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SERIES EXPANSIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

S. M. Verduyn Lunel *

In this paper we prove convergence results for the series expansion of the solution to a linear functional differential equation. The results are consequences of an analysis of eigenfunction expansions for the generator of the solution map. This abstract approach unifies the treatment of retarded and neutral functional differential equations.

INTRODUCTION

Let $\mathcal{C} = C([-h, 0], \mathbb{R}^n)$ denote the Banach space of continuous functions endowed with the supremum norm. Using the Riesz representation theorem, every bounded linear mapping $L : \mathcal{C} \rightarrow \mathbb{R}^n$ can be represented by a matrix function of bounded variation

$$L\varphi = \int_0^h d\zeta(\theta)\varphi(-\theta).$$

The matrix function ζ is unique upon normalization. We call a function of bounded variation $f : [0, \infty) \rightarrow \mathbb{R}$ *normalized* if $f(0) = 0$ and f is continuous from the right on the open interval $(0, \infty)$. Furthermore, we denote for $t \geq 0$ and for a given function $x : [-h, \infty) \rightarrow \mathbb{R}^n$, by x_t the element in $\mathcal{C}$ defined by $x_t(\theta) = x(t + \theta)$, $-h \leq \theta \leq 0$. In the context of FDE, the element x_t is called the *state* at time t .

An initial value problem for a linear functional differential equation (FDE) is given by the following relation

$$(1.1) \quad \begin{cases} \frac{d}{dt} Dx_t = Lx_t, & t \geq 0, \\ x_0 = \varphi, & \varphi \in \mathcal{C}, \end{cases}$$

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where $D, L : \mathcal{C} \rightarrow \mathbb{R}^n$ are continuous and linear and D is atomic at zero; that is, there are normalized $n \times n$ matrix functions of bounded variation μ and ζ such that

$$D\varphi = \varphi(0) - \int_0^h d\mu(\theta)\varphi(-\theta), \quad L\varphi = \int_0^h d\zeta(\theta)\varphi(-\theta)$$

and μ is continuous at zero.

It is a classical result that one can associate with (1.1) a semigroup of solution operators on $\mathcal{C}$. The semigroup is strongly continuous and given by translation along the solution

$$T(t)\varphi = x_t(\cdot; \varphi),$$

where $x(\cdot; \varphi)$ denotes the solution to (1.1). (See [6] for more information.) The infinitesimal generator $A(\mathcal{C} \rightarrow \mathcal{C})$ is given by

$$(1.2) \quad \begin{cases} \mathcal{D}(A) = \{\varphi \in \mathcal{C} \mid \frac{d\varphi}{d\theta} \in \mathcal{C}, D \frac{d\varphi}{d\theta} = L\varphi\} \\ A\varphi = \frac{d\varphi}{d\theta}. \end{cases}$$

It is not difficult to compute the resolvent $R(z, A) := (zI - A)^{-1}$ of A explicitly:

$$(1.3) \quad R(z, A)\varphi = \frac{1}{\det \Delta(z)} P(z)\varphi,$$

where

$$(1.4) \quad \begin{aligned} (P(z)\varphi)(\theta) = & e^{z\theta} \left\{ \text{adj } \Delta(z) \left[D\varphi + \int_0^h [z d\mu(\tau) + d\zeta(\tau)] \int_0^\tau e^{-z\sigma} \varphi(\sigma - \tau) d\sigma \right] \right. \\ & \left. + \det \Delta(z) \int_\theta^0 e^{-z\sigma} \varphi(\sigma) d\sigma \right\} \end{aligned}$$

and

$$(1.5) \quad \Delta(z) = z[I - \int_0^h e^{-zt} d\mu(t)] - \int_0^h e^{-zt} d\zeta(t).$$

Here $\Delta(z)$ is called the *characteristic matrix* and $\text{adj } \Delta(z)$ denotes the matrix of cofactors of $\Delta(z)$.

From this representation, we immediately derive that the spectrum of A consists of point spectrum only, and is given by the zero set of an entire function

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}.$$

Furthermore, for $\lambda \in \sigma(A)$ the space $\mathcal{C}$ decomposes as

$$\mathcal{C} = \text{Im}(\lambda I - A)^{m_\lambda} \oplus \text{Ker}(\lambda I - A)^{m_\lambda}$$

with corresponding projection P_λ onto $\text{Ker}(\lambda I - A)^{m_\lambda}$. This Riesz projection is given by the Dunford integral

$$(1.6) \quad P_\lambda = \frac{1}{2\pi i} \int_{C_\lambda} R(z, A) dz,$$

where C_λ is a simple closed rectifiable curve enclosing λ , but no other points in the spectrum of A .

The range of P_λ is denoted by $\mathcal{M}_\lambda$ and is called the generalized eigenspace at λ . The linear space obtained by the direct sum of all generalized eigenspaces $\mathcal{M}_\lambda$, $\lambda \in \sigma(A)$, will be denoted by $\mathcal{M}(A)$ and is called the *generalized eigenspace* of A . The system of eigenvectors and generalized eigenvectors of A is called *complete* if $\overline{\mathcal{M}(A)} = \mathcal{C}$.

It is easy to see that the eigenvectors and generalized eigenvectors of A are of the form

$$\varphi_j(\theta) = p_j(\theta)e^{\lambda_j\theta}, \quad -h \leq \theta \leq 0,$$

where $\lambda_j \in \sigma(A)$ and p_j is a polynomial. They are precisely the initial conditions corresponding to the elementary solutions of equation (1.1), i.e., solutions of the form

$$x(t) = p(t)e^{\lambda t},$$

where $\lambda \in \mathbb{C}$ and p is a polynomial.

In this paper we shall address the question whether the solution of (1.1) can be represented by a series of elementary solutions. For retarded equations ($\mu \equiv 0$), series expansions in elementary solutions have been studied by Bellman and Cooke [2], Banks and Manitius [1], Levinson and McCalla [13], and Verduyn Lunel [17]. The approach is based on the Laplace transform. In [2] it was proved that for a scalar differential-difference equation (i.e., the kernel ζ is a step function with finitely many discontinuities) the solution of (1.1) can be expanded into a series of elementary solutions for $t > 0$, uniformly on compact t -sets. Later in [1] the result was extended to the matrix case. In [17] the relation with

completeness was made and the results were extended to kernels of bounded variation that have an atom at the endpoint.

Our main theorems deal with the expansion of the state into a linear combination of eigenvectors and generalized eigenvectors:

$$(1.7) \quad x_t(\theta) = \sum_{j=0}^{\infty} p_j(t+\theta) e^{\lambda_j(t+\theta)}, \quad -h \leq \theta \leq 0, \quad t \geq 0,$$

or, in operator language,

$$(1.8) \quad T(t)\varphi = \sum_{j=0}^{\infty} T(t)P_{\lambda_j}\varphi, \quad t \geq 0,$$

where P_{λ_j} denotes the spectral projection corresponding to λ_j . Here the convergence is with respect to the sup norm. Since $x(t; \varphi) = (T(t+h)\varphi)(-h)$, pointwise convergence of the solution on $[0, \infty)$ into a linear combination of eigenvectors and generalized eigenvectors, as considered in [1, 2, 17], corresponds to

$$(T(t)\varphi)(\theta) = \sum_{j=0}^{\infty} (T(t)P_{\lambda_j}\varphi)(\theta), \quad t > h, \quad -h \leq \theta \leq 0.$$

We shall prove the following main results.

THEOREM 1.1. *Let $\mathcal{C} = C([-h, 0], \mathbb{R}^n)$ and let $T(t) : \mathcal{C} \rightarrow \mathcal{C}$ denote the semigroup of solution operators associated with*

$$\dot{x}(t) = L_0 x(t-h) + \int_0^h d\zeta_0(\theta) x(t-\theta),$$

where L_0 is a nonsingular matrix and ζ_0 is an $n \times n$ matrix function of bounded variation which is continuous at h . If A denotes the generator of $T(t)$, then for every $\varphi \in \mathcal{D}(A)$, we have

$$(1.9) \quad \lim_{N \rightarrow \infty} \|T(t)\varphi - \sum_{j=0}^N T(t)P_{\lambda_j}\varphi\| = 0, \quad t > 0$$

uniformly on compact t -sets. Here λ_j , $j = 0, 1, \dots$ denote the eigenvalues of A ordered according to increasing modulus and P_{λ} denotes the spectral projection given by (1.6).

The second result concerns neutral functional differential equations.

