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Application of the Method of Collective Marks to some $M/G/1$ vacation models with exhaustive service*

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We use the Method of Collective Marks to analyze some time-dependent processes in the $M/G/1$ queue with single and multiple vacations. With the server state specified at a fixed time $t \geq 0$, the Laplace transforms with respect to t of mixed transforms for the joint distribution of the number of departures by time t , the queue length, the virtual waiting time, the elapsed and remaining service/vacation times at time t are derived by means of probabilistic interpretations. The Laplace–Stieltjes transform of the virtual waiting time at time t is also given. Some well known results are special cases.

Keywords: Time-dependent queues, single vacation, multiple vacations, Collective Marks.

1. Introduction

The Method of Collective Marks introduced by Van Dantzig [4] (see also [5]) is characterized as a method to derive relations in probability by interpreting the different parts of these relations as probabilities. In queueing problems this method has been used by Kesten and Runnenburg [7], Bloemena [1], Runnenburg [9–11], Gnedenko et al. [6], Sikkil [12], Danieljan [2], Danieljan and Liese [3]. In Runnenburg [9, 10] many important results of the $M/G/1$ queue were derived by interpretation. In Gnedenko et al. [6] the same method was used in the $M/G/1$ queue with priorities. Recently, Danieljan and Liese [3] used this method in a queueing model with time-dependent priorities.

In this paper we use the Method of Collective Marks in some $M/G/1$ vacation models with exhaustive service. We restrict ourself to two basic vacation models studied by Levy and Yechiali [8] and others: the single vacation model and the multiple vacation model. For these vacation models we study the joint time-dependent distribution of the queue length, the number of departures, the virtual waiting time, the elapsed and remaining (service/vacation) times.

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Essential tools in Collective Marks used in the paper are the following auxiliary processes.

- (1) *The catastrophe process.* Catastrophes (bad things) happen according to a Poisson process with parameter ξ . This Poisson process does not depend on the queueing process studied. We are interested in T_ξ , the moment at which the catastrophe happens for the first time. T_ξ is an exponential random variable with parameter ξ .
- (2) *The marking process.* Customers are marked randomly each time they arrive at the system. Each customer has probability $1 - x$ of being marked and probability x of remaining unmarked. x may have any value in $[0, 1]$. Customers are marked independently, and the marking is independent of the queueing process studied. It is worth noting that if customers arrive according to a Poisson process of intensity λ , then marked customers arrive according to a Poisson process of intensity $\lambda(1 - x)$. The marking and catastrophe processes are assumed to be independent.

Throughout the paper the following notations and assumptions will be used. The inter-arrival times a_n are exponentially distributed with parameter λ and the service time b_n of the n th customer has a general distribution with LST (Laplace–Stieltjes transform) $\beta(\xi)$. The number of arrivals in $(0, t)$ is denoted by $A(t)$. The queueing discipline is “first in–first out”. We write (a) “no C_ξ in b_n ” for “no catastrophes happen during the service time b_n ”, (b) “no C_x in b_n ” for “no marked customers arrive during the service time b_n ”, and (c) “no C_x at t ” for “no marked customers are present at t ”.

2. A result concerning the $M/G/1$ queue

We consider the following situation. At time $t = 0$ exactly $m \geq 1$ customers are present at the counter with the first about to be served. Let the random variable L_m denote the length of the busy period that starts with these m customers. For a fixed time t with $0 \leq t < L_m$ we are interested in the random variables

- $B(t)$: the number of departures in $(0, t]$,
- $N(t)$: the number of the customers present at t ,
- $S'(t)$: the elapsed service time of the customer being served at t ,
- $S''(t)$: the remaining service time of the customer being served at t ,
- $W(t)$: the virtual waiting time at t .

Let us write $B_n = \sum_{i=1}^n b_i$, $A_n = \sum_{i=1}^n a_i$, $B_0 = 0$. For $0 \leq t < L_m$ we have the relations

$$\begin{aligned} N(t) &= m + A(t) - B(t), \\ S'(t) &= t - B_{B(t)}, \\ S''(t) &= B_{B(t)+1} - t, \\ W(t) &= S''(t) + \sum_{i=2}^{N(t)} b_{B(t)+i}. \end{aligned}$$

We consider auxiliary random variables T_ω , T_{ω_1} , T_{ω_2} and R_y in which T_ω , T_{ω_1} and T_{ω_2} are exponential with corresponding parameters ω , ω_1 and ω_2 , while R_y are geometrical with $P\{R_y = n\} = y^n(1 - y)$ for $n \geq 0$. Here we have ω , ω_1 , $\omega_2 > 0$ and $0 < y < 1$. Assume that these auxiliary random variables and the marking and catastrophe processes are independent of one another and independent of the original queueing process. We wish to compute the probability

$$\pi_m(\xi, \omega, \omega_1, \omega_2, x, y) = P\{T_\xi < L_m, W(T_\xi) < T_\omega, S'(T_\xi) < T_{\omega_1}, S''(T_\xi) < T_{\omega_2}, \text{no } C_x \text{ at } T_\xi, B(T_\xi) \leq R_y\}. \quad (2.1)$$

