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Diffusion rates in a 4-dimensional mapping model of accelerator dynamics

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This paper is dedicated to the undying memory of Jeff Tennyson: a free spirit, an original thinker, an independent scientist and a dear friend.

We study slow diffusion processes of orbits through thin chaotic layers of $2m$ ($m \geq 2$)-dimensional symplectic maps (often referred to as Arnol'd diffusion) on a 4-dimensional mapping model of accelerator dynamics. Using a method proposed by Chirikov, we compute diffusion rates of vertical displacements, $|y_n|$, near the flat beam case, when the horizontal motion occurs within a thin chaotic layer of the corresponding 2-dimensional x_n, x_{n+1} map. Our computation distinguishes between regions of diffusive and quasiperiodic motion, in which the respective rates are found to differ by 10–12 orders of magnitude for $N = 10^8$ iterations. For diffusion, $|y_n|$ grows on the average with n ($n \leq 10^8$), with a rate $D(R) \propto \exp(aR)$ ($a > 0$, $R^2 = y_0^2 + y_1^2$), which gives accurate estimates of escape times $N_{\text{esc}} \propto D^{-1}$. Modelling synchrotron oscillations by a periodic modulation on the “betatron” frequencies, we find that diffusion rates increase significantly and rapid escape occurs above certain thresholds in the ε, Ω parameter values, ε and Ω being the amplitude and frequency of the modulation respectively.

1. Introduction

Orbital diffusion in Hamiltonian systems and the related subject of beam stability in particle accelerators were among Jeff Tennyson's main research areas, in which he made some important and pioneering contributions [1–5]. In fact, he was among the first ones to realize that many of B.V. Chirikov's ideas, like the resonance overlap criterion [6], could be used to study nonlinear “blow-up” phenomena in the beam-beam interaction problem of colliding particle beams [7].

In this paper, we follow a similar approach and exploit further some techniques developed by Chirikov and co-workers [6,8] to study slow diffusion processes in certain multi-dimensional

mapping models of accelerator dynamics. We shall refer to these processes as Arnol'd diffusion [5,6], as they occur within thin chaotic layers, near unstable resonances of symplectic maps.

In particular, we are interested here in a 4-dimensional mapping model describing the effects of sextupole nonlinearities on a hadron beam passing through a FODO cell composed of a dipole and two quadrupole magnets, which focuses the particles' motion in the horizontal (x)- and vertical (y)-directions. The equations of the mapping, after some appropriate scaling, are seen to reduce to [9]

$$x_{n+1} = 2c_x x_n - x_{n-1} - \rho x_n^2 + y_n^2, \quad (1a)$$

$$y_{n+1} = 2c_y y_n - y_{n-1} + 2x_n y_n, \quad (1b)$$

where

$$\rho = \beta_x s_x / \beta_y s_y \quad (2a)$$

and

$$c_{x,y} \equiv \cos 2\pi q_{x,y}, \quad s_{x,y} \equiv \sin 2\pi q_{x,y}, \quad (2b)$$

q_x, q_y being the so-called betatron frequencies and β_x, β_y the betatron functions of the accelerator [10]. Here, we shall assume that $\beta_{x,y}$ are constant and equal to their mean values, i.e. proportional to $q_{x,y}^{-1}$.

Throughout this paper we shall make use of the following approach put forth by Chirikov and co-workers in [6] and [8]: Placing the initial conditions of the x -motion (x_0, x_1) on some particular orbit of eq. (1a), with $y_n = 0$, and starting with y_0, y_1 near the origin of the y_n, y_{n+1} plane, we shall divide the total number of iterations N of each orbit into N_1 and N_2 subintervals of length ΔN_1 and ΔN_2 respectively.

We then compute, in each case, the diffusion coefficient

$$D_k = \frac{2}{N_k(N_k - 1)} \sum_{m>l} \frac{[|y(m)|_{av} - |y(l)|_{av}]^2}{(\Delta N_k)(m-l)}, \quad (3)$$

$$k = 1, 2, \quad (4)$$

where $|y(m)|_{av}$ is the average value of $|y_n|$ over the m th subinterval, $m, l = 1, 2, \dots, N_k$. In eq. (3), we clearly concentrate on diffusion in the y -direction, since our particular experiment concerns perturbations of a flat (horizontal) beam, for which blow-up phenomena in the vertical direction are significantly more pronounced.