THEOREM 1.2. *Let $\mathcal{C} = C([-h, 0], \mathbb{R}^n)$ and let $T(t) : \mathcal{C} \rightarrow \mathcal{C}$ denote the semigroup of solution operators associated with*

$$\frac{d}{dt} [x(t) - D_0 x(t-h) - \sum_{j=1}^{\infty} D_j x(t-h_j)] = \int_0^h d\zeta(\theta) x(t-\theta),$$

where ζ is an $n \times n$ matrix function of bounded variation, for $j = 1, 2, \dots$, $h_j < h$ and D_0, D_j are matrices such that D_0 is nonsingular, and $|D_j| \rightarrow 0$ as $j \rightarrow \infty$. Then the semigroup $T(t)$ extends to a group of bounded linear operators and for every $\varphi \in \mathcal{C}$, we have

$$(1.10) \quad \lim_{N \rightarrow \infty} \|T(t)\varphi - \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^k T(t) P_{\mu_j} \varphi\| = 0, \quad t \in \mathbb{R}$$

uniformly on compact t -sets. Here μ_j , $j = 0, 1, \dots$ denote the eigenvalues of the infinitesimal generator A of $T(t)$ ordered according to increasing imaginary part and P_λ denotes the spectral projection given by (1.6).

The summation in (1.10) is called Cesàro summation (see Section 4 for more information).

The regularity conditions on L_0 in Theorem 1.1 and D_0 in Theorem 1.2 ensure that the generator of the semigroup $T(t)$ has a complete system of eigenvectors and generalized eigenvectors. This follows from the necessary and sufficient conditions for completeness derived in [18].

The Theorems 1.1 and 1.2 concern the more difficult problem of expansions in eigenvectors and generalized eigenvectors.

The organization of this paper is as follows. In Section 2 we collect some estimates for the inverse of the characteristic matrix $\Delta(z)$ given by (1.5). These estimates extend earlier estimates by Bellman and Cooke [2]. In Section 3 we discuss retarded equations and prove Theorem 1.1. In Section 4 we discuss neutral equations. In general, neutral equations have no smoothing property. Using Cesàro summation, we have a more general Laplace inversion formula. This inversion formula allows us to prove Theorem 1.2. Finally, in Section 5, we use the theory of hyperbolic semigroups to extend the results to general neutral equations. In particular, we prove series expansions of the type (1.9) and

(1.10) for equations of the form

$$(1.11) \quad \frac{d}{dt} [x(t) - D_0 x(t-h) - \sum_{j=1}^{\infty} D_j x(t-h_j)] = L_0 x(t-h) + \int_0^h d\zeta_0(\theta) x(t-\theta),$$

where L_0 is supposed to be regular and D_0 may be singular. Furthermore, for $j = 1, 2, \dots$, $h_j < h$, $|D_j| \rightarrow 0$ as $j \rightarrow \infty$ and ζ_0 is an $n \times n$ matrix function of bounded variation which is continuous at h .

Concluding this section, we would like to mention that many basic ideas about series expansions for delay equations can be found in Bellman and Cooke [2] and that we were influenced by this book.

2. PRECISE ESTIMATES FOR $|\Delta(z)^{-1}|$

In this section we consider both retarded and neutral equations. We start with retarded equations. The characteristic matrix associated with the retarded equation $\dot{x}(t) = Lx_t$ becomes

$$(2.1) \quad \Delta(z) = zI - \int_0^h e^{-z\tau} d\zeta(\tau).$$

To estimate $|\Delta(z)^{-1}|$ in the complex plane outside small circles centered around the zeros of $\det \Delta$, we start with lower bounds for $|\det \Delta|$. (See [2] and [16] for partial results.)

PROPOSITION 2.1. *There exists a constant $C > 0$ such that for $z \in \mathbb{C}$ with $|z| \geq C \max\{1, e^{-h\operatorname{Re} z}\}$*

$$|\det \Delta(z)| \geq \frac{1}{2} |z|^n.$$

PROOF. From the convolution property of the Laplace transform, it follows that

$$\det \Delta(z) = z^n + \sum_{j=1}^n z^{n-j} \int_0^{jh} e^{-z\tau} d\sigma_j(\tau),$$

where σ_j are suitable functions of bounded variation. Let $\operatorname{Var} \sigma_j$ denote the variation of σ_j . Since for $z \in \mathbb{C}$ for which $|z| \geq C \max\{1, e^{-h\operatorname{Re} z}\}$, we have

$$\left| \int_0^{jh} e^{-z\tau} d\sigma_j(\tau) \right| \leq C^{-j} |z|^j \operatorname{Var} \sigma_j,$$

the proposition follows. $\square$

To estimate $|\det \Delta(z)|$ in the complement of $|z| \geq C \max\{1, e^{-h\operatorname{Re} z}\}$, we have to make further restrictions on the mapping L .

(H1) Let $L : \mathcal{C} \rightarrow \mathbb{R}^n$ be given by

$$L\varphi = L_0\varphi(-h) + \int_0^h d\zeta_0(\theta)\varphi(-\theta),$$

where L_0 is a nonsingular matrix and ζ_0 is continuous at h .

If we assume (H1) the characteristic matrix associated with $\dot{x}(t) = Lx_t$ becomes

$$(2.2) \quad \Delta(z) = zI - L_0e^{-zh} - \int_0^h e^{-z\tau} d\zeta_0(\tau).$$

PROPOSITION 2.2. *Let $\Delta(z)$ be given by (2.2). Then there exist positive constants C_1, C_2 and m such that for $z \in \mathbb{C}$ with $|z| \leq C_1e^{-h\operatorname{Re} z}$ and $|z| \geq C_2$*

$$(2.3) \quad |\det \Delta(z)| \geq \frac{1}{2}m|e^{-nhz}|.$$

PROOF. Using representation (2.2) we can write

$$\det \Delta(z) = z^n + \sum_{j=1}^{n-1} z^{n-j} \int_0^{jh} e^{-z\tau} d\sigma_j(\tau) + (-1)^n e^{-nhz} \det L_0 + (-1)^n \int_0^{nh} e^{-z\tau} d\sigma_n(\tau),$$

where σ_n is the determinant of ζ_0 with respect to the convolution product. As ζ_0 is continuous at h , σ_n is continuous at nh . From the continuity of σ_n at nh we conclude that there exists a constant m such that

$$(2.4) \quad |e^{-nhz} \det L_0 + \int_0^{nh} e^{-z\tau} d\sigma_n(\tau)| \geq m|e^{-nhz}|$$

for $\operatorname{Re} z < -\alpha$ and α sufficiently large. For z such that $|z| \leq C_1e^{-h\operatorname{Re} z}$ and $|z| \geq C_2$ with C_1 sufficiently small, it follows that

$$(2.5) \quad \begin{aligned} \left| z^n + \sum_{j=1}^{n-1} z^{n-j} \int_0^{jh} e^{-z\tau} d\sigma_j(\tau) \right| &\leq |e^{-nhz}| |C_1^n + \sum_{j=1}^{n-1} \operatorname{Var}(\sigma_j) C_1^{n-j}| \\ &\leq \frac{1}{2}m|e^{-nhz}|. \end{aligned}$$

Thus if $C_2 = C_1 e^{\alpha h}$ and C_1 is chosen such that (2.5) holds, then

$$|\det \Delta(z)| \geq \frac{1}{2} m |e^{-nhz}|.$$

□

COROLLARY 2.3. *Let $\Delta(z)$ be given by (2.2). Then the zeros of $\det \Delta$ outside the ball of radius C_2 are located in the curvilinear strip*

$$(2.6) \quad V = \{z \in \mathbb{C} : C_1 e^{-h \operatorname{Re} z} \leq |z| \leq C \max\{1, e^{-h \operatorname{Re} z}\}\},$$

where C , C_1 , and C_2 are the constants introduced before.

Next we would like to prove a lower bound for $\det \Delta$ inside V given by (2.6) but outside small circles containing the zeros of $\det \Delta$.

PROPOSITION 2.4. *Let V be given by (2.6) and let $\Delta(z)$ be given by (2.2). For every $\epsilon > 0$, there exists a constant m such that*

$$(2.7) \quad |\det \Delta(z)| \geq \frac{1}{2} m |e^{-nhz}|$$

for values of z outside circles of radius ϵ containing the roots of $\det \Delta$ and inside V .

PROOF. Let $\lambda_1, \lambda_2, \dots$ denote the zeros of $\det \Delta$ and suppose that such a constant m in (2.7) does not exist. Then there exists an $\epsilon > 0$ and a sequence of points $\{z_j\}_{j=1}^\infty$ inside the curvilinear strip V defined by (2.6), but outside the disks $|z - \lambda_j| \leq \epsilon$, $j = 1, 2, \dots$, such that

$$\lim_{j \rightarrow \infty} e^{nhz_j} \det \Delta(z_j) = 0.$$

Since the z_j are inside V , we have

$$\sup_{j \geq 1} \left| \operatorname{Re} \left(z_j + \frac{1}{h} \log z_j \right) \right| < \infty.$$

Therefore, we can assume that

$$\lim_{j \rightarrow \infty} \operatorname{Re} \left(z_j + \frac{1}{h} \log z_j \right) = \bar{z}.$$

Next, we define a sequence of analytic functions by

$$F_j(z) = e^{nhz} \det \Delta(z + i \operatorname{Im} z_j - \frac{1}{h} \operatorname{Re} \log z_j).$$

Since $\det \Delta$ is bounded by $K|z|^n$ in an neighbourhood $\mathcal{U}$ of V and $|e^{zh}| \leq C_0|z|^{-1}$ for $z \in \mathcal{U}$ with $|z|$ sufficiently large, it follows that the sequence $\{F_j\}$ is uniformly bounded on $\mathcal{U}$. So from Montel's theorem [4, VII.2.9], the sequence $\{F_j\}$ forms a normal family. Accordingly, there exists a subsequence $\{F_{j_k}\}$ that converges uniformly on compact subsets of V to a limit function $\overline{F}$. Since

$$F_j(\operatorname{Re}(z_j + \frac{1}{h} \log z_j)) \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

it follows that

$$\overline{F}(\overline{z}) = 0.$$

Since $\overline{F}$ does not vanish identically, Hurwitz's theorem [4, VII.2.5] implies that all but a finite number of the F_j must have a zero inside the disk $\{z : |z - \overline{z}| < \frac{\epsilon}{2}\}$. But this contradicts the fact that $|z_j - \lambda_k| > \epsilon$ for all j and k and the theorem is proved. $\square$

Using the lower bounds for $\det \Delta$, we have precise information about the growth of $|\Delta(z)^{-1}|$ and the density of the zeros of $\det \Delta$.