Note that $\pi_m(\xi, \omega, \omega_1, \omega_2, x, y)$ is nothing but the following transform

$$\xi \int_0^\infty e^{-\xi t} \Pi_m(t, \omega, \omega_1, \omega_2, x, y) dt,$$

with

$$\Pi_m(t, \omega, \omega_1, \omega_2, x, y) = E(1_{\{t < L_m\}} e^{-\omega W(t)} e^{-\omega_1 S'(t)} e^{-\omega_2 S''(t)} x^{N(t)} y^{B(t)}). \quad (2.2)$$

We have

$$\pi_m(\xi, \omega, \omega_1, \omega_2, x, y) = \frac{1}{\beta(\omega)} \pi_m(\xi, 0, \omega_1, \omega + \omega_2, x\beta(\omega), y). \quad (2.3)$$

In fact the result holds for every t (that is, when replacing π_m by Π_m). It is easy to see this from the above formulation, in particular from $W(t) = S''(t) + \sum_{i=2}^{N(t)} b_{n+i}$, $S'(t) = t - B_n$, $S''(t) = B_{n+1} - t$ and $B(t) = n$ on $\{L_m > t\} \cap \{B_n \leq t < B_{n+1}\}$, noting that $\{L_m > t\} \cap \{B_n \leq t < B_{n+1}\} = \{B_n \leq t < B_{n+1}\}$ for $n \leq m-1$ and

$$\{L_m > t\} \cap \{B_n \leq t < B_{n+1}\} = \{B_n \leq t < B_{n+1}\} \cap \bigcap_{k=1}^{n-m+1} \{A_k < B_{m+k-1}\} \\ \text{for } n \geq m,$$

conditioning on $a_1, \dots, a_n, b_1, \dots, b_{n+1}$ and $N(t)$, it all becomes clear.

In $\pi_m(\xi, \omega, \omega_1, \omega_2, x, y)$ the virtual waiting time is included. From the derivation, it is clear that only $\pi_m(\xi, 0, \omega_1, \omega_2, x, y)$ is needed to calculate this quantity. Hence we can forget about the virtual waiting time in the rest of this derivation. Note that $\{t < L_m\} \cap \{B_n \leq t < B_{n+1}\} = \{B(L_m) > n\} \cap \{B_n \leq t < B_{n+1}\}$. We have

$$\begin{aligned}
\pi_m(\xi, 0, \omega_1, \omega_2, x, y) &= \sum_{n=0}^{\infty} y^n P\{B(L_m) > n, B_n \leq T_\xi < B_{n+1}, \text{ no } C_x \text{ at } T_\xi, \\
&\quad T_\xi - B_n < T_{\omega_1}, B_{n+1} - T_\xi < T_{\omega_2}\} \\
&= \sum_{n=0}^{\infty} y^n P\{B(L_m) > n, \text{ no } C_\xi \text{ in } B_n, \text{ no } C_x \text{ at } B_n + 0\} \\
&\quad \times P\{T_\xi < b_1, \text{ no } C_x \text{ in } T_\xi, T_\xi < T_{\omega_1}, b_1 - T_\xi < T_{\omega_2}\} \\
&= \sum_{n=0}^{\infty} y^n \{\epsilon_{m,n}(\xi, x) - \epsilon_{m,n}(\xi, 0)\} \int_0^\infty \int_0^s e^{-\lambda(1-x)t} \\
&\quad \times e^{-\omega_1 t} e^{-\omega_2(s-t)} \xi e^{-\xi t} dt ds P\{b_1 \leq s\}, \tag{2.4}
\end{aligned}$$

where we put

$$\epsilon_{m,n}(\xi, x) = P\{B(L_m) \geq n, \text{ no } C_\xi \text{ in } B_n, \text{ no } C_x \text{ at } B_n + 0\}. \tag{2.5}$$

For $n \geq 0$ we have

$$x\epsilon_{m,n+1}(\xi, x) = \{\epsilon_{m,n}(\xi, x) - \epsilon_{m,n}(\xi, 0)\}\beta(\xi + \lambda(1-x)). \tag{2.6}$$

This relation can easily be proved with or without interpretation (see the appendix for a proof by interpretation). Note that

$$\epsilon_{m,0}(\xi, x) = P\{\text{no } C_x \text{ at } 0\} = x^m, \tag{2.7}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} \epsilon_{m,n}(\xi, 0) y^n &= \sum_{n=1}^{\infty} P\{B(L_m) = n, \text{ no } C_\xi \text{ in } B_n\} y^n = E e^{-\xi L_m} y^{B(L_m)} = \delta(\xi, y)^m, \\
\end{aligned} \tag{2.8}$$

where $\delta(\xi, y) = E e^{\xi L_1} y^{B(L_1)}$. From (2.4)–(2.8) we get

$$\pi_m(\xi, 0, \omega_1, \omega_2, x, y) = \frac{\xi x \{\beta(\omega_2) - \beta(\xi + \omega_1 + \lambda - \lambda x)\}}{\xi + \omega_1 - \omega_2 + \lambda - \lambda x} \cdot \frac{x^m - \delta(\xi, y)^m}{x - y\beta(\xi + \lambda - \lambda x)}. \tag{2.9}$$

$\pi_m(\xi, \omega, \omega_1, \omega_2, x, y)$ is now given by

$$\begin{aligned}
\pi_m(\xi, \omega, \omega_1, \omega_2, x, y) &= \frac{\xi x \{\beta(\omega + \omega_2) - \beta(\xi + \omega_1 + \lambda - \lambda x\beta(\omega))\}}{\xi - \omega + \omega_1 - \omega_2 + \lambda - \lambda x\beta(\omega)} \\
&\quad \times \frac{[x\beta(\omega)]^m - \delta(\xi, y)^m}{x\beta(\omega) - y\beta(\xi + \lambda - \lambda x\beta(\omega))}. \tag{2.10}
\end{aligned}$$

