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Super-Toda lattices

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The Lax formalism, as described by Oevel *et al.* and in an earlier and more fundamental form by Semenov, Kostant, Symes, and Adler, can easily be generalized to the case where anticommuting variables are involved, the so-called supercase. In this article this super-Lax formalism is applied to the well-known associative superalgebra $G = \text{Mat}(m, n, \Lambda)$. Subspaces of G to which the super-Poisson structures can be restricted arise in a natural way. Taking L in one of these subspaces formally leads to superextensions of the hierarchy of nonrelativistic Toda lattices. In the simplest case, where only nearest-neighbor interaction is involved, the equations are explicitly solved. Furthermore, the relevant two super-Hamiltonian structures are explicitly calculated. Finally a superextension of the relativistic Toda lattice with a super-Hamiltonian structure is described herein.

I. INTRODUCTION

In this article we present a super-Lax formalism for finite dimensional systems, in which real and anticommuting variables are involved. This will be done in the spirit of the formalism in the form in Refs. 1–6. For more information about the supercase, the reader is referred to Refs. 7–9. Our starting point is the associative matrix superalgebra $G = \text{Mat}(m, n, \Lambda)$, where Λ is a Grassmann algebra with some unspecified number of odd generators. In Sec. II we describe the structure of G . Obviously G has a super-Lie algebra structure. Furthermore, in complete analogy with the “real case,” we introduce a second super-Lie structure on G using a direct sum decomposition of G . Next we introduce three super-Poisson structures on G . The first one of these structures is simply the super-Lie Poisson structure associated with the original super-Lie bracket on G . The second one, which will be referred to as “the linear super-Poisson structure” is nothing but the super-Lie Poisson structure associated with the second super-Lie bracket on G . The third one, which will be referred to as “the quadratic super-Poisson structure,” is less obvious and can only be defined on the even part of G . Using these structures we define hierarchies of formal super-Lax equations. Subspaces of G to which the linear and the quadratic super-Poisson structure can be restricted arise in a natural way. This will be used in the following. In Secs. III–IV we investigate the case where the super-Lax matrix L is taken in the simplest of these subspaces. In Sec. III we show that in this particular case the formal super-Lax hierarchy is in fact a hierarchy of nonrelativistic super-Toda lattices, where only nearest-neighbor interaction is involved. We describe how this system can be interpreted in terms of real Toda lattices, see also Ref. 10. In Sec. IV we explicitly solve the equations in a way analogous to Ref. 11, using the concept of a superdeterminant or Berezinian. In Secs. V and VI the super-bi-Hamiltonian structure, coming from the linear and the quadratic super-Poisson structure, is investigated. The two super-Hamiltonian matrices involved are explicitly calculated. Finally, in Sec. VII we introduce a different direct sum decomposition of G . Subspaces of G to which the linear Poisson structure can be restricted are defined. Restricting the Lax matrix L to such a subspace, we arrive at a superextension of the relativistic Toda lattice. We explicitly calculate the super-Hamiltonian structure of this system.

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II. THE STRUCTURE OF G

Remark 2.1: Throughout this article we will denote all parities by the same symbol $p()$. It will be clear from the context which parity is involved. As usual in working with graded objects we will in the following often deal with (parity) homogeneous elements only and extend definitions, formulas, etc., by linearity.

Let $G = \text{Mat}(m, n, \Lambda)$ be the superalgebra of block matrices with entries in some Grassmann algebra Λ .¹²⁻¹⁴ Consider the block matrix $L \in G$

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (1)$$

Then we define the supertrace function on G as follows.

Definition 2.2: The even supertrace function $\text{str}: G \rightarrow \Lambda$ is defined in terms of the ordinary traces of the block matrices A and D by

$$\text{str}(L) = \text{tr}(A) - (-1)^{p(L)} \text{tr}(D) = \sum_{i=1}^{m+n} (-1)^{p(i)(p(L)+1)} L^{ii}, \quad (2)$$

where

$$p(i) = \begin{cases} 0, & i = 1, \dots, m \\ 1, & i = m+1, \dots, m+n. \end{cases} \quad (3)$$

Note that the parity of the entries of L is given by $p(L^{ij}) = p(L) + p(i) + p(j)$. Using Eq. (2) one can easily check that $\text{str}([L, M]) = 0$ which also implies that $\text{str}(L)$ is invariant under coordinate transformations.

Definition 2.3: Let $L \in G$ and suppose there exists a $k \in \{1-m-n, \dots, m+n-1\}$ such that

$$L^{ij} \neq 0 \Rightarrow j = i + k.$$

Then k is called the height of L , notation: $\text{ht}(L) = k$.

Definition 2.3 defines a $\mathbb{Z}$ gradation on G , splitting each matrix in its diagonals.

$$G = G_{1-m-n} \oplus \dots \oplus G_{m+n-1},$$

where

$$G_k = \{L \in G : \text{ht}(L) = k\}.$$

Indeed, if $L \in G_k, M \in G_p$ then $(LM)^{ij} = \sum_{s=1}^{m+n} L^{is} M^{sj}$ can only be nonzero if $s = i + k$ and $j = s + p$, hence $j = i + k + p$ and thus $LM \in G_{k+p}$. For $|k+p| > n+m$ obviously $LM = 0$. We will denote the projection of $L \in G$ to G_k by L_k . Hence $L = \sum_k L_k$ where $L_k \in G_k$. Furthermore for $k \leq s$ we denote $G_{k,s} = G_k \oplus \dots \oplus G_s$ and the projection of an element $L \in G$ to $G_{k,s}$ by $L_{k,s} = \sum_{j=k}^s L_j$.

This immediately leads to the decomposition of G in a direct sum $G = G_+ \oplus G_-$ with

$$G_+ = \{L \in G : L^{ij} = 0, \text{ for } i > j\} = G_{1, m+n-1} = \{\text{strictly upper triangular matrices}\},$$

$$G_- = \{L \in G : L^{ij} = 0, \text{ for } i < j\} = G_{1-m-n, 0} = \{\text{lower triangular matrices}\}.$$

It is clear that the bracket $\langle, \rangle$ associated with the supertrace function defined by

$$\langle L, M \rangle = \text{str}(L.M) = \sum_{i,j=1}^{m+n} (-1)^{(p(L)+p(M)+1)p(i)} L^{ij} M^{ji} \quad (4)$$

is nondegenerate on G . Hence we can identify G with G^* . Now one can easily check that

$$G_{k,s}^* = G_{-s,-k}.$$

In particular

$$G_+^* = G_-^\perp = \{L \in G : L^{ij} = 0, \text{ for } i < j\} = G_{1-m-n,-1} = \{\text{strictly lower triangular matrices}\},$$

$$G_-^* = G_+^\perp = \{L \in G : L^{ij} = 0, \text{ for } i > j\} = G_{0,m+n-1} = \{\text{upper triangular matrices}\}.$$

It is clear that for $k \geq 0$ the subspaces $G_{k,n+m-1}$ and $G_{1-m-n,-k}$ are in fact subsuperalgebras. In particular this holds for G_+ , G_- , G_+^* , and G_-^* .

