Mechanical aspects of hearing

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5. **An approximation for the boundary condition**

Abstract. The (mean) height of the cochlear scalae is rather small. This offers the opportunity to approximate the normal derivative of the pressure at the membrane by a term proportional to the second derivative of the pressure along the membrane. The approximation modifies the mixed boundary condition for the pressure at the membrane in a second order equation. This equation belongs to the well-known class of long-wave or transmission-line equations. From a mathematical point of view, the approximation can be justified 'far' from the point of resonance. However, near the point of resonance we are not able to prove the validity of the approximation. The equation is applicable as a substitute for the mixed type boundary condition for the pressure. This simplifies the boundary value problem for the pressure enormously. Because equations of this kind are often used in applications, we will investigate properties of solutions from different points of view. An ordering of the results shows an systematic overview of the main features according to this approach. In the last part of this section we introduced in the equation of the pressure again an additional term, that could simulate consequences of hair cell activity. Both filter characteristics and impulse characteristics for an arbitrarily point of the membrane have been determined.

5.1 **Introduction**

In section 3.4.4 we suggested to approximate the normal derivative of the pressure at the basilar membrane by a term proportional to the second derivative of the pressure along the membrane. In this chapter we shall pay attention to some consequences that follow from this. The approximation forms a bridge between the content of the previous chapters and the class of well-known transmission line models that are known from literature.

These latter models are the result of a period in time, in which the electric analogue as an efficient way of modelling made its entrance in acoustics. Representative results from that time are found in the work of Peterson and Bogert (1950), Fletcher (1951) and Flanagan (1962). In those models attempts have been made to combine the mechanical properties of the basilar membrane and the properties of the surrounding fluid in terms of an equivalent electrical network. The resulting equation of motion is a one-dimensional one. The distance along the membrane is the independent variable. At that time it was customary to calculate results from models that consist of a limited number of sections. Each section represents a cross-section of the cochlea. A typical number of sections is 50. It appears that the transmission line approach essentially corresponds to the early Zwislocki (1948) model that can be derived under the long-wave assumption. According to this assumption the wavelengths of the pressure waves in the cochlea are sufficiently long with respect to a characteristic length dimension of a cross-section. It is difficult to decide a priori on the validity of this assumption. The classical observations according to Von Békésy (1960) are not contradictory to this. However, at that time the methods of measurements were characteristic for the dawn of a long period in which the development of better methods proceeded slowly but successfully. One of the results from this progression is that the long-wave assumption, particularly near the point of resonance, seems to loose its meaning. In consequence of this, the dimensionality in modelling must be extended with at least one dimension perpendicular to the membrane. This automatically leads to the two- or three-dimensional geometry of the models that we described in chapter 3. The two-dimensional
geometry is the starting point of this section. Again we shall model the membrane as a discontinuity between both cochlear scalae.

In this chapter too, we shall assume that all relevant properties can be found from a model with an uneven geometry as has been shown in Fig. 3.1b. Then the formal boundary value problem (3.15) is again the starting point. In addition to this, we shall assume that all normal derivatives coincide with the usual directions of the $x$ and $y$ axis in the $z$ plane. In the approximations that play a role in this chapter the scala height $h$ is the crucial parameter. This height is small compared with the length $l$ of the basilar membrane. In order to express this adequately, the first thing that can be done is to scale the spatial co-ordinates $x$ and $y$ with respect to this length. A second possibility is to model the cochlea as a (half) infinite strip-like model. Then it is without meaning to scale the model with a typical length parameter. Because the cochlea is by nature a system of finite length, we shall avoid the second possibility.

We will start this chapter with two idle attempts to justify an approximation near the point of resonance. The application of this approximation leads to the same equation as follows from the long-wave assumption. This approximation reduces the equation for the pressure at the membrane to an equation that can be solved independent of a boundary value problem for the fluid in the cochlear scalae, both in the time and in the frequency domain. We will pay attention to qualitative considerations that confirm classical asymptotic descriptions. Both descriptions lead to a rather complete image of the wave behaviour of the membrane according to this approximate equation.

In section 5.7 we will extend the equation for the pressure with activity terms. The way in which this will be done is similar to the method that we introduced in section 4.4. Some numerical results and impulse responses will be shown.