THEOREM 2.5. *Let $\Delta(z)$ be given by (2.2). Then there exist constants ϵ, C and K such that*

$$(2.8) \quad |\Delta(z)^{-1}| \leq K \min\left\{\frac{1}{|z|}, |e^{zh}|\right\}$$

for $|z| \geq C$ outside circles of radius ϵ centered around the zeros of $\det \Delta$.

PROOF. From the convolution properties of the Laplace transform, it follows that the entries of $\operatorname{adj} \Delta(z)$ can be represented by

$$(\operatorname{adj} \Delta(z))_{ij} = \sum_{k=0}^{n-1} z^{n-1-k} \int_0^{kh} e^{-z\tau} d\sigma_{ijk}(\tau),$$

where σ_{ijk} are suitable functions of bounded variation. For $|z| \geq C \max\{1, e^{-h\operatorname{Re} z}\}$, there is a constant M_0

$$(2.9) \quad |\operatorname{adj} \Delta(z)| \leq M_0 |z|^{n-1}.$$

So Proposition 2.1 yields for $|z| \geq C \max\{1, e^{-h\operatorname{Re} z}\}$

$$(2.10) \quad |\Delta(z)^{-1}| \leq 2M_0 \frac{1}{|z|}.$$

For $|z| \leq C \max\{1, e^{-h\operatorname{Re} z}\}$, there is a constant M_1

$$(2.11) \quad |\operatorname{adj} \Delta(z)| \leq M_1 |e^{-(n-1)h\operatorname{Re} z}|$$

and Propositions 2.2 and 2.4 yield

$$(2.12) \quad |\Delta(z)^{-1}| \leq 2 \frac{M_1}{m} |e^{h\operatorname{Re} z}|.$$

Therefore a combination of (2.10) and (2.12) proves the theorem. $\square$

THEOREM 2.6. *Let $\Delta(z)$ be given by (2.2) and let $B(a, r) \subset V$ be a disk with center a and radius r with $|a| \geq 2$ and $r \leq |a|/2$. If $\partial B(a, r)$ contains no zeros of $\det \Delta$, then the number of zeros inside $B(a, r)$ can be bounded independently of a and r .*

PROOF. The proof is based on the argument principle [4, V.3.4] and the estimates from Proposition 2.4. Since $\partial B(a, r)$ contains no zeros of $\det \Delta$, there is an $\epsilon > 0$ such that the distance of $\partial B(a, r)$ to the zero set of $\det \Delta$ is at least ϵ . Therefore we can use the estimates from Theorem 2.4.

Let $n(a, r)$ denote the number of zeros of $\det \Delta$ inside $B(a, r)$. Then the argument principle yields

$$(2.13) \quad n(a, r) = \frac{1}{2\pi i} \int_{\partial B(a, r)} \frac{\frac{d}{dz} \det \Delta(z)}{\det \Delta(z)} dz.$$

To estimate $n(a, r)$ we first give estimates for the integrand. Since

$$\frac{d}{dz} \det \Delta(z) = nz^{n-1} + \sum_{j=1}^n \int_0^{jh} e^{-zt}(-t) d\eta_j(t) z^{n-j} + \sum_{j=1}^n \int_0^{jh} e^{-zt} d\eta_j(t)(n-j)z^{n-1-j},$$

we have, for $z \in V$,

$$\begin{aligned} \left| \frac{d}{dz} \det \Delta(z) \right| &\leq (n + B_1^{-1} h \operatorname{TV}(\eta_1)) |z|^{n-1} + \sum_{j=2}^n |z|^{n-j} B_1^{-j} j h \operatorname{TV}(\eta_j) \\ &\quad + \sum_{j=2}^n |z|^{n-j} B_1^{-(j-1)} \operatorname{TV}(\eta_{j-1}). \end{aligned}$$

On the other hand, Theorem 5.6 implies that, for $z \in \partial B(a, r)$, there exists a positive constant m such that

$$|\det \Delta(z)| \geq \frac{1}{2} m |e^{-nz}| \geq \frac{1}{2} m C_0^n |z|^n,$$

where we have used $B(a, r) \subset V$. Therefore, for $z \in \Gamma$,

$$(2.14) \quad \left| \frac{\frac{d}{dz} \det \Delta(z)}{\det \Delta(z)} \right| \leq \frac{2C_0^n}{m|z|} \left((n + B_1^{-1} h \text{TV}(\eta_1)) + \sum_{j=2}^n |z|^{1-j} B_1^{-j} j h \text{TV}(\eta_j) \right. \\ \left. + \sum_{j=2}^n |z|^{1-j} B_1^{-(j-1)} \text{TV}(\eta_{j-1}) \right).$$

Since for $r \leq |a|/2$,

$$\frac{1}{2\pi} \left| \int_{\partial B(a, r)} \frac{1}{|z|} dz \right| = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|a + r e^{i\theta}|} r d\theta \leq \frac{r}{|a| - r} \leq 1$$

and for $z \in \partial B(a, r)$, with $r < |a|/2$,

$$\frac{1}{|z|} \leq \frac{2}{|a|},$$

it follows from (2.13)–(2.14) that, for $|a| \geq 2$,

$$n(a, r) \leq \frac{2C_0^n}{m} \left((n + B_1^{-1} h \text{TV}(\eta_1)) + \sum_{j=2}^n (B_1^{-j} j h \text{TV}(\eta_j) + B_1^{-(j-1)} \text{TV}(\eta_{j-1})) \right).$$

This completes the proof. □

In the neutral case we make the following restriction.

(H2) Let $D : \mathcal{C} \rightarrow \mathbb{R}^n$ be given by

$$D\varphi = \varphi(0) - D_0\varphi(h) - \sum_{j=1}^{\infty} D_j\varphi(h_j),$$

where for $j = 1, 2, \dots$, $h_j < h$ and D_0, D_j are matrices such that D_0 is nonsingular, and $|D_j| \rightarrow 0$ as $j \rightarrow \infty$.

If we assume (H2) the characteristic matrix associated with

$$\frac{d}{dt} D x_t = \int_0^h d\zeta(\theta) x(t - \theta)$$

becomes

$$(2.15) \quad \Delta(z) = z\Delta_0(z) - \int_0^h e^{-z\tau} d\zeta(\tau),$$

where

$$(2.16) \quad \Delta_0(z) = I - D_0 e^{-hz} - \sum_{j=1}^{\infty} D_j e^{-h_j z}.$$

An entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ is of exponential type if and only if

$$\limsup_{r \rightarrow \infty} \frac{1}{r} \log \left(\max_{|z|=r} |f(z)| \right) = E(f),$$

where $0 \leq E(f) < \infty$. In that case $E(f)$ is called the exponential type of f . It is clear that $\det \Delta$ is of exponential type and $E(\det \Delta) \leq nh$.

THEOREM 2.7. *Suppose that $\Delta(z)$ is given by (2.15)–(2.16). Then the zeros of $\det \Delta$ are contained in a strip $\alpha < \operatorname{Re} z < \beta$ and if $n(r)$ denotes the number of zeros of $\det \Delta$ with radius less than r , then*

$$(2.17) \quad n(r) = \frac{E(\det \Delta)}{\pi} r + \omega(r),$$

where ω is a bounded function. Furthermore, there exist constants ϵ, C and K such that

$$(2.18) \quad |\Delta(z)^{-1}| \leq \frac{K}{|z|} \min\{1, |e^{zh}|\}$$

for $|z| \geq C$ outside circles of radius ϵ centered around the zeros of $\det \Delta$.

