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Driven impurity in an ultracold one-dimensional Bose gas with intermediate interaction strength

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We study a single impurity driven by a constant force through a one-dimensional Bose gas using a Lieb-Liniger golden rule with exact transition rates to model the scattering of the impurity on the 1D gas. In contrast to earlier theoretical work [15,20] which found agreement in the strong interaction limit with Tonks-Girardeau modeling, our results quantitatively describe the experimental data for small values of the dimensionless interaction strength $\gamma \sim 1$–3, close to the Bogoliubov limit [24], in keeping with the experimental parameter range (see Table I).

I. INTRODUCTION

The study of quantum mechanical systems out of equilibrium is one of the great frontiers of modern physics. The questions in this field are not only of fundamental interest, but are also of interest to future quantum technologies, as well as to classical technologies on the nanoscale. Cold atomic systems have provided an ideal setting for hand-in-hand theoretical and experimental investigations of this frontier, particularly in low dimensions. Nonetheless, our understanding of the issues involved are sufficiently primitive that it remains useful to consider some of the simplest toy model experiments in order to gain intuition regarding more general questions.

In this paper we focus on the study of driven impurities moving through a one-dimensional (1D) Bose gas. This subject has received much attention of late, thanks to both experimental [1–5] and theoretical progress [6–20]. Our work was inspired by the experimental results in Ref. [1], where the impurity is driven through the gas by a constant force (gravity). This type of experiment has not been extensively investigated at the theoretical level, with the exception of the recent work in Ref. [15] (see also Ref. [20]) which uses a Tonks-gas description [21] (appropriate only in the limit of large interaction strength). We also refer the reader to Ref. [14], where a similar system was studied in presence of a 1D optical lattice.

In this work, we use linear response theory and Fermi’s golden rule with exact transition rates to model the scattering between the driven impurity and the underlying gas (see, e.g., Refs. [1,14,22]). We then model the motion of the impurity as a classical driven stochastic process [23]. Our approach is strictly valid in the limit where the interaction between the impurity and the underlying gas is sufficiently weak (which is not necessarily true in the experiments of Ref. [1]). However, in contrast to prior works attempting to analyze this problem, our approach is valid for any interaction strength between particles in the 1D gas. The main point of this work is to provide a method to analyze the effects of interaction within the 1D gas on a driven impurity—which has previously not been possible for intermediate interaction strength within the gas.

II. METHOD

We find quantitative agreement with the results in Ref. [1] both comparing the center of mass motion as well as the optical density profile of a packet of impurities after they leave the 1D gas. In contrast to earlier theoretical work [15,20] which found agreement in the strong interaction limit with Tonks-Girardeau modeling, our results quantitatively describe the experimental data for small values of the dimensionless interaction strength $\gamma \sim 1$–3, close to the Bogoliubov limit [24], in keeping with the experimental parameter range (see Table I).

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with \(m_{im}\) the impurity mass, \(H_{impurity}\) the Hamiltonian of the impurity, \(H_{1D\text{gas}}\) and \(\bar{\epsilon}_n\) the Hamiltonian and eigenenergies of the 1D gas.

As described above, we assume that only the ground state \(n = 0\) appears in \(|\psi^0\rangle\). Summing over all final states of the 1D gas, we obtain a transition rate for the impurity

\[
W_{k \rightarrow k'} = \frac{g^2_{1D}}{\hbar L} \sum_n \langle n | \rho_{k \rightarrow k'} | 0 \rangle^2 \left[ \bar{\epsilon}_n - \epsilon_0 + \frac{\hbar^2 (k' - k)^2}{2m_{im}} \right]
\]

\[
= \frac{g^2_{1D} N_p}{\hbar L_{\epsilon_F}} S \left( k - k', \frac{\hbar^2 (k' - k)^2}{2m_{im}} \right),
\]

where \(\rho_{k \rightarrow k'} \equiv \sum_n e^{-i(k' - k)\bar{x}_n}\) is the Fourier transform of the density operator of the 1D gas and \(S(k, \omega)\) is the dimensionless dynamic structure factor (DSF) of the 1D gas (note the factor of \(\epsilon_F/N_p\) in our definition of \(S\), \(N_p\), being the number of particles and \(\epsilon_F \equiv \hbar^2 k_F^2/2m\) their Fermi energy; here \(m\) is the mass of the particles in the 1D gas and \(k_F \equiv \pi n\) is the Fermi wave vector with \(n = N_p/L\) the density).