5.2 Approximations in virtue of the scala height

The starting point in this section is again problem (3.15). The length of this system is given by $l$. For several reasons it is useful to introduce dimensionless co-ordinates defined by

$$x' = \frac{x}{l} \quad \text{and} \quad y' = \frac{y}{l}. \quad (5.1)$$

When (5.1) is inserted in (3.15) and primes are omitted, the scaled model for the pressure reads

$$\Delta p = 0 \quad \text{in} \quad D.$$

The pressure is subjected to the boundary conditions

$$p = f \quad \text{at} \quad \partial D_1,$$

$$p = -f \quad \text{at} \quad \partial D_3.$$
At the hard walls of the system the normal derivative of the pressure has to vanish. This condition reads
\[ \frac{\partial p}{\partial y} = 0 \quad \text{at} \quad \partial D_4. \]

The membrane boundary condition is the inhomogeneous condition
\[ \frac{\partial p}{\partial y} - m \frac{2 p}{l} = l \rho \omega_0^2 u \quad \text{at} \quad \partial D_2^+. \]
\[ \frac{\partial p}{\partial y} - m \frac{2 p}{l} = l \rho \omega_0^2 u \quad \text{at} \quad \partial D_2^- . \]

In this model both the membrane length and the scala height have been scaled. The ‘new’ length of the membrane equals unity and because of the scaling procedure, the original height \( h \) has been modified to \( h' = h/l \). Therefore, the region \( D \) and its boundary \( \partial D \) refer to the rectangle \(-1 \leq x \leq 1; 0 \leq y \leq h'\). Along the whole boundary \( \partial D_2 \) holds
\[ \omega_0^2(x) = \omega_0^2(-x) \]
and again the pressure and the deflection are uneven with respect to the point \( x = 0 \). Thus
\[ p(x,0,t) = -p(-x,0,t) \quad \text{and} \quad u(x,0,t) = -u(-x,0,t). \]

We shall first give two arguments that justify the approximation (3.36) up to a certain degree. This can be done because of the small value of the scaled scala height. The length \( l \) of a human cochlea is about 35 mm, whereas the mean height of the scalae is between 1 and 1.4 mm (Yost and Nielson, 1983). Thus the numerical value of \( h' = h/l \) is about 0.03. This small value of \( h' \) makes it possible to approximate \( \partial p/\partial y \) at the membrane, i.e. at the boundary \( y = 0 \). However, when the prescribed time behaviour at the oval window possesses frequencies that cause resonance at the membrane, we know from the results of the preceding sections that \( \partial p/\partial y \) is singular at this boundary. Therefore, we will start the approximation not exactly at but close to the boundary \( y = 0 \). A part of the region \( D \) and its boundary \( \partial D \) are given in Fig. 5.1.

Figure 5.1. A cochlear scala can be considered as a shallow fluid-like strip bounded by a hard wall and the basilar membrane. The basilar membrane is the axis \( y = 0 \). The hard wall is the boundary at \( y = h' \). At this boundary the normal derivative of the pressure vanishes.
Let us consider \( \partial p/\partial y \) at the point \( (x_0, \eta) \). \( \eta \) is positive and \( 0 < \eta \leq h' \). Throughout this section we shall denote \( \partial p/\partial y \) by \( p_y \). For the sake of convenience we will use the shorthand notation \( p_y(x_0, y, t) = p_y(y) \) at a fixed point \( x_0 \) and at an arbitrarily point of time \( t \). Assume that the time \( t \) and at \( x_0 \), \( p_y(y) \) has a Taylor expansion as a function of \( y \) for \( \eta \leq y \leq h' \). Then we may write

\[
p_y(\eta + h' - \eta) = p_y(\eta) + (h' - \eta)p_{yy}(\eta) + O((h' - \eta)^2). \tag{5.3}
\]

In the region \( D \), the pressure \( p \) fulfils Laplace’s equation \( \Delta p = 0 \), so that

\[
p_{yy} = -p_{xx} \text{ at } (x_0, \eta). \tag{5.4}
\]