PROOF. First observe that we have the following representation for $\det \Delta(z)$

$$\det \Delta(z) = z^n \det \Delta_0(z) + \sum_{j=1}^n z^{n-j} \int_0^{nh} e^{-z\tau} d\eta_j(\tau),$$

where η_j , $j = 1, \dots, n$, are functions of bounded variation. From (2.14) it follows that $\det \Delta_0$ is an almost periodic function. In particular, the general theory for almost periodic functions (see Appendix VI of Levin [12]) yields precise information about the zeros of $\det \Delta_0$. The zeros are contained in a horizontal strip $\alpha_0 < \operatorname{Re} z < \beta_0$, and if $n_0(r)$ denotes the number of zeros of $\det \Delta_0$ with radius less than r , then

$$n_0(r) = \frac{E(\det \Delta_0)}{\pi} r + \omega_0(r),$$

where ω_0 is a bounded function. Furthermore, it follows from (2.14) and (H2) that

$$\det \Delta_0(z) = 1 + (-1)^n \det D_0 e^{-nhz} + (-1)^n \int_0^{nh} e^{-z\tau} d\eta_0(\tau),$$

where η_0 is a discrete measure that is continuous at nh . Therefore, there exists an $\epsilon > 0$ and positive constants m, m_1 and M such that outside circles of radius ϵ centered around the zeros of $\det \Delta_0$ one has

$$m \max\{m_1, |e^{-nhz}|\} \leq |\det \Delta_0(z)| \leq M \min\{1, |e^{-nhz}|\}.$$

(The same proof as given to Propositions 2.2 and 2.4 applies.) The Rouché theorem, the estimates for $\det \Delta_0$ and the representation for $\det \Delta$ immediately imply that the zeros of $\det \Delta$ are also contained in a strip $\alpha < \operatorname{Re} z < \beta$ and satisfy (2.17). (Note that $E(\det \Delta_0) = E(\det \Delta)$.) Furthermore, there exist (modified) constants m, m_1 and M such that outside circles of radius ϵ centered around the zeros of $\det \Delta$

$$m|z|^n \max\{m_1, |e^{-nhz}|\} \leq |\det \Delta(z)| \leq M|z|^n \min\{1, |e^{-nhz}|\}.$$

From the representation for $\operatorname{adj} \Delta(z)$:

$$(\operatorname{adj} \Delta(z))_{ij} = \sum_{k=0}^{n-1} \int_0^{(n-1)h} e^{-z\tau} d\sigma_{ijk}(\tau) z^{n-1-k},$$

where σ_{ijk} are suitable functions of bounded variation, it follows that the estimates (2.9) and (2.11) are still valid. Therefore, a similar argument as given in the proof of Theorem 2.5 yields (2.18) and completes the proof. $\square$

Note that in case (H2) holds, the spectrum of the generator A given by (1.2) is contained in a strip $\alpha < \operatorname{Re} z < \beta$. In particular, this will imply (see [11]) that the generator generates a group of linear operators. This is not always the case for neutral equations. See Section 5 for more information.

3. SERIES EXPANSIONS FOR RETARDED EQUATIONS

In this section we shall consider the following retarded FDE

$$(3.1) \quad \begin{cases} \dot{x}(t) = \int_0^h d\zeta(\theta)x(t-\theta), & t \geq 0, \\ x_0 = \varphi, & \varphi \in \mathcal{C}. \end{cases}$$

If $T(t)$ denotes the semigroup associated with (3.1), then the infinitesimal generator A of $T(t)$ is given by (1.2) with $D\varphi = \varphi(0)$.

Let $\omega(A)$ denote the growth bound $T(t)$. Then for every $\epsilon > 0$ there is a constant M such that $\|T(t)\| \leq M e^{(\omega(A)+\epsilon)t}$. Therefore for $\operatorname{Re} z > \omega(A)$ and $\varphi \in \mathcal{C}$, the Laplace transform of $T(t)$ exists and it is known that it equals the resolvent of the generator

$$(3.2) \quad R(z, A)\varphi = \int_0^\infty e^{-zt} T(t)\varphi dt, \quad \operatorname{Re} z > \omega(A).$$

As explained before, the idea is to obtain an explicit representation for $T(t)\varphi$ itself using the inverse of the Laplace transform. The following inversion formula can be found in Hille-Phillips [9, Theorem 11.6.1].

THEOREM 3.1. *Let X be a complex Banach space and let $A(X \rightarrow X)$ be the generator of a C_0 -semigroup. Then, for $\gamma > \omega(A)$ and for every $\varphi \in \mathcal{D}(A)$,*

$$(3.3) \quad T(t)\varphi = \lim_{\omega \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{zt} R(z, A)\varphi dz, \quad t > 0.$$

The limit is uniform for $0 < t_0 < t < t_1 < \infty$ and for $t = 0$ the limit equals

$$(3.4) \quad \frac{1}{2}\varphi = \lim_{\omega \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} R(z, A)\varphi dz.$$

To prove the first main result Theorem 1.1, we shall use Theorem 3.1 and a suitable sequence of contours. Using Theorem 2.6 we can construct a sequence of real numbers ρ_l , such that $\rho_l \rightarrow \infty$ for $l \rightarrow \infty$, and a sequence of closed contours Γ_l , $l = 0, 1, \dots$, such that for some positive constants k, ϵ and δ :

- (i) Γ_l is contained in the interior of Γ_{l+1} and there are at most k zeros between Γ_l and Γ_{l+1} ;
- (ii) the contours have at least distance $\epsilon > 0$ from the set of zeros of $\det \Delta$;
- (iii) the contour Γ_l lies along the circle $|z| = \rho_l$ outside V . Inside V , it lies between the circle $|z| = \rho_l - \delta$ and the circle $|z| = \rho_l + \delta$. Here V is given by (2.6);
- (iv) the length of the portion of Γ_l within V is bounded as $l \rightarrow \infty$.

The part of the sequence of contours Γ_l contained in the left half plane $\operatorname{Re} z \leq \gamma$ will be denoted by Γ_l^- , $l = 0, 1, \dots$

Before we can prove Theorem 1.1 we need a simple observation. If $\lambda \in \sigma(A)$ then λ is a pole of $z \mapsto e^{zt}R(z, A)\varphi$ and

$$(3.5) \quad \operatorname{Res}_{z=\lambda} e^{zt}R(z, A)\varphi = T(t)P_\lambda\varphi,$$

where P_λ denotes the spectral projection onto $\mathcal{M}_\lambda$. (This identity follows from (3.3), (1.6) and the resolvent equation. See [5] for details.)

PROOF OF THEOREM 1.1. The Cauchy theorem implies that

$$(3.6) \quad T(t)\varphi = \lim_{l \rightarrow \infty} \left\{ \sum_{j=0}^{m(l)} T(t)P_{\lambda_j}\varphi - \frac{1}{2\pi i} \int_{\Gamma_l^-} e^{zt}R(z, A)\varphi dz \right\},$$

where $\lambda_0, \dots, \lambda_{m(l)}$ are the zeros of $\det \Delta$ inside the area enclosed by the line $\operatorname{Re} z = \gamma$ and the contour Γ_l^- .

So to prove the theorem, it suffices to prove that for every $\varphi \in \mathcal{D}(A)$

$$(3.7) \quad \lim_{l \rightarrow \infty} \left\| \frac{1}{2\pi i} \int_{\Gamma_l^-} e^{zt}R(z, A)\varphi dz \right\| = 0.$$

To analyse this limit, we shall use the following representation for the resolvent

$$(3.8) \quad R(z, A)\varphi = \Delta(z)^{-1}C(z)\varphi,$$

where

$$\begin{aligned} (C(z)\varphi)(\theta) &= \varphi(0)e^{z\theta} + z \int_0^{-\theta} e^{-zs}\varphi(s+\theta)ds - \int_0^{-\theta} e^{-zs} \int_0^s d\zeta_0(\tau)\varphi(s+\theta-\tau)ds \\ &\quad + \int_{-\theta}^h e^{-zs} \int_s^h d\zeta_0(\tau)\varphi(s+\theta-\tau)ds + L_0 \int_{-\theta}^h e^{-zs}\varphi(s+\theta-h)ds. \end{aligned}$$

Note that for $|z|$ sufficiently large

$$\|C(z)\varphi\| = \sup_{-h \leq \theta \leq 0} |(C(z)\varphi)(\theta)| \leq M|z| \max\{1, |e^{-hz}|\} \|\varphi\|.$$

Next we use the identity

$$R(z, A)\varphi = \frac{1}{z}R(z, A)A\varphi + \frac{1}{z}\varphi, \quad \varphi \in \mathcal{D}(A),$$

to conclude that, for $\varphi \in \mathcal{D}(A)$ and $|z|$ sufficiently large,

$$\|R(z, A)\varphi\| \leq M|z| \max\{1, |e^{-hz}|\} |\Delta(z)^{-1}| \|A\varphi\| + \frac{1}{|z|} \|\varphi\|.$$

Because of Theorem 2.5 and the definition of the contours Γ_l there exists a constant l_0 such that for $z \in \Gamma_l^-$, $l \geq l_0$, we have

$$(3.9) \quad \max\{1, |e^{-hz}|\} |\Delta(z)^{-1}| \leq K \min\left\{\frac{|e^{-zh}|}{|z|}, 1\right\}$$

and hence

$$(3.10) \quad \|R(z, A)\varphi\| \leq C_1 K \min\left\{\frac{|e^{-zh}|}{|z|}, 1\right\} \|A\varphi\| + \frac{1}{|z|} \|\varphi\|.$$

