Mechanical aspects of hearing
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3. Membrane and environment

Abstract. An important part of the direct environment of the basilar membrane is fluid or is fluid-like and incompressible. This implies that the pressure in the environment of the membrane has to obey Laplace's equation. We first formulate a two-dimensional boundary value problem for the pressure in the cochlear fluids. The difference between the equation of motion for the basilar membrane and the equation for the surrounding fluid respectively yields an inhomogeneous mixed type boundary condition. Formal properties of the model and properties of the solution are discussed. Because singularities are neglected the solutions according to the method from this chapter are of limited importance. It appears that a three-dimensional approach hardly leads to new insight.

3.1 Introduction

The purpose of this chapter is to formulate and to solve a model for the basic properties of the motion of the basilar membrane in relation to the surrounding fluid-like medium. This will be done with the help of elementary equations of motion for both the basilar membrane and its environment. These equations and some of their consequences will be discussed in section 3.2 and 3.3. It will appear that it is almost impossible to model the main characteristic of the problem from a still simpler point of departure without omitting essentials. After that, in section 3.4, we shall formulate a boundary value problem for the cochlea. The notion to work with such problems is not new. About twenty-five years ago these ideas made their entrance in literature. For instance, Lesser and Berkley (1972); Viergever (1975); Van Dijk; (1976); Allen (1977) and Shondi (1977) formulated almost equivalent problems in a slightly different way. After taking advantage of some analytical techniques, those models ultimately led to systems of equations that have been solved numerically. There is nothing against the use of models in the shape of boundary value problems and nor is it problematic to apply numerical procedures to them. However, it will appear that the way of modelling was slightly ramshackle and the methods used did not contribute insight into the kernel of the problem. Therefore, it is not astonishing that already before the mid-eighties the development was stagnant. Viergever (1986) gives a review of the research at that time combined with some results.

Now it will be clear that there is a new question. Are there other and more effective methods available to tackle these boundary value problems so that a better insight is acquired on how the ear looks at a sound stimulus? The answer is: yes indeed, there are. However, some care must be taken because the success is closely related to which physical properties are modelled and which mathematical properties are included in the model. Therefore, in section 3.4.1 we shall first pay attention to the formulation of a boundary value problem in which the pressure at the membrane has to obey an inhomogeneous radiation condition. Then we will draw some conclusions from formal properties such as unicity and compatibility. This will be done in the sections 3.4.2 and 3.4.4. In section 3.4.3 we pay attention to the singularities of the problem. In section 3.5 we will solve the problem as if the singularities are negligible. Then it is relatively easy to give a solution and an interpretation of the solution. Because of the lack of attention for the singularities, the validity of the solution is questionable.
In section 3.6 the three-dimensional counterpart will be discussed. There it will appear that from a theoretical point of view, there is no fundamental difference between a two- and a three-dimensional approach to the problem.

A typical property of the motion of the basilar membrane as a result of a sinusoidal sound stimulus is the travelling wave behaviour. Ranké (1931, 1942) was one of the firsts who made a set-up to model this motion of the basilar membrane from a hydrodynamical point of view. His aim was to describe and to explain the early Von Békésy (1928) observations on the motion of the membrane in a fluid-like environment. For that purpose he proposed a description for an undulating potential in the fluid near the membrane. He used expedients from the theory of complex functions so that Laplace’s equation was satisfied and tried to fit the potential to the impedance of the basilar membrane. Ranké’s starting point was rather general and also pioneering in the field of cochlear mechanics. However, the way in which he treated most mathematical aspects of his problem is curious and not very careful. Siebert (1974) renewed the attention to the descriptive style of Ranke with an attempt to support this kind of phenomenology by stronger mathematical foundations. He studied the pressure in a two-dimensional model of the cochlea and proposed to distinguish between ‘short-wave’ and ‘long-wave’ approximations. His short-wave approximation supported Ranke’s description, whereas the second approximation straightforwardly led to the early Zwislocki (1948) approach. The long-wave approximation was Zwislocki’s answer to the work of Ranke. After him a lot of investigators followed his way of approximating the problem. Basically, the derivation of Zwislocki’s long-wave equation is completely ad hoc. This makes this approach both attractive and slightly vulnerable. In spite of Siebert’s proposals it is impossible to support this equation, without additional assumptions, by rigorous mathematical means. We will come to this point in chapter 5.

Zwislocki re-investigated his way of modelling (Zwislocki, 1980) and arrived at a more elegant derivation of his equation. However, from a mathematical point of view, he essentially did not add special information to his original proposals. The Zwislocki approach is extremely popular in cochlear modelling because it corresponds to the concept of travelling waves according to Von Békésy (1960) and to a lot of observations after him. However, this does not yet imply that the equation is right. It rather contributes to the imagination of physicists who prefer to think in terms of travelling waves rather than in terms of a reliable framework for cochlear mechanics. An important part of work in the field of auditory theory is related to this equation. Therefore, in chapter 5, we shall pay much attention to Zwislocki’s equation. Reviews on most topics related to this subject are given in De Boer (1980; 1984; 1991; 1993).

3.2 General dynamics

Let us start with a particular cochlea in which all membranous structures are absent. The fluid in the cavity has to obey the well-known equations of Euler.
In vector notation these equations can be written as

\[
\frac{d\vec{v}}{dt} = -\frac{1}{\rho} \nabla p + \vec{F} .
\]  

(3.1)

The vector \( \vec{v} \) is the velocity of a unit of mass of the fluid. The density of the fluid is \( \rho \) and \( p \) is the pressure. The sum of external forces - per unit of mass - is denoted by \( \vec{F} \). This vector may comprise both classical terms in consequence of the presence of viscosity and terms of quite different origin. The equation expresses the equilibrium between the inertial resistance of a unit of mass of the fluid, the pressure and the sum of all external forces. Van Dijk (1976) and Viergever (1980) showed that in the equations of motion, both non-linear terms and possible viscous effects are of minor importance. In consequence of this we will start with equations in which the viscosity is absent and in which it is allowed to replace the total derivative \( d\vec{v}/dt \) by the local one \( \partial\vec{v}/\partial t \).

Next we assume that in a certain strip-like region of the inner ear cavity a membrane is present in addition to the fluid. The stiffness component of this membrane is not negligible. In order to find the equation of motion for this membranous medium, it is sufficient to add a term to the Euler equation (3.1) that describes the restoring force in consequence of the presence of stiffness. When the stiffness per unit of volume of the strip is given by \( \kappa_m \) and the density of this medium is \( \rho_m \), the restoring force per unit of mass can be written as

\[-\omega_0^2 \ddot{u}_m .\]

Here \( \ddot{u}_m \) is the deflection of a unit of mass of the membranous medium. \( \omega_0^2 \) is defined by \( \omega_0^2 = \kappa_m / \rho_m \). In consequence of this, the modified Euler equations for this medium are

\[
\frac{d\vec{v}_m}{dt} = -\frac{1}{\rho_m} \nabla p_m - \omega_0^2 \ddot{u}_m + \vec{F}_m .
\]  

(3.2)

In this equation subscripts \( m \) are used to distinguish between membrane quantities and the counterparts of the surrounding fluid. From now on the subscript \( n \) will be used to refer to the normal component of the corresponding vector quantity. Then the symbol \( v_{mn} \) denotes the normal component of the vector velocity \( \vec{v}_m \) of a point of the membranous medium. At the boundary between fluid and membrane, the normal component of (3.1) perpendicular to this boundary reads

\[
\frac{d\vec{v}_n}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial n} + \vec{F}_n .
\]  

(3.3)

The equivalent equation for the membranous medium with respect to the same normal direction is

\[
\frac{d\vec{v}_{mn}}{dt} = -\frac{1}{\rho_m} \frac{\partial p_m}{\partial n} - \omega_0^2 \ddot{u}_{mn} + \vec{F}_{mn} .
\]  

(3.4)
3.3 The basilar membrane as a discontinuity

When the cochlear fluid and the membranous medium are in motion, we shall assume that the fluid always follows the membrane and also conversely. Then, the velocities of both media are the same at the common boundary between fluid and membrane. In consequence of this, it holds that $v_n = v_{mn}$. Here, $v_n$ is the normal component of the fluid velocity at this boundary and $v_{mn}$ the normal component of the membrane velocity $\tilde{v}_m$. At this stage it is our aim to avoid unnecessary additions. Therefore we shall assume that normal components of all additional forces at the common boundary between both media are the same. In consequence of this, it holds that $F_n = F_{mn}$. These assumptions make it rather simple to compare the equations of motion (3.3) and (3.4) with each other at the common boundary. Let us subtract (3.4) from (3.3). Then, the difference between the equations can be written as

$$\frac{1}{\rho} \frac{\partial p}{\partial n} - \frac{1}{\rho_m} \frac{\partial p_m}{\partial n} = \omega_{nn}^2 u_{mn}.$$  \hfill (3.5)

This equation expresses that the presence of stiffness in the membranous zone essentially represents a discontinuity for the pressure. In absence of the stiffness there is hardly a reason to distinguish between $p$ and $p_m$. It is easy to verify that the difference between the linear counterparts of (3.3) and (3.4) lead to the same equation. Therefore the validity of (3.5) does not depend on the possible presence of non-linear terms in the Euler equations.

Next we consider the membrane as a thin strip with thickness $\Delta n$. In that case the density $\rho_m$ can be written as

$$\rho_m = \frac{m}{\Delta n},$$  \hfill (3.6)

where $m$ is the mass per unit of area of the membrane. Insertion of this term in (3.5) yields

$$\frac{1}{\rho} \frac{\partial p}{\partial n} - \frac{1}{m} \frac{\partial p_m}{\partial n} \Delta n = \omega_{nn}^2 u_{mn}.$$  \hfill (3.7)

In the second term of the left member of (3.7), the expression $(\partial p_m / \partial n) \Delta n$ is the difference between the pressure at the upper and lower side of the membrane. There is some reason to introduce in models of the cochlea arguments of symmetry with respect to the behaviour of the pressure across the membrane (Van Dijk, 1976; De Boer, 1980). In conformity to those opinions we shall assume that the pressure difference across the membrane equals two times the fluid pressure at the upper side of the membrane. In consequence of this, we introduce the following simplification

$$\frac{\partial p_m}{\partial n} \Delta n = 2p,$$  \hfill (3.8)

where $p$ is the pressure at the upper-side of the membrane.
When this expression is applied, (3.7) is reduced to
\[
\frac{\partial p}{\partial n} - \frac{2\rho}{m} p = \rho \omega^2 u_{mn}.
\] (3.9)

Equation (3.9) holds true at the boundary between fluid and membrane. The equation has the shape of an inhomogeneous radiation condition\(^1\). In this equation the leading parameter is \(2\rho/m\). This quantity determines the ratio between the pressure and its normal derivative. Note that in (3.9) \(\omega^2\) must be considered as the ratio of the stiffness and the mass, both per unit of area of the strip.

In problems concerning oscillations the importance of stiffness depends on the frequency of the vibrations. This can be elucidated with the help of equation (3.4). In absence of external forces, the linear counterpart of (3.4) reads
\[
\frac{\partial^2 u_{mn}}{\partial t^2} = -\omega^2 u_{mn} - \frac{1}{\rho_m} \frac{\partial p_m}{\partial n}.
\] (3.10)

Let us assume that \(u_{mn}\) oscillates with a frequency \(\omega\). Then we have \(\ddot{u}_{mn} = -\omega^2 u_{mn}\), so that the pressure and the deflection obey the expression
\[
-\rho_m \left(1 - \frac{\omega_0^2}{\omega^2}\right) \omega^2 u_{mn} = \frac{\partial p_m}{\partial n}.
\] (3.11)

In chapter 2 we noted that along the basilar membrane the stiffness \(\omega_0^2\) varies as a decreasing function of the distance to the stapes (see for instance Fig. 2.7a. and Fig. 2.7b or de Boer 1980). When the frequency \(\omega\) is fixed the parts of the membrane on either side of the point of resonance, i.e. the point at which \(\omega = \omega_0\), have different properties.

In the first part the stiffness dominates the density, i.e. \(\omega < \omega_0\). It is not difficult to conceive this region as a discontinuity in a fluid-like environment. Near the stapes, the starting point of this region, \(\omega_0^3\) increases rapidly. Then, as follows from (3.11), the deflection tends towards zero. This situation can be considered as a pseudo zero boundary condition for the deflection. We expect that near the stapes, where the membrane is very narrow, the motion of the membrane partly follows from the presence of additional bending stiffness. In that region the membrane and its bony environment are grown into one. Therefore, there is some reason to extend (3.10), especially near the stapes, with a small term of the type
\[
c_0(x) \frac{\partial^4 u_{mn}}{\partial x^4}.
\]

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\(^1\) The homogeneous part of (3.9) has the shape of a classical radiation condition for the pressure. When the right member in (3.9) does not vanish, the solution of this equation expresses the influence of the motion of the membrane on the pressure at particular point. This effect is often called the coupling between membrane oscillators. This meaning differs from the classical notion of radiation. In spite of this difference, we will use the word radiation condition for expressions like (3.9). When (3.9) is used as a boundary condition, this condition could also be called an inhomogeneous mixed boundary condition or an inhomogeneous condition of the third kind.
in which \( c_0(x) \) rapidly diminishes as \( x \) grows. The introduction of this term asks for boundary conditions at the starting point \( x = 0 \) of the membrane. Here we again prefer the well-known clamped boundary condition

\[
u = 0 \text{ and } \frac{\partial u}{\partial x} = 0 \text{ at } x = 0.
\]

In the second region of the membrane, the density of the medium dominates. In this part it holds that \( \omega > \omega_0 \). Then equation (3.11) can be considered as an Euler equation of a medium with a frequency-dependent inertial resistance determined by the effective density

\[
\rho_{\text{eff}} = \rho_m \left(1 - \frac{\omega_0^2}{\omega^2}\right).
\]  

Because in this region every point of the membrane can be replaced by a fluid with an equivalent density \( \rho_{\text{eff}} \), it seems as if the membrane behaves as a fluid-like medium.

Thus, here the boundary between the membrane and the fluid is a line of contact between two different fluids. In other words, in this region the membrane resembles a fluid-like film between the surrounding fluids. At a line of contact between different fluids, cohesive forces are the cause of a pressure difference across that line. It can be shown (for instance Sommerveld, 1964) that these forces only depend on the shape of the surface of separation. The equation of motion for such a thin film is

\[
\rho_m \frac{\partial^2 u_{mn}}{\partial t^2} = \sigma \frac{\partial^2 u_{mn}}{\partial x^2},
\]

in which \( \sigma \) is the surface tension within the medium. In consequence of this, there is from a physical point of view some reason to extend equation (3.10) with a term of the type

\[
c_2 \frac{\partial^2 u_{mn}}{\partial x^2},
\]

in which \( c_2 = \sqrt{\frac{\sigma}{\rho_m}} \). Because at the end of the membrane, i.e. at \( x = l \), again the membrane and its bony environment are grown together, a natural boundary condition is the zero condition for the deflection

\[
u = 0 \text{ at } x = l.
\]

Just after this point, any pressure difference must vanish in consequence of the presence of the helicotrema. This small hole is found close to the end of the membrane. Therefore, it is useful to put as an additional boundary condition

\[
p = 0 \text{ near } x = l.
\]
In summary, the main part of the equation of motion for the membrane is (3.10). The essential distinguishing mark in the equation is the presence of stiffness. Before resonance the stiffness dominates. Near the starting point of the membrane additional forces that model bending stiffness combined with a 'clamped' boundary condition form a rather natural extension. After resonance, the membrane behaves as a fluid-like film. Then the addition of capillary forces to the equation can be argued. This last extension leads automatically to the demand for a zero boundary condition for the deflection at the end of the membrane. In section 3.5 we will come back to (3.10) and in chapter 6 we will discuss an extended version of (3.10) in which is accounted for capillary forces. In the next section we will first formulate a boundary value problem for the pressure in the cochlea. In this problem the radiation condition will take a central place. The model can be considered as stripped to the skin. Thereafter, we will first discuss some global properties of the problem.