Then, when the third term at the right member of (5.3) is omitted and (5.4) is inserted in the resulting expression, we arrive at the approximation

\[
p_y(\eta + h' - \eta) \approx p_y(\eta) - (h' - \eta)p_{xx}(\eta)
\]

which is accurate to the order \( O((h' - \eta)^2) \). Because of the hard wall boundary condition at \( \partial D_1 \), the left member of this expression equals zero. As a result of this, we readily find that

\[
p_y \approx h'p_{xx} \text{ near } \partial D_2. \tag{5.5}
\]

The accuracy of this approximation can be doubtful when \( \eta \) is sufficiently close to zero. Indeed, in general the radius of convergence of a Taylor expansion is restricted to the place of the nearest singularity of that function and it often appears that near such points the convergence of the expansion is very slow. Moreover, at the point of resonance the present expansion fails. Then it is impossible to justify the present approximation. Consequently, when the approximation is applied close to or at a point of resonance there is no guarantee for an accurate approximation.

A second line of reasoning leads to the same approximation and shows the same vulnerability near and at the point of resonance. We will give this method here too. We start with the integral of \( p_{yy} \) over \( y \) for \( \eta < y \leq h' \) at a fixed point \( x_0 \) and at an arbitrarily point of time. When \( \eta \) is a small positive number, the line \( y = \eta \) is close to the membrane boundary at which resonance can take place (Fig. 5.1). The integral can be written as

\[
\int_{\eta}^{h'} p_{yy}(y) \, dy = p_y(y) \bigg|_{\eta}^{h'} = -p_x(\eta). \tag{5.6}
\]

Let us replace in (5.6) the integrand \( p_{yy}(y) \) by \(-p_{xx}(y)\) where we again make use of the potential equation in the shape (5.4).

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1 Here, the order symbol \( O \) means that when \( f(x) \) is of the order \( g(x) \), i.e. \( f(x) = O(g(x)) \), there is a number \( K \) so that \( |f(x)/g(x)| \leq K \) for sufficiently small values of \( x \).
Then, an equivalent reading of (5.6) is
\[ \int_{\eta}^{h'} p_{\alpha}(y) \, dy = p_{\gamma}(\eta). \]

This integral means that at a 'cross section' at \( x_0 \) the integral over \( y \) from \( \eta \) to \( h' \) of the rate of change of forces parallel to the membrane equals the normal derivative \( p_{\gamma} \) at \( (x_0, \eta) \), i.e. the force close to and perpendicular to the membrane. Next we assume that \( p_{\alpha}(y) \) is continuous in \([\eta, h']\). In that case the mean value integral theorem guarantees that there is in the interval \([\eta, h']\) of the 'cross section' at \( x_0 \), a point determined by the number \( \varepsilon \), \( \eta \leq \varepsilon \leq h' \), for which holds
\[ \int_{\eta}^{h'} p_{\alpha}(y) \, dy = (h' - \eta) p_{\alpha}(\varepsilon). \quad (5.7) \]

When \( p_{\alpha}(\varepsilon) \) at this 'cross section' is expanded as a Taylor series with respect to \( \varepsilon - \eta \), we find that
\[ p_{\alpha}(\varepsilon) = p_{\alpha}(\eta) + O(\varepsilon - \eta). \quad (5.8) \]

Then, as follows from (5.6), (5.7) and (5.8), we again find that at the 'cross section' at \( x_0 \) the approximation (5.5) holds true and is accurate up to the order \( O((h' - \eta)(\varepsilon - \eta)) \). In the special case that \( \eta \) tends to zero, this second derivation is not in general applicable. When resonance takes place at the basilar membrane just at the 'cross section' at \( x_0 \), the integrand in the integral of for instance (5.7) is singular at the point \( \eta = 0 \). Then it is impossible to apply the mean value integral theorem and quite formally the present construction again 'collapses' at the point of resonance. This means that, both from a Taylor series expansion and from an application of an elementary integral theorem, it is impossible to justify (5.5) at resonance. We have not been able to develop a different method to prove the validity of (5.5) at resonance. Therefore, according to our opinion it holds that in two- and three-dimensional boundary value problems it is a misconception that the membrane condition can be replaced by a second order equation that follows from the approximation (5.5).