From Jordan's lemma, it follows that for $t > 0$

$$\lim_{l \rightarrow \infty} \left| \int_{\Gamma_l^-} e^{zt} \frac{C_2}{|z|} \|\varphi\| dz \right| = 0.$$

Hence to prove (3.7), it remains to show

$$(3.11) \quad \lim_{l \rightarrow \infty} \left| \int_{\Gamma_l^-} e^{zt} \min\left\{\frac{|e^{-hz}|}{|z|}, 1\right\} dz \right| = 0, \quad \text{for } t > 0.$$

First we prove that inside V the contour can be modified. Using (3.9), we find

$$\begin{aligned} |e^{zt}| \min\left\{\frac{|e^{-hz}|}{|z|}, 1\right\} &\leq K_1 e^{t \operatorname{Re} z} K_1 \leq K_1 e^{-th^{-1} \log(|z|C^{-1})} \\ &\leq K_1 e^{-th^{-1} \log((\rho_l - \delta)C^{-1})}. \end{aligned}$$

Here we have used that for $z \in V$ and $|z|$ sufficiently large, $\operatorname{Re} z \leq -h^{-1} \log(|z|C^{-1})$. So, within V , the integrand tends to zero as $l \rightarrow \infty$ uniformly if $t \geq t_0 > 0$. Since the lengths of the modified arcs inside V are bounded, we can replace the arcs within V by circular arcs.

The integral over Γ_l^- can accordingly be replaced by the integral over the full semi circle $|z| = \rho_l$, $\operatorname{Re} z \leq 0$ and the part of the semi circle in $0 < \operatorname{Re} z < \gamma$. Because of (2.8), the integral in (3.7) along the part of the semi circle in $0 < \operatorname{Re} z < \gamma$ approaches zero, uniformly in $0 < t_0 < t < t_1 < \infty$. Therefore it suffices to prove

$$(3.11) \quad \lim_{l \rightarrow \infty} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{t\rho_l \cos \theta} \min\{\rho_l^{-1} e^{-h\rho_l \cos \theta}, 1\} \rho_l d\theta = 0.$$

Choose θ_0 , $0 \leq \theta_0 \leq \frac{\pi}{2}$ such that

$$(3.12) \quad e^{\rho_l h \sin \theta_0} = \rho_l.$$

Then

$$\begin{aligned} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{t\rho_l \cos \theta} \min\{\rho_l^{-1} e^{-h\rho_l \cos \theta}, 1\} \rho_l d\theta &= 2 \int_0^{\frac{\pi}{2}} e^{-t\rho_l \sin \theta} \min\{e^{h\rho_l \sin \theta}, \rho_l\} d\theta \\ &= 2 \int_0^{\theta_0} e^{(h-t)\rho_l \sin \theta} d\theta + 2 \int_{\theta_0}^{\frac{\pi}{2}} e^{-t\rho_l \sin \theta} \rho_l d\theta. \end{aligned}$$

Note that for $t > h$, we have

$$\int_0^{\theta_0} e^{(h-t)\rho_l \sin \theta} d\theta \leq \theta_0$$

and that for $0 < t < h$, we have using Jordan's inequality ($\frac{2}{\pi}\theta \leq \sin \theta \leq \theta$, $0 \leq \theta \leq \frac{\pi}{2}$),

$$\int_0^{\theta_0} e^{(h-t)\rho_l \sin \theta} d\theta \leq \int_0^{\theta_0} e^{(h-t)\rho_l \theta} d\theta \leq \frac{e^{(h-t)\rho_l \theta_0} - 1}{\rho_l(h-t)}.$$

Also, using Jordan's inequality, we have

$$\int_{\theta_0}^{\frac{\pi}{2}} e^{-t\rho_l \sin \theta} \rho_l d\theta \leq \int_{\theta_0}^{\frac{\pi}{2}} e^{-t\rho_l \frac{2}{\pi}\theta} \rho_l d\theta \leq \frac{\pi}{2t} [e^{-t\rho_l} - e^{-t\rho_l \frac{2}{\pi}\theta_0}].$$

From (3.12) it follows that

$$\theta_0 = \frac{1}{h} \frac{\log \rho_l}{\rho_l} (1 + o(1)) \quad \text{as } l \rightarrow \infty.$$

Together we have that for $t > 0$

$$\lim_{l \rightarrow \infty} \int_0^{\theta_0} e^{(h-t)\rho_l \sin \theta} d\theta = 0, \quad \lim_{l \rightarrow \infty} \int_{\theta_0}^{\frac{\pi}{2}} e^{-t\rho_l \sin \theta} \rho_l d\theta = 0.$$

This proves (3.11) and hence the theorem. □

Since $\text{Im } T(t) \subset \mathcal{D}(A)$ for $t > h$, we have the following corollary.

COROLLARY 3.2. *Suppose that the assumptions from Theorem 1.1 hold.*

Then the solution $x(\cdot; \varphi)$ of RFDE (3.1) has a convergent series expansion

$$x(t) = \sum_{j=0}^{\infty} e^{\lambda_j t} p_j(t), \quad t > 0$$

and

$$(3.13) \quad e^{\lambda_j t} p_j(t) = \operatorname{Res}_{z=\lambda_j} e^{zt} \Delta(z)^{-1} [\varphi(0) + L_0 \int_0^h e^{-z\sigma} \varphi(\sigma - \tau) d\sigma \\ + \int_0^h d\zeta_0(\tau) \int_0^\tau e^{-z\sigma} \varphi(\sigma - \tau) d\sigma].$$

The convergence is uniform for $0 < t_0 < t < t_1 < \infty$.

PROOF. First note that for $t > \epsilon > 0$

$$\begin{aligned} x(t; \varphi) &= (T(t+h)\varphi)(-h) \\ &= (T(t-\epsilon)T(h+\epsilon)\varphi)(-h). \end{aligned}$$

Since $T(h+\epsilon)\varphi \in \mathcal{D}(A)$ for every $\varphi \in \mathcal{C}$, we can apply Theorem 1.1 with respect to $\psi = T(h+\epsilon)\varphi$. Hence

$$\begin{aligned} x(t; \varphi) &= (T(t-\epsilon)\psi)(-h) = \sum_{j=0}^{\infty} (T(t-\epsilon)P_{\lambda_j}\psi)(-h) \\ &= \sum_{j=0}^{\infty} (T(t-\epsilon)P_{\lambda_j}T(h+\epsilon)\varphi)(-h) = \sum_{j=0}^{\infty} (T(t+h)P_{\lambda_j}\varphi)(-h) \\ &= \sum_{j=0}^{\infty} \operatorname{Res}_{z=\lambda_j} e^{z(t+h)} (R(z, A)\varphi)(-h), \end{aligned}$$

where we have used (3.5). Now (3.13) follows from (1.3)–(1.4). $\square$

EXAMPLE 3.3. If a is an integrable function and $b \neq 0$ then every solution of

$$\dot{x}(t) = \int_0^1 a(s)x(t-s) ds + bx(t-1)$$

has a convergent series expansion for $t > 0$ given by

$$x(t) = \sum_{j=0}^{\infty} e^{\lambda_j t} p_j(t),$$

where λ_j , $j = 0, \dots$, denote the roots of

$$z - \int_0^1 e^{-zs} a(s) ds - be^{-z} = 0$$

and

$$e^{\lambda_j t} p_j(t) = \operatorname{Res}_{z=\lambda_j} e^{zt} \frac{\varphi(0) + \int_0^1 a(\tau) \int_0^\tau e^{-z\sigma} \varphi(\sigma - \tau) d\sigma + b \int_0^1 e^{-z\sigma} \varphi(\sigma - \tau) d\sigma}{z - \int_0^1 e^{-zs} a(s) ds - be^{-z}}.$$

Furthermore, for every $\varphi \in \mathcal{C}$ such that φ is continuously differentiable and satisfies the boundary condition

$$\dot{\varphi}(0) = \int_0^1 a(s)\varphi(-s) ds + b\varphi(-1),$$

we have

$$x(t) = \sum_{j=0}^{\infty} e^{\lambda_j t} p_j(t), \quad t > -1,$$

where, as usual, $x(t) = \varphi(t)$, $-1 \leq t \leq 0$.