The main idea in using (3) is the following: If the motion is diffusive, it does not matter if it is averaged over different numbers of subintervals, whence (3) should yield

$$D_1 \approx D_2: \text{ for Arnol'd diffusion.} \quad (4a)$$

On the other hand, if the motion is quasiperiodic, the D_k 's will be a lot smaller than in case (4a) and $|y(m)| \propto (\Delta N_k)^{-1}$, whence one expects

$$\frac{D_1}{D_2} \propto \left(\frac{\Delta N_2}{\Delta N_1} \right)^3: \text{ for quasiperiodic motion.} \quad (4b)$$

Thus, in this paper, we continue and extend an investigation started in [9] by computing diffusion coefficients (3) for several (x_0, x_1, y_0, y_1) choices to study the dependence of diffusion on the number of iterations N and the location of initial conditions of the orbits.

Taking, in section 2, the specific parameter values $q_x = 0.21$ and $q_y = 0.24$, for which there is a large region of bounded orbits in the x_n, x_{n+1} plane, we place our initial x_0, x_1 at different locations along the $x_0 = x_1$ axis and study the way that the vertical displacements $|y_n|$ behave, starting with y_0, y_1 small.

Our main result is that, when (x_0, x_1) lies inside a thin chaotic layer, there is indeed a diffusion rate $D_1 \approx D_2 = D(R)$, as in (4a), averaged over many initial (y_0, y_1) with $y_0^2 + y_1^2 = R^2$. ($R \leq 0.05$). For example, with $R = 0.001$, we find $D(R) \sim 10^{-12}$. On the other hand, for orbits started within regions of quasiperiodic motion, (4b) is nearly obeyed with D_1, D_2 many orders of magnitude smaller than $D(R)$, depending on the total number of iterations N .

Another important result of section 2 is that $D(R)$ appears to increase exponentially with R for x_0, x_1 inside the chaotic layer. In that case, the x - and y -motions appear to be weakly coupled, until the y_n 's grow to about $|y_n| \sim 0.06$ (5% of the x -motion), beyond which most orbits rapidly fly to infinity and slow diffusion processes break down.

In section 3, we have considered the effect of modulational diffusion [5,11] in which the tunes are allowed to vary periodically with n , i.e. q_x, q_y in (1) are replaced by

$$q_{1,2} = q_{x,y} + \varepsilon \cos \Omega n, \quad (5)$$

for $|\varepsilon| \ll 1$ and $\Omega \ll 1$. Such modulations are considered to simulate the effect of synchrotron (longitudinal) oscillations along the beam's motion, caused by the particles being accelerated in

that direction by a series of periodically placed rf cavities [12].

First of all, we find that the mere presence of these modulations, even for ε and Ω as small as 10^{-3} , serves to increase the D_k rates of quasiperiodic motion by several orders of magnitude, compared with the $\varepsilon = \Omega = 0$ case. There also seem to exist certain thresholds in the ε and Ω directions, beyond which diffusion rates grow dramatically and large scale instabilities occur.

This is due to the fact that near stable low-order resonances, the modulations introduced by (5) impose an additional oscillation on the orbits, whose longitudinal and transversal amplitude grows linearly with increasing ε and Ω , respectively. Eventually these oscillations reach the connected chaotic regions of nearby unstable low-order resonances and escape to infinity after only a few thousand iterations.

Section 4 contains our conclusions and a discussion of the results.

2. Diffusion rates near the flat beam case

In this section, we shall concentrate exclusively on the case

$$q_x = 0.21, \quad q_y = 0.24, \quad (6)$$

and study the long term behavior of the orbits of the 4-dimensional symplectic map (1), (2), for different initial conditions x_0, x_1 and $|y_0|, |y_1|$ small. Note that for $y_n = 0$, we recover from (1a) the 2-dimensional Hénon map, which, for $q_x = 0.21$, has a large region of bounded motion with the well-known period 5 islands around the origin shown in fig. 1a.

Taking now initial conditions at a point $P = (x_0, x_1)$ and with y_0, y_1 near $y = 0$ one hopes to find bounded motion, as long as the y_n -displacements remain small. This is indeed what happens, not only near $x = 0$, or inside the major islands, but also for P chosen within the thin

chaotic layer of the period 5 resonance of fig. 1a.

Of course, in the latter case, the respective diffusion coefficients, D_k , cf. (4a), computed from (3), differ from those of the quasiperiodic case, cf. (4b), by many orders of magnitude, see table 1. They certainly do not lead, however, to rapid escape. Their y_n -coordinates are seen to oscillate and slowly grow in magnitude (see fig. 1b) for y_0, y_1 small enough but do not become unbounded even after $N = 10^8$ iterations.