3.4 Properties of a boundary value problem for the cochlea

3.4.1 A boundary value problem

The fluid inside the cochlear scalae is approximately incompressible and non-viscous. Therefore we shall assume that the standard condition for incompressibility, i.e.

\[ \text{div} \bar{v} = 0, \tag{3.13} \]

holds true at every point inside the cochlear scalae and at the membrane. Then, as follows from the Euler equations and (3.13), the pressure \( p \) has to fulfil Laplace's equation

\[ \Delta p = 0. \tag{3.14} \]

For the sake of future convenience we shall assume that the geometry of both cochlear scalae is rectangular and the basilar membrane separates both scalae over almost the whole length of the system. Thus our problem concerns a region \( D \) that is shown in figure 3.1a. In this figure the boundary \( \partial D_1 \) is the entrance of the system. This boundary represents the place of the oval window. The basilar membrane is found at \( \partial D_2 \). The boundary \( \partial D_3 \) is the place of the round window. The rigid bony environment of the cochlea is represented by the boundary \( \partial D_4 \). Between the end of \( \partial D_2 \) and \( \partial D_4 \) there is a small gap \( \partial D_{he} \) that models the helicotrema. Thus, actually we deal with one fluid-filled region with a rather sophisticated boundary. When turning around this boundary we have to distinguish between the upper side \( \partial D_2^+ \) and the lower side \( \partial D_2^- \) of the membrane boundary. Then, both inner fluid normals at the two parts of this boundary point in opposite directions. When the membrane performs its motion under the influence of the pressure in the fluid, this motion is odd with respect to the normals at the membrane. Because the pressure is responsible for this motion it must posses the same odd property. In figure 3.1a the normal at the upper side of the membrane points in the positive \( y \)-direction and the normal at the lower side in the negative \( y \)-direction. In consequence of this, it holds \( p(x, y, t) = -p(x, -y, t) \). This odd behaviour can be depicted in a slightly different geometrical context (Fig. 3.1b). Here, the upper box in Fig. 3.1a has been mapped as the box in the left half plane of Fig. 3.1b.
Figure 3.1. a. Simplified rectangular model of the cochlea. The model consists of two scalae separated from each other over almost the whole length by the membrane at $\partial D_2$. Between the end of $\partial D_2$ and the hard wall at $\partial D_4$ a small gap $\partial D_{he}$ models the helicotrema. b. Mapping of the geometry in a. The purpose of this mapping is to formulate a model for the pressure that is odd with respect to the line $x = 0$. The image of the helicotrema is the line of symmetry at $x = 0$.

The reflection of this last box with respect to the vertical axis in Fig. 3.1b is the image of the lower box in Fig. 3.1a. The technical process of mapping the geometry from Fig. 3.1a to Fig. 3.1b does not alter the shape of Laplace's equation.

Moreover, because this two-dimensional mapping is conformal, we expect that the properties of the pressure that follow from both systems are qualitatively the same. Therefore, we will focus our attention on a model with a geometry as has been given in Fig. 3.1b. Possible quantitative differences as a result of geometrical differences are beyond the scope of this work. From a calculational point of view the geometry in Fig. 3.1b is attractive, especially in consequence of the odd behaviour of the pressure with respect to the vertical axis. Van Dijk (1976) and Allen (1977) have proposed a geometry that profits from odd modelling. Now we are in an appropriate position to formulate a boundary value problem that is relatively easy to handle.

Assume that the pressure at $\partial D_1$ is known and given in the shape of a suitable function $f$. This function depends on time and may depend on the position at $\partial D_1$. However, for the sake of convenience we will ignore this last kind of dependence. Thus the boundary condition at $\partial D_1$ reads

$$p = f,$$

in which $f = f(t)$.

The boundary $\partial D^*_2$ represents the upper side of the membranous zone. The length of this boundary equals the length of the basilar membrane. The length of the human basilar membranes varies between 3 and 3.5 cm. Therefore we choose the length of the membrane in
the model equal to \( \pi \). At this boundary we require that \( p \) obeys the radiation condition (3.9). Thus the boundary condition at \( \partial D^+ \) reads

\[
\frac{\partial p}{\partial n} - \frac{2\rho}{m} p = \rho \omega_0^2(x) u \quad \text{for} \quad -\pi < x < 0.
\]

Here, \( u \) is the membrane deflection and \( \omega_0^2 \) the stiffness along the membrane. Both the deflection and the pressure are odd with respect to the line \( x = 0 \). In consequence of this, the stiffness must be modelled as an even function with respect to \( x = 0 \). Therefore, the boundary condition at \( \partial D^- \) reads

\[
\frac{\partial p}{\partial n} - \frac{2\rho}{m} p = \rho \omega_0^2(-x) u \quad \text{for} \quad 0 < x < \pi.
\]

At the boundary \( \partial D_3 \) we require that in consequence of the odd behaviour of the pressure holds

\[ p = -f. \]

The last boundary \( \partial D_4 \) represents an ideal hard wall. At this boundary we require that the normal fluid velocity vanishes. Therefore, the pressure is subjected to the usual hard wall boundary condition

\[ \partial p / \partial n = 0. \]

In summary, the different requirements for \( p \) lead to the following mathematical model. In this model we are asked for the pressure \( p \), which obeys

\[
\Delta p = 0 \quad \text{in} \quad D.
\]

This pressure is subjected to the boundary conditions

\[
\begin{align*}
p &= f \quad \text{at} \quad \partial D_1, \\
\frac{\partial p}{\partial n} - \frac{2\rho}{m} p &= \rho \omega_0^2 u \quad \text{at} \quad \partial D^+, \\
\frac{\partial p}{\partial n} - \frac{2\rho}{m} p &= \rho \omega_0^2 u \quad \text{at} \quad \partial D^- \quad \text{(3.15)}, \\
p &= -f \quad \text{at} \quad \partial D_3, \\
\frac{\partial p}{\partial n} &= 0 \quad \text{at} \quad \partial D_4.
\end{align*}
\]
Along the whole boundary $\partial D_2$ holds

$$\omega_0^2(x) = \omega_0^2(-x)$$

and

$$p(x,0,t) = -p(-x,0,t) \text{ and } u(x,0,t) = -u(-x,0,t).$$

For the sake of completeness, we assume that zero initial conditions complete the model. Because in this model the pressure is odd with respect to $x = 0$, it follows that $p(0,y,t) = 0$. The line with this property can be considered as a boundary $\partial D_{he}$ that is the image of the helicotrema in Fig. 3.1a with a prescribed boundary condition $p = 0$.

The difference between the present and previous boundary value problems for the pressure in the cochlea is that in (3.15) the radiation condition is explicitly present as an inhomogeneous mixed boundary condition that must hold true at every point in time.

This expresses the basic idea that once the deflection of the membrane is known at a point in time, the new pressure is found as a solution of the present problem. Then, when the pressure at the membrane is inserted in the equation of motion for the membrane as follows from (3.8) and (3.10), the actual deflection is found. This process is self-repeating with infinitesimal small time steps. In other words: the influence of the motion of the membrane on the pressure is the result of an ‘indirect’ radiation process.

A well-posed problem ought to have exactly one solution. In theory, investigations to prove this property have been split up in two parts. The first one concerns the question for the existence of a solution for the problem. In the second part the unicity of a possible solution has been investigated.

Concerning our problem the question for the existence of a solution is readily answered. In the sections 3.5 and 3.6 we will simply construct a solution so that the existence has been fulfilled automatically. Then in the following section we shall prove that once the solution has been found, this solution is unique. However, it will appear that a minor modification of the problem leads to the impossibility to decide on unicity. Therefore, some care must be taken.

In so called Neumann problems, that means problems in which the normal derivative of the pressure have been prescribed at the boundary, a property that must be fulfilled in order to be able to decide on unicity is the condition for compatibility (see for instance Garabedian, 1964). It will appear that for our problem compatibility has the meaning of a global constraint for the sum of all forces over the boundary of the problem. Because there is a discussion on the relation between the influence of hair cell activity in relation to cochlear emissions, it is worthwhile to pay attention to this constraint. This will be done in section 3.4.4. In the next two sections we mainly follow the way that has been pointed out by Garabedian (1964).

### 3.4.2 Unicity

We start this section with the well-known first identity of Green

$$\int_D (\nabla p \cdot \nabla q + p \Delta q) \, d\tau + \int_{\partial D} p \frac{\partial q}{\partial n} \, d\sigma = 0.$$
In this expression $D$ is a region enclosed by the boundary $\partial D$. A volume element of $D$ is denoted by $d\tau$, whereas $d\sigma$ represents a surface element of the boundary. The functions $p$ and $q$ are potential functions that have been defined everywhere in $D$ and at the boundary $\partial D$.

Let $q$ fulfil Laplace's equation $\Delta q = 0$ in $D$. If in addition to this $p$ equals $q$, then Green's identity reduces to the expression

$$\|p\|^2 + \int_{\partial D} p \frac{\partial p}{\partial n} d\sigma = 0 ,$$

in which $\|p\|^2$ is the Dirichlet integral

$$\|p\|^2 = \int_D \nabla p \cdot \nabla p d\tau .$$

For our purpose we shall assume that the boundary $\partial D$ encloses the left half of the rectangular region $D$ (see Fig. 3.1 b) in problem (3.15). As follows from this figure, $\partial D$ is composed of the parts $\partial D_1$, $\partial D_2$, $\partial D_3$ and half the boundary $\partial D_4$. Let us first assume that at an arbitrary moment in time the right member of the radiation condition is known. Suppose that $p_1$ and $p_2$ are two solutions for problem (3.15). Then, the difference $d = p_1 - p_2$ fulfils the equivalent of (3.15) for $d$ with homogeneous boundary conditions. In consequence of this, the counterpart of (3.16) for $d$ reads

$$\|d\|^2 + \frac{2\rho}{m} \int_0^{\pi/2} d\sigma d\sigma = 0 .$$

Here, the integral is taken along $\partial D_2$. Because both terms at the left-hand side are positive or zero we conclude that, both in the region $D$ and at its boundary $\partial D$, only the zero option is valid. Therefore, $p_1 = p_2$ so that a solution for this problem is unique.

The present way of reasoning follows from typical spatial considerations and holds true at every moment in time.

Next we turn our attention to the time domain. Let us assume that the deflection oscillates with a frequency $\omega$. For such vibrations holds that at an arbitrary point $x$ of the membrane boundary $\ddot{u}(x,t)$ equals $-\omega^2 u(x,t)$. The equations of motion, which govern the fluid motion of the neighbouring fluid, are the Euler equations in absence of external forces. In section 3.3 we assumed that the membrane always follows the motion of the fluid. In consequence of this, normal components of comparable physical quantities are the same. Then, as follows from (3.3) when the pressure is known, the deflection at the membrane can be found according to

$$u = \frac{1}{\rho \omega^2} \frac{\partial p}{\partial n} .$$

(3.17)
However, when this corollary is substituted in the radiation condition at $\partial D^*_2$, the originally inhomogeneous condition takes the shape of the homogenous radiation condition
\[
\left(1 - \frac{\omega_0^2}{\omega^2}\right) \frac{\partial p}{\partial n} - \frac{2\rho}{m} p = 0 \text{ for } -\pi < x < 0.
\] (3.18)

Then, a similar way of reasoning as in the first part of this section and application of (3.16) leads to
\[
\|d\|^2 + \frac{2\rho\omega^2}{m} \int_{-\pi}^{0} \frac{d^2}{\omega^2 - \omega_0^2} d\sigma = 0.
\]

Before the point of resonance, i.e. when $\omega < \omega_0$, the denominator in the integrand is negative. Then, apart from the behaviour of the integrand at the point of resonance, i.e. when $\omega = \omega_0(x, \cdot)$, this condition shows that we are not able to decide a priori on unicity. However, when we assume that the deflection is given along the membrane, it is not difficult to decide on unicity of the solution for the pressure from model (3.15).

The modest result of this section is that a problem of the kind (3.15) belongs to the class of well-posed problems. The solution of this problem makes it possible to express the pressure at the membrane in terms of the deflection at the membrane. This property is useful to describe some aspects of time domain modelling and will be applied both in section 3.5 and in section 3.6.

At this stage it is worthwhile to note that in the oscillating case the homogenous condition (3.18) is equivalent to the equation of motion for the membrane (3.10) and (3.8), combined with (3.17). The present considerations show that this combined condition is not suited to serve as a boundary condition for the pressure in a well posed boundary value problem for the pressure.

In the literature different methods are found that can be conceived as trials to overcome this problem. Viergever and Kalker (1975) and Allan and Shondi (1977) looked at the boundary value problem for the pressure as a Neumann problem. In those problems it is assumed that the normal derivative of the pressure at the boundary is a known function. This assumption is sufficient to find a solution for the Neumann problem. However, then there still remains some uncertainty whether this solution is the solution of the problem or not. Proving that the solution under consideration fulfils the condition of compatibility can eliminate this uncertainty. This condition implies that the integral over the whole boundary of the prescribed normal derivative of the pressure at the boundary vanishes (see also section 3.4.4). Van Dijk (1976) removed this intrinsic difficulty by opting for a different formulation of the problem that has to be solved. He used the trick of translating the cochlear fluid problem to a Dirichlet problem. In those problems the boundary function is continuous along the boundary of the problem. In consequence of this, compatibility has been met automatically. In spite of this advantage, the main restriction of this approach is its limitation to only the two-dimensional case.
A typical advantage of a Neumann problem is that it is easy to extend a two-dimensional problem to its three-dimensional counterpart (de Boer, 1980; Viergever, 1980). In recent investigations Mammano and Nobili, (1993) reintroduced the Neumann approach in a three-dimensional model for the pressure in the cochlear scala. In order to solve their problem they applied the Green's function technique to this potential problem. According to our opinion some care must be taken at the interpretation of their results. Quite formally, the function of Green in a Neumann problem does not exist because it is impossible to obey the condition of compatibility (Garabedian, 1963). Therefore, we can never decide on the unicity of a solution for this kind of problems that has been expressed in terms of a ‘simple’ function of Green for this problem.

In section 3.4.4 we will come back to the concept of compatibility. In consequence of the present discussion, we will consider (3.18) as an independent equation for the pressure at the membrane. In order to make the meaning of this equation more realistic, it is useful to introduce some damping in the model. This will be done in the next section combined with a description of some consequences. Properties of the solution of equation (3.18) are the subject of chapter 4 and approximations are found in chapter 5.

3.4.3 The lossy case

Thus far we only considered both the basilar membrane and the surrounding fluid as lossless media. In this section we will enlarge the scope of this description by introducing resistive terms in the equations of motion. This can be done when (3.1) and (3.2) are extended with terms of the kind \(-\lambda \vec{v}\). The positive constant \(\lambda\) denotes an effective coefficient for the damping. The dimensions of such coefficients are \(s^{-1}\). The equation of motion for the fluid, which includes dissipation, is found when (3.1) is replaced by

\[
\frac{d\vec{v}}{dt} = -\frac{1}{\rho} \nabla p - \lambda_1 \vec{v} + \vec{F},
\]

(3.19)

in which \(\lambda_1\) is a measure for the damping of the fluid. In the same manner the equation of motion for the membranous medium (3.2) can be extended to

\[
\frac{d\vec{v}_m}{dt} = -\frac{1}{\rho_m} \nabla p_m - \omega_0^2 \vec{u}_m - \lambda_2 \vec{v} + \vec{F}_m.
\]

(3.20)

Here, \(\lambda_2\) determines the damping for this medium. In general it is not necessary that both coefficients are equal.