Now, obviously G is a Lie superalgebra with Lie bracket $[,]$ defined by $[L, M] = LM - (-1)^{p(L)p(M)}ML$. Consider the projection operators $\Pi = \Pi_+ - \Pi_-$ and $\Pi^* = \Pi_+^* - \Pi_-^*$. By means of these projection operators we can define a second Lie bracket $[,]_1$ on G by $[L, M]_1 = [\Pi L, M] + [L, \Pi M] = 2[L_+, M_+] - 2[L_-, M_-]$. These two Lie brackets define two super-Lie Poisson structures on G

$$\{f, g\}(L) = \langle L, [df, dg] \rangle, \quad \{f, g\}_1(L) = \langle L, [df, dg]_1 \rangle,$$

where we assume the functions to be polynomial in the entries of L . For the formal definition of $df \in G$ with parity $p(f)$ see Eq. (6) below. Note that the parity of $\{f, g\}_i$ is given by $p(f) + p(g)$. One can easily check that the bracket (4) satisfies $\langle \Pi L, M \rangle = \langle L, \Pi^* M \rangle$ and has the invariance property $\langle [L, M], N \rangle = \langle L, [M, N] \rangle$. We use this to rewrite the two super-Lie Poisson structures in the form $\{f, g\}_i = \langle P_i(L)df, dg \rangle$, where $P_i(L)df$ is called the super-Hamiltonian vector field associated with Hamiltonian f

$$\{f, g\}(L) = \langle [L, df], dg \rangle,$$

$$\{f, g\}_1(L) = \langle [L, \Pi df] + \Pi^*[L, df], dg \rangle.$$

Note that the parity of the super-Hamiltonian vector fields $P_i(L)df$ is given by $p(L) + p(f)$ and hence the parity of $P_i(L)$ is equal to the parity of L . Following Ref. 2, on the even part G_0 of G we define

$$\{f, g\}_2(L) = \langle [L, \Pi(Ldf + df.L)] + L.\Pi^*([L, df]) + \Pi^*([L, df]).L, dg \rangle. \quad (5)$$

It is not *a priori* obvious that the quadratic bracket (5) defines a super-Poisson structure on the even part G_0 of G . If we restrict ourselves to even functions on G_0 , according to the classical lemma of Ref. 2 we have to check that the projection operator $\Pi_+ - \Pi_+^*$ indeed satisfies the (Yang-Baxter) YB (1) equation. Decomposing G in the direct sum of strictly upper, diagonal, and strictly lower matrices, i.e., $G = G_+ \oplus G_0 \oplus G_+^*$, this follows from a short calculation.

For polynomial functions f the formal definition for the differential $df = df(L) \in G$ of parity $p(f)$ is given by

$$\langle M, df(L) \rangle = \left(\frac{d}{d\epsilon} \right) \bigg|_{\epsilon=0} f(L + \epsilon M), \quad (6)$$

where ϵ is a formal parameter of parity $p(L) + p(M)$.

Using Eqs. (4), (6) and the relation

$$(\epsilon M)^{ij} = (-1)^{p(i)p(\epsilon)} \epsilon M^{ij},$$

one can check that for functions f on G , $df(L) \in G$ is represented by

$$df^{ij} = (-1)^{p(j)(p(L)+p(f)+1)} \frac{\partial f}{\partial L^j},$$

which obviously reduces to

$$df^{ij} = (-1)^{p(j)(p(f)+1)} \frac{\partial f}{\partial L^j} \quad (7)$$

for functions on G_0 .

Hence we have two linear super-(Lie) Poisson structures on G and a quadratic Poisson structure on G_0 . The super-Hamiltonian vector fields $P_1(L)df$ and $P_2(L)df$ are given by

$$\begin{aligned} P_1(L)df &= [L, \Pi df] + \Pi^*[L, df], \\ P_2(L)df &= [L, \Pi(Ldf + df \cdot L)] + L \cdot \Pi^*([L, df]) + \Pi^*([L, df]) \cdot L. \end{aligned} \quad (8)$$

Without mentioning it explicitly we will in the sequel always assume that $L \in G_0$ when dealing with the quadratic bracket. Now, the formal equation for the flow of the super-Hamiltonian vector fields (8) is given by

$$L_t = P_i(L)df. \quad (9)$$

Let us now consider the functions $f_k(L) = (1/k) \text{str}(L^k)$ for $k \geq 1$. For all k the function f_k is even. For odd k this is obvious. For even k this follows from the fact that the supertrace function vanishes on the commutators and for odd L we have $L^{2k} = \frac{1}{2}[L^{2k-1}, L]$. Using Eq. (8), for $f_k(L)$ Eq. (9) gets the super-Lax form

$$L_t = P_1(L)df_{k+1} = \frac{1}{2}P_2(L)df_k = [L, \Pi(L^k)] = 2[L, (L^k)_+] = -2[L, (L^k)_-]. \quad (10)$$

Furthermore, note that all Hamiltonians f_k commute with respect to the brackets $\{, \}_1$ and $\{, \}_2$.

As in the nonsupercase, due to the fact that the supertrace function vanishes on the commutators, we have $\text{str}(P_i(L)dC) = 0$ for all Casimirs and so in particular $0 = \text{str}(L_t) = (\text{str}(L))_t$. Hence to describe the super-Lax equations, after rescaling it is obviously possible to set $\text{str}(L) = 0$ and we could actually work with the following sub-Lie superalgebra of G :

$$\{L \in G : \text{str}(L) = 0\} \subset G,$$

which is the Lie superalgebra usually denoted by $\mathfrak{sl}(m, n, \Lambda)$. This Lie superalgebra or its even part is often referred to as the Grassmann hull of the Lie superalgebra $\mathfrak{sl}(m, n, \mathbb{R})$, which is simple for $m \neq n$, see, e.g., Refs. 12–14. If one imposes the assumption that $n \neq m$ the restriction of the bracket $\langle, \rangle$ to this Lie algebra is also nondegenerate. In the sequel we will not make use of this restriction.

In the next lemma we will show that the above two super-Poisson structures can be restricted to all subspaces $G_{k,s}$ for $k < 0 \leq s$.

Lemma 2.4: *Let $L \in G_{k,s}$ with $k < 0 \leq s$ then*

$$P_1(L)df = P_1(L)(df)_{-s, -k-1} \in G_{k+1, s}, \quad (11)$$

$$P_2(L)df = P_2(L)(df)_{-s, -k} \in G_{k, s}. \quad (12)$$

Proof: Let $L \in G_{k, s}$. We can rewrite the super-Hamiltonian vector fields as follows:

$$P_1(L)df = 2([L, df_+] - \Pi_-^*[L, df]) = -2([L, df_-] - \Pi_+^*[L, df]), \quad (13)$$

$$\begin{aligned} P_2(L)df &= 2([L, (L \cdot df + df \cdot L)_+] - L \cdot \Pi_-^*[L, df] - \Pi_-^*[L, df] \cdot L) \\ &= -2([L, (L \cdot df + df \cdot L)_-] - L \cdot \Pi_+^*[L, df] - \Pi_+^*[L, df] \cdot L). \end{aligned} \quad (14)$$

Obviously $[L, df_+] \in G_{k+1, m+n-1}$ and $\Pi_-^*[L, df] \in G_{0, m+n-1} \subset G_{k+1, m+n-1}$. Hence it follows from Eq. (13) that $P_1(L)df \in G_{k+1, m+n-1}$. On the other hand we have $[L, df_-] \in G_{1-m-n, s}$ and $\Pi_+^*[L, df] \in G_{1-m-n, -1} \subset G_{1-m-n, s}$. So it follows from the second equality in (13) that $P_1(L)df \in G_{1-m-n, s}$. Combining this we get

$$P_1(L)df \in G_{1-m-n, s} \cap G_{k+1, m+n-1} = G_{k+1, s}.$$

For $P_2(L)df$ we can reason in the same way using Eq. (14) which proves the lemma. $\square$

Remark 2.5: It is obvious from Lemma 2.4 that considering super-Lax equations for $L \in G_{k, s}$ we may assume that L_k is constant and furthermore that the linear super-Poisson structure can be restricted to $G_{k+1, s}$. The quadratic Poisson structure however according to this lemma may only be restricted to $G_{k, s}$. Note that for Casimirs we actually do have $P_2(L)dC = P_2(L)(dC)_{-s, -k-1} \in G_{k+1, s}$. Now with the same reduction procedure as used in the case of the (super-Korteweg-de Vries) sKdV(3) equation, see Ref. 7 and 9, we will see later on that the quadratic Poisson bracket can in fact also be restricted to $G_{k+1, s}$.