By contrast, the y_n -coordinates of quasiperiodic motion, for the same y_0, y_1 values, are always seen to execute perfect oscillations, as shown in fig. 1c. In fact, the number of rings in the y_n, y_{n+1} plane equals the order of the resonance in which the point P is placed in the x_n, x_{n+1} plane. For an orbit started within a period 37 island, 37 rings were distinctly observed in the corresponding y_n, y_{n+1} plane.

We are interested in the diffusion properties of the orbits of (1):

- (a) When P varies along a straight line between points A and B of fig. 1a, and
- (b) As the magnitude of the y_0, y_1 initial conditions increases for each choice of (x_0, x_1) .

We shall use Chirikov's formula (3) and divide our total number of iterations $N = 10^8$ in several subintervals of length $\Delta N_k, k = 1, 2, \dots$. For a true (Arnol'd) diffusion process (like the one occurring within the chaotic layer of the 5 islands of fig. 1a) we find, indeed, that the D_k values are all of the same order of magnitude (see table 1).

Varying now the point P along the straight line between points A, B of fig. 1a, we plot in fig. 2a the logarithm (base 10) of the D_2 diffusion rate vs. the $x_0 (=x_1)$ coordinate of the point P , for $y_0 = 0.01, y_1 = 0$ and $N = 10^6$ iterations. As we see, D_2 grows by several orders of magnitude, as P passes through the thin chaotic layers present in fig. 1a. There is also an interesting small increase in D_2 for P values near $x = 0$, which we find as yet difficult to understand. We suggest

Fig. 1. (a) The x_n, x_{n+1} motion of eq. (1) with $q_x = 0.21$, $q_y = 0.24$ and $y_0 = 0.01$, $y_1 = 0$. By P_1, P_2 we designate the two points chosen for quasiperiodic motion and thin chaotic layer diffusion, respectively. (b) The y_n, y_{n+1} motion for $P = P_2$ and y_0, y_1 as in (a). (c) Same as (b) for $P = P_1$.

Table 1

Initial conditions at $P = (x_0, x_1)$, $(y_0, y_1) = 10^{-3} (\cos(\frac{1}{3}n\pi), \sin(\frac{1}{3}n\pi))$; $N = 10^8$, $N_1 = 10$, $N_2 = 100$, $\Delta N_1 = 10^7$, $\Delta N_2 = 10^6$.

n	Arnol'd diffusion $P_2 = (-0.0049, -0.5329)$		Quasiperiodic motion $P_1 = (-0.0019, -0.4569)$	
	$\log D_1$	$\log D_2$	$\log D_1$	$\log D_2$
1	-12.8	-12.63	-26.2	-24.5
2	-11.4	-11.34	-26.4	-24.0
3	-12.18	-12.15	-27.4	-24.0
4	-13.15	-12.9	-26.2	-24.1
5	-11.91	-11.92	-26.7	-24.0

that this may be due to an increased coupling, in that region, between the x - and y -oscillations, but clearly a more detailed analysis of this phenomenon is required.

To examine more carefully the y -dependence of our diffusion rates, we shall place sets of 100 initial conditions, y_0, y_1 on circles of different radii,

$$y_0^2 + y_1^2 = R^2, \quad 0.001 \leq R \leq 0.1.$$

The initial conditions of each set are equally distributed along the upper ($y_1 > 0$) semicircle. Due to the symmetry of our equations (1) under the transformation $y_n \rightarrow -y_n$, an initial point on the lower semicircle behaves identically as its conjugate point on the upper semicircle. Averaging now these diffusion rates

$$D(R) = \frac{1}{100} \sum_{i=1}^{100} D^{(i)}(R) = \frac{1}{100} \sum_{i=1}^{100} D_2^{(i)} \quad (7)$$

over the initial conditions $y_0^{(i)}, y_1^{(i)}$, $i = 1, 2, \dots, 100$, we plot, in fig. 2b, $\log D(R)$ and find that it increases quasi-linearly with R , as

$$\log D(R) = aR + b, \quad (8)$$

i.e. that $D(R)$ has an exponential-like dependence on R , at least over the interval $0.001 \leq R \leq 0.06$. This is quite different than the behavior of the corresponding $D(R)$ for quasiperiodic motion, shown by $\oplus$ in fig. 2b.

In fig. 3, we have drawn on 12 such semicircles in the y_0, y_1 plane solid “boxes” whose size is proportional to the number of initial conditions leading to bounded orbits. Boxes of y_0, y_1 values for which the orbits have escaped to infinity after $N = 10^6$ iterations are left blank. Note that there is a clear asymmetry in the figure, indicating that orbits with $y_0 y_1 < 0$ tend to be more stable, on the average, than those with $y_0 y_1 > 0$. The explanation of this phenomenon, however, remains still an open question.