Similar considerations as have been pointed out in section 3.3, lead to the membrane condition in the lossy case. This condition again has the shape of an inhomogeneous radiation condition and reads

\[
\frac{\partial p}{\partial n} - 2\rho_m \frac{\partial u_{mn}}{\partial n} = \rho \omega_0^2 \dot{u}_{mn} + \rho (\lambda_2 - \lambda_1) \dot{u}_{mn}.
\]
With the assumption that zero initial conditions hold true, the Laplace transform is
\[
\frac{\partial p}{\partial n} + \frac{2\rho}{m} \tilde{p} = \rho \omega_0^2 \tilde{u}_{nn} + \rho (\lambda_2 - \lambda_1) \tilde{s} \tilde{u}_{nn}.
\]  
(3.21)

In absence of external forces, the Laplace transform of the linear counterpart of (3.19) for the deflection normal to the boundary between fluid and membranous region is
\[
(s^2 + \lambda_1 s) \tilde{u}_{nn} = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial n}.
\]  
(3.22)

Then, when \( \tilde{u}_{nn} \) is eliminated from (3.21) and (3.22) we arrive at the equation of motion of the membrane in terms of the transformed pressure. Thus
\[
(s^2 + \lambda_2 s + \omega_0^2) \frac{\partial \tilde{p}}{\partial n} - \frac{2\rho}{m} (s^2 + \lambda_1 s) \tilde{p} = 0.
\]  
(3.23)

The last equation is the analogue of (3.18) in the lossy case. The coefficient of the pressure possesses two zeroes, namely \( s = 0 \) and \( s = -\lambda_1 \). Because we shall assume that \( \lambda_1 \) is relatively small, both zeros can be replaced by one second order zero. This means that the surrounding fluid is considered as effectively lossless. In consequence of this (3.23) is replaced by
\[
(s^2 + \lambda_2 s + \omega_0^2) \frac{\partial \tilde{p}}{\partial n} - \frac{2\rho}{m} s^2 \tilde{p} = 0.
\]  
(3.24)

The coefficient of the normal derivative has two complex conjugate zeros in the \( s \)-plane. Because in our applications \( \lambda_2 \) is positive, the real parts of these zeroes are in the left half plane \( \text{Re} \ s < 0 \), so that stability in the time domain has been ensured. The net result of the addition of damping is that at every moment the right member of the radiation condition in problem (3.15) is free from resonant behaviour in a mathematical sense. Then, several techniques can be applied without restrictions in view of singular behaviour. In section 3.5 and 3.6 we will profit from this behaviour and apply the Fourier sine transform to the deflection of the membrane.

In the cochlea, the normalised stiffness \( \omega_0^2 \) depends on the length parameter \( x \) along the membrane. Thus, in general it holds that \( \omega_0^2 = \omega_0^2(x) \). This dependence on \( x \) determines points near the membrane axis at which resonance takes place. These points are the zeros of the equation
\[
s^2 + \lambda_2 s + \omega_0^2(x) = 0.
\]  
(3.25)

We first assume that the stiffness behaves as a linearly increasing function along the membrane. Thus \( \omega_0^2(x) = ax + b \), in which \( a \ (>0) \) and \( b \) are constants. Note that this expression is only meaningful as far as the stiffness is positive. In view of our interest in oscillatory behaviour, we again put \( s = \pm i\omega \) and insert this in (3.25).
Then we have that the points of resonance in the place domain follow from

\[ ax + b = \omega^2 \mp i\lambda_2 \omega . \]

The minus sign corresponds to positive frequencies\(^3\), whereas the plus sign determines points of resonance for negative frequencies. This means that the mathematical point of resonance for positive frequencies lies just below the membrane axis and for negative frequencies at the upper side of this axis. This division implies that we have to distinguish between an upper and a lower plane approximation to the membrane axis. The upper plane approximation is valid for positive frequencies. The lower plane approximation holds true for negative frequencies. When the damping tends to zero, the respective points of resonance tend to coincide with the membrane axis. Because the lossless case is always the limiting case of the lossy one, we conclude that even in the lossless case there must be a distinction between an upper and a lower plane approximation to the membrane axis. These differences are induced by the sign of the frequency under consideration and are necessary to conserve the image of complex conjugate amplitudes that correspond to complex conjugate time behaviour of the kind \( \exp(\pm i\omega t) \).

Up to now we considered the case of an increasing stiffness along the membrane. When the stiffness decreases the constant \( a \) is negative. Then it is useful to replace \( a \) by \( -|a| \). In consequence of this, the place of the points of resonance at the membrane axis has been mirrored with respect to the real axis compared with the increasing case. Thus the role of the upper and lower plane approximation in relation to the sign of the frequency must be changed. Therefore, when we deal with negative frequencies we arrive at an upper plane approximation to the membrane axis and at a lower plane approximation when the frequencies under consideration are positive. The linear behaviour of the stiffness is not typical for the present considerations. Only the monotonic behaviour of the stiffness in relation to the sign of the frequency is the characteristic feature at the choice of the appropriate approximation. We will not work out similar considerations for this more general case.

Let us reconsider (3.24) for complex oscillations with positive frequencies. In oscillating problems it is attractive to model the coefficient for the damping proportional to the resonance frequency of an oscillator. In our case this means that \( \lambda_2 \) is chosen proportional to \( \omega_0(x) \). Let the constant of proportionality be \( 2\sin \varepsilon \) in which \( \varepsilon \) is a small positive number. Then, when \( \lambda_2 = 2\sin \varepsilon \omega_0(x) \) and \( s = +i\omega \) are inserted in (3.24) it appears that this equation can be written as

\[
\frac{\partial \bar{p}}{\partial n} + \frac{2\rho}{m} q(x, \omega)|\bar{p}| = 0 ,
\]

in which

\[
q(x, \omega) = \frac{1}{2\cos \varepsilon} \left( \frac{\omega}{\omega_0(x) - \omega \exp(-i\varepsilon)} - \frac{\omega}{\omega_0(x) + \omega \exp(+i\varepsilon)} \right).
\]

\(^2\) Here \( x \) must be conceived as the real axis of the complex \( z \) plane.

\(^3\) Positive frequencies are points of the positive frequency axis of the \( s \) plane; negative frequencies correspond to points at the negative part of this axis.
Let us first assume that the positive $x$ axis represents the membrane axis and that this time not the stiffness but the resonance frequency $\omega_0(x)$ increases linearly along this axis. Thus $\omega_0(x) = x$. In consequence of this, it follows that resonance takes place at the points $x = \pm \omega \exp(\pm i \varepsilon)$. Then it holds that for almost the whole positive axis the main contribution to $q(x, \omega)$ is determined by the first term between the brackets.

When $\omega$ is negative it is useful to write $\omega = -|\omega|$ and to insert this in $q(x, \omega)$. In this case the second term between the brackets is the leading one. For this reason and in consequence of the small value of $\varepsilon$, the behaviour of $q(x, \omega)$ near the membrane axis is determined by the approximation

$$q(x, \omega) \approx \begin{cases} \frac{1 + \omega}{2 x - \omega \exp(-i \varepsilon)} & ; x > 0 \text{ and } \omega > 0 \\ \frac{1 - |\omega|}{2 x - |\omega| \exp(+i \varepsilon)} & ; x > 0 \text{ and } \omega < 0 \end{cases}$$

The first part shows that for positive frequencies the point of resonance reads $x = \omega \exp(-i \varepsilon)$. This point is found just below the membrane axis. From the second part it follows that for negative frequencies the point of resonance equals $x = |\omega| \exp(+i \varepsilon)$ and lies at the upper side of the membrane axis. Therefore, again we have to distinguish between an upper and a lower plane approximation to the membrane axis.

It is worthwhile to note that essentially only the place-to-frequency map that follows from $x = |\omega|$ determines the behaviour of $q(x, \omega)$.

From a mathematical point of view, the present result can be modelled in a slightly different way. Let us start with the lossless case in which $x > 0$ and $\omega$ is positive. Near the point $x = \omega$, the denominator can be replaced by a small complex quantity defined by the arc of a half-circle around this point. When the radius of this circle is $\varepsilon$, $q(x, \omega)$ can be modified to

$$q(x, \omega) \approx \begin{cases} \frac{1 + \omega}{2 x - \omega} & , |x - \omega| > \varepsilon \\ \frac{1 + \omega}{2 \varepsilon \exp(i \varphi)} & , |x - \omega| \leq \varepsilon \end{cases}$$

The radius of the half-circle models the magnitude of the damping in an artificial way. When this radius depends on $\omega$, i.e. $\varepsilon = \varepsilon(\omega)$, the damping depends on the frequency.
The angle $\varphi$ is given by

$$\varphi = \begin{cases} \arctan \left( \frac{\sqrt{\varepsilon^2 - (x-\omega)^2}}{x-\omega} \right), & x-\omega > 0 \\ \frac{\pi}{2}, & x-\omega = 0 \\ \pi + \arctan \left( \frac{\sqrt{\varepsilon^2 - (x-\omega)^2}}{x-\omega} \right), & x-\omega < 0. \end{cases}$$

As we mentioned in chapter 2, in practice the stiffness is often considered as an exponentially varying function along the membrane axis. In this case it is sufficient to write for the resonance frequency $\omega_0(x) = \exp(x)$. Then, according to (3.26) $q(x,\omega)$ can be written as

$$q(x,\omega) = \frac{1}{2\cos \varepsilon} \left( \frac{\exp(i\varepsilon)}{\exp(x-\ln \omega+i\varepsilon)-1} - \frac{\exp(-i\varepsilon)}{\exp(x-\ln \omega-i\varepsilon)+1} \right).$$

Quite formally, the first term between the brackets is resonant when

$$x - \ln \omega + i\varepsilon = \pm i2k\pi, \quad k = 0,1,...$$

and the second term when $x$ obeys the expression

$$x - \ln \omega - i\varepsilon = \pm i(2k + 1)\pi, \quad k = 0,1,... .$$

The only singular term that is close to the membrane axis is the singularity for $k = 0$. This term reads

$$x - \ln \omega + i\varepsilon = 0 .$$

All other singular points are ‘far’ from the membrane axis. Thus again we may expect that the behaviour of $q(x,\omega)$ near the membrane axis is determined by this singularity. Because we again assume that the damping $\varepsilon$ is rather small, the behaviour of $q(x,\omega)$ is highly determined by the approximation

$$q(x,\omega) \approx \frac{1}{2} \frac{1}{\exp(x-\ln \omega + i\varepsilon)-1} . \quad (3.28)$$

In this expression again the frequency-to-place map $x = \ln \omega$ determines the qualitative behaviour of $q(x,\omega)$.

An approximation of $q(x,\omega)$ that has been based upon this map can be found when we put

$$\zeta = x - \ln \omega + i\varepsilon . \quad (3.29)$$
When we consider $\xi$ as a new independent variable, the approximation (3.28) of $q(x,\omega)$ can be written in terms of this variable. This yields

$$q(\xi) \approx \frac{1}{4} \exp\left(-\frac{\xi}{\sinh \frac{\xi}{2}}\right).$$

The Laurent series expansion for the hyperbolic cosecant (Abramowitz and Stegun; 4.5.65) is

$$\frac{1}{\sinh z} = \frac{1}{z} - \frac{1}{z^3} + \frac{7}{360} \frac{z}{z^3} - \frac{2(2^{2n-1} - 1)}{(2n)!} B_{2n} z^{2n-1} + \ldots ; |z| < \pi.$$

Here, the numbers $B_{2n}$ are the Bernoulli numbers are defined by the expression (Abramowitz and Stegun; 23.1.1)

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} ; |t| < 2\pi.$$

From this expansion follows that near resonance it is allowed to replace the hyperbolic cosecant by the first term of its series expansion. In consequence of this, the near resonance approximation for $q(\xi)$ reads

$$q(\xi) \approx \frac{1}{2\xi},$$

or, which is equivalent to this,

$$q(x,\omega) \approx \frac{1}{2} \frac{1}{x - \ln \omega + i\epsilon}.$$

Again from a mathematical point of view, it is sufficient to replace $q(x,\omega)$ by its lossless counterpart

$$q(x,\omega) \approx \frac{1}{2} \frac{1}{x - \ln \omega}$$

and replace the term $x - \ln \omega$ near resonance by a complex quantity determined by the arc of a half-circle around the point $x = \ln \omega$. When again the radius of this circle is $\epsilon$, it is readily found that $q(x,\omega)$ reads

$$q(x,\omega) \approx \frac{1}{2} \frac{1}{\exp(\xi) - 1}, \quad (3.30)$$

in which
\[
\begin{align*}
\zeta &= \begin{cases} x - \ln \omega , & |x - \ln \omega| > \varepsilon \\
\varepsilon \exp(i\phi) , & |x - \ln \omega| \leq \varepsilon \end{cases} \\
\phi &= \begin{cases} \arctan \left( \frac{\sqrt{e^2 - (x - \ln \omega)^2}}{x - \ln \omega} \right) , & x - \ln \omega > 0 \\
\frac{\pi}{2} & \text{, } x - \ln \omega = 0 \\
\pi + \arctan \left( \frac{\sqrt{e^2 - (x - \ln \omega)^2}}{x - \ln \omega} \right) , & x - \ln \omega < 0 \end{cases}
\end{align*}
\]

and

In this case too, \( \varepsilon \) may depend on \( \omega \) so that a frequency dependent damping can be modelled. Similar considerations hold true for strictly decreasing functions for the stiffness in relation to negative frequencies.

It should be noted that \( \zeta \), as defined by (3.29), is the independent variable in the main part of \( q(x, \omega) \) that is given by (3.28). This variable can be written as \( \zeta = (x + i\varepsilon) - \ln \omega \). From this simple expression it is easy to see that we actually deal with a problem defined at the straight line \( -\infty < x < +\infty; y = \varepsilon \) parallel to the original membrane axis.

From the present considerations we conclude that, when we deal with a linearly increasing relation between the frequency and the place of resonance at the membrane and in addition to this with positive frequencies, it is sufficient to replace \( q(x, \omega) \) in equation (3.26) by (3.27). In this case the positive x-axis represents the membrane axis. The negative real axis is appropriate to model a decreasing linear place-to-frequency map. In this last case the point of resonance is just below the membrane axis for negative frequencies.

In the case of an exponentially increasing relation between resonance frequency and place of resonance at the membrane, \( q(x, \omega) \) can be chosen according to (3.30). In this case too, we first assume that we deal with positive frequencies. In contrast with the linear case, the whole real axis can be considered as the membrane axis. When the resonance frequency decreases exponentially, the imaginary part of the point of resonance is negative for negative frequencies.

When physical damping is present, it is not necessary to take special measures to avoid any point of resonance. Then we deal with a problem in which the distance of a point of the membrane to the mathematical point of resonance is at least \( \varepsilon \).

### 3.4.4 Compatibility

In problem (3.15), the pressure \( p \) is the potential from which forces can be deduced in any direction by taking derivatives in the appropriate direction. This implies that when the
solution of (3.15) is available, we are in the position to study the global equilibrium of forces that plays a role in our problem.

In this section we will first derive a condition for this equilibrium from a quite formal point of view. Then it will be argued that the same condition can be found as the result of an ad hoc way of reasoning. After that we will give plausible arguments that in the case that the deflection of the membrane partly results from additional forces, for instance outer hair cell forces between the oval window and the point of resonance, the motion of the oval window can ‘reflect’ these additions.

We start with a formal method that has been given by Garabedian (1964). Let us turn our attention to the identity of Green in the section 3.4.2. When in this expression \( p \) and \( q \) are interchanged and the result is subtracted from the original identity we arrive at what is called the second shape of Green’s identity

\[
\int_D (p \Delta q - q \Delta p) \, d\tau + \int_{\partial D} \left( p \frac{\partial q}{\partial n} - q \frac{\partial p}{\partial n} \right) \, d\sigma = 0.
\]

After insertion of \( q = 1 \), this identity takes the shape

\[
\int_D \Delta p \, d\tau + \int_{\partial D} \frac{\partial p}{\partial n} \, d\sigma = 0.
\]

In the region \( D \) of our problem, the potential \( p \) obeys Laplace’s equation \( \Delta p = 0 \). Therefore, this expression is reduced to

\[
\int_{\partial D} \frac{\partial p}{\partial n} \, d\sigma = 0.
\]

This means that the integral of normal derivatives of the pressure over the whole boundary of the system must vanish. However, according to the Euler equations (3.3) a normal derivative of the pressure represents a force per unit of mass. Thus this condition expresses the law of conservation of forces over the boundary of the system. In the literature, this expression is called the condition for compatibility. This condition is a global constraint for our problem too. So far the formal side of the problem.

Now we turn our attention to the informal side. The boundary \( \partial D \) of one of the cochlear scalae encloses an incompressible fluid in the region \( D \). Let \( u_n \) be the normal component of the deflection at any point of this boundary. According to the law of conservation of mass and the incompressibility of the fluid we have

\[
\int_{\partial D} u_n \, d\sigma = 0 \quad (3.31)
\]

at every point in time. Let us differentiate this expression two times with respect to the time and change the order of differentiation and integration. Then, when the equation of Euler equation (3.3) in absence of external forces is applied, we almost immediately arrive again at
the formal shape of the condition for compatibility. Therefore, for our problem the formal condition of compatibility and \((3.31)\) are equivalent. Let us look for some consequences.