Remark 1

$\pi_1(\xi, 0, \omega_1, \omega_2, x, y)$ can be obtained from (4.18) from Runnenburg [11]. One can make use of this result to obtain $\pi_m(\xi, 0, \omega_1, \omega_2, x, y)$ for $m \geq 2$ by observing

$$\pi_m(\xi, 0, \omega_1, \omega_2, x, y) = x^{m-1} \pi_1(\xi, 0, \omega_1, \omega_2, x, y) + \delta(\xi, y) \pi_{m-1}(\xi, 0, \omega_1, \omega_2, x, y). \quad (2.11)$$

To see this we write

$$\pi_m(\xi, 0, \omega_1, \omega_2, x, y) = P(A) = P(A \cap \{T_\xi < L_1^*\}) + P(A \cap \{L_1^* \leq T_\xi < L_m\}),$$

where

$$A = \{T_\xi < L_m, S'(T_\xi) < T_{\omega_1}, S''(T_\xi) < T_{\omega_2}, \text{no } C_x \text{ at } T_\xi, B(T_\xi) \leq R_y\}$$

and L_1^* is the first time at which the number of customers in the system drops to $m - 1$.

Then $(A \cap \{T_\xi < L_1^*\})$ is equal to the first summand on the right hand side of (2.11), while $P(A \cap \{L_1^* \leq T_\xi < L_m\})$ is equal to the second one.

Remark 2

One of the referees suggested to derive $\pi_1(\xi, 0, \omega_1, \omega_2, x, y)$ by using a martingale argument. In fact his martingale argument is used to calculate

$$\sum_{n=0}^{\infty} \{\epsilon_{1,n}(\xi, x) - \epsilon_{1,n}(\xi, 0)\} y^n = E \sum_{n=0}^{B(L_1)-1} e^{-\xi B_n} x^{1+A(B_n)} \left(\frac{y}{x}\right)^n$$

by observing that

$$M_n = \left[1 - \frac{y}{x} \beta(\xi + \lambda(1-x))\right] \sum_{k=1}^{n-1} e^{-\xi B_k} x^{1+A(B_k)} \left(\frac{y}{x}\right)^k + e^{-\xi B_n} x^{1+A(B_n)} \left(\frac{y}{x}\right)^n - 1$$

is a zero mean martingale and $B(L_1)$ is a stopping time.

3. M/G/1 with multiple vacation

In a multiple vacation model with exhaustive service, the server takes a vacation as soon as the system becomes empty. On returning from a vacation if the server finds at least one customer waiting in the system, he immediately starts exhaustive service. Otherwise he takes another vacation. Vacations are taken repeatedly until at least one customer arrives. The lengths of vacations are i.i.d. non-negative random variables with the common LST $V^*(\xi) \neq 1$. Assume that at time 0 there are $m \geq 0$ customers in the system. If $m = 0$, the first vacation starts at time 0. If $m \geq 1$, the service of the first customer starts at 0.

For this vacation model we study the joint time-dependent distribution for a number of random variables including the virtual waiting time, the number of customers present in the system, the number of departures, the elapsed and remaining service/vacation times. We shall use (2.10) to compute (3.2).

The server state at time t is specified by the random variable

$$I(t) = \begin{cases} 0 & \text{if the server is on vacation at } t, \\ 1 & \text{if the server is busy at } t. \end{cases} \quad (3.1)$$

For a fixed $t \geq 0$ we are interested in the random variables

$B(t)$: the number of departures in $(0, t]$,

$N(t)$: the number of customers present at time t ,

$S'(t)$: the elapsed service time of the customer being served at t if $I(t) = 1$ or 0 otherwise,

$S''(t)$: the remaining service time of the customer being served at t if $I(t) = 1$ or 0 otherwise,

$V'(t)$: the elapsed vacation time of the server on vacation at t if $I(t) = 0$ or 0 otherwise,

$V''(t)$: the remaining vacation time of the server on vacation at t if $I(t) = 0$ or 0 otherwise,

$W(t)$: the virtual waiting time at t .

We intend to calculate the probabilities

$$P(\xi, \omega, \omega_1, \omega_2, x, y) = P\{I(T_\xi) = 1, W(T_\xi) < T_\omega, S'(T_\xi) < T_{\omega_1}, S''(T_\xi) < T_{\omega_2}, \text{no } C_x \text{ at } T_\xi, B(T_\xi) \leq R_y\}, \quad (3.2)$$

$$Q(\xi, \omega, \omega_1, \omega_2, x, y) = P\{I(T_\xi) = 0, W(T_\xi) < T_\omega, V'(T_\xi) < T_{\omega_1}, V''(T_\xi) < T_{\omega_2}, \text{no } C_x \text{ at } T_\xi, B(T_\xi) \leq R_y\}. \quad (3.3)$$

Assumptions about auxiliary random variables $T_\omega, T_{\omega_1}, T_{\omega_2}$ and R_y are the same as those in section 2.