Let us plot now the proportion of orbits $S(N)$, which remain bounded after N iterations, as a function of N for different R , with x_0, x_1 near the limits of bounded motion in the x_n, x_{n+1} plane. What we find is that $S(N)$ has an algebraic-like decay, $S(N) \propto n^{-\nu}$, with a power $\nu > 0$ which increases as R increases, see fig. 3b. What is important here is that $S(N)$ curves appear to converge as $N \rightarrow \infty$, indicating the existence of a definite proportion of bounded orbits ($\approx 10\%$) which survive in that limit, even for an elliptic beam with a vertical width which is comparable to its horizontal width.

Fig. 2. (a) The logarithm of the diffusion rate, for $y_0 = 0.01$, $y_1 = 0.0$, vs. $x_0 (=x_1)$, chosen along the AB line segment of fig. 1a. (b) $\log D$ vs. R , averaged over 100 points y_0, y_1 on $y_0^2 + y_1^2 = R^2$ for $P = P_1$ (lower curve, marked with $\oplus$) and $P = P_2$ (upper curve), see fig. 1a.

3. The effect of modulational diffusion

We now turn to the investigation of certain additional instabilities, which occur along the direction of the beam's motion and are caused by the accelerating force felt by the particles as they speed through a series of rf cavities. In principle, this effect should be taken into account by considering, instead of (1), a 6-dimensional

Fig. 3. (a) Solid boxes representing escape times are drawn in the radial direction for 1200 initial conditions on 12 semicircles in the y_0, y_1 plane. Maximum box size corresponds to orbits remaining bounded after $N = 10^7$ iterations, $P = P_1$ as in fig. 1a. (b) Percentage of orbits which remain bounded as a function of N for various $R (=10^{-2} \times$ the number on each curve). $P = P_1$, as in fig. 1a.

mapping with the appropriate force terms included in the direction along the orbit.

A common approach, however, is to model these longitudinal (synchrotron) oscillations by a periodic modulation of the betatron tunes, replacing $q_{x,y}$ in (1) by

$$q_{1,2} = q_{x,y} + \varepsilon_{x,y} \cos(\Omega n + \delta_{x,y}), \quad (9)$$

with $|\varepsilon_{x,y}| \ll 1$ and $0 < \Omega \ll 1$. This is what brings about the additional effect of modulational diffusion [5,11], which is expected to enhance the blow-up properties of the particles and significantly decrease the beam's lifetime.

Of course, the amplitude of these modulations depends on the amount of energy lost by the particles through synchrotron radiation, which, in the case of protons, is usually considered negligible. However, for the very high energy machines currently under construction, like the SSC, synchrotron radiation is expected to be important even for hadronic beams [13].

To simplify matters we have taken here

$$\varepsilon_x = \varepsilon_y = \varepsilon, \quad \delta_x = \delta_y = 0 \quad (10)$$

in (9) and have studied the same diffusion rates as in the previous section with q_x and q_y given by (6). We have found that even for small values of ε and Ω , orbits starting inside the stable resonances have D_k coefficients which are larger, by many orders of magnitude, than in the $\varepsilon = 0$ unmodulated case (see fig. 4 and table 1).

Modulation causes an additional oscillation within the resonances of the 4-dimensional case, which shows up as a spiralling motion along certain tube-like surfaces in the x_n, x_{n+1} plane of fig. 5a. On the other hand, orbits starting within a chaotic layer wander about in a connected region surrounding these "tubes".

Interestingly enough, the length of these tube-like surfaces of quasiperiodic motion depends linearly on ε , while their width is also found to grow linearly with increasing Ω , for fixed ε . This may be explained by the fact that these are

Fig. 4. $\log D_2$ vs ε , showing the effects of modulational diffusion on quasiperiodic motion, i.e. with $P = P_1$, see fig. 1a.

projections of oscillations occurring in 2 additional phase space dimensions, in one of which displacements are proportional to the oscillation amplitude ε , while in the other they are proportional to the velocity amplitude $\varepsilon\Omega$. Beyond certain critical values of ε and Ω , however, these "tubes" are seen to merge with the chaotic layer of the associated unstable resonances and the orbits quickly escape to infinity (see fig. 5b).

This strong instability and rapid escape, beyond a certain critical value of ε , is depicted here, in fig. 4, as a sharp increase of the diffusion coefficient at $\varepsilon \approx \varepsilon_c$. The value of ε_c also depends on Ω , but not too sensitively. In fact, all the results presented here are the same for Ω in the range (0.01, 0.02).