We assume our 1D gas is made of spinless bosons and has short ranged interactions of the form \(g_{1D} \sum_{i \neq j} d(x_i - x_j)\). For convenience, we introduce the standard dimensionless interaction strength \(\gamma \equiv g_{1D} m / (\hbar^2 n)\). Analytical solutions for \(S(k, \omega)\) are available in the weakly or strongly interacting limit. For intermediate values of \(\gamma\) one may use the exact Lieb-Liniger (LL) solution for the DSF which can be obtained numerically for any values of \(k\) and \(\omega\). A description of this numerical procedure can be found in Ref. [25].

Once we can calculate the transition rate, we need to account for the driven motion of the impurity. To simulate both the driving force and the scattering, we discretize time and momentum and write a scattering transition probability \(P_{k \rightarrow k'} \equiv W_{k \rightarrow k'} \delta t \Delta k\) and we define the probability of not scattering to be \(P_{k \rightarrow k} \equiv 1 - \sum_{k' \neq k} P_{k \rightarrow k'}\). For every time interval \(\delta t\), we evolve the position and velocity of the particles deterministically. In the present case (inspired by the experiments of Ref. [1]) we are concerned with the impurity being accelerated (driven) by gravity \(g\) (assume acceleration in the +x direction) so we have

\[
x(t + \delta t) = x(t) + v(t) \delta t + g \delta t^2 / 2,
\]

\[
v(t + \delta t) = v(t) + g \delta t.
\]

After each time interval \(\delta t\) we then allow for a stochastic scattering attempt \(v_1 \equiv v(t + \delta t) \rightarrow v_2\) with probability per unit wave vector \(P_{v_1 \rightarrow v_2} = W_{k \rightarrow k} \delta t\), where \(v_1 = \hbar k/m_{im}\) and \(v_2 = \hbar k'/m_{im}\). This allows for efficient simulation of the impurity motion. In the large and small \(\gamma\) regime, we have used analytic forms of the DSF to test our numerical algorithm.

### III. EXPERIMENTAL PARAMETERS

As demonstration of our method we apply it to the experimental situation from Ref. [1]. A Bose condensate of \(^8\text{Rb}\) atoms is confined into an ensemble of harmonic traps with long axis aligned with the Earth gravitational field. The transverse radius of each trap and temperature are such that each system is in an effective zero-temperature 1D regime. The parameters of the 1D traps vary with position—both between different 1D systems in the ensemble and within each individual 1D system. It should be noted that the value of \(\gamma\) is expected to vary significantly across the system. Private communication with the authors of Ref. [1] led to the estimates reported in Table I.

For simplicity we crudely neglect the nonuniformity of the system, considering only the case of a homogeneous 1D system with fixed density [27].

A radio-frequency (RF) pulse is used to change the hyperfine ground state of some (up to three) atoms near the center of the 1D system so as to decouple them from the trap. The pulse is Fourier limited in width [full width at half maximum (FWHM) \(\sim 2.3 \mu\text{m}\)] and has a velocity distribution of width \(\sim 2 \times 10^{-3} \text{ m/s}\) (close to the uncertainty limit). Therefore, we consider a wave packet \(\psi(x) \propto \sin(\alpha x)/x\), where \(\alpha = 2\pi/\Delta x \sim 2.73 \mu\text{m}^{-1}\) (although we find that \(\alpha \approx 1.8 - 2.0 \mu\text{m}^{-1}\) produces a better fit to the width of the unscattered peak in the experimental density profile of the falling atoms at long times, illustrated in Fig. 2).

We model these initial conditions using a Gaussian-smoothed [28] Wigner function [29] for the position and momentum distribution of the falling atoms at time \(t = 0\):

\[
G(x, p) \propto \int_{-\infty}^{\infty} W(x', p') e^{-\frac{p'^2}{2\sigma^2} + \frac{x'^2}{2\bar{B}^2}} dx' dp',
\]

\[
W(x, p) \propto \int_{-\infty}^{\infty} \psi^*(x + y)\psi(x - y)e^{2\mu y/\hbar} dy,
\]

where \(\sigma\) and \(\bar{B}\) are positive real constants that satisfy the condition \(\sigma \bar{B} \leq 1\), i.e., the smoothing area is \(\geq \hbar\). We choose the least possible smoothing that yields a positive semidefinite probability, namely, \(\sigma \bar{B} = 1\). The value of \(\sigma\) is then set to equal \(\hbar a^2\). After a few lines of algebra one obtains

\[
G(x, p) \propto e^{-a x^2} \left| \text{Erf} \left( \frac{1 + \frac{p}{\sigma a} + i\alpha x}{\sqrt{2}} \right) + \text{Erf} \left( \frac{1 - \frac{p}{\sigma a} - i\alpha x}{\sqrt{2}} \right) \right|^2,
\]

where \(\text{Erf}\) is the Gaussian error function extended to the complex plane. This distribution was sampled using the rejection sampling technique.