The safest interpretation of the present result is: if a problem of the kind (5.2) has been solved, then useful approximations for the normal derivative near the membrane can be found when (5.5) is applied. However, when (5.5) is gratuitously substituted in the membrane condition, a new equation is defined that can lead to an essential different behaviour for the pressure. Therefore, we will consider the 'new' equation as a different model for the pressure at the membrane that takes the place of the original radiation condition. Let us insert (5.5) in the membrane condition from problem (5.2). Then this condition takes the shape
\[ \frac{\partial^2 p}{\partial x^2} - 2 \frac{\rho l^2}{m h} p = \frac{\rho l^2}{h} \omega_0^2 u \quad \text{at} \quad \partial D_2, \]
where we replaced \( h' \) by \( h/l \).
It appears to be useful to multiply both members of this equation by $2/m$ and to introduce the quantity $p_m$ defined by

$$p_m = 2 \frac{P}{m}.$$  (5.9)

Then it is readily seen that $p_m$ obeys the equation

$$\frac{\partial^2 p_m}{\partial x^2} - a^2 p_m = a^2 \omega_0^2 u \quad \text{at } \partial D_2$$

in which

$$a = l \sqrt{\frac{2 \rho}{mh}}.$$

The physical dimension of $p_m$ follows from the sections 3.2 and 3.3 and is a force per unit of mass of the membrane. The present equation has the shape of a second order inhomogeneous equation for the pressure $p_m$. This contrasts with the original equation in problem (5.2), which is of the first order. When its solution has been found, the pressure can be inserted in the equation of motion for the membrane. After (3.4), (3.6), (3.8) and (5.9) this equation can be written as

$$\frac{d^2 u}{dt^2} = -\omega_0^2(x)u - p_m + F,$$  (5.11)

where

$$p_m(x,t) = \frac{2}{m} p(x,0,t) ; u(x,t) = u_{mn}(x,0,t) \quad \text{and} \quad F(x,t) = F_{imn}(x,0,t).$$

We consider the pair (5.10) and (5.11) as a model for the motion of the basilar membrane that follows from an ‘improper’ application of (5.5) to the membrane condition. In the next sections some properties of (5.10) will be investigated.

### 5.3 An integral equation

Equation (5.11) is the equation of motion for the basilar membrane. Here, $u = u(x,t)$ is the deflection normal to the membrane. The stiffness $\omega_0^2(x)$ varies as a monotonic function of $x$ for $0 \leq x \leq 1$. The deflection can be found when the pressure $p_m$ is known. This pressure obeys equation (5.10) and can be solved from (5.10) when appropriate boundary conditions are prescribed. The simplest conditions are zero boundary conditions. In that case $p_m$ is the solution of the problem

$$\frac{\partial^2 p_m}{\partial x^2} - a^2 p_m = a^2 \omega_0^2(x)u, \quad 0 < x < 1,$$  (5.12)
with boundary conditions

\[ p_m = 0 \quad \text{at} \quad x = 0 \]

\[ p_m = 0 \quad \text{at} \quad x = 1. \]

The constant \( a \) is given by

\[ a = l \sqrt{\frac{2\rho}{mh}} . \tag{5.13} \]

We shall assume that the known force \( F = F(x,t) \) in equation (5.11) has the shape

\[ F(x,t) = -f(t) \frac{\sinh(a(1-x))}{\sinh a} . \tag{5.14} \]

It can be verified that minus this function is a particular solution of the equation for \( p_m \) that defines a prescribed pressure \( f(t) \) at the point \( x = 0 \) of the system. Then the remaining part of the pressure from problem (5.12) obeys the homogenous problem at every point of time. For the sake of completeness we again assume that zero initial conditions are supplementary to both equations in this section. The solution of problem (5.12) can be written explicitly in terms of a function of Green and the right-hand member of the equation for \( p_m \). This solution reads

\[ p_m(x,t) = -a^2 \int_0^1 G(x,\xi,a) \omega_0^2(\xi)u(\xi,t)d\xi . \tag{5.15} \]

Here, \( G(x,\xi,a) \) is the solution of the problem

\[ G_{xx} - a^2 G = -\delta(x-\xi), \quad 0 < x < 1 \]

\[ G = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1. \tag{5.16} \]