In problem (3.15) the boundary \(\partial D\) of the left box in Fig. 3.2b consists of the four parts \(\partial D_1, \partial D_2^*, \partial D_4\) and half the boundary \(\partial D_4\). Normal components of the deflection vanish at \(\partial D_4\). Then \((3.31)\) requires that

\[
\int_{\partial D_1} u_n \, d\sigma + \int_{\partial D_2} u_n \, d\sigma + \int_{\partial D_4} u_n \, d\sigma = 0.
\]

It is generally accepted that, except for the lowest frequencies, the deflection of the membrane near the helicotrema is negligible. Thus, in most cases, the last integral in the left side of this expression is without meaning. Therefore, the sum of the integrals reduces effectively to

\[
\int_{\partial D_1} u_n \, d\sigma \approx -\int_{-\pi}^{0} u \, dx,
\]

where \(u_{ov}\) is the deflection normal to the oval window at \(\partial D_1\). In this case the membrane starts at \(x = -\pi\) and ends at \(x = 0\). Normal components of the membrane deflection are denoted by \(u\). The formal ‘surface’-element \(d\sigma\) along the membrane has been replaced by \(dx\). Note that if the geometry of the model should be given by fig. 3.2a and \(\partial D_2^*\) should meet the hard wall \(\partial D_4\), then the present expression would be exact.

Next we introduce the mean deflection at the oval window according to

\[
\bar{u}_{ov} = \frac{1}{h} \int_{\partial D_1} u_{ov} \, d\sigma,
\]

where \(h\) is the ‘surface’ of the oval window. In consequence of this we have that at every point in time

\[
\bar{u}_{ov} \approx -\frac{1}{h} \int_{-\pi}^{0} u \, dx.
\]  \((3.32)\)

This expression shows that the integral of all that happens along the membrane is ‘reflected’ at the oval window and conversely.

In auditory theory, it is usual to speak of passive models when in models of the cochlea only forces at the oval window are the cause of the deflections. When in addition to these forces, forces in consequence of hair cell activity also have an influence on the motion of the membrane, it is customary to call those models active. Independent of the presence of activity, both kind of models have to fulfil the formal condition of compatibility or the approximation \((3.32)\). Therefore, we may expect that in active models effects in consequence of additional forces at the membrane can be present at the oval window. In the next part of this section we shall elucidate this more clearly with the help of an approximation.

Let us turn our attention to the steady-state oscillating behaviour. Then, when the membrane oscillates with frequency \(\omega\), it holds that the deflection \(u\) equals \(-\omega^2 \ddot{u}\). For the sake of convenience we will restrict ourselves to only one frequency. However, when the
membrane should perform a complex vibration pattern, extensions can be made in a similar
way.
When again the Euler equation is applied to the motion normal to \( u \), (3.32) can be written as

\[
\bar{u}_{ov} \approx \frac{1}{\rho \omega^2 h} \int_{-\pi}^{0} \frac{\partial p}{\partial y} \, dx ,
\]

(3.33)
in which \( y \) represents the normal direction. In consequence of this, the mean deflection at the oval window is proportional to the integral of all forces along the membrane.

A typical parameter of cochlear models is the scala height \( h \). In the human cochlea the mean scala height is about \( 3\% \) of the length of the basilar membrane. Thus it is evident that \( h \) is a small parameter. Small values of \( h \) can be used to introduce approximations for the derivative of the pressure in the normal direction near the membrane.

The pressure in a cochlear scala is a function of place and time. Therefore, in view of the present two-dimensional approach we write \( p = p(x, y, t) \). The scala can be considered as a thin strip with height \( h \). The upper side of the strip is a hard wall where the usual hard wall boundary condition holds true. The lower side of the strip is the basilar membrane (Fig. 3.2).

\[
\partial D_4
\]

\[
\partial D_2
\]

\[
x_r
\]

\[
h
\]

\[
\varepsilon
\]

Figure 3.2. The height \( h \) of the cochlear scalae is about \( 3\% \) of the length of the basilar membrane. When \( h \) is considered as a small parameter, the normal derivative of the pressure at the membrane can be approximated by the second derivative of the pressure near the membrane. This approximation can be validated for points that are sufficiently far from the point of resonance at the membrane.

Let us consider what happens at a line \( y = \varepsilon \) parallel to the membrane. \( \varepsilon \) is a small positive number so that \( \varepsilon << h \). Because \( h \) is small, the second derivative with respect to \( y \) at \( y = \varepsilon \) can be approximated by

\[
\frac{\partial^2 p(x, \varepsilon, t)}{\partial y^2} \approx \frac{p_y(x, h, t) - p_y(x, \varepsilon, t)}{h - \varepsilon} .
\]

(3.34)

Here \( x \) is an arbitrary point at the membrane. The derivative of the pressure normal to the boundary \( \partial D_4 \), i.e. the boundary at \( y = h \), vanishes because this boundary represents a hard wall. In consequence of this (3.34) is reduced to

\[
\frac{\partial^2 p(x, \varepsilon, t)}{\partial y^2} \approx \frac{p_y(x, \varepsilon, t)}{h - \varepsilon} .
\]

(3.35)
Next we assume that Laplace's equation holds true between the boundaries of the strip and at every point in time. This equation can be written in the shape

\[ \frac{\partial^2 p(x, \varepsilon, t)}{\partial y^2} = -\frac{\partial^2 p(x, \varepsilon, t)}{\partial x^2}. \]

When in (3.35) the second derivative of the pressure in the \( y \) direction is replaced by minus the second derivative in the \( x \) direction, it follows that

\[ (h - \varepsilon)\frac{\partial^2 p(x, \varepsilon, t)}{\partial x^2} \approx \frac{\partial p(x, \varepsilon, t)}{\partial y}. \]

Because \( \varepsilon \) is very small, we approximate this expression by

\[ h \frac{\partial^2 p(x, 0, t)}{\partial x^2} \approx \frac{\partial p(x, 0, t)}{\partial y} \text{ at } \partial D_2. \]

(3.36)

However, some care must be taken. At the boundary \( \partial D_2^* \) resonance can take place. In mathematical terms this means that in case of resonance we meet singular behaviour. Then, the validity of the approximations is restricted to points of the membrane that are sufficiently far from the point of resonance or to points in the surrounding fluid at a suitable distance to the point of resonance. What can happen when a point of resonance is included as a regular point is not the topic of this section but will be discussed in chapter 5.

Throughout this section we will assume that the present approximation is applicable near the membrane. Then, when (3.36) is inserted in (3.34) and the integration is carried out, the mean deflection at the oval window takes the shape

\[ u_{ow} \approx \frac{1}{\rho \omega^2} \left( \frac{\partial p(-\pi, 0, t)}{\partial x} - \frac{\partial p(0, 0, t)}{\partial x} \right). \]

There is evidence to neglect the second term between the brackets because this term describes effects near the helicotrema. A second argument could be that when the geometry of the model corresponds to Fig. 3.1a, the hard wall boundary condition at \( \partial D_4 \) implies a vanishing normal derivative. Therefore, we approximate the mean value of the deflection at the oval window by

\[ u_{ow} \approx \frac{1}{\rho \omega^2} \frac{\partial p(-\pi, 0, t)}{\partial x}. \]

(3.37)

Assume that resonance takes place at the point \( x_r; -\pi < x_r < 0 \). At present it is believed that near a point of resonance additional forces as a result of outer hair cell activity have substantial influence on the motion of the membrane. It seems as if this motion is related to the typical undulatory behaviour of the membrane before resonance. In consequence of this, (3.32) and (3.33) must be extended with terms that model this motion.
Because the behaviour of the cochlear fluid obeys linear equations of motion, the extended version of (3.32) can be written as

$$\bar{u}_{ov} \approx -\frac{1}{h}\left(\int_{-\pi}^{0} u_p \, dx + \int_{-\pi}^{\pi} u_a \, dx\right).$$

Here, $u_p$ is the passive part and $u_a$ is that part of the membrane motion in consequence of hair cell activity. The extended version of (3.33) takes the shape

$$\bar{u}_{ov} \approx -\frac{1}{\rho \omega^2 h}\left(\int_{-\pi}^{0} \frac{\partial p_p}{\partial y} \, dx + \int_{-\pi}^{\pi} \frac{\partial p_a}{\partial y} \, dx\right),$$

where again the subscripts $p$ and $a$ refer to the passive and active part of the pressure respectively. When (3.36) is applied to this last expression and again the integration is carried out, the extended version of (3.37) is found and reads

$$\bar{u}_{ov} \approx -\frac{1}{\rho \omega^2} \frac{\partial p_p(-\pi,0,t)}{\partial x} + \frac{1}{\rho \omega^2} \left(\frac{\partial p_a(-\pi,0,t)}{\partial x} - \frac{\partial p_a(x_r,0,t)}{\partial x}\right).$$

(3.38)

At this stage it is worthwhile to formulate some preliminary conclusions that only hold true for the model of this section. In section 3.8 we will come back to this point. First of all, when additional forces are present in the region of the membrane in which the motion is undulating, i.e. $-\pi < x < x_r$, the motion at the oval window consists of two parts. The first part is the effect of a prescribed pressure at the stapes and the second part is the difference between the change of rates of the additional pressure at the oval window and at the point of resonance.

As is known from the literature, the first part is responsible for a travelling wave from the oval window to the point of resonance. The additional part of the motion at the oval window essentially follows from considerations concerning compatibility, which means considerations on the conservation of forces along the whole boundary of the system. The present expression shows that in addition to the prescribed pressure $p_p(-\pi,0,t)$ at the oval window, there is a pressure $p_a(-\pi,0,t)$ at this window.

Both the deflection and the pressure at the oval window may have unknown components as the result of additional forces at the membrane. Both quantities will be present in energy considerations. Therefore, it is better to look at the balance of energy in the system. This will be done in section 3.7.

3.5 On the solution of the problem

3.5.1 Introduction

In section 3.4.3 we showed that resonance at the basilar membrane is the cause of singularities in a spatial boundary problem for the pressure in the cochlea. However, both in this section and in section 3.6 we will act the part of the naïve mathematician and neglect the implications that follow from the presence of these singularities. Then there are no stings of conscience to look in problem (3.15) for a function of Green that obeys the radiation condition for the
pressure at the membrane and follows from a regular expression that fits the other boundary conditions. In section 3.5.2 we will construct that function quite formally.

The function of Green offers the opportunity to express the pressure at the membrane in terms of the stiffness forces of the membrane oscillators. This pressure and the equation of motion of the membrane yield an integral equation for the deflection of the membrane. The equation is given in section 3.5.3. We will invert this equation and again derive the equation for the pressure at the membrane. This process clearly emphasises that the singularities of the problem are neglected. This is unacceptable from a mathematical point of view. Because of this it holds that the solutions from section 3.5 and section 3.6, and solutions that are equivalent to this, cannot lay any claim to be a solution of a spatial boundary value problem for the cochlea. We will come back to this point in the discussion in section 3.8.

### 3.5.2 A function of Green

The purpose of this section is to investigate some properties of the forced motion of the basilar membrane. This motion has to obey an equation of motion that follows from the previous sections. According to (3.4), (3.6) and (3.8) and in absence of external forces this equation of motion can be written as

\[
\frac{d^2 u}{dt^2} = -\omega_o^2(x) u - p_m ,
\]

in which along the basilar membrane

\[
p_m(x, t) = p(x,0,t) ; u(x, t) = u_{mn}(x,0,t).
\]

In this expression the subscripts for the membrane deflection have been omitted for the sake of clarity. The pressure \( p(x,0,t) \) is the pressure at the membrane and follows from a model of the kind (3.15).

Let us return to problem (3.15) and let us write the pressure \( p(x, y,t) \) so that

\[
p(x, y,t) = p^{part}(x, y,t) + q(x, y,t) ,
\]

(3.40)

It is our aim that the particular solution \( p^{part}(x, y,t) \) includes all effects that follow from the prescribed pressure \( f(t) \) at the stapes. In consequence of this, the second term in this expression, the pressure \( q(x, y,t) \), only depends on the stiffness force \( \omega_o^2(x) u(x,t) \) at the membrane. The place dependent part of the particular solution is composed of two terms. The first term is a linearly decreasing pressure that equals unity at the oval window and vanishes at the helicotrema. It will be clear that this pressure does not fulfil the homogenous part of the membrane condition. In order to reach that goal, we add to this pressure a sine series in the shape

\[
\sum_{n=1}^{\infty} b_n \sin n x \cosh n(h - y) .
\]
The separate terms of this series obey Laplace’s equation for the pressure and fulfil the hard wall boundary condition at \( y = h \). In addition to this, the series vanishes both at the oval window and the helicotrema. When the series is inserted in the boundary condition at the membrane, the coefficients \( b_n; \ n = 1, 2, \ldots \) can be chosen so that the known part of the pressure obeys the homogenous membrane condition. The result is

\[
p_{\text{part}}(x, y, t) = f(t)\left(-\frac{x}{\pi} - \sum_{n=1}^{\infty} c_n \frac{a \cosh(y - h)}{n \sinh nh + a \cosh nh} \sin nx\right),
\]

where the constant \( a \) is defined by

\[
a = \frac{2 \rho}{m}.
\]

The coefficients \( c_n; \ n = 1, 2, \ldots \) in this expression are coefficients according to the Fourier series expansion

\[
-\frac{x}{\pi} = \sum_{n=1}^{\infty} c_n \sin nx ; -\pi < x < \pi .
\]

It can be verified that

\[
c_n = (-1)^n \frac{1}{n}.
\]

The particular solution (3.41) fits the prescribed pressure at the stapes, i.e. the pressure at the boundaries \( x = \pm \pi \) and obeys the hard wall boundary condition at \( y = h \). Moreover, it can be verified that at the membrane axis, the axis \( y = 0 \), this solution satisfies the homogenous radiation condition

\[
\frac{\partial p_{\text{part}}}{\partial y} - a p_{\text{part}} = 0; \ -\pi < x < \pi .
\]

In consequence of this homogenous boundary condition, the requirements for the remaining part of the pressure \( q(x, y, t) \) at the membrane axis \( \partial D_2 \) are

\[
\frac{\partial q}{\partial y} - aq = \rho \omega_0^2 u ; \ -\pi < x < \pi , \ y = 0.
\]

The problem for \( q(x, y, t) \) that follows from (3.15), (3.40) and (3.41) concerns the question for the pressure \( q \) that is the solution of

\[
\Delta q = 0 \quad \text{in} \ D.
\]

The function \( q \) is subjected to boundary conditions.
These conditions are

\[ q = 0 \quad \text{at } \partial D_1 \text{ and } \partial D_3 \]

\[ \frac{\partial q}{\partial n} - aq = \rho \omega_0^2 u \quad \text{at } \partial D_2 \]  \tag{3.43}

\[ \frac{\partial q}{\partial n} = 0 \quad \text{at } \partial D_4 . \]

The constant \( a \) is defined in (3.41). In the same way as in problem (3.15) we have here
that the stiffness \( \omega_0^2(x) \) is an even function along the membrane. Therefore along the whole
boundary \( \partial D_2 \) holds

\[ \omega_0^2(x) = \omega_0^2(-x) . \]

In this case too, at the membrane both the pressure \( q \) and the deflection of the membrane
are uneven with respect to the line \( x = 0 \). Therefore, these quantities fulfil the symmetry
relation

\[ q(x,0,t) = -q(-x,0,t) \quad \text{and} \quad u(x,0,t) = -u(-x,0,t) . \]

The particular solution (3.41) at the membrane \( y = 0 \) completely describes the propulsion
of the membrane caused by the pressure at the stapes. The next figure shows a plot of (3.41).

![Graph of the known terms in (3.41) at the membrane. The dashed straight line is the first term
of the place dependent part of (3.41). The second term in this expression 'corrects' the first one. This
yields the actual distribution of the pressure, the solid line, along the membrane. The plot shows that the
basilar membrane essentially performs its motion under the influence of a prescribed pressure (solid line)
along the whole membrane. However, just after the oval window the amplitude of this pressure rapidly
diminishes.](image)

Figure 3.3. Graph of the known terms in (3.41) at the membrane. The dashed straight line is the first term
of the place dependent part of (3.41). The second term in this expression 'corrects' the first one. This
yields the actual distribution of the pressure, the solid line, along the membrane. The plot shows that the
basilar membrane essentially performs its motion under the influence of a prescribed pressure (solid line)
along the whole membrane. However, just after the oval window the amplitude of this pressure rapidly
diminishes.
In (3.41) the term \(-f(t)x/x\) can be associated with the potential of a uniform time dependent fluid flow parallel to a hard wall at the place of the membrane. The second term between the brackets is the correction to the place dependent part of this potential in consequence of the presence of the radiation condition at the membrane. The influence of this correction grows as \(y\) tends to the membrane axis. In Fig. 3.3 the straight dashed line gives the place dependent part of the 'original' potential at the membrane. The actual pressure distribution as a result of radiation is shown by the second curve. This pressure is the net effect along the membrane in consequence of the prescribed pressure \(f(t)\) at the oval window, At the membrane the correction almost suppresses the 'original' potential function. The leading parameter in this process is \(2\rho/m\). This follows from the radiation condition in problem (3.15), in which the parameter \(2\rho/m\) controls the ratio in which the stiffness force is distributed over the pressure and its normal derivative. From Fig. 3.3 it follows that the basilar membrane performs its motion under the influence of a prescribed pressure along the whole membrane. However, just after the oval window the amplitude of this pressure rapidly diminishes. Thus, it is as if we deal with a classical transmission line where the driving force has been defined at the point \(x = -\pi\), the entrance of the system.