The decoupled impurity atoms are then allowed to accelerate under the constant driving force of the gravitational field. In our simulations we assume only a single impurity atom is decoupled from the trap (i.e., we neglect interactions between multiple falling impurity atoms; we also disregard possible effective interactions between the decoupled atoms that may be mediated by the condensate).
In this experiment, the falling (impurity) atoms are identical to the atoms in the trap (gas) up to their spin state. Hence, $m_{im} = m$ and all interactions (impurity-gas and gas-gas) are described by the same $\delta$ function potential, $g_{im} = g_{1D}$.

**IV. CENTER OF MASS**

We start by considering the position of the overall center of mass of the packet of falling atoms as a function of time, which was measured experimentally and reported in Fig. 3 of Ref. [1]. The parameters used in this experiment are those listed as case (3) in Table I.

Figure 1 shows a comparison between the experimental curves from Ref. [1] and the results from our stochastic simulations, using $n = N_p/L \approx 1.2 \mu m^{-1}$ and different values of $\gamma$ (top panel), as well as using $\gamma = 2.6$ and different values of $n$ (bottom panel). In the simulations we consider an infinite 1D gas of uniform density. The experimental time $t = 0$ in Ref. [1] was chosen to correspond to the middle of the RF pulse that creates the packet of falling atoms. Accordingly, we chose $t = 0$ in our simulations as the time when the Tonks limited packet starts moving through the 1D gas.

The numerical results appear to be very sensitive to the values of $\gamma$ and $n$. This allows us to determine that the combination $n \approx 1.2 \mu m^{-1}$ and $\gamma = 2.6$ provides the best fit to the experimental data. (We note that particles begin to leave the 1D trapped gas after about 2 ms, which corresponds to the longest time reported in Fig. 1. Such an effect may be responsible for the discrepancy that we observe between our numerics and the very last data point in the experiment.)

Our results are in contrast with earlier theoretical modeling [1,15,20] which achieved a similarly good fit to the experimental results by using the strongly interacting Tonks-Girardeau (TG) approximation [21] corresponding to $\gamma = \infty$ within the 1D gas and then treating the interaction between the impurities and the 1D gas at mean field level with an intermediate interaction strength $\gamma = 7$ (see also Appendix B).

**V. DENSITY PROFILE**

In order to further test our approach, we computed the profile of the falling atoms at long times after they exit the 1D gas, which can be compared with the experimental results reported in Fig. 5 of Ref. [1]. Experimental results are available [1] for all three cases in Table I. Unfortunately, a similar comparison was not carried out in earlier theoretical modeling [1,15,20].

In our simulations, we approximate the 1D gas to have uniform density and fixed length, with parameters $N_p$ and $L$ as in the experiments. Once again, we find that the resulting density profile of the falling packet has a significant sensitivity on the value of $\gamma$, which allows us to readily identify which one gives the best fit.

The outcome is shown in Fig. 2. The experimental and simulation curves were normalized so that the area under the profiles equals 1 (after subtraction of a background [30]). The main peak in the figure is due to the fraction of particles that fall freely through the 1D gas without scattering [31].

Both the overall shape of the curves and the ratio between scattered and free-falling contributions are in reasonable agreement between numerics and experiments for $\gamma = 1.1, 1.9, and 2.1$, respectively. These results suggest that the relevant values of $\gamma$ in the experiments are those from the central region of the condensate.

We note that for these values of $\gamma$ we find very good agreement between the exact LL solution and the Bose gas (BG) approximation [24]. In the BG limit, we studied also 1D gases with static position-dependent density [27]. We found that the resulting effects are minor and do not alter the best fit values of $\gamma$.

