmetry algebra of the monopole is in fact a *non-linear algebra*.

2. Our discussion is based, firstly, on a general method [6] which uses Killing-Yano tensors [8] to generate additional supersymmetries in a theory which already possesses some known supersymmetries of a standard type, and, secondly, on the realisation that vital theoretical information can stem from the employment, where appropriate, of fermionic phase spaces of odd dimension. We have seen that the last circumstance does apply to the monopole. Also, it is true that the flat background of the monopole admits the simplest available non-trivial example of a Killing-Yano tensor, viz.

$$f_{ij} = \varepsilon_{ijk} x_k. \quad (5)$$

This satisfies trivially the conditions generally obeyed [7,6] by such a tensor, namely,

$$f_{ij} = -f_{ji}, \quad f_{ij,k} + f_{ik,j} = 0. \quad (6)$$

Further, we can use it, as in [6], to define a hermitian supercharge $\tilde{Q} = \tilde{Q}^\dagger$, which obeys (2) and (3). We develop these matters below starting from the superfield formulation. Our treatment may be compared with the recent work of [9], which added electromagnetic interactions to the curved space analysis of spinning particles in d dimensions in [6]. Although [9] does not address the special properties and simplifications that occur in the specific case of $d = 3$, our work evidently has links with the general results found there.

We use scalar superfields

$$\Phi_i = x_i + i\theta\psi_i \quad (i = 1, 2, 3), \quad (7)$$

involving one real Grassmann variable such that $\theta^2 = 0$, $\theta = \theta^*$, a real coordinate x_i and Majorana fermions ψ_i . The most general Lagrangian we can build is of the form

$$S = \int dt d\theta \left[\frac{1}{2} i \dot{\Phi}_i D \Phi_i + i e D \Phi_i A_i(\Phi) + \frac{1}{6} i k \varepsilon_{ijk} D \Phi_i D \Phi_j D \Phi_k \right], \quad (8)$$

where the operator $D = \partial_\theta - i\theta\partial_t$. This contains kinetic, electromagnetic and torsion terms, but no pure potential term can be built without the use of spinor superfields (see e.g. [4] for more details on this point). However, we do not need the latter here and will simplify (8) by taking the coupling constant for the torsion term to be zero at first.

Writing the action out in component fields one obtains

$$L = \frac{1}{2} \dot{x}_i \dot{x}_i + \frac{1}{2} i \dot{\psi}_i \dot{\psi}_i + e \dot{x}_i A_i(x) - \frac{1}{2} i e F_{ij} \psi_i \psi_j, \quad (9)$$

where now $A_i(\mathbf{r})$ is seen to be the vector potential of the magnetic field $\mathbf{B}$, with components $B_i = \varepsilon_{ijk} A_{k,j} = \frac{1}{2} \varepsilon_{ijk} F_{jk}$. As usual [1],

$$p_i = \dot{x}_i + e A_i, \quad [x_i, p_j] = i \delta_{ij}, \quad \{\psi_i, \psi_j\} = \delta_{ij}, \quad (10)$$

and $H = Q^2$ for the supercharge

$$Q = (p_i - e A_i(x)) \psi_i. \quad (11)$$

Following the procedure of [6], we seek a new supercharge $\tilde{Q}$, which obeys (2) and hence describes a new supersymmetry of H , in the form

$$\tilde{Q} = (p_i - e A_i) f_{ij} \psi_j + \frac{1}{6} i b \varepsilon_{ijk} \psi_i \psi_j \psi_k, \quad (12)$$

with the tensor f_{ij} given by (5), and some parameter b to be determined by demanding that (2) holds. Using the basic commutators (10), one finds that for arbitrary A ,

$$\{Q, \tilde{Q}\} = -2(1+b)(\mathbf{p} - e\mathbf{A}) \cdot \mathbf{S} + e\mathbf{B} \times \mathbf{r} \cdot \mathbf{S}, \quad (13)$$

where $S_i = -\frac{1}{2} i \varepsilon_{ijk} \psi_j \psi_k$ defines the spin operator [1]. We chose $b = -1$, and see that the anticommutator

vanishes for radial B , and therefore for the monopole, for which

$$B = g \frac{r}{r^3}. \quad (14)$$

The condition on B here is a special case of the more general condition

$$f^\lambda{}_{[\mu} F_{\nu]\lambda} = 0 \quad (15)$$

found in [9].