This function reads

\[ G(x,\xi,a) = \begin{cases} 
\frac{\sinh ax \sinh a(1-\xi)}{a \sinh a} & \text{if} \quad 0 \leq x < \xi < 1 \\
\frac{\sinh a\xi \sinh a(1-x)}{a \sinh a} & \text{if} \quad 0 < \xi < x \leq 1 .
\end{cases} \tag{5.17} \]

Insertion of (5.15) in (5.11) shows that the motion of the membrane obeys the equation

\[ \ddot{u}(x,t) = -\omega_0^2(x)u(x,t) + a^2 \int_0^1 G(x,\xi,a) \omega_0^2(\xi)u(\xi,t)d\xi + F(x,t) , \tag{5.18} \]
where $0 < x < 1$ and $t \geq 0$. $F(x,t)$ is given by (5.14). In the methods that we used to solve this equation, we frequently make use of time scaling. It is then useful to consider the resonance frequency at the point $x = 1$ as the unity. This can be accomplished by introducing a new time scale $\tau$ defined by

$$t = \frac{\tau}{\omega_0(1)} .$$

(5.19)

Indeed, insertion of (5.19) in (5.18) yields

$$\ddot{u}(x, \tau) = -\Omega_0^2(x)u(x, \tau) + a^2 \int_0^1 G(x, \xi, a)\Omega_0^2(\xi)u(\xi, \tau)d\xi + \Gamma(x, \tau) ,$$

(5.20)

in which

$$\Gamma(x, \tau) = \frac{F(x, t)}{\omega_0^2(1)} \text{ and } \Omega_0^2(x) = \frac{\omega_0^2(x)}{\omega_0^2(1)} .$$

### 5.4 Zwislocki's equation

In 1948 Zwislocki proposed a second order differential equation for the pressure in an inner ear model as an alternative for Ranke's potential problem for the cochlea. His starting point is the assumption that the length of waves along the basilar membrane are relatively long with respect to the cross section of the cochlear scalae. Therefore, theories that have been based on this assumption are often called 'long waves' theories. After him several investigators (Petersen and Bogert, 1950; Fletcher, 1951) derived transmission line models in which essentially the same equation for the pressure plays the central role. In 1980 Zwislocki has given a new and more elegant derivation of his long wave equation. However, he did not modify the basic assumptions for this equation. In this section we shall show that, in spite of the differences between Zwislocki's starting point and ours, the dynamical equations (5.11) and (5.12) or the integral counterpart (5.18) comprise Zwislocki's equation for the pressure.

In order to show this we introduce the expression $L(x)$ defined by

$$L(x) = -\int_0^1 G(x, \xi, a)\Phi(\xi)d\xi .$$

(5.21)

in which $G(x, \xi, a)$ is the function of Green that obeys (5.16). As follows from (5.21) and (5.16), the function $L(x)$ can be considered as the solution of the problem

$$L_{xx} - a^2L(x) = \Phi(x) ; \quad 0 < x < 1 ,$$

(5.22)

$L(0) = 0$ and $L(1) = 0$.
At an arbitrarily but fixed point in time we replace $\Phi(x)$ by $-a^2\omega_0^2(x)u(x,t)$. Then we have

$$\Phi(x) = -a^2\omega_0^2(x)u(x,t), \quad (5.23)$$

in which the time $t$ is only a parameter. With the help of (5.21) and (5.23), the integral equation (5.18) can be written as

$$L(x) = \dot{u}(x,t) + \omega_0^2(x)u(x,t) - F(x,t). \quad (5.24)$$

When (5.24) and (5.23) are used and inserted in (5.22) a differential equation is found that reads

$$\frac{\partial^2}{\partial x^2} \left( \dot{u}(x,t) + \omega_0^2(x)u(x,t) \right) - a^2\ddot{u}(x,t) = H(x,t) \quad (5.25)$$

for $0 < x < 1$. The function $H(x,t)$ is a shorthand abbreviation for

$$H(x,t) = \frac{\partial^2 F(x,t)}{\partial x^2} - a^2 F(x,t). \quad (5.26)$$

In addition to this the boundary conditions that follow from (5.24) and (5.22) are