We notice that a small dip between the scattered peak and the free-falling peak appears in the experimental data (most
noticeably at larger values of $\gamma$) whereas it is nearly absent in the numerical simulations. We conjecture that this dip might be due to the fact that in the actual experiment two or more impurities might fall though the trap at the same time. An effective attractive interaction between impurities could bind together nearby impurities and enhance the main peak at the expense of weight on either sides of the main peak. This effect is beyond our approximation and must be relegated to future research.

VI. CONCLUSIONS

Using linear response theory and Fermi’s golden rule with exact transition rates to model the scattering between the driven impurity and the underlying 1D Bose gas, we have been able to obtain a quantitative description of the experimental results in Ref. [1]: center of mass, profile of driven packet with time-of-flight measurements, and tomography. It appears that our crude approach is entirely sufficient to describe the vast majority of the observed physics. If desired, the approach taken here could be systematically improved by considering corrections of higher order in the coupling between the Bose gas and the impurity. The detection of finer quantum mechanical effects beyond our description may, however, require higher experimental accuracy.

In the range of $\gamma$ values considered here, the need for an exact LL solution was limited and the results would have been in large part the same had we used the BG approximation instead. However, sizable differences between LL and BG arise already for $\gamma \gtrsim 3$, which is within experimental reach.

The case considered here can be viewed as an extremely simple example of handling a nonequilibrium situation in the presence of strong correlations. While the impurity is driven, the physics of the Bose gas can still be understood as remaining at equilibrium. Going further, one could try to understand how putting the gas itself out of equilibrium affects the impurity dynamics. Moreover, besides cold atom settings, one could also consider driven quantum magnets, for which the necessary exact correlators are also available. We will return to these issues in future work.

Statement of compliance with EPSRC policy framework on research data: This publication reports theoretical work that does not require supporting research data.

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APPENDIX A: DEPENDENCE ON 1D GAS DENSITY PROFILE

For the values of $\gamma$ of relevance to the experiments in Ref. [1], we find that our simulations give similar results whether we use the DSF from LL or in the BG approximation. We can therefore use the latter to test how the results are affected by (static) changes in the 1D gas density profile.

The DSF of a 1D Bose gas can be determined directly from its spectrum [24]

$$h\omega = \frac{\hbar^2 k^2}{2m} \sqrt{1 + \frac{4\gamma k_F^2}{\pi^2 k^2}}, \quad (A1)$$

using the f-sum rule

$$S(k, \omega) = \frac{N_p}{\hbar} \left[ 1 + \frac{4\gamma}{\pi^2 (k/k_F)^2} \right]^{-1/2} \delta(\omega - \omega_k). \quad (A2)$$

After a few lines of algebra, following the steps outlined in Sec. II, one obtains that the only allowed outgoing wave vector is $k' = \gamma k_F^2/(\pi^2 k)$, with probability

$$P_{k \rightarrow \gamma k_F^2/\pi^2} = \begin{cases} \frac{2\gamma^2}{\pi^2} \frac{\pi^2}{\hbar} \delta t \left| \frac{k}{k_F} \right| \frac{1}{\delta(\omega - \omega_k)} & |k|/k_F > \sqrt{\gamma}/\pi, \\ 0 & \text{otherwise}. \end{cases} \quad (A3)$$

Notice that Eq. (A3) can be interpreted as a probability only if it is $\lesssim 1$, which in turn is satisfied if we choose

$$\delta t \lesssim \frac{\pi^2}{2\gamma^{1/2} k_F}, \quad (A4)$$

where we used explicitly the condition $|k|/k_F > \sqrt{\gamma}/\pi$.

Using Eq. (A3) one can straightforwardly adapt the simulations to a position-dependent (static) density profile of the underlying 1D gas. For concreteness we fix the average density at the experimentally relevant value of 1.3278 particles/$\mu m$ (corresponding, in the case of uniform density, to an average $\gamma = 1.9$). We then contrast the following cases: (i) a uniform condensate of finite length $L = 24.1$ $\mu m$; (ii) a uniform condensate of the same length with a square depletion to half...
the density near its center (defined as $-2.0 < x < 2.0 \mu m$); (iii) a parabolic condensate of the same length and average density; and finally (iv) a parabolic condensate with a central depletion obtained by subtracting the Gaussian-smoothed Wigner function density near its center (defined as $-2.0 < x < 2.0 \mu m$).

APPENDIX B: TONKS-GIRARDEAU LIMIT

In our work we have found quantitative agreement with the experimental results in Ref. [1] for small values of $\gamma$ where the BG approximation is reasonably accurate. This is in contrast with the modeling presented in that very same reference [1], as well as the work done in Refs. [15] and [20], which make use of the TG limit.