It remains to compute and interpret the conserved quantity K that one obtains by squaring this new supercharge. A direct computation gives the answer (3) announced in the introduction, where the total angular momentum J can also be written as

$$\begin{aligned} J &= r \times (p - eA) + S - eg \frac{1}{r} r \\ &= L - er \times A + S - eg \frac{1}{r} r. \end{aligned} \quad (16)$$

The first term L in the last formula, shows that the part of K quadratic in the momenta: $\frac{1}{2} p_i K_{ij} p_j$, is determined by a well-known Killing-tensor

$$K_{ij} = \delta_{ij} r^2 - x_i x_j, \quad (17)$$

with a well-understood interpretation [10]. In fact, we write

$$K_{ij} = f_{ik} f_{jk} \quad (18)$$

in order to suggest an interpretation of it as a metric to which we will return below.

We wish also to comment on our extra supersymmetry in relation to a distinct approach to such matters, to be found in [11]. This requires to search for these transformations in a superfield form. Our result can be readily cast into such a form. Using (12) to compute $\delta\Phi_i = [\tilde{\epsilon}\tilde{Q}, \Phi_i]$, we are soon led to the result

$$\tilde{\delta}\Phi_i = -\epsilon\epsilon_{ijk} D\Phi_j \Phi_k, \quad (19)$$

which is of the form used in [11] for the antisymmetric quantity $I_{ij} = -\epsilon_{ijk} D\Phi_k$. Of course, here we do not find that I_{ij} defines a complex structure since we demand only invariance of the action under $\tilde{\delta}$ and not closure of the supersymmetry algebra generated by $\tilde{\delta}$ in the same way as in [11]. Our result however

satisfies the relevant subset of conditions laid down in [11] for the application in hand.

Apart from the (super)conformal symmetry that was discussed in [12] and used in [1], the symmetry algebra of the magnetic monopole therefore becomes:

$$\begin{aligned} Q^2 &= H, \quad \{Q, \tilde{Q}\} = 0, \quad \tilde{Q}^2 = \frac{1}{2} (J^2 - e^2 g^2 + \frac{1}{4}), \\ [Q, H] &= 0, \quad [\tilde{Q}, H] = 0, \quad [J_i, H] = 0, \\ [Q, J_i] &= 0, \quad [\tilde{Q}, J_i] = 0, \\ [J_i, J_j] &= i\epsilon_{ijk} J_k. \end{aligned} \quad (20)$$

Such a structure in which the Killing-Yano tensor is related to a square root of total angular momentum is familiar from other examples [6,13]. It represents the particular kind of non-linearity familiar also from finite-dimensional W -algebras [14].

3. Recently it was shown by Rietdijk and one of us [15], that there exists a certain duality which relates two theories in which the role of the Killing-Yano tensor and the vierbein and the role of the supercharges Q and $\tilde{Q}$ are interchanged. Comparison of (11) and (12) indicates that similar ideas are relevant in the theory studied here. Thus one is led to consider not only our original theory but also the one in which K and $\tilde{Q}$ are Hamiltonian and “first” supersymmetry. One knows the Hamiltonian of the latter theory and the canonical equations for the variables that occur in it, so that its dynamical content can be completely determined. In fact we can show that it describes a particle of spin $\frac{1}{2}$ confined to a sphere of radius ρ , for each fixed $\rho > 0$, centered at the position of a monopole.