III. A SUPER-TODA LATTICE

In this section we will investigate the super-Lax equations (10) for a general even Lax matrix $L = L(m, n) \in G_{-1, 1}$ defined by

$$L(m, n) = \begin{pmatrix} p_1 & a_1 & 0 & \cdot & \cdot & 0 \\ c_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_{m+n-1} \\ 0 & \cdot & \cdot & 0 & c_{m+n-1} & p_{m+n} \end{pmatrix}. \quad (15)$$

Obviously all $p_i = p_i(t) = L^{ii}$ are even elements of Λ and $a_i = a_i(t) = L^{i, i+1}$ is even for $i \neq m$ and odd for $i = m$. Following Remark 2.5 we will assume here that the c_i 's are constant, where the even constants $c_i = L^{i+1, i}$ for $i \neq m$ are assumed to be real numbers. In contrary to the odd constant c_m they play an inessential role and can without loss of generality be set equal to 1.

Formally we have the hierarchy of super Lax equations (10). For $k=1$ we calculate explicitly the corresponding equations for the entries p_i and a_i :

$$\begin{aligned} \dot{p}_i &= 2(c_{i-1}a_{i-1} - a_i c_i), \\ \dot{a}_i &= 2(p_i - p_{i+1})a_i, \quad 1 \leq i \leq m+n, \quad a_0 = c_0 = a_{m+n} = c_{m+n} = 0. \end{aligned} \quad (16)$$

Note that by setting $n=0$ and $\Lambda=\mathbf{R}$ these equations reduce (after rescaling) to the equations for the ordinary Toda lattice. Hence this construction yields indeed a generalization of the classical case. The corresponding even Hamiltonian H is given by

$$H = \frac{1}{2} \text{str}(L^2) = \frac{1}{2} \sum_{i,j=1}^{m+n} (-1)^{p(i)} L^{ij} L^{ji} = \sum_{i=1}^{m+n} (-1)^{p(i)} \left(\frac{1}{2} p_i^2 + a_i c_i \right),$$

which can be rewritten as

$$H = \left(\sum_{i=1}^m \frac{1}{2} p_i^2 + \sum_{i=1}^{m-1} a_i c_i \right) - \left(\sum_{i=m+1}^{m+n} \frac{1}{2} p_i^2 + \sum_{i=1}^{m+n-1} a_i c_i \right) + a_m c_m.$$

We will now briefly describe the interpretation of this super-Toda lattice where we set $c_i=1$ for $1 \leq i \leq m+n$, $i \neq m$ and $c_m=\xi$, for more details see Ref. 10. We set

$$a_i = e^{q_i - q_{i+1}}, \quad 1 \leq i \leq m+n-1, \quad i \neq m,$$

$$a_m = \alpha e^{q_m - q_{m+1}},$$

where $q_i \in \Lambda_{\bar{0}}$, $\alpha \in \Lambda_{\bar{1}}$ and $e^{q_i - q_{i+1}} \in \Lambda_{\bar{0}}$ is defined by its Taylor expansion. Then, omitting the inessential factor 2, Eqs. (16) in the coordinates p_i, q_i take the form $\dot{L}(t) = [L, L_+]$. We can rewrite them as follows:

$$\begin{aligned} \dot{q}_i &= p_i, \quad 1 \leq i \leq m+n, \quad q_0 = -q_{m+n+1} = -\infty, \\ \dot{p}_i &= e^{q_{i-1} - q_i} - e^{q_i - q_{i+1}}, \quad m \neq i \neq m+1, \\ \dot{p}_m &= e^{q_{m-1} - q_m} + \gamma e^{q_m - q_{m+1}}, \\ \dot{p}_{m+1} &= \gamma e^{q_m - q_{m+1}} - e^{q_{m+1} - q_{m+2}}, \end{aligned} \tag{17}$$

where $\gamma = \xi \alpha \in \Lambda_{\bar{0}}$, being the product of two odd elements, obviously satisfies $\gamma^2=0$. The system (17) is equivalent to the following:

$$\begin{aligned} \ddot{q}_i &= e^{q_{i-1} - q_i} - e^{q_i - q_{i+1}}, \quad i \neq m, m+1, \\ \ddot{q}_m &= e^{q_{m-1} - q_m} + \gamma e^{q_m - q_{m+1}}, \\ \ddot{q}_{m+1} &= \gamma e^{q_m - q_{m+1}} - e^{q_{m+1} - q_{m+2}}. \end{aligned} \tag{18}$$

We can expand q_i in powers of γ . Since $\gamma^2=0$ we get $q_i = q_i^{(0)} + \gamma q_i^{(1)}$. Substituting this in Eq. (18) and equating the coefficients of 1 and γ we can rewrite Eq. (18) as

$$\begin{aligned} \ddot{q}_i^{(0)} &= e^{q_{i-1}^{(0)} - q_i^{(0)}} - e^{q_i^{(0)} - q_{i+1}^{(0)}}, \quad i \neq m, m+1, \\ \ddot{q}_m^{(0)} &= e^{q_{m-1}^{(0)} - q_m^{(0)}}, \quad \ddot{q}_{m+1}^{(0)} = -e^{q_{m+1}^{(0)} - q_{m+2}^{(0)}}, \end{aligned} \tag{19}$$

$$\ddot{q}_i^{(1)} = (q_{i-1}^{(1)} - q_i^{(1)}) \cdot e^{q_{i-1}^{(0)} - q_i^{(0)}} - (q_i^{(1)} - q_{i+1}^{(1)}) \cdot e^{q_i^{(0)} - q_{i+1}^{(0)}}, \quad i \neq m, m+1,$$

$$\ddot{q}_m^{(1)} = (q_{m-1}^{(1)} - q_m^{(1)}) \cdot e^{q_{m-1}^{(0)} - q_m^{(0)}} + e^{q_m^{(0)} - q_{m+1}^{(0)}}, \tag{20}$$

$$\ddot{q}_{m+1}^{(1)} = e^{q_m^{(0)} - q_{m+1}^{(0)}} - (q_{m+1}^{(1)} - q_{m+2}^{(1)}) \cdot e^{q_{m+1}^{(0)} - q_{m+2}^{(0)}}.$$

Obviously setting $q_i^{(0)}, q_i^{(1)} \in \mathbb{R}$, system (19) is simply a closed system of two noninteracting ordinary Toda lattices T_m and T_n of m and n particles, respectively. System (20) consists of additional equations of motion for $m+n$ particles depending on system (19). The asymptotic behavior of the particles in system (19) is well known, see, e.g., Ref. 11. For $t \rightarrow -\infty$ and $t \rightarrow +\infty$, all particles behave freely and line up in order of, respectively, decreasing and increasing velocity. The asymptotic velocities $p_i^{(0)}(\pm) = \dot{q}_i^{(0)}(\pm)$ are given by the (distinct) time independent eigenvalues λ_i, μ_i of the underlying Lax matrices of T_m and T_n . More specific

$$p_i^{(0)}(-) = \dot{q}_i^{(0)}(-) = \lambda_{m-i+1},$$

$$p_i^{(0)}(+) = \dot{q}_i^{(0)}(+) = \lambda_i, \quad i = 1, \dots, m,$$

$$p_{m+i}^{(0)}(-) = \dot{q}_{m+i}^{(0)}(-) = \mu_{n-i+1},$$

$$p_{m+i}^{(0)}(+) = \dot{q}_{m+i}^{(0)}(+) = \mu_i, \quad i = 1, \dots, n,$$

where $\lambda_1 < \lambda_2 < \dots < \lambda_m$ and $\mu_1 < \mu_2 < \dots < \mu_n$.