In this section we investigate the motion of the center of mass of the falling packet in the TG limit using our method. A reasonable agreement with the experimental results can be obtained only in the small $\gamma$ limit, which is in contradiction with the TG approximation. According to our simulations, already at intermediate values of the coupling strength (namely, $\gamma \gtrsim 10$) the falling atoms reach terminal velocity well within the time of the experiment, in contrast with the observed behavior.

The dynamic structure factor of a 1D gas in the TG limit [21] ($\gamma \to \infty$) can be written as

$$S(k,\omega) = \frac{N_p}{4e^2 \hbar k} \left[ \Theta(\omega - \omega_+) \Theta(\omega_+ - \omega) \right. \left. - \Theta(\omega - \omega_-) \Theta(\omega_- - \omega) \right],$$

where

$$\omega_{\pm}(k) = \frac{\hbar^2 k^2}{2m} \pm \frac{k^2}{2k_F^2},$$

Introducing the dimensionless wave vector notation $\tilde{k} = k/k_F$, after the usual substitution $k = k' - k$ and $\omega = \hbar^2 k'^2/2m - \hbar^2 k^2/2m$, a few lines of algebra show that the scattering...
probability density per unit of dimensionless wave vector, from $k$ to $k'$, is given by

$$
P_{k \to k'} = \begin{cases} \frac{\gamma^2 \varepsilon_F}{\pi^3 \hbar} \delta t \frac{1}{|k - k'|} & \text{if } |k| > 1 \text{ and } |k'| < 1 \\ 0 & \text{otherwise.} \end{cases} \quad (B3)$$

The expression above, which is correct to leading order in $\gamma$, presents the intrinsic problem that the total scattering probability at a given time,

$$\int_{-\infty}^{+\infty} P_{k \to k'} \, dk' = \frac{\gamma^2 \varepsilon_F}{\pi^3 \hbar} \delta t \left| \ln \left( \frac{k - 1}{k + 1} \right) \right|, \quad (B4)$$

diverges in the limit $k \to k_F$. For the stochastic approach to be valid, a necessary condition is that $\delta t$ be small enough so that the integrated probability at any given time remains smaller than 1, which thus requires $\delta t$ to be vanishingly small for $k$ arbitrarily close to $k_F$.

The singularity is directly related to the limit $\gamma \to \infty$. However, it cannot be easily resolved by including the subleading correction in $1/\gamma$ because the expansion of $S$ becomes negative in some range of $k$ and $\omega$ [32].

A compromise to obtain a nonnegative, nondivergent probability is to use the expansion of $S$ to leading order, as in Eq. (B1), but to replace the Heaviside $\Theta$ functions with those from the random phase approximation (RPA). Namely, we use the TG form of the DSF, but with support in $k$ and $\omega$ from RPA. This in turn means that the probability $P_{k \to k'}$ retains the same form as in Eq. (B3), but it is set to zero identically outside the range:

$$1 - \frac{4}{\gamma} < k \leq 1, \quad \frac{\gamma - 4 - \bar{k}(\gamma - 2)}{2} < k' < 1 - \frac{2(1 + \bar{k})}{\gamma - 2},$$

or

$$k > 1 - \frac{\gamma - 4 + 2\bar{k}}{\gamma - 2} < k' < 1 - \frac{2(1 + \bar{k})}{\gamma - 2},$$

and similarly for negative values of $\bar{k}$. In the limit of $\gamma \to \infty$ this tends to Eq. (B3), as one would expect. We tested our choice of probability regularization in the TG limit by comparing its results with RPA and LL simulations for large values of $\gamma$ and we found good quantitative agreement (not shown).

Using the new boundaries in Eq. (B5), the probability that a particle with wave vector $\bar{k}$ scatters with the condensate in a time interval $\delta t$ (to any allowed $k'$) remains finite for all allowed values of $\bar{k}$. Namely,

$$\int_{1 - 4/\gamma < \bar{k} \leq 1}^{1 - 2(1 + \bar{k})/\gamma(\gamma - 2)} \frac{\gamma^2 \varepsilon_F}{\pi^3 \hbar} \delta t \ln \left( \frac{\gamma - 2}{2} \right)$$