4. SERIES EXPANSIONS FOR NEUTRAL EQUATIONS

In this section we consider FDE (1.1) when $D\varphi \neq \varphi(0)$. In general, when $D\varphi \neq \varphi(0)$, the semigroup $T(t)$ associated with FDE (1.1) is no longer differentiable after finite time, i.e., $\text{Im} T(t) \not\subset \mathcal{D}(A)$ for t larger than some number t_0 . So the condition $\varphi \in \mathcal{D}(A)$ in Theorem 1.1 is not natural. Since the inversion formula in Theorem 3.1 only holds for $\varphi \in \mathcal{D}(A)$, we first give a weaker inversion formula. To formulate the result, we have to introduce the notion of the Cesàro mean of integrals

$$\int_{h-il}^{h+il} f(z) dz$$

which by definition, is given by

$$\begin{aligned} (C, 1)\text{-} \int_{h-i\infty}^{h+i\infty} f(z) dz &= \lim_{N \rightarrow \infty} i \int_{-N}^N f(h+i\nu) \left(1 - \frac{|\nu|}{N}\right) d\nu \\ &= \lim_{N \rightarrow \infty} \frac{i}{N} \int_0^N \left(\int_{-l}^l f(h+i\nu) d\nu \right) dl. \end{aligned}$$

THEOREM 4.1. *Let X be a complex Banach space and let $A(X \rightarrow X)$ be the generator of a C_0 -semigroup. Then for $\gamma > \omega(A)$ and for every $\varphi \in X$*

$$(4.1a) \quad T(t)\varphi = \frac{1}{2\pi i} (C, 1)\text{-} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt} R(z, A) \varphi dz, \quad t > 0.$$

The limit is uniform for $0 < t_0 < t < t_1 < \infty$ and for $t = 0$ the limit equals

$$(4.1b) \quad \frac{1}{2} \varphi = \frac{1}{2\pi i} (C, 1)\text{-} \int_{\gamma-i\infty}^{\gamma+i\infty} R(z, A) \varphi dz, \quad \gamma > \omega(A).$$

To prove the theorem, apply a continuous linear functional to both sides of (4.1), use the scalar version of (4.1) from Widder [19], Theorem II.9.2 and apply the Hahn-Banach theorem.

We start with a lemma that relates the Cesàro summation of infinite integrals to the Cesàro summation of infinite sums.

LEMMA 4.2. *Let A denote the infinitesimal generator corresponding to FDE (1.1). If (H2) holds, then*

$$(4.2) \quad \lim_{R \rightarrow \infty} \frac{1}{R} \int_0^R \left(\sum_{|\operatorname{Im} \lambda_j| \leq l} \operatorname{Res}_{z=\lambda_j} R(z, A) \varphi \right) dl = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^n \operatorname{Res}_{z=\lambda_j} R(z, A) \varphi.$$

Here λ_j with $|\operatorname{Im} \lambda_0| \leq |\operatorname{Im} \lambda_1| \leq |\operatorname{Im} \lambda_2| \leq \dots$ denote the zeros of $\det \Delta$ (ordered according to increasing imaginary part).

PROOF. From representation (1.3)–(1.4) for the resolvent of A , it follows that the set of possible poles of $R(z, A)$ coincides with the set of zeros of $\det \Delta$. Let

$$n(l) := \#\{\lambda : |\operatorname{Im} \lambda| \leq l, \det \Delta(\lambda) = 0\}.$$

Then Theorem 2.7 implies

$$n(l) = \frac{E(\det \Delta)}{\pi} l + \omega(l),$$

where ω is a bounded function. Set

$$b = \frac{E(\det \Delta)}{\pi}.$$

If $b = 0$, then the number of residues is finite and there is nothing to prove. So in the remaining part of the proof we assume $b > 0$. Define

$$\begin{aligned} I(R) &:= \frac{1}{R} \int_0^R \left(\sum_{|\operatorname{Im} \lambda_j| \leq l} \operatorname{Res}_{z=\lambda_j} R(z, A) \varphi \right) dl \\ &= \frac{1}{R} \int_0^R \left(\sum_{j=0}^{n(l)-1} \operatorname{Res}_{z=\lambda_j} R(z, A) \varphi \right) dl \\ &= \frac{1}{R} \int_0^R \left(\sum_{j=0}^{bl-1} \operatorname{Res}_{z=\lambda_j} R(z, A) \varphi \right) dl + M(R), \end{aligned}$$

where

$$M(R) := \frac{1}{R} \int_0^R \left(\sum_{j=0}^{\omega(l)-1} \operatorname{Res}_{z=\lambda_{bl+j}} R(z, A) \varphi \right) dl.$$

First note that

$$\lim_{R \rightarrow \infty} \frac{1}{R} \int_0^R \left(\sum_{j=0}^{bl-1} \operatorname{Res}_{z=\lambda_j} R(z, A) \varphi \right) dl = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^n \operatorname{Res}_{z=\lambda_k} R(z, A) \varphi.$$

Hence to prove the lemma, it remains to show that

$$(4.3) \quad \lim_{R \rightarrow \infty} \|M(R)\| = 0.$$

The proof of (4.3) will be given in two steps. First we prove that, if, for every $\varphi \in \mathcal{C}$,

$$(4.4) \quad \lim_{j \rightarrow \infty} \|\operatorname{Res}_{z=\lambda_j} R(z, A) \varphi\| = 0,$$

then (4.3) follows. In the second step we use the representation of the resolvent of A to prove (4.4).

Fix φ and assume that (4.4) holds. Then for all positive ϵ , there exists a positive integer j_0 such that

$$\|\operatorname{Res}_{z=\lambda_j} R(z, A) \varphi\| < \frac{\epsilon}{4} \quad \text{for all } j \geq j_0.$$

Next we choose R_0 such that, for all integers $j \geq j_0$, we have $|\operatorname{Im} \lambda_j| > R_0$. Then, there exists a positive integers R_1, R_2 such that

$$M_1(R) := \left\| \frac{1}{R} \int_0^{R_0} \left(\sum_{j=0}^{\omega(l)-1} \operatorname{Res}_{z=\lambda_{bl+j}} R(z, A) \varphi \right) dl \right\| < \frac{\epsilon}{2} \quad \text{for all } R \geq R_1,$$

and

$$M_2(R) := \left\| \frac{1}{R} \int_{R_0}^R \left(\sum_{j=0}^{\omega(l)-1} \operatorname{Res}_{z=\lambda_{bl+j}} R(z, A) \varphi \right) dl \right\| < \frac{\epsilon}{2} \quad \text{for all } R \geq R_2.$$

So for $R > \max\{R_0, R_1, R_2\}$ we obtain

$$\|M(R)\| \leq \|M_1(R)\| + \|M_2(R)\| < \epsilon.$$

This shows (4.3) and it remains to prove (4.4). In order to do so we first rewrite the representation for the resolvent of A . A straightforward calculation yields the following representation

$$R(z, A)\varphi = \Delta(z)^{-1}C(z)\varphi,$$

where

$$(4.5) \quad \begin{aligned} (C(z)\varphi)(\theta) &= D\varphi e^{zh} + z \int_0^{-\theta} e^{-z\sigma} \varphi(\sigma + \theta) d\sigma \\ &\quad - \int_0^{-\theta} e^{-zs} \int_0^s [zd\mu(\tau) + d\zeta(\tau)] \varphi(s + \theta - \tau) ds \\ &\quad + \int_{-\theta}^h e^{-zs} \int_s^h [zd\mu(\tau) + d\zeta(\tau)] \varphi(s + \theta - \tau) ds. \end{aligned}$$

Note that from the Riemann-Lebesgue lemma

$$(4.6) \quad \frac{\|C(z)\varphi\|}{|z|} \rightarrow 0 \quad \text{as } |\operatorname{Im} z| \rightarrow \infty.$$

Let C_λ denote a small circle with radius r_0 surrounding only λ of the roots of $\det \Delta$. From (4.6) and the estimates for $|\Delta(z)^{-1}|$ given in Theorem 2.7, it follows that for all positive δ there are an integer N_0 and a positive number ϵ such that, if $C_\lambda \subset \{z \mid |\operatorname{Im} z| > N_0\}$ and $r_0 \geq \epsilon$, then

$$\max_{z \in C_\lambda} \|R(z, A)\varphi\| < \frac{\delta}{r_0}.$$

Let j_0 be chosen such that $|\operatorname{Im} \lambda_j| > N_0$ for $j \geq j_0$. Then for $j \geq j_0$, we obtain

$$\| \operatorname{Res}_{z=\lambda_j} R(z, A)\varphi \| \leq \frac{1}{2\pi} \left\| \int_{C_{\lambda_j}} R(z, A)\varphi dz \right\| \leq r_0 \max_{z \in C_\lambda} \|R(z, A)\varphi\| < \delta.$$

Therefore (4.4) holds and the proof of the lemma is complete. $\square$

Armed with Theorem 4.1 and Lemma 4.2 we can prove Theorem 1.2

PROOF OF THEOREM 1.2. Hypothesis (H2) implies that the spectrum of A is contained in a strip $\rho < \operatorname{Re} z < \gamma$. Now Theorem 3.1 and the proof of Theorem 6.1 of [11] yield that $A(\mathcal{C} \rightarrow \mathcal{C})$ is the generator of a group of linear operators $\{T(t)\}_{t \in \mathbb{R}}$ and the backward extension to $-\infty$ is given by

$$(4.7) \quad T(-t)\varphi = -\frac{1}{2\pi i} (C, 1) \cdot \int_{\rho-i\infty}^{\rho+i\infty} e^{-zt} R(z, A)\varphi dz, \quad t > 0.$$

Furthermore,

$$(4.8) \quad \frac{1}{2\pi i}(C, 1) \cdot \int_{\rho-i\infty}^{\rho+i\infty} R(z, A)\varphi dz = -\frac{1}{2}\varphi.$$

From the Laplace inversion formula given in Theorem 4.1, we find for $t > 0$

$$(4.9) \quad T(t)\varphi = \frac{1}{2\pi i}(C, 1) \cdot \int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt} R(z, A)\varphi dz$$

and for $t = 0$

$$(4.10) \quad \frac{1}{2\pi i}(C, 1) \cdot \int_{\gamma-i\infty}^{\gamma+i\infty} R(z, A)\varphi dz = \frac{1}{2}\varphi.$$

We shall first prove (1.10) for $t = 0$. From (4.8) and (4.10) it follows that

$$\varphi = \frac{1}{2\pi i}(C, 1) \cdot \int_{\gamma-i\infty}^{\gamma+i\infty} R(z, A)\varphi dz - \frac{1}{2\pi i}(C, 1) \cdot \int_{\rho-i\infty}^{\rho+i\infty} R(z, A)\varphi dz.$$

Next we would like to write the right hand side as a Cesàro integral of a closed contour.