Let us begin with the case $N(0) = 0$ and the first vacation of length V starts at time 0. Let C be the time between 0 and the moment at which the second vacation starts. Clearly $C = V$ if there are no customers arriving during V . If $k \geq 1$ customers arrive during V , then C is equal to V plus the length of the busy period that starts with these k customers. We denote by $P^*(\xi, \omega, \omega_1, \omega_2, x, y)$ and $Q^*(\xi, \omega, \omega_1, \omega_2, x, y)$, respectively, the probabilities of the following events

$$\{T_\xi < C, I(T_\xi) = 1, W(T_\xi) < T_\omega, S'(T_\xi) < T_{\omega_1}, S''(T_\xi) < T_{\omega_2}, \text{no } C_x \text{ at } T_\xi, B(T_\xi) \leq R_y\}, \quad (3.4)$$

$$\{T_\xi \leq C, I(T_\xi) = 0, W(T_\xi) < T_\omega, V'(T_\xi) < T_{\omega_1}, V''(T_\xi) < T_{\omega_2}, \text{no } C_x \text{ at } T_\xi, B(T_\xi) \leq R_y\}. \quad (3.5)$$

As for the derivation, it suffices to compute $P^*(\xi, \omega, \omega_1, \omega_2, x, y)$ and $Q^*(\xi, \omega, \omega_1, \omega_2, x, y)$. The reason is that one can write

$$\begin{aligned} P(\xi, \omega, \omega_1, \omega_2, x, y) &= \sum_{n=0}^{\infty} [P\{C < T_\xi, B(C) \leq R_y\}]^n P^*(\xi, \omega, \omega_1, \omega_2, x, y) \\ &= \frac{P^*(\xi, \omega, \omega_1, \omega_2, x, y)}{1 - P\{C < T_\xi, B(C) \leq R_y\}}, \end{aligned} \quad (3.6)$$

and similarly

$$Q(\xi, \omega, \omega_1, \omega_2, x, y) = \frac{Q^*(\xi, \omega, \omega_1, \omega_2, x, y)}{1 - P\{C < T_\xi, B(C) \leq R_y\}}, \quad (3.7)$$

with

$$P\{C < T_\xi, B(C) \leq R_y\} = V^*(\xi + \lambda - \lambda\delta(\xi, y)). \quad (3.8)$$

Now we compute $P^*(\xi, \omega, \omega_1, \omega_2, x, y)$ and $Q^*(\xi, \omega, \omega_1, \omega_2, x, y)$. These are given by

$$\begin{aligned} P^*(\xi, \omega, \omega_1, \omega_2, x, y) &= \sum_{k=1}^{\infty} \pi_k(\xi, \omega, \omega_1, \omega_2, x, y) \int_{0-}^{\infty} e^{-\lambda v} \frac{(\lambda v)^k}{k!} e^{-\xi v} d_v P\{V \leq v\} \\ &= \frac{\xi x \{\beta(\omega + \omega_2) - \beta(\xi + \omega_1 + \lambda - \lambda x \beta(\omega))\}}{\xi - \omega + \omega_1 - \omega_2 + \lambda - \lambda x \beta(\omega)} \\ &\quad \times \frac{V^*(\xi + \lambda - \lambda x \beta(\omega)) - V^*(\xi + \lambda - \lambda \delta(\xi, y))}{x \beta(\omega) - y \beta(\xi + \lambda - \lambda x \beta(\omega))}, \end{aligned} \quad (3.9)$$

$$\begin{aligned} Q^*(\xi, \omega, \omega_1, \omega_2, x, y) &= \sum_{k=0}^{\infty} \int_{0-}^{\infty} \int_0^v e^{-\lambda t} \frac{(\lambda t)^k}{k!} e^{-\omega(v-t)} \beta(\omega)^k e^{-\omega_1 t} e^{-\omega_2(v-t)} x^k \xi e^{-\xi t} dt d_v P\{V \leq v\} \\ &= \frac{\xi \{V^*(\omega + \omega_2) - V^*(\xi + \omega_1 + \lambda - \lambda x \beta(\omega))\}}{\xi - \omega + \omega_1 - \omega_2 + \lambda - \lambda x \beta(\omega)}, \end{aligned}$$

where $\pi_k(\xi, \omega, \omega_1, \omega_2, x, y)$ is given in (2.10).

From (3.6)–(3.10) we have for the case that $N(0) = 0$ and the first vacation starts at 0

$$\begin{aligned} P(\xi, \omega, \omega_1, \omega_2, x, y) &= \frac{1}{1 - V^*(\xi + \lambda - \lambda \delta(\xi, y))} \\ &\quad \times \frac{\xi x \{\beta(\omega + \omega_2) - \beta(\xi + \omega_1 + \lambda - \lambda x \beta(\omega))\}}{\xi - \omega + \omega_1 - \omega_2 + \lambda - \lambda x \beta(\omega)} \\ &\quad \times \frac{V^*(\xi + \lambda - \lambda x \beta(\omega)) - V^*(\xi + \lambda - \lambda \delta(\xi, y))}{x \beta(\omega) - y \beta(\xi + \lambda - \lambda x \beta(\omega))}, \end{aligned} \quad (3.11)$$

$$\begin{aligned}
Q(\xi, \omega, \omega_1, \omega_2, x, y) \\
= \frac{1}{1 - V^*(\xi + \lambda - \lambda\delta(\xi, y))} \cdot \frac{\xi\{V^*(\omega + \omega_2) - V^*(\xi + \omega_1 + \lambda - \lambda x\beta(\omega))\}}{\xi - \omega + \omega_1 - \omega_2 + \lambda - \lambda x\beta(\omega)}.
\end{aligned} \quad (3.12)$$