if $1 - 4/\gamma < \bar{k} \leq 1$, and

$$\int_{-1 / \gamma + 4(1 + \bar{k})/\gamma(\gamma - 2)}^{1 - 2(1 + \bar{k})/\gamma(\gamma - 2)} \frac{\gamma^2 \varepsilon_F}{\pi^3 \hbar} \delta t \ln \left( \frac{\bar{k} + 1 - 4/\gamma}{\bar{k} - 1 + 4/\gamma} \right).$$

if $\bar{k} > 1$. Notice that the maximum over $\bar{k} > 1$ of the logarithmic contribution in the second case is in fact the same as the first case: $\ln(\gamma - 2)/2$. Our stochastic approach is therefore valid, provided that we choose

$$\delta t \lesssim \frac{\pi \hbar}{\gamma^2 \varepsilon_F} \ln \left( \frac{\gamma - 2}{2} \right)^{-1}. \quad (B6)$$

For the typical system parameters considered in this work, the upper bound for $\delta t$ scales as $(\gamma^2 \ln \gamma)^{-1}$ ms. This is satisfied for instance by choosing $\delta t \lesssim 0.01 \mu$s up to $\gamma = 100$.

We can then implement our stochastic approach using the inverse transform sampling analytically in the TG limit. Figure 5 shows the resulting behavior of the center of mass (c.m.) motion from TG simulations for a uniform 1D gas of density $n = 1.2 \, \mu m^{-1}$ and $\gamma = 4.1, 5, 6, 8, 10, 25, 100$ (red, blue, green, magenta, cyan, yellow, and black, respectively). The black dashed line represents the free-fall curve. The black solid line corresponds to the expected behavior in the $\gamma \to \infty$ limit (i.e., terminal velocity $v_F$). Blue open squares (joined by a dotted line) represent the experimental data from Fig. 3 in Ref. [1].

We notice that the c.m. motion becomes asymptotically linear in time within the simulation time window for $\gamma \gtrsim 7$, suggesting that the falling atoms reach terminal velocity. The value of the terminal velocity is nonmonotonic in $\gamma$: it initially decreases (in agreement with Ref. [15]) with increasing $\gamma$, and later increases and tends asymptotically to $v_F$ in the $\gamma \to \infty$ limit, as expected.

Reasonable agreement with the experimental results can only be obtained in the weak coupling limit ($\gamma \sim 5$), which is in contradiction with the TG limit (and even with the RPA approximation, which has a hard limit of applicability of $\gamma > 4$, and is known to begin to fit reasonably well the LL DSF only for $\gamma \gtrsim 10$ [32]).


[23] In support of our choice of approach, we point out that Ref. [14] shows how a perturbative approach to study these systems appears to be valid beyond the perturbative regime. Moreover, both Refs. [9] Ref. [14] demonstrate the appropriateness and effectiveness of a (semiclassical) stochastic description.


[26] C. Sias and M. Köhl (private communication).

[27] For the relatively small values of $\gamma$ that produce the best agreement with the experiments according to our simulations, we find that the exact DSF from LL calculations yields results similar to the DSF from the BG approximation. Within the BG approximation, we were then able to test the effects of a static nonuniform 1D gas density profile on our results and demonstrate that they are negligible (see Appendix A).

[28] Smoothing of the Wigner function is required (over appropriately wide regions) to ensure nonnegativity, as we want to interpret it as a probability distribution function (see Ref. [29]).


[30] In the experimental data for $g = 1.1$ and $g = 1.9$, we subtracted a background (0.0046 and 0.023, respectively, in optical intensity units) so that the right tail of the position distribution drops down to approximately zero. This is justified by the fact that such a background ought to be spurious, as it could only arise from atoms accelerating faster than $g = 9.813 \text{ m/s}^2$ (unphysical).

[31] We note that the time reported in Ref. [1], 18.7 ms, appears to be inconsistent with the position of the main peak. Assuming $g = 9.813 \text{ m/s}^2$, the main peak of free falling atoms starting at $t = 0$ in a Gaussian packet centered at $x = 0$ should appear at $x = 1716 \mu m$ after 18.7 ms. We find instead that a time of 18.6 ms is in better agreement with the data and we used it in our simulations. This is consistent with the fact that the falling packet is created with a laser pulse of finite duration. Indeed, comparing the noninteracting data from Ref. [15] with the analytical free-fall behavior $g(t - t_0)^2/2$, we find that a shift $t_0 \approx 90 \mu s$ is needed to achieve good agreement, which is consistent with a difference between 18.6 and 18.7 ms.