There are several ways to express the solution of problem (3.43) at the membrane. One of them is to look for the pressure \(q(x, y, t)\) in the shape

\[
q(x, y, t) = \sum_{n=1}^{\infty} a_n \sin nx \cosh n(y - h), \quad -\pi < x < \pi.
\]  

(3.44)

Then, when at an arbitrarily point in time \(t\) the stiffness force \(\rho \omega_0^2 u\) is considered as a known force along the membrane, this function can be written as a Fourier series with known coefficients. In model (3.43) we assumed that \(\omega_0^2(x)\) is even and \(u(x, t)\) is uneven with respect to the point \(x = 0\). Therefore, \(\rho \omega_0^2(x) u(x, t)\) is an uneven function in the place domain. In consequence of this, the Fourier series of this expression reduces to the sine series

\[
\rho \omega_0^2(x) u(x, t) = \sum_{n=1}^{\infty} b_n \sin nx,
\]

(3.45)

with

\[
b_n = \frac{2}{\pi} \int_0^\pi \rho \omega_0^2(\xi) u(\xi) \sin n\xi d\xi.
\]

Let us insert (3.44) and (3.45) in the membrane condition (3.42). When in the series at both sides of the equal sign the corresponding terms are compared with each other, it is found that

\[
a_n = \frac{b_n}{n \sinh nh + a \cosh nh},
\]  

(3.46)

In consequence of this, the solution \(q(x, y, t)\) as follows from (3.44) is completely known.
At the axis \( y = 0 \) this solution reads

\[
q(x,0,t) = -\sum_{n=1}^{\infty} \frac{b_n}{n \tanh nh + a} \sin nx .
\]  

(3.47)

Next we pay attention to the special case in which the stiffness force at the membrane is replaced by a unit force \(-\delta(x - \xi)\) at the point \( x = \xi \); \( 0 < \xi < \pi \). Clearly, the density of this force is \(-1\). An uneven reflection with respect to \( x = 0 \) makes it possible to express \(-\delta(x - \xi)\) as a Fourier sine-series in the interval \( 0 < x < \pi \). The series reads

\[
-\delta(x - \xi) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \sin n\xi \sin nx .
\]

In this special case the coefficients \( b_n \) are given by

\[
b_n = -\frac{2}{\pi} \sin n\xi \quad ; \quad n = 1, 2, \ldots \quad (3.48)
\]

When the coefficients are inserted in (3.47), it first follows that this solution does not depend on the time. Thus it holds \( q = q(x,0) \). Second, it appears that \( q \) fulfils the boundary condition

\[
\frac{\partial q}{\partial y} - \frac{2\rho}{m} q = -\delta(x - \xi) \quad ; \quad 0 < x < \pi ,
\]

at the membrane axis and vanishes at the ends \( x = 0 \) and \( x = \pi \). In consequence of these properties, we will consider this solution as the function of Green for the membrane condition at the positive part of the membrane axis. For the sake of simplicity we will write this function as \( G(x,\xi,a) \). In consequence of this, it holds that \( G(x,\xi,a) \) obeys the special shape of the membrane condition

\[
\frac{\partial G}{\partial y} - aG = -\delta(x - \xi) \quad ; \quad a = \frac{2\rho}{m} \quad , \quad 0 < x < \pi ,
\]

with boundary conditions

\[
G = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad G = 0 \quad \text{at} \quad x = \pi .
\]

The explicit shape of \( G(x,\xi,a) \) is found when the coefficients (3.48) are inserted in (3.47).

---

4 An analytical treatment of singular functions in relation to Fourier theory can be found in Papoulis (1962) or Papoulis (1984).
This yields
\[ G(x, \xi, a) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\xi \sin nx}{n \tanh nh + a} ; \quad a = \frac{2\rho}{m} ; \quad 0 < x < \pi . \] (3.49)

The present function of Green according to (3.49) is applicable to the positive part of the model that has been given by (3.43). In that part of the model the basilar membrane starts at \( x = \pi \) whereas the helicotrema is found at the origin. The (anti-)symmetrical properties in the model have their influence on the description of the function of Green. However, from a physical point of view, there is no reason to perform this exercise because no new results will be found. Therefore, here we will consider a basilar membrane model in which the helicotrema is found at \( x = 0 \) and the stapes at the point \( x = \pi \).

Then, the unknown part of the pressure as follows from model (3.43) at the membrane boundary \( \partial D^*_x \) can be written in the concise shape
\[ q(x,0,t) = -\int_{0}^{\pi} G(x, \xi, a) \rho \omega_0^2(\xi) u(\xi,t) d\xi . \] (3.50)

This expression shows that the stiffness of an arbitrary point of the membrane contributes to the pressure at every point of the membrane. The magnitude of this influence is controlled by the function of Green. Note that when (3.49) should be inserted in (3.50) and terms in the resulting expressing are slightly rearranged, we arrive at (3.47) in which the coefficients \( b_n ; n=1, 2, ... \) are given by (3.45). This means that a solution in the shape (3.50) is equivalent with the series representation (3.44). In the next section we will frequently profit from the solution in the shape (3.50).

3.5.3 Free and forced vibrations

Now we are in a position to formulate the explicit shape of equation (3.39). According to (3.41) the known part of the pressure at the membrane is found when \( y = 0 \) is inserted in this expression. The unknown part, which is the result of the presence of the stiffness force, is given by (3.50). When both expressions are inserted in (3.39), the result is an integral equation of the kind
\[ \ddot{u}(x,t) = -\omega_0^2(x) u(x,t) + a \int_{0}^{\pi} G(x, \xi, a) \omega_0^2(\xi) u(\xi,t) d\xi + F(x,t) , \] (3.51)

\[ 0 < x < \pi , \quad t \geq 0 . \] 

\( F(x,t) \) follows from (3.41) and reads
\[ F(x,t) = \frac{2}{m} f(t) \left( \sum_{n=1}^{\infty} c_n \frac{n \sinh nh}{n \sinh nh + a \cosh nh} \sin nx \right) . \]

In (3.41) we defined the constant \( a \).
This constant reads
\[
a = \frac{2\rho}{m}.
\]
The equation (3.51) must be fulfilled at every moment of time. At the time \( t = 0 \) appropriate initial conditions complete this equation.

Equation (3.51) has the shape of an equation in which oscillators at neighbouring points of the membrane are forced to move under the influence of an external force \( F(x,t) \). All oscillators are ‘coupled’ with each other. The nature of the ‘coupling’ is expressed by the function of Green. Here we note that the coupling does not depend on time.

In absence of the external force \( F(x,t) \), equation (3.51) is a homogenous equation for the deflection of the membrane. The solutions of this equation are the free vibrations of the system. Therefore, quite in accordance with the general properties of linear vibration theory, the solution of (3.51) consists of free and forced vibrations.

In equation (3.51) the function of Green expresses the influence of membrane oscillators on each other in consequence of the presence of fluid. However, this function depends on the parameter \( a \). Therefore, we shall pay some attention to this dependency.

It is worthwhile to note that typical values of model parameters are \( m = 0.05 \) g/cm\(^2\); \( h = 0.1 \) cm (de Boer, 1980). Besides, the cochlear fluids are watery. This implies that the density \( \rho \) of these fluids is about 1 g/cm\(^3\). In consequence of this, the value of \( a \) in a model with membrane length \( l = \pi \) is about 40. Moreover, when the length of the model under consideration should be scaled to unity, the value of \( a \) even exceeds 125. Therefore, in our opinion we deal with a large value of the parameter \( a \). Let us look for consequences. When the parameter \( a \) is large, it follows from (3.49) that approximately
\[
aG(x,\xi,a) \approx \frac{2}{\pi} \sum_{n=1}^{\infty} \sin n\xi \sin nx \quad 0 < x < \pi.
\]
The right hand side of this expression is the Fourier sine expansion of a delta function placed at the point \( x = \xi \) in the interval \( 0 < x < \pi \). Therefore, in the interval \( 0 < x < \pi \) it holds that
\[
\lim_{a \to \infty} aG(x,\xi,a) = \delta(x - \xi).
\]
However in that case we have
\[
a\int_{0}^{\pi} G(x,\xi,a) \omega_{0}^{2}(\xi) \omega(\xi,\xi,\xi) \, d\xi \approx \omega_{0}^{2}(x) u(x,t).
\]
In consequence of this, when \( a \) is sufficiently large, the integral equation (3.51) reduces to
\[
\ddot{u}(x,t) = -F(x,t),
\]
and the model ‘fails’ to describe adequately expected basilar membrane behaviour.
A second problem with respect to the choice of model parameters may be the cause of numerical discomfort. The function \( \omega_0^2(x) \) represents the stiffness at the boundary \( \partial D_2 \). According to Békésy's (1960) classical observations, the stiffness can be modelled as an exponential function along the basilar membrane. In our case, in which the point \( x = 0 \) represents the helicotrema and \( x = \pi \) the end of the membrane near the stapes, has \( \omega_0^2(x) \) the shape

\[
\omega_0^2(x) = Q \exp(bx),
\]

in which \( 0 \leq x \leq \pi \). Both the constants \( Q \) and \( b \) are positive. It is suggestive to associate in a model the range of resonance with the human audible frequency range, i.e. with the frequency range \( 20 \, \text{Hz} < f < 20000 \, \text{Hz} \). Then, when the length of the membrane is again scaled to unity, it follows that the magnitude of \( b \) must be about 13. Here we again meet a large value of a model parameter that can be the reason for (numerical) imperfections.

In our approach we will frequently make use of time scaling. When this is applied the factor \( Q \) drops out the problem. Besides, in general we prefer to consider the resonance frequency at the helicotrema as the unity. Because in this section the helicotrema corresponds to the point \( x = 0 \), time scaling with respect to the resonance frequency at this point is expressed by

\[
t = \frac{\tau}{\omega_0(0)}.
\]

When (3.53) is inserted in (3.51) the integral equation takes the shape

\[
iu(x, \tau) = -\Omega_0^2(x)u(x, \tau) + a \int_0^\pi G(x, \xi, a) \Omega_0^2(\xi)u(\xi, \tau)d\xi + \Gamma(x, \tau),
\]

in which

\[
\Gamma(x, \tau) = \frac{F(x, \tau)}{\omega_0^2(0)} \quad \text{and} \quad \Omega_0^2(x) = \frac{\omega_0^2(x)}{\omega_0^2(0)}.
\]

The range of \( \Omega_0^2(x) \) is \( 1 \leq \Omega_0^2(x) \leq \exp(b\pi) \) for \( 0 \leq x \leq \pi \). In our opinion the only solution to overcome the problem of the large parameter value of \( b \) is to determine the admissible actual value of this parameter in (3.52) experimentally.

3.5.4 The equation for the pressure

Let us introduce the expression \( L(x) \) defined by

\[
L(x) = \int_0^\pi G(x, \xi, a) \Phi(\xi)d\xi.
\]

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In this expression the function of Green $G(x, \xi, a)$ is the limiting value of the sum of eigenfunctions of the kind
\[ \sin nx \cosh n(y - h) ; \quad n = 1, 2, \ldots, \] (3.56)
at the boundary $\partial D^z$ of problem (3.15). A detailed description of this function is found in section 3.5.2. Here it is sufficient to note that at this axis the function of Green\(^5\) obeys
\[ \frac{\partial G}{\partial y} - aG = -\delta(x - \xi) ; \quad a = \frac{2\rho}{m}, \quad 0 < x < \pi. \]

Quite formally, when it is allowed to change the order of differentiation and integration in expressions of the kind
\[ -\int \sum_{n=1}^{\infty} c_n \sin nx \cos n(y - h) dx, \]
so that
\[ \int \sum_{n=1}^{\infty} c_n \sin nx \cosh n(y - h) dx = \int \sum_{n=1}^{\infty} nc_n \sin nx \sinh n(y - h) dx, \]
it can be shown that $L(x)$ obeys the equation
\[ \frac{\partial L}{\partial y} - aL = -\Phi(x) ; \quad 0 < x < \pi, \] (3.57)
at the boundary of our problem. Conditions that have to be satisfied in order to validate the change of the order of differentiation and integration can be found in literature (for instance: Ledermann and Vajda, 1982b). We shall assume that those conditions have been met. In addition to this, the construction of the function of Green from building bricks of the kind (3.56) shows that when the point $x$ at the boundary $\partial D^z$ coincides with the ends of the membrane $L[x]$ must vanish. Therefore, we have
\[ L(0) = 0 \quad \text{and} \quad L(\pi) = 0 \] (3.58)
at the ends $(0, 0)$ and $(\pi, 0)$ of the boundary $\partial D^z$ of the problem (3.15).

Next we define the function $\Phi(x)$ so that
\[ \Phi(x) = -a\omega_0^2(x)u(x,t), \] (3.59)
in which $t$ will be considered as a parameter. The meaning of (3.59) is that at an arbitrarily point in time the right member of this expression is called $\Phi(x)$.\(^5\)

---

\(^5\) We mention that in general both the function of Green and its derivative depend on $y$. At the membrane axis holds $y = 0$. In the notation we ignore the dependence on the $y$ direction.
With the help of (3.55) and (3.59), the integral equation (3.51) can be written as

\[ L(x) = \bar{u}(x,t) + \omega_0^2(x)u(x,t) - F(x,t) \]  

(3.60)

The right member of (3.60) is determined by the limiting value of the pressure at the membrane. This pressure is the limiting value of a function that depends on \( y \). Here too, we will not denote this dependence explicitly.

When (3.55) and (3.60) are inserted in (3.57), a differential equation for the deflection of the membrane is found. This equation reads

\[ \frac{\partial}{\partial y} \left( \bar{u}(x,t) + \omega_0^2(x)u(x,t) \right) - a\bar{u}(x,t) = H(x,t) \]  

(3.61)

for \( 0 < x < \pi \) at the boundary \( y = 0 \) of our problem. The function \( H(x,t) \) is the shorthand notation for

\[ H(x,t) = \frac{\partial F(x,t)}{\partial y} - aF(x,t) \]

at the boundary under consideration. The function \( F(x,t) \) represents the prescribed pressure at the membrane and is given in (3.51). The origin of this term is the particular solution (3.41) at the boundary \( \partial D^2 \) of our problem. This particular pressure has been discussed in section 3.5.2. In that section we mentioned that this solution obeys a homogeneous boundary condition of the kind \( H(x,t) = 0 \). In consequence of this, equation (3.61) reduces to a homogeneous equation. Additionally, the boundary conditions that belong to (3.61) and follow from (3.58) and (3.60) are

\[ \bar{u}(0,t) + a\bar{u}(0)u(0,t) = F(0,t) \quad \text{and} \]

\[ \bar{u}(\pi,t) + a\bar{u}(\pi)u(\pi,t) = F(\pi,t) . \]

The function \( F(\pi,t) \) represents the prescribed pressure at the entrance of the system. At the point \( x = \pi \) holds \( F(\pi,t) = -f(t) \). At the other side of the membrane, i.e. the point \( x = 0 \), holds \( F(0,t) = 0 \). Thus, the ultimate model becomes a model for the deflection and reads

\[ \frac{\partial}{\partial y} \left( \bar{u}(x,t) + \omega_0^2(x)u(x,t) \right) - a\bar{u}(x,t) = 0 \quad , \quad 0 < x < \pi . \]  

(3.62)

with boundary conditions

\[ \bar{u}(0,t) + \omega_0^2(0)u(0,t) = 0 \quad \text{at} \quad x = 0 \]

\[ \bar{u}(\pi,t) + \omega_0^2(\pi)u(\pi,t) = -\frac{2}{m} f(t) \quad \text{at} \quad x = \pi . \]
The second boundary condition is the equation of motion for the deflection of the oscillator at $x = \pi$, the entrance of the system. This oscillator moves essentially under the influence of a known ‘force’ $f(t)$. In consequence of this it holds that the mechanical impedance of this system is determined by only the mechanical properties of the oscillator at $x = 0$. The first boundary condition is the equation of motion for the oscillator at $x = 0$ in absence of a driving ‘force’. Because we assumed that all initial conditions are zero, there is exactly one solution for this equation, namely the vanishing one. In consequence of this, we can replace this last boundary condition by $u(0, t) = 0$.