The asymptotic behavior of the $m+n$ particles in system (20) can easily be obtained by substituting the asymptotic behavior of the particles in system (19) in the equations of system (20). It follows that this behavior depends on the mutual disposition of the segments $[\lambda_1, \lambda_m]$ and $[\mu_1, \mu_n]$. For example, if $\lambda_{i_0} < \mu_1 < \lambda_{i_0+1}$ and $\mu_{i_1} < \lambda_m < \mu_{i_1+1}$ then we have for $t \rightarrow +\infty$ that those particles corresponding with eigenvalues in the intersection fly exponentially to infinity

$$q_i^{(1)}(t) \sim e^{(\lambda_i - \mu_1)t}, \quad i = i_0 + 1, \dots, m,$$

$$q_{m+i}^{(1)}(t) \sim e^{(\lambda_m - \mu_i)t}, \quad i = 1, \dots, i_1,$$

whereas the rest of the particles behave freely. For more details about this asymptotic behavior we refer to Ref. 10.

Remark 3.1: One can also consider the case $L \in G_{-l,s}$ for $s > 1$ which obviously leads to superextensions of Toda systems with a higher order of interaction. We will not go into this here, neither will we explore the case where $L \in G_{k,s}$ for $k < -1$.

IV. SOLVING THE SYSTEM

One can solve the system (16) [or (17)] explicitly using the method described in Ref. 11 adjusted to the supercase. In order to do so we need the well-known definition of the super-analog of the determinant.

Definition 4.1: For $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_0$ such that A and D are invertible we define the Berezinian (or superdeterminant) by Ber: $G_0 \rightarrow \Lambda$.

$$\text{Ber}(L) = \det(A - BD^{-1}C) \cdot \det(D^{-1}). \quad (21)$$

One can check, see, e.g., Refs. 15, 13, that Ber has the essential property

$$\text{Ber}(L.M) = \text{Ber}(L) \cdot \text{Ber}(M). \quad (22)$$

Note that in the definition of $\text{Ber}(L)$ only the inverse of D is involved. One could also have defined

$$\text{Ber}'(L) = \det(D - CA^{-1}B) \cdot \det(A^{-1}), \quad (23)$$

where we have the relation

$$\text{Ber}(L) = \text{Ber}'(L)^{-1}.$$

In fact $\text{Ber}'(L)$ is the Berezinian of the matrix obtained from L by switching the parity of the rows and columns. Furthermore, note that the definition of the Berezinian strongly resembles the ordinary determinant of block matrices (1) with real entries which is given by $\det(A - BD^{-1}C) \cdot \det(D)$. Taking the inverse in the second factor in the definition of the Berezinian is necessary to ensure property (22).

To be able to describe minors of L we introduce the following notation: Let $1 \leq k \leq m+n$. Then consider a set $i_1, \dots, i_k, j_1, \dots, j_k$ such that $1 \leq i_r, j_s \leq m+n$ for $r, s = 1, \dots, k$. Now we denote the $k \times k$ matrix M with $M^{rs} = L^{i_r j_s}$ by

$$L_{j_1 \dots j_k}^{i_1 \dots i_k}.$$

Hence $L_{j_1 \dots j_k}^{i_1 \dots i_k}$ is the matrix obtained by deleting all rows and columns from L , except rows $i_1, \dots, i_k$ and columns $j_1, \dots, j_k$. In particular we will focus on the following minors of L :

- (1) the principal minors $L_{i+1 \dots m+n}^{i+1 \dots m+n}$,
- (2) the minors $L_{j+1 \dots m+n}^{i+1 \dots m+n}$, $j < i$,
- (3) the minors $L_{j+1 \dots m+n}^{ij+1 \dots m+n}$, $i < j$.

Consider the following cases:

- (i) $p(i) = p(j) = 0$. In this case minors (1), (2), and (3) are even and as L have a block structure.
- (ii) $p(i) = p(j) = 1$. In this case minors (1), (2), and (3) are even and are in fact minors of D .
- (iii) $p(i) = 0, p(j) = 1$. In this case the parity of minors (3) is undefined.
- (iv) $p(i) = 1, p(j) = 0$. In this case the parity of minors (2) is undefined.

For the minors in cases (i) and (ii) both Ber and Ber' exist. In case (i) they are given by Eq. (21) and (23), respectively. In case (ii) Ber' coincides with the ordinary determinant and Ber with its inverse. In cases (iii) and (iv) only the ordinary determinant can be calculated and will be denoted by Ber' . Its value is odd. We furthermore denote

$$\begin{aligned} \Delta_i^{\bar{0}}(L) &= \text{Ber}(L_{i+1 \dots m+n}^{i+1 \dots m+n}), \\ \Delta_i^{\bar{1}}(L) &= \text{Ber}'(L_{i+1 \dots m+n}^{i+1 \dots m+n}), \\ \Delta_{ji}^{\bar{0}}(L) &= \text{Ber}(L_{ji+1 \dots m+n}^{i+1 \dots m+n}), \quad j < i, \quad p(i) = p(j), \\ \Delta_{ji}^{\bar{1}}(L) &= \text{Ber}'(L_{ji+1 \dots m+n}^{i+1 \dots m+n}), \quad j < i \\ \Delta_{ji}^{\bar{0}}(L) &= \text{Ber}(L_{jj+1 \dots m+n}^{ij+1 \dots m+n}), \quad i < j, \quad p(i) = p(j) \\ \Delta_{ji}^{\bar{1}}(L) &= \text{Ber}'(L_{jj+1 \dots m+n}^{ij+1 \dots m+n}), \quad i < j. \end{aligned} \quad (24)$$

Now we have the following proposition, the proof of which we will not present here since it is completely analogous to the well-known Gauss factorization in the nonsupercase.

Proposition 4.2: Let D be a diagonal invertible matrix and let Z_u and Z_l be upper, respectively, lower triangular matrices both with ones on the diagonal. Then, setting $E = Z_u \cdot D \cdot Z_l$ we have

$$D^{ii} = \frac{\Delta_i^{p(i)}(D)}{\Delta_{i+1}^{p(i)}(D)} = \frac{\Delta_i^{p(i)}(E)}{\Delta_{i+1}^{p(i)}(E)},$$

$$Z^{ij} = \frac{\Delta_{ji}^{p(i)}(E)}{\Delta_i^{p(i)}(E)}, \quad i > j,$$

$$Z^{ij} = \frac{\Delta_{ji}^{p(j)}(E)}{\Delta_j^{p(j)}(E)}, \quad i < j.$$

Now consider the diagonal matrix $D = D(t)$ with diagonal entries $D^{ii} = e^{q_i}$. Furthermore let Z_u and Z_l be uniquely defined by

$$\dot{Z}_u(t) = Z_u(t) L_+(t), \quad Z_u(0) = I, \quad (25)$$

$$\dot{Z}_l(t) = D(t)^{-1} L_+^*(t) D(t) Z_l(t), \quad Z_l(0) = I, \quad (26)$$

where L is the super-Lax matrix for system (17). It follows from Eq. (26) that $\dot{Z}^{i+1i} = e^{q_i - q_{i+1}}$ for $i \neq m$ and $\dot{Z}^{m+1m} = \xi \cdot e^{q_m - q_{m+1}}$ and hence

$$\dot{p}_i = e^{q_{i-1} - q_i} - e^{q_i - q_{i+1}} = \dot{Z}^{ii-1} - \dot{Z}^{i+1i}, \quad i \neq m, m+1,$$

$$\dot{p}_m = e^{q_{m-1} - q_m} + \gamma \cdot e^{q_m - q_{m+1}} = \dot{Z}^{mm-1} - \alpha \cdot \dot{Z}^{m+1m},$$

$$\dot{p}_{m+1} = \gamma \cdot e^{q_m - q_{m+1}} - e^{q_{m+1} - q_{m+2}} = \dot{Z}^{m+1m} \cdot \alpha - \dot{Z}^{m+2m+1},$$