From (4.5), (4.6) and the estimates for $|\Delta(z)^{-1}|$ given in Theorem 2.7, it follows that there is an $\epsilon > 0$ such that for every l , there is a continuous function $\nu_l : [\rho, \gamma] \rightarrow \mathbb{R}$ with $\nu_l(\rho) = \nu_l(\gamma) = l$, $\max_{\rho < \omega < \gamma} |\nu_l(\omega) - l| < \epsilon$, the portion of ν_l that is different from l has finite length and

$$(4.11) \quad \max_{\rho < \omega < \gamma} |R(\omega \pm i\nu_l(\omega), A)\varphi| \rightarrow 0 \quad \text{as } l \rightarrow \infty.$$

This implies

$$\lim_{N \rightarrow \infty} \left| \frac{1}{2\pi i N} \int_0^N \left(\int_{\rho}^{\gamma} R(\omega \pm i\nu_l(\omega))\varphi d\omega \right) dl \right| = 0,$$

and we can write

$$\varphi = \lim_{N \rightarrow \infty} \frac{1}{2\pi i N} \int_0^N \left(\int_{\Pi_l} R(z, A)\varphi dz \right) dl,$$

where Π_l is the contour given by

$$\begin{cases} z = \omega + i\nu_l(\omega), & \gamma > \omega > \rho, \\ z = \rho + i\lambda, & l > \lambda > -l, \\ z = \omega - i\nu_l(\omega), & \rho < \omega < \gamma, \\ z = \gamma + i\lambda, & -l < \lambda < l, \end{cases}$$

Since the residues of $R(z, A)$ are precisely the spectral projections corresponding to A , it follows from Lemma 4.2 that (1.10) holds for $t = 0$.

The proofs of representation (1.10) for $t > 0$ and $t < 0$ are very similar. We shall prove the representation for $t < 0$. First we use (4.7) to claim that

$$\begin{aligned} T(-t)\varphi &= \frac{1}{2\pi i}(C, 1) \cdot \int_{\rho-i\infty}^{\rho+i\infty} e^{-zt} R(z, A)\varphi dz \\ &= \frac{1}{2\pi i} \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N \left(\int_{\Pi_l} e^{-zt} R(z, A)\varphi dz \right) dl, \quad t > 0. \end{aligned}$$

To prove this claim, we have to verify that

$$\lim_{N \rightarrow \infty} \left| \frac{1}{2\pi i} \frac{1}{N} \int_0^N \left(\int_{\rho}^{\gamma} e^{-(\omega+il)t} R(\omega \pm i\nu_l(\omega), A)\varphi d\omega \right) dl \right| = 0$$

and

$$\frac{1}{2\pi i}(C, 1) \cdot \int_{\gamma-i\infty}^{\gamma+i\infty} e^{-zt} R(z, A)\varphi dz = 0.$$

The first identity follows from (4.11). To prove the second identity, it suffices to show

$$(4.12) \quad \lim_{N \rightarrow \infty} \left| \int_0^N \left(\int_{C_l} e^{-zt} R(z, A)\varphi dz \right) dl \right| = 0,$$

where C_l is the arc of the circle of radius l in the right half plane $\operatorname{Re} z > \gamma$. As $\|R(z, A)\varphi\| \rightarrow 0$, for $|z|$ tending to infinity in the half plane $\operatorname{Re} z \geq \gamma$, the limit in (4.12) follows from Jordan's lemma.

Therefore,

$$\begin{aligned} T(-t)\varphi &= \frac{1}{2\pi i} \lim_{N \rightarrow \infty} \frac{1}{N} \int_0^N \left(\int_{\Pi_l} e^{-zt} R(z, A)\varphi dz \right) dl \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^k \operatorname{Res}_{z=\lambda_j} e^{-zt} R(z, A)\varphi \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^k T(t)P_{\lambda_j}\varphi. \end{aligned}$$

This completes the proof. $\square$

COROLLARY 4.4. *Suppose that the assumptions in Theorem 4.3 hold.*

Then the solution $x(\cdot; \varphi)$ of FDE (1.1) has a convergent Cesàro series expansion

$$x(t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^k e^{\lambda_j t} p_j(t), \quad t \in \mathbb{R}$$

and

$$(4.13) \quad e^{\lambda_j t} p_j(t) = \operatorname{Res}_{z=\lambda_j} e^{zt} \Delta(z)^{-1} [D\varphi + \int_0^h [z d\mu(\tau) + d\zeta(\tau)] \int_0^\tau e^{-z\sigma} \varphi(\sigma - \tau) d\sigma].$$

The convergence is uniform for $-\infty < t_0 < t < t_1 < \infty$.

EXAMPLE 4.5. Every solution of

$$\dot{x}(t) - \dot{x}(t-1) = x(t-1)$$

has a convergent Cesàro sum expansion

$$x(t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^k e^{\lambda_j t} p_j, \quad t \in \mathbb{R},$$

where λ_j is a root of

$$z(1 - e^{-z}) - e^{-z} = 0$$

and

$$p_j = \lim_{z \rightarrow \lambda_j} (z - \lambda_j) \frac{\varphi(0) - \varphi(-1) + (z+1) \int_0^1 e^{-z\sigma} \varphi(\sigma - \tau) d\sigma}{z(1 - e^{-z}) - e^{-z}}.$$

5. SERIES EXPANSIONS FOR NEUTRAL EQUATIONS CONTINUED

In this section we discuss some generalizations of the results presented in Sections 3 and 4. From the proofs of Theorem 1.1 and Theorem 1.2 it is clear that all we need are good estimates for the resolvent of A . The objective of the following results is to formalize these observations.

THEOREM 5.1. *Let $A(X \rightarrow X)$ be the generator of a C_0 -semigroup on a complex Banach space X such that for $\varphi \in X$, the resolvent $z \mapsto R(z, A)\varphi$ is a meromorphic function. Suppose that there exists a sequence of real numbers ρ_l such that $\rho_l \rightarrow \infty$ for $l \rightarrow \infty$ and a sequence of closed contours Γ_l , $l = 0, 1, \dots$, such that for some positive constants k, ϵ and δ :*

- (i) Γ_l is contained in the interior of Γ_{l+1} and there are at most k zeros between Γ_l and Γ_{l+1} ;
- (ii) the contours have at least distance $\epsilon > 0$ from the set of zeros of $\det \Delta$;

- (iii) there is a set V such that the contour Γ_l lies along the circle $|z| = \rho_l$ outside V . Inside V , it lies between the circle $|z| = \rho_l - \delta$ and the circle $|z| = \rho_l + \delta$;
- (iv) the length of the portion of Γ_l within V is bounded as $l \rightarrow \infty$.

If the resolvent of A satisfies the following estimate

$$(5.1) \quad \|R(z, A)\varphi\| \leq C|z|^m \min\left\{\frac{|e^{-zh}|}{|z|}, 1\right\}, \quad \text{for } z \in \Gamma_l$$

for some integer m and constant C , then, for $\varphi \in \mathcal{D}(A^m)$, we have

$$T(t)\varphi = \sum_{j=0}^{\infty} T(t)P_{\lambda_j}\varphi, \quad t > 0,$$

where λ_j denote the eigenvalues of A ordered according to increasing modulus.