In section 3.5.2 we started with the equation of motion for a single point of the basilar membrane. This equation is given by formula (3.39). When additional forces are present, the equation of motion (3.39) must be extended. Let $F_{\text{ext}}$ denote the sum of all additional forces at the point $x$ of the membrane. Then equation (3.39) must be replaced by

$$\frac{d^2 u}{dt^2} = -\omega_0^2(x) u + F_{\text{ext}} - p_m .$$

In the present notation, the physical dimension of $F_{\text{ext}}$ is a force per unit of mass. Components of $F_{\text{ext}}$ may depend on the deflection, the velocity, the place at the membrane and the time. It can be shown that in consequence of the presence of this external force the integral equation (3.51) must be extended to

$$\bar{u}(x, t) = -\omega_0^2(x) u(x, t) + F_{\text{ext}} + a \int_0^\pi G(x, \xi, a) \left[ \omega_0^2(x) u(\xi, t) - F_{\text{ext}} \right] d\xi + F(x, t) .$$

When we follow the same line that has been pointed out in this section, the model for the deflection in terms of a first order differential equation for the pressure reads

$$\frac{\partial}{\partial y} \left( \bar{u}(x, t) + \omega_0^2(x) u(x, t) - F_{\text{ext}} \right) - a \bar{u}(x, t) = 0 \ , \ 0 < x < \pi , \ 6$$

with boundary conditions

$$\bar{u}(0, t) + \omega_0^2(0) u(0, t) - F_{\text{ext}} = 0 \quad \text{at} \quad x = 0$$

$$\bar{u}(\pi, t) + \omega_0^2(\pi) u(\pi, t) - F_{\text{ext}} = -\frac{2}{m} f(t) \quad \text{at} \quad x = \pi .$$

An obvious extension for this model is the introduction of damping. This can be done in the same way as has been pointed out in section 3.4.3.

\[\text{Note that in the first term of the left member of (3.63) the expression between the brackets represents the pressure at the membrane. This pressure is the limiting value of the pressure in the scalae. In consequence of this it is reasonable to assume that the derivative in the } \gamma \text{ direction of this expression exists. For the sake of convenience we will not denote this dependence explicitly.}\]
Therefore we write

\[ F_{\omega} = -\lambda_2 \dot{u}(x,t) \]  

(3.64)

where \( \lambda_2 \) is a coefficient for the damping and \( \dot{u}(x,t) \) the velocity at the point \( x \). When this term is inserted in (3.63), the first term of the resulting equation contains the expression

\[ \ddot{u}(x,t) + \lambda_2 \dot{u}(x,t) + \omega_0^2(x) u(x,t) \].

The Laplace transform of this term reads

\[ \left( s^2 + \lambda_2 s + \omega_0^2(x) \right) \bar{u}(x,s) \],

in which \( \bar{u}(x,s) \) is the transform of the deflection \( u(x,t) \) of the membrane at \( x \). The coefficient of the transformed deflection possesses zeros, the points of resonance. These points are the solutions of equation (3.25) from section 3.4.3. In that section it appeared that, in order to solve equations of this kind adequately, we have to distinguish between an upper and a lower plane approximation to the membrane. The monotonic behaviour of the stiffness along the membrane and the sign of the frequency under consideration determine this approximation. In chapter 4 we will come back to this point extensively and distinguish systematically between an upper-plane and a lower-plane approximation.

At this stage we put for the sake of simplicity \( s = -i\omega \). This suggests that we are interested in complex vibrations of the kind

\[ u(x,t) = \bar{u}(x) \exp(-i\omega t) \].

When this expression is inserted in (3.63) and (3.64), we arrive at a first order ordinary homogeneous differential equation. This equation reads

\[ - (A(x,\omega)\bar{u}) + a\omega^2 \bar{u} = 0 \]  

(3.65)

in which

\[ A(x,\omega) = -\omega^2 - i\lambda_2 \omega + \omega_0^2(x) \].

This term is the cause of effects in consequence of the presence of resonance. Therefore, the behaviour of this equation highly depends on \( A(x,\omega) \). In the next section we will specify this function so that it is possible to determine the main properties of the solution of (3.65). Here we put quite formally

\[ \bar{p}(x,\omega) = A(x,\omega) \bar{u}(x,\omega) \],

so that (3.65) can be written as

\[ A(x,\omega) \frac{\partial \bar{p}}{\partial y} + a\omega^2 \bar{p} = 0 \]  

(3.66)
In this last equation we recognise the prototype (3.18) of the membrane condition. The expression (3.66) can be conceived as an equation for the pressure at the membrane. It is attractive to solve this equation directly, because knowledge of this pressure means that the kernel of the problem has been solved. In the next chapter we will make an attempt to this.

3.6 The three-dimensional case.

3.6.1 Introduction

In this section we will pay attention to the three-dimensional counterpart of the two-dimensional model that has been discussed in section 3.5. The intention of this section is to show that in the three-dimensional case we meet similar sorts of snags, properties and solution techniques as in the two-dimensional case. In consequence of this we may expect that, at least from a qualitative point of view, a three-dimensional approach does not lead to additional significant results.

Let us consider a block like model of the cochlea. The model is the three-dimensional equivalent of the model in figure 3.1b. Again the length direction of the membrane is given by $x$. Just as in the two-dimensional case, here the length of a single cochlear scala is again $\pi$. The place of the membrane is found at the plane $y = 0$. The height of a scala is again $h$.

The direction parallel to the membrane fibres is $z$. Values of $z$ are restricted to the width $b$ of the model. The co-ordinate $z$ starts at $z = 0$ and ends at $z = b$. The co-ordinates $x$, $y$ and $z$ constitute a system according to the ‘right hand’ rule. The model is shown is Fig. 3.4.

In this section we shall assume, mainly for the sake of simplicity, that the motion of an arbitrary point of the membrane obeys a second order equation of the kind (3.39). The system again contains an incompressible fluid in which possible losses have been neglected. All walls parallel to the length direction of the membrane are hard walls where the usual hard wall boundary condition holds true.
The wall \( \partial D_1 \) is the place of the oval window. There we shall again assume that the pressure is a simple prescribed function \( f(t) \) that only depends on the time. The wall \( \partial D_{he} \) is parallel to \( \partial D_1 \) and models the 'enlarged' helicotrema. At this wall we again assume that the pressure and the deflection must vanish. In consequence of this, the pressure in the whole system must be uneven with respect to \( x = 0 \). All these requirements for \( p \) are similar to those that have been pointed out in section 3.4.1. Thus, we are led to the three-dimensional equivalent of the two-dimensional model (3.15).

In this model we are asked for the pressure \( p = p(x, y, z, t) \). The pressure has to obey the three-dimensional Laplace equation

\[
\Delta p = 0 \quad \text{in } D.
\]

This pressure is subjected to the boundary conditions

\[
p = f \quad \text{at } \partial D_1,
\]

\[
\frac{\partial p}{\partial n} - \frac{2p}{m} = \rho \omega_0^2 u \quad \text{at } \partial D^+_2,
\]

\[
\frac{\partial p}{\partial n} - \frac{2p}{m} = \rho \omega_0^2 u \quad \text{at } \partial D^-_2
\]

and

\[
p = -f \quad \text{at } \partial D_3,
\]

\[
\frac{\partial p}{\partial n} = 0 \quad \text{at } \partial D_4, \partial D_5 \text{ and } \partial D_6.
\]

Along the whole boundary \( \partial D_2 \) holds

\[
\omega_0^2(x, z) = \omega_0^2(-x, z)
\]

and

\[
p(x, 0, z, t) = -p(-x, 0, z, t) ; u(x, 0, z, t) = -u(-x, 0, z, t).
\]

At an arbitrary point of the membrane the deflection follows from the equation

\[
\ddot{u} = -\omega_0^2(x, z) u - p_m ,
\]

where

\[
p_m = \frac{2}{m} p(x, 0, z, t).
\]
In this case too, zero initial conditions complete the model. It is our aim to find from the solution of problem (3.67) an expression for the pressure at the membrane so that (3.68) becomes accessible. The technique that we will apply is the three-dimensional extension of the two-dimensional approach that has been discussed in the previous sections.

3.6.2 A solution for the pressure

The idea of this section is to treat problem (3.67) in a way similar to the two-dimensional approach. This approach has been given in section 3.5. There, we proposed to look at the solution of the problem as the sum of a particular solution, which comprises all direct effects in consequence of the prescribed pressure at the stapes, and a term that originates from the presence of stiffness at the membrane. Therefore, in this case too, we write the pressure as

\[ p(x, y, z, t) = p^\text{part}(x, y, z, t) + q(x, y, z, t). \]

Again, the particular solution \( p^\text{part}(x, y, z, t) \) will be chosen proportional to the prescribed pressure \( f(t) \) at the stapes. The second term will be a function of the stiffness force \( \alpha_0^2(x)u(x, t) \) at the membrane. We propose to write in the three-dimensional case too the particular solution in the shape

\[ p^\text{part}(x, y, z, t) = f(t) \left[ -\frac{x}{\pi} - \sum_{n=1}^{\infty} c_n \frac{a \cosh n(y - h)}{n \sinh nh + a \cosh nh} \sin nx \right], \]

(3.69)

with

\[ a = \frac{\omega^2}{m}. \]

The coefficients \( c_n ; n = 1, 2, \ldots \) are defined by the Fourier series expansion

\[ -\frac{x}{\pi} = \sum_{n=1}^{\infty} c_n \sin nx ; -\pi < x < \pi, \]

with

\[ c_n = (-1)^n \frac{1}{n}. \]

This solution does not depend on the co-ordinate \( z \). In consequence of this, the pressure according to (3.69) is constant as a function of \( z \) for fixed values of \( x \) and \( y \). This pressure fits the prescribed pressure at the stapes, i.e. the pressure at the boundaries \( x = \pm \pi \) of the system and all hard wall boundary conditions. At the membrane plane - i.e. the plane \( y = 0, 0 < z < b \) - this solution satisfies the homogeneous radiation condition

\[ \frac{\partial p^\text{part}}{\partial y} - ap^\text{part} = 0 ; -\pi < x < \pi, \ y = 0, \ 0 < z < b. \]
The requirements for the remaining part of the pressure \( q(x,y,z,t) \) at the membrane plane \( \partial D_2 \) are

\[
\frac{\partial q}{\partial y} - aq = \rho \omega_0^2 u \quad ; \quad -\pi < x < \pi \ , \quad y = 0 \ , \quad 0 < z < b . \tag{3.70}
\]

In this expression the pressure \( q \) is the limiting value of \( q(x,y,z,t) \) at the membrane in the plane \( y = 0 \). In the surrounding fluid the pressure \( q(x,y,z,t) \) has to obey the three-dimensional equation of Laplace. This equation reads

\[
\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2} = 0 .
\]

The boundary conditions for \( q \) in this problem that follow from model (3.67) are

\[
q = 0 \quad \text{at} \quad \partial D_1 \text{ and } \partial D_3
\]

\[
\frac{\partial q}{\partial n} - \frac{2\rho}{m} p = \rho \omega_0^2 u \quad \text{at} \quad \partial D_2^z
\]

\[
\frac{\partial q}{\partial n} = 0 \quad \text{at} \quad \partial D_4 \text{, } \partial D_5 \text{ and } \partial D_6 .
\]

In this problem too, we have that the stiffness is an even function along the membrane. Thus

\[
\omega_0^2(x,z) = \omega_0^2(-x,z)
\]

along the whole boundary \( \partial D_2 \). Besides, the pressure \( q \) at the membrane and the deflection of the membrane are uneven with respect to the line \( x = 0 \). Therefore, for these quantities holds

\[
q(x,0,z,t) = -q(-x,0,z,t) \text{ and } u(x,z,t) = -u(-x,z,t) .
\]

When rather classical tools are applied, solutions for the problem of the pressure \( q \) can be found. The basic idea is to look first for special solutions of the kind

\[
q(x,y,z,t) \sim \Phi(x,z) \cosh k(y-h) ,
\]

in which \( k \) is a still undetermined constant. Solutions of this kind meet the hard wall boundary condition at \( y = h \), i.e. the boundary condition at \( \partial D_4 \). Besides, in consequence of these special solutions, Laplace's equation is reduced to the two-dimensional Helmholtz's equation

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} + k^2 \Phi = 0 .
\]
In order to find a solution for this last equation, we will apply the well-known method of ‘separation of variables’. Let us put
\[ \Phi(x, z) = \Psi(x) \Theta(z). \]

When \( \Phi \) is inserted in the Helmholtz equation, it appears that \( \Psi \) and \( \Theta \) have to obey
\[ \Psi_{xx} + \alpha^2 \Psi = 0 \quad \text{and} \quad \Theta_{zz} + \beta^2 \Theta = 0. \]

Here, \( \alpha \) and \( \beta \) are still undetermined constants and \( k^2 = \alpha^2 + \beta^2 \). The constants \( \alpha \) and \( \beta \) follow from the general solutions for \( \Psi \) and \( \Theta \) and the boundary conditions of the model.

Ultimately, it appears that the three-dimensional analogue of the pressure (3.44) can be written as
\[ q(x, y, z, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_{mn} \sin mx \cosh k_{mn} (y - h) \cos n \pi \frac{z}{b}, \quad (3.71) \]
in which
\[ k_{mn} = \sqrt{m^2 + \left( \frac{n \pi}{b} \right)^2} \]

This pressure obeys almost all the boundary conditions of the problem and reduces at the membrane to
\[ q(x, 0, z, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_{mn} \sin mx \cosh k_{mn} h \cos n \pi \frac{z}{b} \quad (3.72) \]

At the membrane plane \( y = 0 \) the normal derivative of \( q \) takes the shape
\[ \frac{\partial q(x, 0, z, t)}{\partial y} = -\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} k_{mn} a_{mn} \sin mx \sinh k_{mn} h \cos n \pi \frac{z}{b}. \quad (3.73) \]

Next we write the deflection at the membrane as the two-dimensional Fourier series
\[ \rho \omega_0^2 u = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} b_{mn} \sin mx \cos n \pi \frac{z}{b}. \quad (3.74) \]
with
\[ b_{mn} = \frac{4}{\pi b} \int_{0}^{h} \rho \omega_0^2 (\xi) u(\xi) \sin m \xi \cos n \pi \frac{z}{b} d\xi d\eta. \]

Let us insert (3.72), (3.73) and (3.74) in (3.70) and compare the corresponding terms in the series with each other.
Then we find that

\[ a_{mn} = \frac{b_{mn}}{k_m \sinh k_m h + a \cosh k_m h} ; \quad m = 1, 2, ..., n = 0, 1, ... . \]

The constants \( k_m ; m = 1, 2, ..., n = 1, 2, ... \) have been given in (3.71). From the coefficients

\[ a_{mn} \] and (3.72) follows that at the membrane the pressure \( q \) can be written as

\[ q(x,0,z,t) = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{b_{mn}}{k_m \tanh k_m h + a} \sin nx \cos n\pi \frac{z}{b} . \]  

(3.75)

This expression is useful to express the pressure \( q(x,0,z,t) \) in a slightly different way, namely again in terms of an appropriate function of Green. In order to do that we first pay attention to the special case in which a unit force \( -\delta(x-x')\delta(z-z') \) at the point \( x = x', z = z' ; 0 < x < \pi, 0 < z < b \) replaces the stiffness force at that point. Clearly, the density of this force is \(-1\).

Let us first reflect this force uneven with respect to the line \( x = 0 \) in the membrane plane. After that we subject the result between the lines \( x = \pi \) and \( x = -\pi \) to a periodic continuation with period \( 2\pi \). In virtue of this procedure this force can be written as a Fourier sine series with respect to \( x \).

An even reflection followed by an appropriate continuation in the \( z \) direction offers the opportunity to consider this force as a cosine series with respect to \( z \).