We consider a supersymmetric model for a particle on a sphere of fixed radius ρ , in the background field of a magnetic monopole located at the center of the sphere. The constraints can be imposed using a spinorial supermultiplet of Lagrange multipliers $(\lambda + \theta a)$:

$$\Delta L_{\text{constr}} = -\frac{1}{2} a (x_i^2 - \rho^2) - i\lambda\psi_i x_i. \quad (21)$$

The full set of constraints (primary and secondary) then becomes

$$x_i^2 = \rho^2, \quad x_i \psi_i = 0, \quad x_i (p_i - eA_i) = 0. \quad (22)$$

After introducing Dirac brackets, the constraints can be used to write the classical Hamiltonian as

$$H^* = \frac{1}{2\rho^2} (p_i - eA_i) K_{ij} (p_j - eA_j) - e \mathbf{B} \cdot \mathbf{S}. \quad (23)$$

Using (18), it is seen that this is equivalent to

$$H^* = \frac{M^2}{2\rho^2} - e \mathbf{B} \cdot \mathbf{S}, \quad (24)$$

where $M_i = -f_{ij} (p_j - eA_j) = \epsilon_{ijk} x_j (p_k - eA_k)$. Clearly, this supersymmetric model describes a subclass of the solutions of the original model (those with spherical symmetry), and involves precisely the Stackel-Killing tensor as its metric and the Killing-Yano tensor as the dreibein.

Interestingly, the modification introduced here changes the new Killing-Yano-based supersymmetry of the original model into a standard supersymmetry of the constrained Hamiltonian. In the quantum theory, to find the corresponding result, we rescale the new supercharge (12) by a factor ρ

$$\tilde{Q} = \frac{1}{\sqrt{2}\rho} (-i\epsilon_{ijk}\sigma_i x_j D_k + \frac{1}{2}) \quad (25)$$

(where $D_k = \partial_k - ieA_k$), so that the square of this supercharge for a magnetic monopole field can be written in the form

$$\tilde{Q}^2 = \frac{1}{2} \left[-D_i \left(\delta_{ij} - \frac{x_i x_j}{\rho^2} \right) D_j - e \boldsymbol{\sigma} \cdot \mathbf{B} + \frac{1}{4\rho^2} \right], \quad (26)$$

As a result, the supercharge (25) is naturally identified with the proper restriction of $\tilde{Q}/\rho$ of the original model to the sphere with radius ρ ; however, restricted to the sphere it plays the role of the *ordinary* supercharge Q : it is a square root of the (constrained) Hamiltonian. Thus we see, how the new non-standard supercharge of one model is related to the standard supersymmetry of another model. Moreover, the second model is here a constrained version of the original model.

We remark on the appearance of (25) in the constrained model. In the constrained model, the representation of the fermions ψ_i and of the momentum p_i requires an extra projection operator K_{ij} in comparison with the representation in the unconstrained model. However, the extra terms do not contribute to the representation of $\tilde{Q}$ in the constrained model, basically owing to the zero mode of the Killing-Yano tensor f_{ij} .

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References

- [1] E. d'Hoker and L. Vinet, Phys. Lett. B 137 (1984) 72.
- [2] M. Sakimoto, Phys. Lett. B 151 (1985) 115.
- [3] R.H. Rietdijk, Applications of supersymmetric quantum mechanics, Ph.D. thesis Amsterdam (1992).
- [4] A.J. Macfarlane, Nucl. Phys. B 438 (1995) 455.
- [5] J.L. Martin, Proc. Roy. Soc. 251 (1959) 536, 543.
- [6] G.W. Gibbons, R.H. Rietdijk and J.W. van Holten, Nucl. Phys. B 404 (1993) 42.
- [7] B. Carter and R.G. McLenaghan, Phys. Rev. D 19 (1979) 1093.
- [8] K. Yano, Ann. Math. 55 (1952) 328.
- [9] M. Tanimoto, TIT/HEP-277/COSMO-50, gr-qc/9501006.
- [10] J.W. van Holten and R.H. Rietdijk, Class. Quantum Grav. 11 (1993) 559.
- [11] R.A. Coles and G. Papadopoulos, Class. Quantum Grav. 7 (1990) 427.
- [12] R. Jackiw, Ann. Phys. 129 (1980) 183.
- [13] J.W. van Holten, Phys. Lett. B 342 (1995) 47.
- [14] J. de Boer, F. Harmsze and T. Tjin, preprint ITP-SB-95-07 and ITFA-03-2/95.
- [15] R. Rietdijk and J.W. van Holten, Durham/NIKHEF preprint (July 1995).