If $L(0) = m \geq 1$ and a service starts at time 0, then it takes a time L_m (length of the busy period starting at 0 with m customers) until the first vacation starts. In this case we have

$$\begin{aligned}
P(\xi, \omega, \omega_1, \omega_2, x, y) &= \pi_m(\xi, \omega, \omega_1, \omega_2, x, y) \\
&+ P\{L_m < T_\xi, B(L_m) \leq R_y\} \frac{P^*(\xi, \omega, \omega_1, \omega_2, x, y)}{1 - V^*(\xi + \lambda - \lambda\delta(\xi, y))},
\end{aligned} \quad (3.13)$$

$$Q(\xi, \omega, \omega_1, \omega_2, x, y) = P\{L_m < T_\xi, B(L_m) \leq R_y\} \frac{Q^*(\xi, \omega, \omega_1, \omega_2, x, y)}{1 - V^*(\xi + \lambda - \lambda\delta(\xi, y))}. \quad (3.14)$$

Substituting (2.10), (3.9) and (3.10) into (3.13) and (3.14), noting that $P\{L_m < T_\xi, B(L_m) \leq R_y\} = \delta(\xi, y)^m$, we get

$$\begin{aligned}
P(\xi, \omega, \omega_1, \omega_2, x, y) &= \left[[x\beta(\omega)]^m + \delta(\xi, y)^m \frac{V^*(\xi + \lambda - \lambda x\beta(\omega)) - 1}{1 - V^*(\xi + \lambda - \lambda\delta(\xi, y))} \right] \\
&\times \frac{\xi x\{\beta(\omega + \omega_2) - \beta(\xi + \omega_1 + \lambda - \lambda x\beta(\omega))\}}{\xi - \omega + \omega_1 - \omega_2 + \lambda - \lambda x\beta(\omega)} \cdot \frac{1}{x\beta(\omega) - y\beta(\xi + \lambda - \lambda x\beta(\omega))},
\end{aligned} \quad (3.15)$$

$$\begin{aligned}
Q(\xi, \omega, \omega_1, \omega_2, x, y) &= \frac{\delta(\xi, y)^m}{1 - V^*(\xi + \lambda - \lambda\delta(\xi, y))} \\
&\times \frac{\xi\{V^*(\omega + \omega_2) - V^*(\xi + \omega_1 + \lambda - \lambda x\beta(\omega))\}}{\xi - \omega + \omega_1 - \omega_2 + \lambda - \lambda x\beta(\omega)}.
\end{aligned} \quad (3.16)$$

Remark 1

Some known time-dependent results can be obtained from (3.11), (3.12), (3.15) and (3.16). If we write

$$\Omega(\xi, \omega_2, x) = \int_0^\infty e^{-\xi t} E(\mathbf{1}_{\{I(t)=0\}} e^{-\omega_2 V''(t)} x^{N(t)}) dt, \quad (3.17)$$

$$\Pi(\xi, \omega_2, x) = \int_0^\infty e^{-\xi t} E(\mathbf{1}_{\{I(t)=1\}} e^{-\omega_2 S''(t)} x^{N(t)}) dt, \quad (3.18)$$

then

$$\Omega(\xi, \omega_2, x) = \frac{1}{\xi} Q(\xi, 0, 0, \omega_2, x, 1), \quad (3.19)$$

$$\Pi(\xi, \omega_2, x) = \frac{1}{\xi} P(\xi, 0, 0, \omega_2, x, 1), \quad (3.20)$$

which are known results in Takagi [14] (there he uses a supplementary variable technique). The time-dependent result for the virtual waiting time is given by

$$\begin{aligned} W^*(\xi, \omega) &= \int_0^\infty e^{-\xi t} E e^{-\omega W(t)} dt = \frac{1}{\xi} [P(\xi, \omega, 0, 0, 1, 1) + Q(\xi, \omega, 0, 0, 1, 1)] \\ &= \frac{\beta(\omega)^m}{\xi - \omega + \lambda - \lambda\beta(\omega)} - \frac{[1 - V^*(\omega)]\delta(\xi, 1)^m}{\{\xi - \omega + \lambda - \lambda\beta(\omega)\}\{1 - V^*(\xi + \lambda - \lambda\delta(\xi, 1))\}}, \end{aligned} \quad (3.21)$$

which is a known result in Takagi [15] (formula (3.4)).

Remark 2

The steady-state limit of the joint distribution of the random variables $N(t)$, $S'(t)$, $S''(t)$, $V'(t)$, $V''(t)$ and $W(t)$ can be obtained from (3.15) and (3.16) by applying an Abelian theorem.

4. $M/G/1$ with single vacation

In a single vacation, the server takes a vacation once after each busy period. The lengths of vacations are i.i.d. non-negative random variables that are independent of the arrival and service processes. We denote the LST of a vacation time by $V^*(\xi)$. Assume that at time 0 there are $m \geq 0$ customers in the system. We consider two cases:

- (i) $m = 0$ and the first vacation starts at time 0,
- (ii) $m \geq 1$ and the service of the first customer starts at 0.