Both expansions can be combined. In consequence of this, the function \( -\delta(x-x')\delta(z-z') \) can be considered as a double Fourier series in the region \( 0 < x < \pi, 0 < z < b \). The series reads

\[ -\delta(x-x')\delta(z-z') = -\frac{4}{\pi b} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin m\xi \cos n\pi \frac{\eta}{b} \sin nx \cos n\pi \frac{z}{b} . \]

In this special case the coefficients \( b_{mn} \) are given by

\[ b_{mn} = -\frac{4}{\pi b} \sin m\xi \cos n\pi \frac{\eta}{b} ; \quad m = 1, 2, ..., n = 1, 2, ... . \]  

(3.76)

Let us insert these coefficients in (3.75). In the first place, we note again that this solution does not depend on time. Thus, in this special case it holds that \( q = q(x,0,z) \). Secondly, it appears that \( q \) fulfills the boundary condition

\[ \frac{\partial q}{\partial y} - \frac{2\rho}{m} q = -\delta(x-x')\delta(z-z') ; \quad 0 < x < \pi , 0 < z < b . \]

at the membrane plane \( y = 0 \) and vanishes at the ends \( x = 0 \) and \( x = \pi \) of this plane. In consequence of these properties, we will consider this solution as a function of Green for the membrane condition at the positive part of the membrane axis.
For the sake of simplicity we will write this function as \( G(x, z, \xi, \eta, a) \). It holds that \( G(x, z, \xi, \eta, a) \) obeys the special shape of the membrane condition

\[
\frac{\partial G}{\partial y} - aG = -\delta(x - \xi)\delta(z - \eta) ; \quad a = \frac{2\rho}{m} , \quad 0 < x < \pi , \quad 0 < z < b
\]

with boundary conditions \( G = 0 \) at the ends \( x = 0 \) and \( x = \pi \) of the plane \( y = 0 \). Besides, at the boundaries \( z = 0 \) and \( z = b \) of the membrane plane, the normal derivative of this function vanishes too.

The explicit shape of \( G(x, z, \xi, \eta, a) \) follows readily when the coefficients \((3.76)\) are inserted in \((3.75)\). This yields

\[
G(x, z, \xi, \eta, a) = \frac{4}{\pi b} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin m\xi \cos n\pi \eta / b \sin m\pi z / b}{k_{mn} \tanh k_{mn} b + a}
\]

for \( 0 < x < \pi \) and \( 0 < z < b \). The constant \( a \) reads

\[
a = \frac{2\rho}{m} .
\]

This shape is useful to express the unknown part \((3.75)\) of the pressure at the boundary \( \partial D_2^+ \) in the concise shape

\[
q(x, 0, z, t) = -\int_0^b \int_0^h G(x, z, \xi, \eta, a) \rho \omega^2_0(\xi, \eta) u(\xi, \eta, t) d\xi d\eta .
\]

(3.78)

Note that when \((3.77)\) is inserted in \((3.78)\) and the terms in the resulting expression are rearranged, we arrive at \((3.75)\) with coefficients \( b_{mn} ; m = 1, 2, ..., n=1, 2, ... \) given by \((3.74)\). This means that a solution in the shape \((3.78)\) is equivalent with the series representation \((3.75)\). In the next section we will profit from the solution in the shape \((3.78)\).

3.6.3 Forced vibrations

As follows from the previous section, the pressure at the membrane can be built up from two different parts. The first part is the known function \( p^{\text{part}}(x, 0, z, t) \), which is proportional to the time behaviour of the pressure at the oval window, and the second part is the expression \( q(x, 0, z, t) \) that follows from the presence of stiffness at the basilar membrane. The complete pressure is found by the superposition of these parts. Therefore, we write

\[
p(x, 0, z, t) = p^{\text{part}}(x, 0, z, t) + q(x, 0, z, t) .
\]
The function \( p(x,0,z,t) \) is found when \( y = 0 \) is inserted in (3.69). The remaining part is given by (3.78). From these expressions and (3.68) follows that the deflection of the membrane has to fulfil the following integral equation

\[
\ddot{u}(x,z,t) = -\omega_0^2(x,z)u(x,z,t) + \int_0^b G(x,z,\xi,\eta,a)\omega_0^2(\xi,\eta)u(\xi,\eta,t)\,d\xi\,d\eta + F(x,z,t) .
\]

(3.79)

Here, \( 0 < x < \pi \) and \( 0 < z < b \). For the time \( t \) holds \( t \geq 0 \). The function \( F(x,z,t) \) follows from (3.69) and reads

\[
F(x,z,t) = \frac{2}{m} f(t) \left[ -\frac{x}{\pi} - \sum_{n=1}^\infty c_n \frac{a \cosh nh}{n \sinh nh + a \cosh nh} \sin nx \right].
\]

The constant \( a \) is given by

\[
a = \frac{2\rho}{m} .
\]

The deflection must obey this equation at every point in time. Appropriate initial conditions complete this equation.

Equation (3.79) has the shape of an equation in which oscillators at points of the membrane are forced to move under the influence of an external force \( F(x,t) \). In absence of the external force \( F(x,z,t) \), equation (3.79) is a homogenous equation for the deflection of the membrane. The solutions of this equation are the free vibrations of the system. Therefore, as follows in this case too from general vibration theory, the solution of (3.79) consists of the superposition of free and forced vibrations.

All oscillators are ‘coupled’ with each other. This coupling is expressed by the function of Green and does not depend on time. However, this function depends on the parameter \( a \). Therefore, in this case too we shall pay some attention to this dependency.

In section 3.5.2 we noted that typical values of model parameters are \( m = 0.05 \) g/cm\(^2\); \( h = 0.1 \) cm and that the density \( \rho \) of the cochlear fluid about 1 g/cm\(^3\). In consequence of this, the magnitude of \( a \) in the present model is about 40. However, in the model the length of the membrane equals \( \pi \). When this length should be scaled to the unity, the value of \( a \) even exceeds 125. Therefore, here again, notes on large values of the parameter \( a \) must be taken seriously. When \( a \) is large, it follows from (3.77) that approximately

\[
aG(x,z,\xi,\eta,a) \approx \frac{4}{\pi b} \sum_{m=1}^\infty \sum_{n=1}^\infty \sin m\xi \cos n\pi \frac{\eta}{b} \sin nx \cos n\pi \frac{z}{b} ; 0 < x < \pi , 0 < z < b .
\]

The right hand side of this expression is the Fourier expansion of a delta function placed at the point \( x = \xi , z = \eta \) in the region \( 0 < x < \pi , 0 < z < b \).
Therefore, in this region it holds that
\[ \lim_{a \to \infty} a G(x, z, \xi, \eta, a) = \delta(x - \xi) \delta(z - \eta). \]

However in that case we have
\[ a \int_0^\pi \int_0^b G(x, z, \xi, \eta, a) \omega_0^2(\xi, \eta) u(\xi, \eta, t) \, d\xi \, d\eta \approx \omega_0^2(x, z) u(x, z, t). \]

In consequence of this, when \( a \) is sufficiently large, the integral equation (3.79) reduces to
\[ \ddot{u}(x, z, t) = -F(x, z, t), \]

and the model degenerates to the present equation.

### 3.6.4 The equation for the pressure

In section 3.5.4 we derived in a two-dimensional model for the cochlea a differential equation for the pressure at the membrane. In this section we will do this again but this time in the three-dimensional case. In order to reach our goal, we introduce the expression \( L(x, z) \) defined by

\[ L(x, z) = -\int_0^\pi \int_0^b G(x, z, \xi, \eta, a) \Phi(\xi, \eta) \, d\xi \, d\eta. \]  \hspace{1cm} (3.80)

The function of Green in this expression is given by (3.77). Essentially, this function is the limiting value of the sum of eigenfunctions of the kind

\[ \sin m\pi \cosh \frac{k_m(y - h)}{b} \cos \pi \frac{z}{b}; \quad m = 1, 2, \ldots, \quad n = 1, 2, \ldots \]  \hspace{1cm} (3.81)

at the membrane plane \( y = 0 \) of our problem. In this case too, we shall assume that \( \partial L/\partial y \) exists at this plane (see section 3.5.4) and that the validity of the equation

\[ \frac{\partial L}{\partial y} - aL = -\Phi(x, z); \quad 0 < x < \pi, \quad 0 < z < b \]  \hspace{1cm} (3.82)

at the plane \( y = 0 \) can be verified. In addition to this, the construction of the function of Green from building bricks of the kind (3.81) shows that when a point \( (x, z) \) of the membrane plane falls together with one of the straight lines \( x = 0 \) or \( x = \pm \pi \), the expression \( L(x, z) \) vanishes. Therefore, we have

\[ L(0, z) = 0 \quad \text{and} \quad L(\pi, z) = 0, \quad 0 < z < b \]  \hspace{1cm} (3.83)

at the boundaries \( x = 0 \) and \( x = \pi \) of the membrane plane in problem (3.67).
Let us define $\Phi(x,z)$ by

$$\Phi(x,z) = -a\omega_0^2(x,z)u(x,z,t),$$

(3.84)

in which $t$ is considered as a parameter. This means that at an arbitrarily moment of time the right member of (3.84) is called $\Phi(x,z)$. By application of (3.80) and (3.84), the integral equation (3.79) can be written as

$$L(x,z) = \ddot{u}(x,z,t) + \omega_0^2(x,z)u(x,z,t) - F(x,z,t).$$

(3.85)

Just as in (3.60) the pressure at the membrane determines the right member of this equation. This pressure is the limiting value of a function of $y$. At this place too, and in the rest of this section we will not denote this dependence explicitly.

After insertion of (3.84) and (3.85) in (3.82), a differential equation for the deflection of the membrane is found. This equation has the shape

$$\frac{\partial}{\partial y} \left( \ddot{u}(x,z,t) + \omega_0^2(x,z)u(x,z,t) \right) - a\dddot{u}(x,z,t) = H(x,z,t).$$

(3.86)

Here, the values $0 < x < \pi, 0 < z < b$ determine points in the boundary plane $y = 0$ of our problem. For the sake of convenience we introduced the function $H(x,z,t)$ according to

$$H(x,z,t) = \frac{\partial F(x,z,t)}{\partial y} - aF(x,z,t).$$

at the boundary in question. $F(x,z,t)$ represents the prescribed pressure at the membrane and is given in (3.79). The origin of this term is the particular solution (3.69). In section 3.6.2 we discussed some properties of this pressure and argued that this solution obeys a homogenous boundary condition of the kind $H(x,z,t) = 0$. In consequence of this, equation (3.86) reduces to a homogenous one. In addition to this, the boundary conditions that belong to (3.86) and that follow from (3.83) and (3.85) are

$$\ddot{u}(0,z,t) + \omega_0^2(0,z)u(0,z,t) = F(0,z,t) \quad \text{and}$$

$$\ddot{u}(\pi,z,t) + \omega_0^2(\pi,z)u(\pi,z,t) = F(\pi,z,t).$$

The function $F(\pi,z,t)$ represents the prescribed pressure at the entrance of the system. At the line $x = \pi$ holds $F(\pi,z,t) = -2/mf(t)$. At the other side of the membrane, i.e. the line $x = 0$, we have that $F(0,z,t) = 0$. Thus, the model is a model for the deflection and reads

$$\frac{\partial}{\partial y} \left( \dddot{u}(x,z,t) + \omega_0^2(x,z)u(x,z,t) \right) - a\dddot{u}(x,z,t) = 0; 0 < x < \pi, 0 < z < b.$$ 

(3.87)

\[7\] For the $y$ dependency of the term between the brackets in the left member of (3.86), we refer to the arguments which have been given in the note at the expression (3.63).
with boundary conditions

\[
\ddot{u}(0, z, t) + \omega_0^2(0, z)u(0, z, t) = 0 \quad \text{at } x = 0, \ 0 < z < b
\]

\[
\ddot{u}(\pi, z, t) + \omega_0^2(\pi, z)u(\pi, z, t) = -\frac{2}{m}f(t) \quad \text{at } x = \pi, \ 0 < z < b.
\]

The second boundary condition is the equation of motion for the deflection of the oscillators at the line \( x = \pi \), the entrance of the system near the stapes. These oscillators move under the influence of a known 'force' proportional to \( f(t) \). Therefore, the mechanical 'input' impedance of this system is determined by the mechanical properties of the oscillators at the line \( x = 0 \).

The first boundary condition is an equation of motion for oscillators at the line \( x = 0 \). In this equation a driving 'force' is absent. Because we assumed that all initial conditions are zero, the only solution of this equation is the vanishing one. In consequence of this, we can replace the last boundary condition by the zero condition: \( u(0, z, t) = 0, \ 0 < z < b \) at every point in time.

Equation (3.87) and the boundary condition are the three-dimensional analogue of the two-dimensional approach to the pressure as has been given by (3.62).

In section 3.6.1 the deflection at a point of the membrane follows from the equation of motion (3.68). In this equation only the inertial resistance, the stiffness and the pressure difference at a point of the membrane determine the dynamical equilibrium. When additional forces are present, this equation must be extended. This can be done in the same way as has been pointed out in section 3.5.4. Let again \( F_{\text{ext}} \) be an additional force at the point \((x, z)\) of the membrane. Then we have to replace equation (3.68) by

\[
\frac{d^2u}{dt^2} = -\omega_0^2(x, z)u + F_{\text{ext}} - p_m.
\]

Here, the force \( F_{\text{ext}} \) denotes a force per unit of mass. This force may depend for instance on the deflection, the velocity, the place at the membrane and the time. In consequence of the presence of this external force the integral equation (3.79) must be extended to

\[
\dot{u}(x, z, t) = -\omega_0^2(x, z)u(x, z, t) + F_{\text{ext}} + a \int_0^\pi \int_0^b G(x, z, \xi, \eta, a)(\omega_0^2(\xi, \eta)u(\xi, \eta, t) - F_{\text{ext}})d\xi d\eta + F(x, z, t).
\]

Then, when we follow the same line of the first part of this section, the model for the deflection in terms of a first order differential equation for the pressure reads

\[
\frac{\partial}{\partial y} \left( \dot{u}(x, z, t) + \omega_0^2(x, z)u(x, z, t) - F_{\text{ext}} \right) - a\dot{u}(x, z, t) = 0; \ 0 < x < \pi, \ 0 < z < b \quad (3.88)
\]
with boundary conditions

\[ \ddot{u}(0, z, t) + \omega_0^2(0, z)u(0, z, t) - F_{eu} = 0 \] at \( x = 0 \)

\[ \ddot{u}(\pi, z, t) + \omega_0^2(\pi, z)u(\pi, z, t) - F_{eu} = -\frac{2}{m} f(t) \] at \( x = \pi \).

A natural way to extend this model is the introduction of some damping. Therefore we write

\[ F_{eu} = -\lambda_2 \dot{u}(x, z, t), \] (3.89)

where again \( \lambda_2 \) is a coefficient for the damping and \( \dot{u}(x, t) \) denotes the velocity at the point \( x \). Let us insert this term in (3.83). Then, the first term of the resulting equation contains the expression

\[ \ddot{u}(x, z, t) + \lambda_2 \dot{u}(x, z, t) + \omega_0^2(x, z)u(x, z, t) . \]

The Laplace transform of this last term reads

\[ (s^2 + \lambda_2 s + \omega_0^2(x, z))\bar{u}(x, z, s) , \]

where we assumed that zero initial conditions hold true. Here, \( \bar{u}(x, z, s) \) is the transform of the deflection \( u(x, z, t) \) of the membrane at the point \( (x, z) \). The coefficient of the transformed deflection possesses zeros, the points of resonance. These points are the solutions of equation (3.25) from section 3.4.3. In that section we argued that in order to solve equations of this kind adequately, we have to distinguish between an upper-plane and a lower-plane approximation to the membrane. Thus here again we meet the same mathematical requirement. The cause of this demand is the monotonic behaviour of the stiffness along the membrane and the sign of the frequency under consideration.

At this stage we again put \( s = -i\omega \). This suggests that we are interested in complex vibrations of the kind

\[ u(x, z, t) = \bar{u}(x, z)\exp(-i\omega t) . \]

Insertion of this expression in (3.87) yields a first order homogeneous differential equation

\[ \frac{\partial}{\partial y} (A(x, z, \omega)\bar{u}) + a\omega^2\bar{u} = 0 \] (3.90)

in which

\[ A(x, z, \omega) = -\omega^2 - i\lambda_2 \omega + \omega_0^2(x, z) . \]

The behaviour of this equation highly depends on \( A(x, z, \omega) \). This is because in the three-dimensional case too, this term determines the effects when resonance takes place.
Let us put
\[ p(x, z, \omega) = A(x, z, \omega)u(x, z, \omega), \]
so that equation (3.90) can be written as
\[ A(x, z, \omega) \frac{\partial p}{\partial y} + a\omega^2 p = 0. \]
This last equation is again the prototype of the membrane condition for the complex pressure.