from which we conclude

$$p_i(t) = p_i(0) + Z^{ii-1} - Z^{i+1i}, \quad i \neq m, m+1,$$

$$p_m(t) = p_m(0) + Z^{mm-1} - \alpha \cdot Z^{m+1m}, \quad (27)$$

$$p_{m+1}(t) = p_{m+1}(0) + Z^{m+1m} \cdot \alpha - Z^{m+2m+1}.$$

Thus, using Proposition 4.2, if we define

$$E(t) = Z_u(t) \cdot D(t) \cdot Z_l(t) \quad (28)$$

we can rewrite Eq. (27) as

$$\begin{aligned}
p_i(t) &= p_i(0) + \frac{\Delta_{i-1i}^{p(i)}(E)}{\Delta_i^{p(i)}(E)} - \frac{\Delta_{ii+1}^{p(i+1)}(E)}{\Delta_{i+1}^{p(i+1)}(E)}, \quad i \neq m, m+1, \\
p_m(t) &= p_m(0) + \frac{\Delta_{m-1m}^{\bar{0}}(E)}{\Delta_m^{\bar{0}}(E)} - \alpha \cdot \frac{\Delta_{mm+1}^{\bar{1}}(E)}{\Delta_{m+1}^{\bar{1}}(E)}, \\
p_{m+1}(t) &= p_{m+1}(0) + \frac{\Delta_{mm+1}^{\bar{1}}(E)}{\Delta_{m+1}^{\bar{1}}(E)} \cdot \alpha - \frac{\Delta_{m+1m+2}^{\bar{1}}(E)}{\Delta_{m+2}^{\bar{1}}(E)}.
\end{aligned} \tag{29}$$

Furthermore, using $p(i) = p(i+1)$ for $i \neq m$ it is obvious that we have for $i \neq m$

$$a_i = e^{q_i - q_{i+1}} = \frac{D^{ii}}{D^{i+1i+1}} = \frac{\Delta_i^{p(i)}(E)}{\Delta_{i+1}^{p(i)}(E)} \cdot \frac{\Delta_{i+2}^{p(i+1)}(E)}{\Delta_{i+1}^{p(i+1)}(E)} = \frac{\Delta_i^{p(i)}(E) \cdot \Delta_{i+2}^{p(i)}(E)}{\Delta_{i+1}^{p(i)}(E) \cdot \Delta_{i+1}^{p(i)}(E)}. \tag{30}$$

Since $0 = p(m)$ and $1 = p(m+1)$ and the fact that $\text{Ber} = (\text{Ber}')^{-1}$ we have for a_m

$$a_m = \alpha \cdot e^{q_m - q_{m+1}} = \alpha \cdot \frac{D^{mm}}{D^{m+1m+1}} = \alpha \cdot \frac{\Delta_m^{\bar{0}}(E)}{\Delta_{m+1}^{\bar{0}}(E)} \cdot \frac{\Delta_{m+2}^{\bar{1}}(E)}{\Delta_{m+1}^{\bar{1}}(E)} = \alpha \cdot \frac{\Delta_m^{\bar{0}}(E)}{\Delta_{m+2}^{\bar{0}}(E)}. \tag{31}$$

Using formulas (25), (26) and the fact that $\dot{D}(t) = L_0(t) \cdot D(t)$, where $L_0(t)$ is the projection of $L(t)$ to G_0 [i.e., $L_0(t)$ is the diagonal matrix with entries $(L_0(t))^{ii} = p_i$], one can easily derive the following expression for the time derivative of $E(t)$ given by Eq. (28):

$$\dot{E}(t) = Z_u(t) \cdot L(t) \cdot D(t) \cdot Z_l(t).$$

Obviously we have $\dot{E}(t) \cdot E(t)^{-1} = Z_u(t) \cdot L(t) \cdot Z_u(t)^{-1}$. Using Eq. (25) and the super-Lax equation $\dot{L}(t) = [L, L_+]$ one easily concludes that the time derivative of the expression $\dot{E}(t) \cdot E(t)^{-1}$ vanishes. Hence we have $\dot{E}(t) = C \cdot E(t)$ for some constant matrix C , from which we conclude that $E(t) = e^{tC} \cdot E(0)$. Now, since obviously $E(0) = D(0)$ and $C = L(0)$ we finally have

$$E(t) = e^{tL(0)} \cdot D(0).$$

Now since $D(0)$ is a diagonal matrix the expressions (29), (30), and (31) for p_i and a_i in terms of minors of $E(t)$ can easily be shown to be equal to the same expressions rewritten in terms of minors of $e^{tL(0)} = M$ multiplied by factors of the form $(D(0))_{ii} \cdot (D(0))_{i+1i+1}^{-1} = e^{q_i(0) - q_{i+1}(0)}$. If we take into account the fact that for $i \neq m$, $a_i(0) = e^{q_i(0) - q_{i+1}(0)}$ and furthermore $a_m(0) = \alpha \cdot e^{q_m(0) - q_{m+1}(0)}$ it follows that the solutions $p_i(t), a_i(t)$ of Eqs. (17) are given by

$$p_i(t) = p_i(0) + \frac{\Delta_{i-1i}^{p(i)}(M)}{\Delta_i^{p(i)}(M)} \cdot a_{i-1}(0) - a_i(0) \cdot \frac{\Delta_{ii+1}^{p(i+1)}(M)}{\Delta_{i+1}^{p(i+1)}(M)}, \tag{32}$$

$$a_i(t) = a_i(0) \cdot \frac{\Delta_i^{p(i)}(M) \cdot \Delta_{i+2}^{p(i)}(M)}{\Delta_{i+1}^{p(i)}(M) \cdot \Delta_{i+1}^{p(i)}(M)}, \tag{33}$$

$$a_m(t) = a_m(0) \cdot \frac{\Delta_m^{\bar{0}}(M)}{\Delta_{m+2}^{\bar{0}}(M)}. \tag{34}$$

V. THE REDUCED LINEAR SUPER-POISSON STRUCTURE

In this section we will take a closer look at the linear super-Hamiltonian structure given by Eq. (8) of the super-Toda lattice (16).

For $L \in G_{-1,1}$ of the form (15), according to Lemma 2.4, in the calculation of $P_1(L)df$ we can take $df(L) = df \in G_{-1,0}$ with entries given by Eq. (7)

$$df^{ii} = (-1)^{p(i)(p(f)+1)} \frac{\partial f}{\partial p_i}, \quad df^{i+1i} = (-1)^{p(i)(p(f)+1)} \frac{\partial f}{\partial a_i}.$$

Now we can easily calculate the super-Hamiltonian vector field

$$P_1(L)df = 2([L, df_+] - \Pi_-^*[L, df]) \in G_{0,1}.$$

Since $df \in G_{-1,0} \subset G_-$ and hence $df_+ = 0$ the first term cancels. It follows that $P_1(L)df$ up to a factor -2 is equal to the matrix $P = \Pi_-^*[L, df]$ for which the only nonzero entries are given by

$$\begin{aligned} P^{ii} &= (-1)^{p(i)(p(f)+1)} \left(a_i \cdot \frac{\partial f}{\partial a_i} - a_{i-1} \cdot \frac{\partial f}{\partial a_{i-1}} \right), \\ P^{ii+1} &= (-1)^{p(i)(p(f)+1)} \left(a_i \cdot \frac{\partial f}{\partial p_{i+1}} - a_i \cdot \frac{\partial f}{\partial p_i} \right), \quad i \neq m, \\ P^{mm+1} &= (-1)^{p(f)+1} \left(a_m \cdot \frac{\partial f}{\partial p_{m+1}} + a_m \cdot \frac{\partial f}{\partial p_m} \right). \end{aligned}$$

In general, for even L , using representation (7) of $df \in G$ and the explicit form (4) of the bracket $\langle, \rangle$, the super-Poisson bracket $\{f, g\}_1$ can be rewritten as

$$\{f, g\}_1(L) = \langle P, dg \rangle = \sum_{i,j} (-1)^{p(f)+p(g)+1} p(i) P^{ij} dg^{ji} = \sum_{i,j} (-1)^{p(f)p(i)} P^{ij} \frac{\partial g}{\partial L^{ij}}.$$

Hence, representing the differential $dg \in G_{-1,0}$ and the matrix $P \in G_{0,1}$, respectively, by the vectors $dg = (\partial g / \partial p_i, \partial g / \partial a_i)$ and $P = ((-1)^{p(i)p(f)} P^{ii}, (-1)^{p(i)p(f)} P^{ii+1})$ with $2m+2n-1$ entries, this last expression equals

$$\sum_i (-1)^{p(f)p(i)} \left(P^{ii} \frac{\partial g}{\partial p_i} + P^{ii+1} \frac{\partial g}{\partial a_i} \right) = \sum_k P_k dg_k = \langle \langle P, dg \rangle \rangle, \quad (35)$$

where $\langle \langle, \rangle \rangle$ stands for the ordinary inner product of two vectors.