The server state is specified by

$$I(t) = \begin{cases} 0 & \text{if the server is on vacation at } t, \\ 1 & \text{if the server is busy at } t, \\ 2 & \text{if the server is idle at } t. \end{cases}$$

Random variables $B(t)$, $N(t)$, $S'(t)$, $S''(t)$, $V'(t)$, $V''(t)$ and $W(t)$ are defined in the same way as in section 3. We wish to find the probabilities

$$P_0(\xi, y) = P\{I(T_\xi) = 2, B(T_\xi) \leq R_y\}, \quad (4.1)$$

$$\begin{aligned} P(\xi, \omega, \omega_1, \omega_2, x, y) &= P\{I(T_\xi) = 1, W(T_\xi) < T_\omega, S'(T_\xi) < T_{\omega_1}, \\ &\quad S''(T_\xi) < T_{\omega_2}, \text{no } C_x \text{ at } T_\xi, B(T_\xi) \leq R_y\}, \end{aligned} \quad (4.2)$$

$$\begin{aligned} Q(\xi, \omega, \omega_1, \omega_2, x, y) &= P\{I(T_\xi) = 0, W(T_\xi) < T_\omega, V'(T_\xi) < T_{\omega_1}, \\ &\quad V''(T_\xi) < T_{\omega_2}, \text{no } C_x \text{ at } T_\xi, B(T_\xi) \leq R_y\}. \end{aligned} \quad (4.3)$$

Following the reasoning of section 3, we obtain for both cases (i) and (ii) (we do not give the details)

$$P_0(\xi, y) = \frac{\delta(\xi, y)^m}{1 - C^*(\xi, y)} V^*(\xi + \lambda) \frac{\xi}{\xi + \lambda}, \quad (4.4)$$

$$\begin{aligned} P(\xi, \omega, \omega_1, \omega_2, x, y) &= \frac{\xi x \{ \beta(\omega + \omega_2) - \beta(\xi + \omega_1 + \lambda - \lambda x \beta(\omega)) \}}{\xi - \omega + \omega_1 - \omega_2 + \lambda - \lambda x \beta(\omega)} \\ &\quad \times \frac{1}{x \beta(\omega) - y \beta(\xi + \lambda - \lambda x \beta(\omega))} \\ &\quad \times \left\{ [x \beta(\omega)]^m - \delta(\xi, y)^m + \frac{\delta(\xi, y)^m}{1 - C^*(\xi, y)} \left[V^*(\xi + \lambda - \lambda x \beta(\omega)) \right. \right. \\ &\quad \left. \left. - V^*(\xi + \lambda - \lambda \delta(\xi, y)) \right. \right. \\ &\quad \left. \left. + V^*(\xi + \lambda) \frac{\lambda}{\xi + \lambda} [x \beta(\omega) - \delta(\xi, y)] \right] \right\}, \end{aligned} \quad (4.5)$$

$$Q(\xi, \omega, \omega_1, \omega_2, x, y) = \frac{\delta(\xi, y)^m}{1 - C^*(\xi, y)} \cdot \frac{\xi \{ V^*(\omega + \omega_2) - V^*(\xi + \omega_1 + \lambda - \lambda x \beta(\omega)) \}}{\xi - \omega + \omega_1 - \omega_2 + \lambda - \lambda x \beta(\omega)}, \quad (4.6)$$

where

$$C^*(\xi, y) = V^*(\xi + \lambda) \frac{\lambda}{\xi + \lambda} \delta(\xi, y) + V^*(\xi + \lambda - \lambda \delta(\xi, y)) - V^*(\xi + \lambda). \quad (4.7)$$

Remark

Using (4.4)–(4.7) we obtain some time-dependent results. For example

$$\begin{aligned} W^*(\xi, \omega) &= \int_0^\infty e^{-\xi t} E e^{-\omega W(t)} dt = \frac{1}{\xi} [P_0(\xi, 1) + P(\xi, \omega, 0, 0, 1, 1) + Q(\xi, \omega, 0, 0, 1, 1)] \\ &= \frac{\beta(\omega)^m}{\xi - \omega + \lambda - \lambda \beta(\omega)} - \frac{1}{\xi - \omega + \lambda - \lambda \beta(\omega)} \\ &\quad \times \frac{\delta(\xi, 1)^m \{ (\xi + \lambda) [1 - V^*(\omega)] + \omega V^*(\xi + \lambda) \}}{(\xi + \lambda) [1 - V^*(\xi + \lambda - \lambda \delta(\xi, 1))] + V^*(\xi + \lambda) [\xi + \lambda - \lambda \delta(\xi, 1)]}. \end{aligned} \quad (4.8)$$

The result for $W^*(\xi, \omega)$, case (ii) and $m = 1$, given in Takagi [15] is incorrect. This can be easily checked by observing (we assume the DF of a service time and the DF of a vacation time are continuous at 0)

$$P_0(t) = P\{\text{server is idle at } t\} = P\{W(t) = 0\} = \lim_{\omega \rightarrow \infty} Ee^{-\omega W(t)}.$$

We have

$$P_0^*(\xi) = \int_0^{\infty} e^{-\xi t} P_0(t) dt = \lim_{\omega \rightarrow \infty} \int_0^{\infty} e^{-\xi t} Ee^{-\omega W(t)} dt = \lim_{\omega \rightarrow \infty} W^*(\xi, \omega).$$

P_0^* is easy to calculate. It is given by (5.4) (see below).