### 3.7 Distribution and transport of energy

Transport of energy can take place throughout the whole cochlear fluid. When energy is applied at the oval window and we only consider equations of motion for the membrane of the kind that follows from (3.10) and (3.8), we deal with systems that in auditory theory are called passive. When additional forces are applied at the membrane - for instance as the result of outer hair cell activity - the system is called active. Because of considerations concerning the conservation of energy throughout the cochlear fluid, it is unimportant whether or not we deal with an active or passive system. The reason for this is that inside the cochlear fluid there are no energy sources or sinks. All peculiar effects originate from the boundary of the problem.

Let us look for the energy contained in a small fixed volume element \( d\tau \) of the region \( D \) in problem (3.15). The amount of energy contained in this element is the sum of the kinetic and the internal energy and reads
\[ \frac{1}{2} \rho v^2 + \rho U. \]
Here, \( v^2 = \vec{v} \cdot \vec{v} \) and \( \rho U \) is the internal energy per unit of volume. \( \rho \) is the density of the fluid. This quantity changes as a function of time. The rate of change per unit of time of this quantity is
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \rho U \right). \]
Landau and Lifchitz noted that in 1874 the Russian physicist N. Oumov was the first who expressed this time derivative at a point of the fluid as a spatial local rate of change as
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \rho U \right) = -\text{div} \vec{v} \left( \frac{v^2}{2} + H \right), \]
(3.91)
in which the enthalpy \( H \) is defined by \( H = U + pV \). \( V \) is the specific volume and given by \( V = 1/\rho \). A concise proof, which holds true for an ideal fluid and that is mainly based on fundamental laws of thermodynamics and some vector operations, is found in Landau and
Lifchitz (1971). This way of writing is important because it offers the opportunity to express the whole amount of temporal changes over the fluid in one of the cochlear scalae as an integral over the boundary of this region.

Indeed, in section 3.4.2 we started with Gauss's divergence theorem and this theorem can be applied fruitfully to the right member of the present expression. When this is done, it is found that

$$\frac{\partial}{\partial t} \int_D \left( \frac{1}{2} \rho v^2 + \rho U \right) d\tau = -\int_{\partial D} \left( \frac{v^2}{2} + H \right) d\sigma .$$

Here, $d\tau$ is the volume element of the region $D$ and $d\sigma$ is a surface element of the boundary. The component of the velocity normal to the boundary is $v_n$.

Let us assume that the region $D$ is again the left rectangle $D$ in figure 3.1b. The boundary $\partial D$ consists of the parts $\partial D_1$, $\partial D_2^\perp$, $\partial D_{he}$ and half the boundary $\partial D_4$. In all our problems the fluid is incompressible. This implies that the internal energy $U$ is constant for each element of the fluid. In consequence of this, the term $U$ drops out the left member of this equation whereas in the right member $H$ must be replaced by $p/\rho$. This yields

$$\frac{\partial}{\partial t} \int_D \frac{\rho v^2}{2} d\tau = -\int_{\partial D} \left( \frac{\rho v^2}{2} + p \right) d\sigma .$$

This expression expresses the law of conservation of energy for the region $D$ and its boundary $\partial D$ and holds true at every moment. Let us look for the vector

$$I = \bar{v} \left( \rho \frac{v^2}{2} + p \right).$$

This vector is the density of the energy flow per unit of time. In other words, the vector $I$ is the intensity of the flow and consists of two parts.

The first part, the term $\rho \bar{v} v^2 / 2$, is the contribution to the intensity in consequence of the motion of the fluid. This is the kinetic part of the intensity. The second part, the term $p \bar{v}$, is the density of the work carried out by the pressure per unit of time. At the boundary the vector takes the shape

$$I_n = v_n \left( \rho \frac{v^2}{2} + p \right).$$

The intensity is a vector that describes properties of a flow of energy. Because this flow always takes place between different levels of energy it can be compared with the velocity vector of a fluid flow. Then it is natural to define an energy potential function $\psi = \psi(x, y, t)$ so that

$$\bar{I} = -\text{grad} \psi .$$
When \( \ddot{f} \) is introduced in (3.91) it follows that

\[
\Delta \Psi = \frac{\partial}{\partial t}\left(\frac{1}{2} \rho \dot{v}^2\right).
\]

The right member of this equation can be modified slightly. According to the linear Euler equations (3.1) in absence of external forces the equation of motion for the cochlear fluid is

\[
\frac{\partial \ddot{v}}{\partial t} = -\frac{1}{\rho} \nabla p.
\]

Because

\[
\frac{1}{2} \rho \frac{\partial \dot{v}^2}{\partial t} = \rho \ddot{v} \cdot \frac{\partial \ddot{v}}{\partial t},
\]

it follows that

\[
\frac{1}{2} \rho \frac{\partial \dot{v}^2}{\partial t} = -\ddot{v} \cdot \nabla p.
\]

Then, the equation for \( \Psi \) can be written as

\[
\Delta \Psi = -\ddot{v} \cdot \nabla p.
\]

This equation is an inhomogeneous equation of Laplace. The solution of this equation, subjected to appropriate boundary conditions, determines the intensity distribution in a cochlear model at a fixed time. In our case, normal derivatives of the potential \( \Psi \) must be prescribed at the boundary \( \partial D \) of the region \( D \). When this is done, we arrive at the problem

\[
\Delta \Psi = -\ddot{v} \cdot \nabla p \quad \text{in} \quad D
\]

\[
\frac{\partial \Psi}{\partial n} = I_{ev} \quad \text{at} \quad \partial D_i
\]

\[
\frac{\partial \Psi}{\partial n} = I_{mem} \quad \text{at} \quad \partial D_{m}^i
\]

\[
\frac{\partial \Psi}{\partial n} = 0 \quad \text{at} \quad \partial D_{3,4}.
\]

This problem is a Neumann problem. The problem cannot be solved unless again the prescribed values of the intensity at the boundary satisfy the condition of compatibility at every moment in time.
In terms of the intensity this condition reads

\[ \int_{\partial D} I_n d\sigma = 0. \]

Insertion of the definition of intensity yields the similar shape

\[ \int_{\partial D} \left( \rho \frac{\partial^2}{\partial t^2} + p \right) d\sigma = 0. \]

at every point in time. Because the boundaries \( \partial D_{3,4} \) are hard walls where the velocity normal to a boundary must vanish, this condition reduces to

\[ \int_{0}^{h} I_{ov} d\sigma = -\int_{0}^{\pi} I_{memb} d\sigma. \]

Let us assume that exactly at one point of the membrane an additional intensity at the time \( t = 0 \) is prescribed. The cause of this activity could be the influence of an outer hair cell in a small region of resonance. This activity can be modelled as \( I_a \delta(x_r)\delta(t) \), in which \( x_r \) is the point where this intensity has been concentrated and \( I_a \) is the density of the activity. Then we have to replace \( I_{memb} \) by \( I_{memb} + I_a \delta(x_r)\delta(t) \) in the condition for compatibility. When this is done the result can be written as

\[ \int_{0}^{h} I_{ov} d\sigma = -\int_{0}^{\pi} I_{memb} d\sigma - I_a \delta(t). \]

This expression implies that at the same moment at which activity at the membrane takes place, an effect of this intensity is noticeable at the oval window. Therefore, we expect that there will be a negligible delay between outer hair cell activity and its consequences at the oval window. In the next section this point will be reconsidered.

3.8 Discussion

The traditional way to study the motion of the basilar membrane in relation to the surrounding cochlear fluid is to start from the concept of membrane impedance. This concept follows from the study of complex oscillations and is defined as the ratio of the complex pressure difference across the membrane and the complex velocity of the membrane. The notion of the impedance is only applicable when we deal with linear equations. We assumed that the linear approximation to the basic constituents in our problem is valid. This assumption is sufficiently supported because the order of magnitude of the motion of the basilar membrane is rather small (Van Dijk, 1976; Viergever, 1980, 1986). It is common practice to eliminate the membrane velocity from the expression for the impedance in favour of the normal derivative of the pressure at the membrane. The result is an expression of the kind (3.18) that is essentially a translation of the original equation of motion of the membrane for oscillating
time behaviour. The equation has been given in terms of the pressure at the membrane and sets boundaries to the behaviour of the pressure at the membrane. At first glance it seems to be reasonable to consider this equation as a special boundary condition in a boundary value problem for the cochlea. However, when this is done it is impossible to decide on the unicity of a solution for this problem. This points to an improperly posed problem. There are several ways to avoid this difficulty.

The first one is to conceive this equation as an equation for the pressure that can be solved directly in terms of travelling waves. This idea is attractive because in auditory theories the concept of travelling waves is quite current. In the next chapter we shall pay attention to that approach under the title ‘cochlear phenomenology’.

A second possibility is to change the shape of the equation and again to solve the resulting equation directly. When the equation of motion for the membrane is written in terms of the pressure, we deal with a first order equation in which the normal derivative of the pressure forms an unpractical detail. An often-used simplification is the well-known second order equation of Zwislocki (1948, 1980). The prototype of this equation is found when the normal derivative of the pressure at the membrane is approximated by a term proportional to the second derivative of the pressure along the membrane. The approximation was given by (3.36). In section 3.4.4 we discussed a possibility to support this approximation.

From a physical point of view it can be dangerous to replace a first order equation by a second order one. In general a second order equation possesses two independent solutions whereas the first order equation has only one solution. Then, when a physical meaning is attributed to the wrong part of the solution, properties that are not present in the original problem are introduced artificially. This is what we try to avoid.

Zwislocki’s equation is still popular even in the case that both independent solutions are used to explain physical properties of the cochlea (De Boer, 1993; Kanis and De Boer, 1993). The reason for this success is mainly because its shape is relativity easy to handle. In chapter 5 we will study at length the properties of this equation and determine the price we have to pay for making a priori approximations.

A third possibility is to consider the normal derivatives at the boundary of a boundary value problem for the pressure as prescribed functions. These notions represent the way of thinking in the eighties and lead to Neumann problems. After analytical exercises at any depth, there always results an equation for the pressure or the normal derivative at the membrane that must be solved numerically. The numerical methods applied often lead to cumbersome calculations (Diependaal, 1988) and scarcely contribute to insight in the main properties of the cochlea.

In this chapter we formulated a fourth possibility. We began with a simple second order equation in the time domain. The shape of this equation is similar to an Euler equation in which an additional term expresses the influence of the presence of stiffness. We modified this equation by assuming that the basilar membrane actually represents a discontinuity in a fluid-like environment. This equation models the membrane in a rather primitive way. However, the presence of stiffness is an essential point. The stiffness term is not present in the equation for the surrounding fluid. The difference between both equations has been written as an inhomogeneous radiation condition. The inhomogeneous term is the stiffness term. This
condition must hold true at every moment in time. The condition expresses that a deflection of
the membrane leads to a stiffness force that has an influence on both the pressure and its
normal derivative in the surrounding fluid. The ratio in which this effect is smeared out over
the pressure and its normal derivative is determined by one single model parameter. We called
this constant the radiation constant of the problem.

In a general radiation problem not only the object that radiates but also the direct
environment of the object plays a characteristic role. In the cochlea this environment is the
incompressible fluid in the cochlear channels. Therefore, we constructed a boundary value
problem in which the combined action of the fluid in a box-like scalae and the radiation
condition determine the pressure in the model. The basic idea behind the model is that a
change of the stiffness force immediately has its influence on the pressure and on forces in the
fluid. Because a force is the gradient of the pressure, we actually deal with a problem for the
pressure. When the problem for the pressure has been solved, the motion of the membrane can
be determined in several ways.

An important aspect of this model is that the radiation condition, as an inhomogeneous
boundary condition of the third kind, determines what happens in the model. We first proved
that when a solution for this problem exists, this solution is unique. After that, we paid
attention to the condition of compatibility from a physical point of view. In mathematics this
condition is a necessary requirement for the unicity of Neumann problems and for boundary
value problems with mixed condition. In physical terms this condition expresses a global
constraint for the integral of forces along the boundary of the problem.

Both in section 3.4.4 and in section 3.7 we used arguments, based on the formal condition
of compatibility, to show that when outer hair cells exert forces at the membrane, effects in
consequence of this must be immediately observable near the oval window. However, in the
models of this section symmetry considerations offered the opportunity to restrict ourselves to
only one cochlear scala.

When the condition of compatibility is applied in a one-scala model the ‘oval’ window is
both the entrance for usual sound signals and the exit for signals that have been generated at
the basilar membrane. Let us now assume that activity in the organ of Corti modifies the
pressure at the upper side of the membrane. This will disturb the assumed symmetry. The
‘broken’ symmetry could lead to observable effects in the middle ear cavity. However, there
are no observations that lead to evidence for this. Because the ear emits sound produced by
hair cells, the ‘outlet’ of the system apparently differs from the entrance. The current opinion
is that the fluid in the cochlear duct transports the main contribution of those signals to the
saccule in the vestibule (Fig. 1.1). Then it is easy to imagine that the ossicular chain is
responsible for the transport of emitted energy to the ear canal.

In section 3.5 we ‘solved’ the boundary value problem under consideration. We expressed
the pressure at the membrane in two terms. The first one is proportional to the time behaviour
of the prescribed pressure at the stapes. The second term has the shape of an integral over the
length of the membrane. In this integral the momentary stiffness force is weighted for
numbers that follow from a function of Green. This function is given by (3.49) and has the
shape of a Fourier series. It is easy to prove that this function is integrable and therefore
applicable in numerical exercises.
When the pressure according to this solution is inserted in the equation of motion for the membrane, the result is an integral equation that must be solved at every time. The integral in the equation expresses the coupling between the membrane oscillators in consequence of the presence of the surrounding fluid. We showed that the pressure, which follows from this equation, indeed obeys the homogenous radiation condition (3.18). This special shape of the equation of motion for the membrane is unsuitable to serve as a boundary condition in a boundary value problem.

In section 3.6 we showed that in the three-dimensional case a similar method for solving the three-dimensional counterpart of the original problem leads to essentially the same solution of the problem as in the two-dimensional case.

In the oscillating case, special attention has been paid to the place of a point of resonance in the plane (section 3.4.3). In mathematical terms, a point of resonance is a singularity for the equation of the pressure at the membrane. In that section we showed that in the lossy case the pressure is singular at points near the basilar membrane. However, both in section 3.5 and in section 3.6 we neglected the presence of those points and constructed a function of Green as if the box-like cochlear scalae are free from singularities. From a mathematical point this way of working cannot be justified. Therefore, a solution that obeys one of the integral equations (3.51) or (3.79) cannot be a solution for the motion of the basilar membrane in a spatial model of the cochlea. Moreover, methods that lead to equations similar to (3.51) or (3.79) suffer from the same defect.

In mathematics there is a common consensus that solutions of an equation are determined by the singularities that are present in the equation. Therefore, in the next chapter we shall first pay attention to this aspect of the problem.
When the condition of compatibility is applied to the membrane under consideration, the stresses for normal and shear loads and the strain for the displacements that have been assumed in the elastic membrane. Let us now assume that activity of the system changes the stress at the upper side of the membrane. This will disturb the assumed symmetry. The"tanker" symmetry could lead to observable effects in the elastic membrane. However, there are no observations that lead to evidence for this. Because the elas-mic system generated by some cells, the"tanker" of the system apparently differs from the"tanker" of the tank. The membrane opinion is that the fluid in the cochlear duct transmits the main contribution of these signals to the saddle in the vestibule (Fig. 1.1). Then it is easy to imagine that the oscillator alone is responsible for the transport of emitted energy to the ear canal.

In section 3.5 we "solved" the boundary value problem under consideration. We expressed the pressure of the membrane in two terms. The first term is proportional to the fluid behavior of the prescribed pressure at the angles. The second term has the shape of an integral over the length of the membrane. In this integral the necessary stiffness terms is weighted for numbers that follow from a function of Green. This function is given by (3.49) and has the shape of a Fourier series. It is easy to prove that this function is integrable and therefore amenable to numerical exercises.