We can rewrite P as follows:

$$P_s = ((-1)^{p(s)p(f)} H_1 df)_s = \sum_t (-1)^{p(s)p(f)} H_1^{st}(df)_t, \quad (36)$$

where in Eq. (36) $p(s) = 1$ if $s = 2m+n$ and 0 otherwise. This definition of parity may seem strange, but is due to the fact that we did not represent the vector df in standard form. To obtain the standard form of df we have to permute the entries of the vector df to make sure that the entry $\partial f / \partial a_m$ comes last, in that case $p(s) = 1$ for $s = 2m+2n-1$ and 0 otherwise. Obviously for $n=1$ this is already the case. H_1 is the following $(2m+2n-1) \times (2m+2n-1)$ matrix:

$$H_1 = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix},$$

where A is a $(m+n) \times (m+n-1)$ matrix, B is a $(m+n-1) \times (m+n)$ matrix for which the only nonzero entries are given by

$$\begin{aligned} A^{ii} &= H_1^{im+n+i} = (-1)^{p(i)} a_i, & A^{i+1i} &= H_1^{i+1m+n+i} = (-1)^{p(i+1)+1} a_i, \\ B^{ii} &= H_1^{m+n+ii} = (-1)^{p(i)+1} a_i, \\ B^{ii+1} &= H_1^{m+n+ii+1} = (-1)^{p(i+1)} a_i, \quad i=1, \dots, m+n-1. \end{aligned} \quad (37)$$

Hence, combining Eqs. (35) and (36), the reduced super-Poisson bracket on $G_{0,1}$, parametrized by the coordinates (p_i, a_i) , for even super-Hamiltonian matrix H has the form

$$\{f, g\}_1 = \langle \langle P, dg \rangle \rangle = \langle \langle (-1)^{p(s)p(f)} H_1 df, dg \rangle \rangle \quad (38)$$

$$= \sum_{s,t} (-1)^{p(s)p(f)} H_1^{st} (df)_t (dg)_s, \quad (39)$$

where the bracket $\langle \langle, \rangle \rangle$ stands for the ordinary inner product of two vectors. If the vectors df and dg are in standard form, form (38) will be called the standard form for a super-Poisson bracket for even H . For a bracket in standard form (38) we can formulate general conditions on H to ensure that the bracket is a super-Poisson bracket: First of all one can easily check that, if H satisfies the skew- P -symmetry condition

$$H^{ij} = (-1)^{p(i)p(j)+1} H^{ji}, \quad (40)$$

bracket (38) satisfies

$$\{f, g\} = (-1)^{p(f)p(g)+1} \{g, f\}.$$

The condition on H concerning the super-Jacobi identity can be formulated in terms of the vanishing of a superanalog of the so-called Schouten–Nijenhuis bracket $[H, H]$ of H with itself as defined in Ref. 16. In general, for two matrices H and K satisfying the skew-symmetry property (40) this super-Schouten–Nijenhuis bracket is defined as follows:

$$[H, K]_{ijk} = \sum_s ((-1)^{p(i)p(j)} (H^{is} \cdot (dK^{kj})_s + K^{is} \cdot (dH^{kj})_s)) + \text{cycl}(i, j, k).$$

A rather long, but straightforward calculation which we leave to the reader shows that indeed, for H satisfying the condition

$$[H, H] = 0 \quad (41)$$

the super-Jacobi identity

$$(-1)^{p(f)p(h)} \{f, \{g, h\}\} + \text{cycl}(f, g, h) = 0$$

for bracket (38) is fulfilled.

We will call a matrix H satisfying Eqs. (40) and (41) a super-Hamiltonian matrix.

In particular, setting $m=2$ and $n=1$ in Eqs. (38) and (37) H_1 takes the form of the following 5×5 super Hamiltonian matrix:

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & -a_1 & a_2 \\ 0 & 0 & 0 & 0 & a_2 \\ -a_1 & a_1 & 0 & 0 & 0 \\ 0 & -a_2 & -a_2 & 0 & 0 \end{pmatrix}. \quad (42)$$

VI. THE REDUCED QUADRATIC SUPER-POISSON STRUCTURE

We are now going to investigate in the same way the quadratic super-Hamiltonian vector fields $P_2(L)df$ which are up to a factor 2 given by

$$[L, (L.df + df.L)_+] - L.\Pi_-^*[L, df] - \Pi_-^*[L, df].L.$$

Using the fact that $G_-^* = G_+ \oplus G_0$ one can easily check that this expression equals

$$[df, L]_0.L + L.[df, L]_0 + 2L(df.L)_+ - 2(L.df)_+.L. \quad (43)$$

For general $L \in G_{-1,1}$ given by Eq. (15) we can according to Lemma 2.4 in the calculation of $P_2(L)df$ take $df(L) = df \in G_{-1,1}$ with entries

$$\begin{aligned} df^{ii} &= (-1)^{p(i)(p(f)+1)} \frac{\partial f}{\partial p_i}, \\ df^{i+1i} &= (-1)^{p(i)(p(f)+1)} \frac{\partial f}{\partial a_i}, \\ df^{ii+1} &= (-1)^{p(i+1)(p(f)+1)} \frac{\partial f}{\partial c_i}. \end{aligned} \quad (44)$$

Note that here we are forced to incorporate the c_i 's as variables, see also Remark 2.5. Substituting Eq. (44) into Eq. (43) one obtains, after a long but straightforward calculation, the explicit form of the super-Hamiltonian matrix $\bar{H}_2$ such that the reduction of the quadratic super-Poisson bracket to $G_{-1,-1}$ takes the form (38) where df is the vector $(\partial f/\partial p_i, \partial f/\partial a_i, \partial f/\partial c_i)$ with $3m+3n-2$ components and $p(s)=1$ for $s=2m+n$ and $s=3m+2n-1$ and 0 otherwise. In particular, for $m=2, n=1$, representing df in standard form as the vector

$$df = \left(\frac{\partial f}{\partial p_1}, \frac{\partial f}{\partial p_2}, \frac{\partial f}{\partial p_3}, \frac{\partial f}{\partial a_1}, \frac{\partial f}{\partial c_1}, \frac{\partial f}{\partial a_2}, \frac{\partial f}{\partial c_2} \right)$$

one obtains for $\bar{H}_2$ the following 7×7 matrix:

$$\bar{H}_2 = \begin{pmatrix} 0 & -2a_1c_1 & 0 & -2p_1a_1 & 0 & 0 & 0 \\ 2a_1c_1 & 0 & 2a_2c_2 & 2p_2a_1 & 0 & -2p_2a_2 & 0 \\ 0 & -2a_2c_2 & 0 & 0 & 0 & -2p_3a_2 & 0 \\ 2p_1a_1 & -2p_2a_1 & 0 & 0 & 0 & -a_1a_2 & -a_1c_2 \\ 0 & 0 & 0 & 0 & 0 & -c_1a_2 & c_1c_2 \\ 0 & 2p_2a_2 & 2p_3a_2 & a_1a_2 & c_1a_2 & 0 & 2a_2c_2 \\ 0 & 0 & 0 & a_1c_2 & -c_1c_2 & 2a_2c_2 & 0 \end{pmatrix}.$$