$$\dot{u}(0,t) + \omega_0^2(0)u(0,t) = F(0,t) \quad \text{and} \quad (5.27)$$

$$\dot{u}(1,t) + \omega_0^2(1)u(1,t) = F(1,t).$$

The function $F(x,t)$ is a prescribed pressure and is given by (5.14). Because $F(x,t)$ has this special shape it follows from (5.26) that $H(x,t)$ vanishes for $0 < x < 1$. Then (5.25) reduces to a homogenous one. From (5.27) and (5.14) we are led to an explicit shape of the boundary conditions at the ends $x = 0$ and $x = 1$. The ultimate model for the deflection becomes

$$\frac{\partial^2}{\partial x^2} \left( \dot{u}(x,t) + \omega_0^2(x)u(x,t) \right) - a^2\ddot{u}(x,t) = 0, \quad 0 < x < 1,$$

$$\ddot{u}(0,t) + \omega_0^2(0)u(0,t) = -f(t) \quad \text{at} \quad x = 0 \quad (5.28)$$

$$\ddot{u}(0,t) + \omega_0^2(1)u(0,t) = 0 \quad \text{at} \quad x = 1.$$

The first boundary condition is the equation of motion for the oscillator at $x = 0$. The oscillator moves under the influence of a known ‘force’ $-f(t)$. 

141
Thus the mechanical impedance of this system is determined only by the mechanical properties of the oscillator at \( x = 0 \). The second boundary condition is the equation of motion for the oscillator at \( x = 1 \) in absence of a driving ‘force’. Because we assumed zero initial conditions the solution of this equation is the vanishing one. Therefore, we can replace this last condition by \( u(1, t) = 0 \).

In (5.28) the second order equation is a differential equation for the motion of the basilar membrane. This equation is linear, thus it is natural to deduce some of its properties by applying complex harmonic oscillations. Let us put

\[
u(x, t) = \tilde{u}(x) \exp(-i \omega t),\]

and insert this in the membrane equation. This leads to the ordinary homogeneous differential equation

\[
\frac{d^2}{dx^2} (A(x, \omega) \tilde{u}) + a^2 \omega^2 \tilde{u} = 0
\]

(5.29)
in which

\[
A(x, \omega) = \omega_0^2(x) - \omega^2.
\]

(5.30)

The behaviour of this equation highly depends on \( A(x, \omega) \). In the next section this function will be specified so that it is possible to draw some conclusions from (5.29). At this stage we put

\[
\bar{p}(x, \omega) = A(x, \omega) \tilde{u}(x, \omega),
\]

so that (5.29) can be written as

\[
\frac{d^2 \bar{p}}{dx^2} + \frac{a^2 \omega^2}{A(x, \omega)} \bar{p} = 0
\]

(5.31)

In this last equation we recognise the prototype of Zwislocki’s equation for the pressure in the cochlea. In hearing theory the well-known ‘long wave’ approximations that lead to this kind of equations are very popular, mainly because simple numerical methods are available to solve them. Unfortunately, in two- or three-dimensional models of the cochlea, this equation cannot be justified at or near a point of resonance. However, this can never be a reason to avoid the study of this equation. Therefore, in section 5.5 we will investigate the main properties of (5.31) from a general point of view.

In Appendix 5-A we summarise the main asymptotic properties of the solution of this equation. It will appear that both views lead to a rather complete image of properties of the solutions according to this equation.
5.5 On oscillating and non-oscillating behaviour

5.5.1 Introduction

Equation (5.31) can be written as

\[ \frac{d^2 p}{dx^2} + q(x, \omega) \tilde{p} = 0, \quad (5.32) \]

in which

\[ q(x, \omega) = \frac{a^2 \omega^2}{\omega_0^2(x) - \omega^2}. \quad (5.33) \]

The behaviour of the solution of (5.32) is determined by properties of (5.33). In this expression denotes \( \omega_0^2(x) \) the stiffness function of the membrane that varies as a function of the length parameter \( x \).

Usually, all that happens at the basilar membrane or in the cochlear scalae is described in terms of the distance to the stapes. However, in virtue of the symmetry of the skull it is self-evident to choose a co-ordinate system that covers this property of symmetry.

Let us look at a listener and assume that the axis \( x = 0 \) of an \( x-y \) plane coincides with the mid-line of his head. Next we imagine that a strong thread connects the apical ends of the separate cochleae with each other. When we pull with sufficient strength at both the listeners' ears, both cochleae are extracted from his head. Because the apical ends have been connected so strongly, the cochleae uncoil automatically. Then, what remains is a symmetrical picture of what has ever been a system of two coiled cochleae. The listener's right ear has been depicted in the left half-plane. His left ear is found at the other side of the vertical axis \( x = 0 \).