As we see in table 2, for ε small enough ($\varepsilon = 0.001$), the diffusion rates of orbits lying in the chaotic regions do not differ very much from the unmodulated $\varepsilon = 0$ case (cf. table 1). As ε approaches, however, the critical value $\varepsilon_c \approx 0.002$, diffusion rates in these regions grow significantly (see table 2) and orbits run away to infinity, after only a few thousand iterations.

Fig. 5. Tube-like surfaces of oscillations produced by modulational diffusion on quasiperiodic motion, $P = P_1$, $y_0 = y_1 = 0.01$, $\varepsilon = 0.003$ and $\Omega = 0.01$. (b) Large scale instability occurring, as motion on the resonance “tubes” becomes connected for $\varepsilon = 0.011$ and $\Omega = 0.001$.

4. Concluding remarks

The problem of diffusion of orbits through chaotic layers of higher dimensional Hamiltonian systems has a long history of analytical as well as numerical studies [5,6,11]. In particular, the investigation of slow (Arnol’d) diffusive processes, which can have significant effects after

Table 2

Modulational diffusion rates for $N = 10^6$; $\Omega = 0.01$, $(x_0, x_1) = P_2 = (-0.0049, -0.5329)$, $y_0 = y_1 = 0.01$.

$\varepsilon = 0.001 (N_1 = 10, N_2 = 100)$		$\varepsilon = 0.0015$ ($N_1 = 100, N_2 = 1000$)	
$\log D_1$	$\log D_2$	$\log D_1$	$\log D_2$
-11.2	-11.2	-8.62	-8.54
-11.5	-11.4	-8.52	-8.39
-12.0	-11.9	-8.67	-8.61
-11.4	-11.3	-8.57	-8.45

very long times, has attracted the attention of many researchers, due to its great importance in many fields [14,15].

In this paper, we applied a simple technique due to Chirikov and co-workers [6,8] to calculate diffusion rates for a 4-dimensional mapping model of accelerator dynamics. This model describes the passage of a hadron beam through a focusing element (consisting of one dipole and two quadrupole magnets), whose sextupole nonlinearities can bring about serious instabilities and significantly decrease the beam’s lifetime [9].

We have concentrated on the nearly flat beam case, in which the initial horizontal displacements x_0, x_1 are chosen from the chaotic layer, or the islands of a period-5 resonance of the 2-dimensional case, while the initial vertical displacements y_0, y_1 have very small amplitudes.

We have computed diffusion coefficients $D(R)$ for motion within the chaotic layer (with $y_0^2 + y_1^2 = R^2$) and have found that $D(R)$ grows sharply at some value of R , which is nearly 20% of the “radius” of the horizontal bounded motion. Above this value of R , the proportion of particles that have not escaped after N iterations appears to converge to 10% of the original beam, as $N \rightarrow \infty$.

In our calculations of diffusion through thin chaotic layers, we observed that, knowing the diffusion rates $D(R)$ for a particular set of orbits, we obtain an estimate of their “escape time” N_{esc} from

$$D(R) N_{\text{esc}} \sim K = \text{const.} \quad (11)$$

Fig. 6. Quasilinear dependence of $\log D$ on $\log N_{\text{esc}}$ of the form $\log D = -\log N_{\text{esc}} - 3$ for many orbits with $y_0^2 + y_1^2 = R^2$, $0 < R \leq 0.05$ and $P = P_2$, see fig. 1a.

As can be seen from fig. 6, where we have plotted $\log D(R)$ vs. $\log N_{\text{esc}}$ for many orbits, an expression of the form (11) is nearly satisfied with $K \cong 10^{-3}$.

Finally, we began to study the effect of modulational diffusion on the orbits of our map, caused by adding a periodic modulation of the form $\varepsilon \cos(\Omega n)$ to the harmonic (betatron) frequencies q_x and q_y . These modulations were found to significantly affect the overall stability properties of the motion.

Even for small values of ε (≤ 0.002) and Ω (≈ 0.01), diffusion rates were measured to be considerably larger (and escape times much shorter, according to (11)). This is due to motion through resonances of a higher than 4-dimensional space, in a way that is reminiscent of what Tennyson had termed “resonance streaming” in similar m (≥ 3) degree of freedom systems [3].

Eventually, the orbits are led via a network of overlapping resonances to a connected large scale chaotic region, through which they escape to infinity at an extremely rapid rate.

Of course, all our results so far have been obtained for a particular beam geometry and specific q_x and q_y parameter values. Work is

currently in progress to determine whether similar diffusion rates and with analogous properties exist in the more general case of an “elliptic” beam with different betatron “tune” values, in eqs. (1) and (2).

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