Now, as in the (infinite dimensional) case of the sKdV(3) equation,⁷ for all Hamiltonians of hierarchy (10) the c_1 and c_2 components vanish, see also remark 2.5. Using the reduction procedure as described in Ref. 2, for the nonsupercase we set

$$-c_1a_2 \frac{\partial f}{\partial a_2} + c_1c_2 \frac{\partial f}{\partial c_2} = 0, \quad (45)$$

$$a_1c_2 \frac{\partial f}{\partial a_1} - c_1c_2 \frac{\partial f}{\partial c_1} + 2a_2c_2 \frac{\partial f}{\partial a_2} = 0. \quad (46)$$

Using these equations we can eliminate $\partial f/\partial c_1$ and $\partial f/\partial c_2$ from the super-Hamiltonian vector fields. Omitting an inessential factor 2 the reduced quadratic super-Poisson bracket on $G_{0,1}$ gets the standard form (38) where now $p(s)=1$ for $s=5$ and 0 otherwise, df is given in standard form by the vector $(\partial f/\partial p_i, \partial f/\partial a_i)$, and H_2 is the following 5×5 super-Hamiltonian matrix:

$$H_2 = \begin{pmatrix} 0 & -a_1c_1 & 0 & -p_1a_1 & 0 \\ a_1c_1 & 0 & a_2c_2 & p_2a_1 & -p_2a_2 \\ 0 & -a_2c_2 & 0 & 0 & -p_3a_2 \\ p_1a_1 & -p_2a_1 & 0 & 0 & -a_1a_2 \\ 0 & p_2a_2 & p_3a_2 & a_1a_2 & 0 \end{pmatrix}. \quad (47)$$

Hence we have found two super-Hamiltonian descriptions of hierarchy (10) of super-Toda lattices associated with two super-Poisson structures of the form (38) obtained by reduction of the two super-Poisson structures on G . The super-Hamiltonian matrices involved are given by (42) and (47).

VII. A SUPER-RELATIVISTIC TODA LATTICE

Now we are going to describe in the same way superextensions of the relativistic Toda lattice. We introduce another more complicated splitting of the superalgebra $G = G_+ \oplus G_-$ which is to be compared with the description as given in the nonsuper-relativistic case in Ref. 2. In order to do so we choose for convenience $n=1$, hence $G = \text{Mat}(m, 1, \Lambda)$. We define G_- as the subset of the upper triangular block matrices L such that for all $m+1$ columns the sum of the elements is equal to zero. Note that in particular this means that $L^{11}=0$ and furthermore $L^{m+1, m+1}=0$ since it is the only even/odd element in the $m+1$ -th column

$$L = \begin{pmatrix} 0 & L^{12} & \cdot & L^{1m} & L^{1m+1} \\ 0 & L^{22} & & & \\ \cdot & \cdot & \cdot & & \\ \cdot & & \cdot & L^{mm} & L^{mm+1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (48)$$

where in Eq. (48) $\sum_i L^{ij} = 0$ for all j .

Obviously G_- is a subsuperalgebra. In fact, the property that for each column the sum of the elements vanishes is preserved under right multiplication with an arbitrary matrix due to

$$\sum_s (LM)^{si} = \sum_{s,t} L^{st} M^{ti} = \sum_t \left(\sum_s L^{st} \right) M^{ti} = 0$$

and hence for $L, M \in G_-$ it follows trivially that $LM \in G_-$, $ML \in G_-$.

We define G_+ as the subset of G of "almost lower triangular" matrices L with one nonzero entry directly above the main diagonal in the $m+1$ -th column

$$L = \begin{pmatrix} L^{11} & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & 0 & 0 \\ \cdot & & & L^{mm} & L^{mm+1} \\ L^{m+11} & \cdot & \cdot & \cdot & L^{m+1m+1} \end{pmatrix}.$$

It is obvious that G_+ is also a subsuperalgebra of G whereas $G = G_+ \oplus G_-$. For $L = (L^{ij}) \in G$ the projections $\Pi_+(L) = L_+ \in G_+$ and $\Pi_-(L) = L_- \in G_-$ are given by the following formulas:

$$L_-^{ij} = L^{ij}, \quad i < j, \quad (i, j) \neq (m, m+1),$$

$$L_-^{mm+1} = - \sum_{j < m} L^{jm+1},$$

$$L_-^{ij} = 0, \quad i > j,$$

$$L_-^{m+1m+1} = 0, \quad L_-^{ii} = - \sum_{j < i} L^{ji}, \quad i \neq m+1,$$

$$L_+^{ij} = 0, \quad i < j, \quad (i, j) \neq (m, m+1),$$

$$L_+^{mm+1} = \sum_{j < m} L^{jm+1},$$

$$L_+^{ij} = L^{ij}, \quad i > j,$$

$$L_+^{m+1m+1} = L^{m+1m+1}, \quad L_+^{ii} = \sum_{j < i} L^{ji}, \quad i \neq m+1.$$

Using the nondegenerate bilinear form $\langle, \rangle$ we also have a dual (orthogonal) splitting $G = G_+^* \oplus G_-^*$. Using $G_+^* = G_-^{\perp}$ it can easily be seen that G_+^* consists of matrices L for which

the entries below and on the main diagonal of each of the first m rows are equal, whereas in the $m+1$ -th row all entries below the main diagonal are equal. Hence an element of G_+^* has the general form

$$\begin{pmatrix} L^{11} & L^{12} & \cdot & \cdot & \cdot & L^{1m+1} \\ L^{22} & L^{22} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L^{mm} & \cdot & \cdot & L^{mm} & L^{mm+1} & \cdot \\ L^{m+1m} & \cdot & \cdot & L^{m+1m} & L^{m+1m+1} & \cdot \end{pmatrix}.$$

One can also easily check that $G_-^* = G_+^\perp$ consists of strictly lower triangular matrices L for which also $L^{m+1m}=0$. Hence a general element of G_-^* has the following form:

$$\begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ L^{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & L^{mm-1} & 0 & \cdot & \cdot \\ L^{m+11} & \cdot & L^{m+1m-1} & 0 & 0 & 0 \end{pmatrix}.$$

For $L=(L^{ij}) \in G=G^*$ the projections $\Pi_+^*(L) = L_+^* \in G_+^*$ and $\Pi_-^*(L) = L_-^* \in G_-^*$ are given by

$$(L_+^*)^{ij} = L^{ij}, \quad i \leq j,$$

$$(L_+^*)^{ij} = L^{ii}, \quad m+1 \neq i > j,$$

$$(L_+^*)^{m+1j} = L^{m+1m}, \quad j \leq m,$$

$$(L_-^*)^{ij} = 0, \quad i \leq j,$$

$$(L_-^*)^{ij} = L^{ij} - L^{ii}, \quad m+1 \neq i > j,$$

$$(L_-^*)^{m+1j} = L^{m+1j} - L^{m+1m}, \quad j \leq m.$$

Note that G_+^* is not a subsuperalgebra of G whereas G_-^* is. Obviously, using Eq. (13) the linear super-Poisson structure can be restricted to G_+^* (see also Ref. 2). The following lemma states that also reductions can be made to the subspaces $G(k)$, $k \geq 2$ of G defined by

$$G(k) = \{L \in G_{-m,k} : L^{jj+k} = 0, \quad \text{for } j \leq m-k\} \quad (49)$$

$$= \{L \in G : L^{jj+s} = 0, \quad \text{for } j \leq m-k, s \geq k\}. \quad (50)$$

Lemma 7.1: Let $L \in G(k)$. Then $P_1(L)df \in G(k)$.