The listener's right ear has been depicted in the left half-plane. His left ear is found at the other side of the axis \( x = 0 \). We shall assume that the stiffness \( \omega_0^2(x) \) behaves as an exponential function that increases as a function of the distance to the origin. This is described by \( \omega_0^2(x) = \exp(\theta |x|) \). From (5.19) and (5.20) follows that a possible multiplicative factor of the stiffness can be met within time scaling. Therefore this factor will be ignored here. When this stiffness function is used, it follows from (5.33) that resonance takes place at points \( x \), so that \( bx - 2 \ln \omega = 0 \) for \( x > 0 \) and \( bx + 2 \ln \omega = 0 \) when \( x < 0 \).
In the successive parts of this chapter we will again fix our attention to only one ear, namely the left ear at the right half-plane. In this case the stiffness increases along the positive $x$ axis. For reasons of convenience we will extend this behaviour over the whole $x$ axis. Thus we consider a stiffness function that is given by

$$\omega_0^2(x) = e^{bx}, \quad -\infty < x < \infty.$$  

In section 3.7 we discussed the influence of the damping on the mathematical point of resonance. There it appeared that for positive frequencies combined and an increasing stiffness the equation (5.33) is the limiting case of an upper-plane approach to the membrane axis. Therefore, we again restrict ourselves in this chapter to an upper-plane approximation. It appears to be useful to scale the $x$ axis with the factor $b$ and to carry out a translation so that the point of resonance coincides with the origin of the real axis of a complex $z$ plane. This is accomplished when we put

$$z = bx - 2 \ln \omega.$$  

(5.34)

Next we insert the increasing exponential stiffness in (5.33) and apply (5.34) to both (5.32) and (5.33). As a result of this, Zwislocki's equation takes the shape

$$\frac{d^2p}{dz^2} + q(z)\bar{p} = 0, \quad -\infty < z < +\infty.$$  

(5.35)

Here

$$q(z) = \frac{c^2}{e^z - 1}; \quad c = \frac{a}{b}. $$  

(5.36)

Equation (5.35), in which $q(z)$ has been defined by (5.36), is the subject of the next sections.

### 5.5.2 General properties

Equation (5.35) has some properties in common with the reduced wave equation. Indeed, when in this equation the function $q(z)$ is replaced by a positive constant, (5.35) is the reduced wave equation and we are ensured of travelling waves. When $q(z)$ is a positive function we might expect that the solution of equation (5.35) is oscillatory too. However, in general this is not the case. The question how we can decide on possible oscillatory behaviour of the solution of equation (5.35) is the topic of this section.

In order to give an answer to this question we shall profit from the rich literature on qualitative aspects of second order differential equations. For a comprehensive overview we refer to Ledermann and Vajda (1982).

Let us consider equation (5.35). The point of resonance coincides with the origin. The behaviour of the solution of this equation is controlled by $q(z)$. This function is given by (5.36) and can be either positive or negative, or both positive and negative in the region of interest.
We start the discussion with the case that $q(z)$ is negative at the negative part of the $z$ axis. At first we will show that as a result of this, the solution of (5.35) can never oscillate in that region. In order to prove this we note that every function that oscillates in any part of the negative axis must have at least two zero crossings in that region. Then it is sufficient to show that because $q(z)<0$, this is impossible for every non-trivial solution of (5.35). In order to show this it is useful to write equation (5.35) as

$$\frac{d^2\tilde{p}}{dz^2} = -q(z)\tilde{p}. \quad (5.37)$$

Let us first assume that $\tilde{p}(z)$ is a solution of (5.37) so that at a point $\zeta$ of the negative part of the $z$ axis holds that $\tilde{p}(\zeta)=0$. If at $\zeta$ holds $d\tilde{p}(\zeta)/dz = 0$, then $\tilde{p}(z)$ is the trivial solution. This solution certainly does not oscillate but falls outside the scope of our interest.