From Takagi [15] we get ($\delta_{11} = 1$ and $\delta_{m1} = 0$ for $m \neq 1$)

$$\lim_{\omega \rightarrow \infty} W^*(\xi, \omega) = \frac{\delta(\xi, 1)^m V^*(\xi + \lambda) + \delta_{m1} \beta(\xi + \lambda) [1 - V^*(\xi + \lambda - \lambda \delta(\xi, 1))]}{(\xi + \lambda) [1 - V^*(\xi + \lambda - \lambda \delta(\xi, 1))] + [\xi + \lambda - \lambda \delta(\xi, 1)] V^*(\xi + \lambda)},$$

which differs from $P_0^*(\xi)$ if $m = 1$.

5. Some additional results

In sections 3 and 4 we were able to find the LST of the virtual waiting time at the exponential time T_ξ for the multiple vacation model and the single vacation model. In this section we give a representation for the LST of the virtual waiting time at a fixed time t for the above models.

LEMMA 5.1

For the multiple vacation model described in section 3, the Laplace–Stieltjes transform of the virtual waiting time $W(t)$ is given by

$$Ee^{-\xi W(t)} = e^{(\xi - \lambda + \lambda \beta(\xi))t} \left[\beta(\xi)^m - [1 - V^*(\xi)] \int_{0-}^t e^{-(\xi - \lambda + \lambda \beta(\xi))u} dF(u) \right], \quad (5.1)$$

where $F(t)$ is a non-decreasing, right-continuous function and

$$F^*(\xi) = \int_{0-}^{\infty} e^{-\xi t} dF(t) = \frac{\delta(\xi, 1)^m}{1 - V^*(\xi + \lambda - \lambda \delta(\xi, 1))}. \quad (5.2)$$

LEMMA 5.2

For the single vacation model described in section 4, the Laplace–Stieltjes transform of the virtual waiting time $W(t)$ is given by

$$Ee^{-\xi W(t)} = e^{(\xi - \lambda + \lambda\beta(\xi))t} \left[\beta(\xi)^m - \xi \int_0^t e^{-(\xi - \lambda + \lambda\beta(\xi))u} P_0(u) du \right. \\ \left. - [1 - V^*(\xi)] \int_{0-}^t e^{-(\xi - \lambda + \lambda\beta(\xi))u} dF(u) \right], \quad (5.3)$$

where $P_0(t) = P\{\text{server is idle at } t\}$, $F(t)$ is a non-decreasing, right-continuous function and

$$P_v^*(\xi) = \int_0^\infty e^{-\xi t} P_0(t) dt \\ = \frac{\delta(\xi, 1)^m V^*(\xi + \lambda)}{(\xi + \lambda)[1 - V^*(\xi + \lambda - \lambda\delta(\xi, 1))] + V^*(\xi + \lambda)[\xi + \lambda - \lambda\delta(\xi, 1)]}, \quad (5.4)$$

$$[1 - V^*(\xi + \lambda - \lambda\delta(\xi, 1))]F^*(\xi) = \delta(\xi, 1)^m - [\xi + \lambda - \lambda\delta(\xi, 1)]P_0^*(\xi). \quad (5.5)$$

Proof

We only give a proof for lemma 5.2. Lemma 5.1 can be proved similarly. We introduce the following events which depend on t :

- A: no C_ξ happens during the service time of the customers who arrive before t (including $m \geq 0$ customers present at 0),
- B: catastrophe happens for the first time at a time u at which the server is idle and after that no C_ξ happens during the remaining service time of the customers who arrive before t , $0 \leq u < t$,
- C: catastrophe happens for the first time during a vacation that starts at a time u and after that no C_ξ happens during the remaining service time of the customers who arrive before t , $0 \leq u < t$,
- D: no C_ξ happens in $[0, t + W(t)]$.

As event A is clearly the conjunction of the disjoint events B, C and D, we have

$$P(A) = P(B) + P(C) + P(D). \quad (5.6)$$

For event A we have

$$P(A) = \beta(\xi)^m \sum_{i=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^i}{i!} \beta(\xi)^i = \beta(\xi)^m e^{-\lambda(1-\beta(\xi))t}, \quad (5.7)$$

because during a service time with probability $\beta(\xi)$ no C_ξ happens.

As the probability that no C_ξ happens during the service time of the customers who arrive in the interval (u, t) is equal to $e^{-\lambda(1-\beta(\xi))(t-u)}$ (see (5.7)), we have

$$P(B) = \int_0^t P_0(u) e^{-\lambda(1-\beta(\xi))(t-u)} \xi e^{-\xi u} du. \quad (5.8)$$

Let X_n be the time at which the n th vacation starts. We define

$$F(u) = \sum_{n=1}^{\infty} P\{X_n \leq u\}. \quad (5.9)$$

If a vacation starts at time $u < t$, the first catastrophe happens during this vacation with probability $e^{-\xi u}[1 - V^*(\xi)]$, while $e^{-\lambda(1-\beta(\xi))(t-u)}$ is the probability that no C_ξ happens during the service time of the customers who arrive in (u, t) . Therefore

$$P(C) = [1 - V^*(\xi)] \int_{0-}^t e^{-\xi u} e^{-\lambda(1-\beta(\xi))(t-u)} dF(u). \quad (5.10)$$

Finally for event D we have

$$P(D) = P\{t + W(t) < T_\xi\} = \int_{0-}^{\infty} e^{-\xi(t+w)} dP\{W(t) \leq w\} = e^{-\xi t} E e^{-\xi W(t)}. \quad (5.11)$$

From (5.6)–(5.8), (5.10) and (5.11) we get (5.3).