Proof: We have to check that $[L, df_+] - \Pi^*[L, df] \in G(k)$ for $L \in G(k)$. Since G^* consists of strictly lower triangular matrices which obviously lie within $G(k)$ for all k it remains to prove that $[L, df_+] \in G(k)$ which comes down to showing that the relevant entries above the main diagonal of $[L, df_+]$ are equal to 0. More specific we have to prove that for $s > k$, $j < m - k$

$$[L, df_+]^{jj+s} = 0,$$

which follows from the fact that $[L, df_+]^{jj+s}$ is equal to

$$\sum_p L^{jp} df_+^{pj+s} - (-1)^{p(L)p(f)} \sum_q df_+^{jq} L^{qj+s}.$$

Now $L^{jp} = 0$ for $p > j + k$. Furthermore for $p < j + k \leq j + s$ we have in particular $p < m$ and hence $df_+^{pj+s} = 0$ since df_+ contains no nonzero terms above the main diagonal except in the m th row. Thus the first sum vanishes. Furthermore note that $j < m - k < m$ and hence $df_+^{jq} = 0$ for $q > j$ whereas for $q < j$ we have $L^{qj+s} = 0$. Hence also the second sum vanishes and the lemma is proven. $\square$

Hence we can also restrict the linear Hamiltonian vector fields $P_1(L)df$ to $G_+^* \cap G(k)$.

As an example, consider the hierarchy of Lax equations $L_t = [L, (L^k)_+]$ for even $L \in G_+^* \cap G(2)$ of the form

$$L(m, n) = \begin{pmatrix} p_1 & a_1 & 0 & \cdot & \cdot & 0 \\ p_2 & p_2 & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & a_{m-1} & \beta \\ p_m & & \cdot & \cdot & p_m & \alpha \\ \xi & \cdot & \cdot & & \xi & p_{m+1} \end{pmatrix}. \quad (51)$$

For $k=1$ we calculate the corresponding equations

$$\dot{p}_i = p_{i+1}a_i - p_i a_{i-1},$$

$$\dot{a}_i = a_i(p_{i+1} - p_i + a_i - a_{i-1}), \quad i = 1, \dots, m-2,$$

$$\dot{p}_{m-1} = p_m a_{m-1} - p_{m-1} a_{m-2} + \beta \xi, \quad \dot{a}_{m-1} = a_{m-1}(p_m - p_{m-1} + a_{m-1} - a_{m-2}) + \beta \xi,$$

$$\dot{p}_m = -p_m a_{m-1} + \xi \beta, \quad \dot{\alpha} = -\alpha a_{m-1} - \beta p_{m+1},$$

$$\dot{\beta} = \beta(p_{m+1} - p_{m-1} + a_{m-1} - a_{m-2}) + \alpha a_{m-1}, \quad \dot{p}_{m+1} = 0, \quad \dot{\xi} = 0.$$

Obviously setting $n=0$ these equations reduce to the equations for the classical relativistic Toda lattice as described in Ref. 2. Furthermore, note that the odd variable ξ and the even variable p_{m+1} turn out to be constant along the flow. The explicit form of the corresponding Hamiltonian $H = \frac{1}{2} \text{str}(L^2)$ is given by

$$H = \frac{1}{2} \sum_{i=1}^{m+1} (-1)^{p(i)} p_i^2 + \sum_{i=1}^{m-1} a_i p_{i+1} + (\beta + \alpha) \xi.$$

In order to find the linear super-Hamiltonian vector fields $P_1(L)df$ we represent the differential $df = df(L)$ for L given by Eq. (51) in the dual of $G_+^* \cap G(2)$ by the matrix df^{ij} with the nonzero entries given by

$$df^{ii} = (-1)^{p(i)(p(f)+1)} \frac{\partial f}{\partial p_i}, \quad i=1, \dots, m+1,$$

$$df^{i+1i} = \frac{\partial f}{\partial a_i}, \quad i=1, \dots, m-1,$$

$$df^{m+1m} = \frac{\partial f}{\partial \alpha}, \quad df^{mm+1} = (-1)^{p(f)+1} \frac{\partial f}{\partial \xi}, \quad df^{m+1m-1} = \frac{\partial f}{\partial \beta}.$$

Note that considering the super-Poisson bracket we have to incorporate p_{m+1} and ξ as variables. Now $P_1(L)df = -2([L, df_-] - \Pi_+^*[L, df])$. Since $df_- = 0$ the first term vanishes, hence we have to calculate the matrix $P = \Pi_+^*[L, df]$. The only nonzero entries of P are given by

$$P^{ij} = a_i \frac{\partial f}{\partial a_i} - a_{i-1} \frac{\partial f}{\partial a_{i-1}}, \quad j \leq i \leq m-2,$$

$$P^{m-1j} = a_{m-1} \frac{\partial f}{\partial a_{m-1}} - a_{m-2} \frac{\partial f}{\partial a_{m-2}} + \beta \frac{\partial f}{\partial \beta}, \quad j \leq m-1,$$

$$P^{mj} = \alpha \frac{\partial f}{\partial \alpha} - a_{m-1} \frac{\partial f}{\partial a_{m-1}} - \xi \frac{\partial f}{\partial \xi}, \quad j \leq m,$$

$$P^{m+1m+1} = (-1)^{p(f)} \left(\alpha \frac{\partial f}{\partial \alpha} + \beta \frac{\partial f}{\partial \beta} - \xi \frac{\partial f}{\partial \xi} \right),$$

$$P^{ii+1} = a_i \frac{\partial f}{\partial p_{i+1}} - a_i \frac{\partial f}{\partial p_i}, \quad i \leq m-2,$$

$$P^{m-1m} = a_{m-1} \frac{\partial f}{\partial p_m} - a_{m-1} \frac{\partial f}{\partial p_{m-1}} + \beta \frac{\partial f}{\partial \alpha},$$

$$P^{mm+1} = (-1)^{p(f)+1} \left(\alpha \frac{\partial f}{\partial p_{m+1}} + \alpha \frac{\partial f}{\partial p_m} + \beta \frac{\partial f}{\partial a_{m-1}} + p_m \frac{\partial f}{\partial \xi} - p_{m+1} \frac{\partial f}{\partial \xi} \right),$$

$$P^{m-1m+1} = (-1)^{p(f)+1} \left(a_{m-1} \frac{\partial f}{\partial \xi} + \beta \frac{\partial f}{\partial p_{m+1}} + \beta \frac{\partial f}{\partial p_{m-1}} \right),$$

$$P^{m+1j} = \xi \frac{\partial f}{\partial p_m} + \xi \frac{\partial f}{\partial p_{m+1}} + p_{m+1} \frac{\partial f}{\partial \alpha} - a_{m-1} \frac{\partial f}{\partial \beta} - p_m \frac{\partial f}{\partial \alpha}, \quad j \leq m.$$

If we represent the differential df by the vector

$$df = \left(\frac{\partial f}{\partial p_i}, \frac{\partial f}{\partial a_i}, \frac{\partial f}{\partial \alpha}, \frac{\partial f}{\partial \beta}, \frac{\partial f}{\partial \xi} \right),$$

with $2m+3$ components in standard form one can easily extract from P , in the same way as we did in the preceding sections the super-Hamiltonian matrix H_1 , with respect to which the linear super-Poisson structure gets the standard form (38) where $p(s)=1$ for $s \geq 2m+1$ and 0 otherwise. In particular, setting $m=2$ one obtains the super-Hamiltonian matrix

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & a_1 & 0 & \beta & 0 \\ 0 & 0 & 0 & -a_1 & \alpha & 0 & -\xi \\ 0 & 0 & 0 & 0 & \alpha & \beta & -\xi \\ -a_1 & a_1 & 0 & 0 & \beta & 0 & 0 \\ 0 & -\alpha & -\alpha & -\beta & 0 & 0 & p_3-p_2 \\ -\beta & 0 & -\beta & 0 & 0 & 0 & -a_1 \\ 0 & \xi & \xi & 0 & p_3-p_2 & -a_1 & 0 \end{pmatrix}.$$

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