PROOF. The proof follows the same lines as the proof of Theorem 1.1. Choose γ such that $\gamma > \omega(A)$ and let Γ_l^- denote the arc of Γ_l contained in the left half plane $\operatorname{Re} z < \gamma$. It suffices to prove that for $\varphi \in \mathcal{D}(A^m)$

$$\lim_{l \rightarrow \infty} \left\| \frac{1}{2\pi i} \int_{\Gamma_l^-} e^{zt} R(z, A)\varphi dz \right\| = 0.$$

For $\varphi \in \mathcal{D}(A^m)$, we have

$$R(z, A)\varphi = \frac{1}{z^m} R(z, A)A^m\varphi + \sum_{j=1}^m \frac{1}{z^j} A^{j-1}\varphi.$$

From Jordan's lemma it follows that

$$\lim_{l \rightarrow \infty} \left\| \frac{1}{2\pi i} \int_{\Gamma_l^-} e^{zt} \sum_{j=1}^m \frac{1}{z^j} A^{j-1}\varphi dz \right\| = 0, \quad \text{for } t > 0.$$

Moreover, (5.1) implies that

$$\lim_{l \rightarrow \infty} \left\| \frac{1}{2\pi i} \int_{\Gamma_l^-} \frac{e^{zt}}{z^m} R(z, A)A^m\varphi dz \right\| \leq \lim_{l \rightarrow \infty} \frac{1}{2\pi i} \int_{\Gamma_l^-} |e^{zt} \min\left\{\frac{|e^{-zh}|}{|z|}, 1\right\}| dz| = 0,$$

where we have used (3.11). □

It is also not difficult to state an abstract version of Theorem 1.2.

THEOREM 5.2. *Let $A(X \rightarrow X)$ be the generator of a group of bounded linear operators on a complex Banach space X such that for every $\varphi \in X$ the resolvent $z \mapsto R(z, A)\varphi$ is a meromorphic function such that:*

- (i) If $\rho < \operatorname{Re} z < \gamma$ denotes the strip in which the poles $\mu_j + i\nu_j$, $j = 0, \dots$, of $R(z, A)\varphi$ are located, then there exists an $\epsilon > 0$ such that, for every l , there is a continuous function $\nu_l : [\rho, \gamma] \rightarrow \mathbb{R}$ with $\nu_l(\rho) = \nu_l(\gamma) = l$, $\max_{\rho < \omega < \gamma} |\nu_l(\omega) - l| < \epsilon$, the portion of ν_l that is different from l has finite length and

$$\max_{\rho < \omega < \gamma} |R(\omega \pm i\nu_l(\omega), A)\varphi| \rightarrow 0 \quad \text{as } l \rightarrow \infty.$$

- (ii) If $n(l)$ denotes the number of eigenvalues of radius less than l , then there is a strictly positive constant b and a bounded function ω such that

$$n(l) = bl + \omega(l).$$

- (iii) For $\operatorname{Re} z \geq \gamma$

$$\|R(z, A)\varphi\| \rightarrow 0 \quad \text{as } |z| \rightarrow \infty.$$

Then

$$T_-(t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^k T(t) P_{\lambda_j} \varphi, \quad t \in \mathbb{R},$$

where λ_j denote the eigenvalues of A ordered to increasing imaginary part.

The proof follows precisely the same lines as the proof of Theorem 1.2 and is omitted.

As an illustration of these generalizations we shall discuss general FDE. Let A be given by (1.2). If we assume (H1) the system of eigenvectors and generalized eigenvectors of A is complete (see [18]), but the spectrum of A is no longer contained in a strip. In the remaining part of this section we show that we can use the results on hyperbolic semigroups to reduce this more general situation to Theorem 5.1 and Theorem 5.2.

We shall consider the following FDE

$$(5.2) \quad \frac{d}{dt} [x(t) - D_0 x(t-h) - \sum_{j=1}^{\infty} D_j x(t-h_j)] = L_0 x(t-h) + \int_0^h d\zeta_0(\theta) x(t-\theta),$$

where for $j = 1, 2, \dots$, $h_j < h$ and L_0, D_0, D_j are $n \times n$ matrices such that L_0 is nonsingular, $|D_j| \rightarrow 0$ as $j \rightarrow \infty$ (D_0 will now be singular). Furthermore, ζ_0 is an $n \times n$ matrix function of bounded variation which is continuous at h .

We can split the (point) spectrum of A into two parts, one infinite part is contained in a strip $\alpha < \operatorname{Re} z < \beta$, the other infinite part is contained in a half plane $\operatorname{Re} z < \alpha_0 < \alpha$ and has the property that there are finitely many eigenvalues in every strip $\gamma < \operatorname{Re} z < \alpha_0$, $\gamma < \alpha_0$. According to the spectrum in the strip we can make a spectral decomposition (see [11] or [8])

$$\mathcal{C} = \mathcal{C}_- \oplus \mathcal{C}_+$$

with corresponding projection P onto $\mathcal{C}_-$ along $\mathcal{C}_+$, such that the spectrum of the part of A in $\mathcal{C}_-$, $A_- := PA : \mathcal{D}(A) \cap \mathcal{C}_- \rightarrow \mathcal{C}_-$, is given by $\sigma(A) \cap \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, and the spectrum of A_+ , the part of A in $\mathcal{C}_+$, is given by $\sigma(A) \cap \{z \in \mathbb{C} : \operatorname{Re} z < \alpha_0\}$. The spaces $\mathcal{C}_-$ and $\mathcal{C}_+$ are invariant under the semigroup generated by, respectively, A_- and A_+ . Furthermore, the semigroup $T_-(t)$ generated by A_- extends to a group of bounded operators, and the semigroup $T_+(t)$ is differentiable for t sufficiently large (use [15,; Theorem 4.7 of Chapter 2] and estimates for $\det \Delta$ similar to those given in Section 2). The resolvents of $R(z, A_-)$ and $R(z, A_+)$ correspond to the parts of $R(z, A)$ to, respectively, $\mathcal{C}_-$ and $\mathcal{C}_+$.

It now becomes clear how to continue. We can apply Theorem 5.2 to the generator A_- defined on $\mathcal{C}_-$. For the generator A_+ defined on $\mathcal{C}_+$ we can prove similar but weaker estimates for $|\Delta(z)^{-1}|$. To be precise, using the techniques presented in Section 2, we can prove

$$(5.3) \quad \|R(z, A_+)\varphi\| \leq C|z|^m \min\left\{\frac{|e^{-zh}|}{|z|}, 1\right\} \|\varphi\|,$$

where m is an integer less than or equal to n . Therefore we can apply Theorem 5.1 to the generator A_+ on $\mathcal{C}_+$.

We summarize the results.

THEOREM 5.3. *Let $\mathcal{C} = C([-h, 0], \mathbb{R}^n)$ and let $T(t)$ denote the semigroup associated with*

$$(5.4) \quad \frac{d}{dt}[x(t) - D_0 x(t-h) - \sum_{j=1}^{\infty} D_j x(t-h_j)] = L_0 x(t-h) + \int_0^h d\zeta_0(\theta) x(t-\theta),$$

where for $j = 1, 2, \dots$, $h_j < h$ and L_0, D_0, D_j are $n \times n$ matrices such that L_0 is nonsingular, $|D_j| \rightarrow 0$ as $j \rightarrow \infty$. Furthermore, ζ_0 is an $n \times n$ matrix function of bounded variation which

is continuous at h . If $A(\mathcal{C} \rightarrow \mathcal{C})$ denotes the generator of the semigroup $T(t)$ associated with (5.4), then one can decompose the space

$$\mathcal{C} = \mathcal{C}_- \oplus \mathcal{C}_+$$

into invariant subspaces, such that $A_- : \mathcal{D}(A) \cap \mathcal{C}_-$ has $\sigma(A_-) = \sigma_p(A_-) \subset \{z \in \mathbb{C} : \operatorname{Re} z > \alpha\}$ and $A_+ : \mathcal{D}(A) \cap \mathcal{C}_+$ has $\sigma(A_+) \subset \{z \in \mathbb{C} : \operatorname{Re} z < \alpha_0\}$. Here $\alpha_0 < \alpha$ is such that the intersection of $\sigma(A_+)$ with a strip is finite. The part of $T(t)$ in $\mathcal{C}_-$ extends to a group and

$$(5.5) \quad T_-(t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^n T(t) P_{\mu_k} \varphi, \quad \varphi \in \mathcal{C}_-, t \in \mathbb{R},$$

where μ_k , $k = 0, 1, \dots$, denote the eigenvalues of A in the right half plane $\operatorname{Re} z > \alpha$. The part of $T(t)$ in $\mathcal{C}_+$, $T_+(t)$ is differentiable for $t > t_0$, where t_0 is sufficiently large and for $\varphi \in \mathcal{C}_+$

$$(5.6) \quad T_+(t)\varphi = \sum_{j=0}^{\infty} T_+(t) P_{\lambda_j} \varphi, \quad t > t_0,$$

or, for $\varphi \in \mathcal{D}(A^n)$,

$$(5.7) \quad T_+(t)\varphi = \sum_{j=0}^{\infty} T_+(t) P_{\lambda_j} \varphi, \quad t > 0,$$

where λ_j , $j = 0, \dots$, denote the eigenvalues of A in the left half plane $\operatorname{Re} z < \alpha_0$.

REMARK 5.4. The eigenvectors and generalized eigenvectors of A can be computed from the Jordan chains of the characteristic matrix $\Delta(z)$ (see [10]).

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