To prove relation (5.4) we note that $\xi P_0^*(\xi)$ can be interpreted as the probability that the server is idle at T_ξ . It is easy to see that

$$\xi P_0^*(\xi) = \sum_{i=0}^{\infty} \delta(\xi, 1)^m C^*(\xi)^i V^*(\xi + \lambda) \frac{\xi}{\xi + \lambda} = \frac{\delta(\xi, 1)^m}{1 - C^*(\xi)} V^*(\xi + \lambda) \frac{\xi}{\xi + \lambda}, \quad (5.12)$$

where $C^*(\xi)$ is the probability that no C_ξ happens in the time between two successive moments at which vacations start. Probability $C^*(\xi)$ is given by

$$C^*(\xi) = V^*(\xi + \lambda) \frac{\lambda}{\xi + \lambda} \delta(\xi, 1) + V^*(\xi + \lambda - \lambda \delta(\xi, 1)) - V^*(\xi + \lambda). \quad (5.13)$$

From (5.12) and (5.13) we get (5.4). Relation (5.5) can be obtained from (5.3) and (5.4) without (5.9). To see this we first replace ξ in (5.3) by $\xi + \lambda - \lambda \delta(\xi, 1)$, noting that $\delta(\xi, 1) = \beta(\xi + \lambda - \lambda \delta(\xi, 1))$, then multiply both sides by $e^{-\xi t}$, and finally let $t \rightarrow \infty$. We then have

$$[1 - V^*(\xi + \lambda - \lambda \delta(\xi, 1))] F^*(\xi) = \delta(\xi, 1)^m - [\xi + \lambda - \lambda \delta(\xi, 1)] P_0^*(\xi). \quad (5.14)$$

This ends the proof of lemma 5.2. □

Remark 1

If $V^*(\xi) \equiv 1$, equation (5.3) becomes the well known Takács equation (see e.g. Takács [13, p. 51]). A probabilistic interpretation for this equation was given by Runnenburg [9].

Remark 2

From (5.3) we find

$$\begin{aligned}
 W^*(\xi, \omega) &= \int_0^\infty e^{-\xi t} E e^{-\omega W(t)} dt \\
 &= \frac{1}{\xi - \omega + \lambda - \lambda\beta(\omega)} \{ \beta(\omega)^m - \omega P_0^*(\xi) - [1 - V^*(\omega)] F^*(\xi) \} \\
 &= \frac{\beta(\omega)^m}{\xi - \omega + \lambda - \lambda\beta(\omega)} - \frac{1}{\xi - \omega + \lambda - \lambda\beta(\omega)} \\
 &\quad \times \frac{\delta(\xi, 1)^m \{ (\xi + \lambda)[1 - V^*(\omega)] + \omega V^*(\xi + \lambda) \}}{(\xi + \lambda)[1 - V^*(\xi + \lambda - \lambda\delta(\xi, 1))] + V^*(\xi + \lambda)[\xi + \lambda - \lambda\delta(\xi, 1)]},
 \end{aligned} \tag{5.15}$$

which is identical with (4.8).

Appendix*Proof of relation (2.6)*

$$\begin{aligned}
 &\text{Because of the independence of } b_{n+1} \text{ and } (1_{\{B(L_m) \geq n\}}, B_n) \text{ we have for } n \geq 0 \\
 &P\{B(L_m) \geq n, \text{ no } C_\xi \text{ in } B_n, \text{ no } C_x \text{ at } B_n + 0, \text{ no } C_\xi \text{ in } b_{n+1}, \text{ no } C_x \text{ in } b_{n+1}\} \\
 &= P\{B(L_m) \geq n, \text{ no } C_\xi \text{ in } B_n, \text{ no } C_x \text{ at } B_n + 0\} P\{\text{no } C_\xi \text{ in } b_{n+1}, \text{ no } C_x \text{ in } b_{n+1}\} \\
 &= \epsilon_{m,n}(\xi, x) \beta(\xi + \lambda(1 - x)).
 \end{aligned} \tag{*}$$

On the other hand,

$$\begin{aligned}
 &P\{B(L_m) \geq n, \text{ no } C_\xi \text{ in } B_n, \text{ no } C_x \text{ at } B_n + 0, \text{ no } C_\xi \text{ in } b_{n+1}, \text{ no } C_x \text{ in } b_{n+1}\} \\
 &= P\{B(L_m) = n, \text{ no } C_\xi \text{ in } B_n\} P\{\text{no } C_\xi \text{ in } b_{n+1}, \text{ no } C_x \text{ in } b_{n+1}\} \\
 &\quad + P\{B(L_m) \geq n + 1, \text{ no } C_\xi \text{ in } B_{n+1}, \text{ no } C_x \text{ at } B_{n+1} + 0, (n + 1)\text{st customer} \\
 &\quad \text{is unmarked}\} \\
 &= \epsilon_{m,n}(\xi, 0) \beta(\xi + \lambda(1 - x)) + x P\{B(L_m) \geq n + 1, \text{ no } C_\xi \text{ in } B_{n+1}, \text{ no } C_x \text{ at } B_{n+1} + 0\} \\
 &= \epsilon_{m,n}(\xi, 0) \beta(\xi + \lambda(1 - x)) + x \epsilon_{m,n+1}(\xi, x).
 \end{aligned} \tag{**}$$

Relation (2.6) follows from (*) and (**).

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