Next we suppose that $d\tilde{p}(\zeta)/dz < 0$. In consequence of this, there will be a region at the left of $z = \zeta$ in which $\tilde{p}(z)$ is positive. Because $q(z)$ is negative and $\tilde{p}(z)$ is positive in that region, $d^2\tilde{p}/dz^2$ is positive. Then, $\tilde{p}(z)$ is increasing for $z < \zeta$ and there will never be a second zero crossing. In the same manner it can be shown that when $d\tilde{p}(\zeta)/dz > 0$, $\tilde{p}(z)$ is decreasing for $z < \zeta$. When both results are combined, we are led to the conclusion that at the left of resonance a solution of (5.35) does not oscillate, because every non-trivial solution has at most one zero in that region.

Next we consider some aspects of the solution of eq. (5.35) for positive values of $z$. Assume that $q(z)$ is positive and tends to zero when $z$ tends to infinity. The local shape of this equation resembles that of the reduced wave equation. Then we could expect that solutions of (5.35) will be oscillatory. In general this is not a guarantee for oscillatory behaviour. It will appear that the way in which $q(z)$ tends to zero has its influence on this behaviour. In order to show this, we consider the next example.

$$\frac{d^2\tilde{w}}{dz^2} + \frac{k^2}{(z+1)^2} \tilde{w} = 0 , \quad z \geq 0 . \quad (5.38)$$

The number $k$ is a still undetermined constant. Equation (5.38) is a specific example of an Euler equation. When the standard solution technique for this kind of equations is applied, it appears that the general solution can be written as

$$\tilde{w}(z) = \sqrt{z+1} \left( Ae^{i\beta(z+1)} + Be^{-i\beta(z+1)} \right) , \quad (5.39)$$

in which

$$\beta = \sqrt{k^2 - \frac{1}{4}} \quad k^2 \neq \frac{1}{4} ;$$

and

$$\tilde{w}(z) = \sqrt{z+1} \left( A + B \ln(z+1) \right) \quad \text{when} \quad k = \frac{1}{4} .$$

145
From this expression it is easy to conclude that $w(z)$ is oscillatory for $k^2 > 1/4$ and not oscillatory for $k^2 \leq 1/4$. This means that when the function $k^2 / (z + 1)^2$ tends faster to zero than the function $1/4(z + 1)^2$ for $z \to \infty$, the solution of equation (5.5.3) is not oscillatory. This example strongly suggests that when in (5.35) the function $q(z)$ tends too rapidly to zero as $z$ tends to infinity, there will be no oscillations at all. In that case the notion of a travelling wave loses its meaning.

In the qualitative theory of differential equations, it is sometimes useful to compare an unknown differential equation with a known one. Then it may happen that aspects of the solutions of the unknown equation can be predicted from properties of the known equation. In the present case it is our aim to compare the (5.35) and (5.38) with each other and to draw some conclusions from this. A usual way to do this is to apply Sturm's comparison theorem.

**Theorem.** Let $\bar{p}_1$ and $\bar{p}_2$ be solutions of the equations

$$\frac{d^2 \bar{p}_1}{dz^2} + q_1(z)\bar{p}_1 = 0$$

and

$$\frac{d^2 \bar{p}_2}{dz^2} + q_2(z)\bar{p}_2 = 0 ,$$

respectively. If $q_1(z)$ and $q_2(z)$ are continuous functions and if $0 < q_1(z) < q_2(z)$ for $a \leq z \leq b$, then $\bar{p}_2$ has at least one zero between any two successive zeros of $\bar{p}_1$ in $a \leq z \leq b$.

**Corollary.** Assume that the conditions of the theorem have been fulfilled and that $\bar{p}_1$ is oscillatory for $z > a$, then $\bar{p}_2$ is oscillatory too for $z > a$.

In the next section we will apply the theorem to equation (5.35) and the special example (5.38). Then we will arrive at a test that can be used to predict whether the solution of equation (5.35) is oscillatory or not.

### 5.5.3 A test for oscillating behaviour

Sturm's comparison theorem can be applied to investigate whether the solution of the equation for the pressure along the basilar membrane is oscillatory or not. Let $l$ be the length of the basilar membrane. The pressure along the membrane has to obey (5.37), so that

$$\frac{d^2 \bar{p}}{dx^2} + q(x, \omega)\bar{p} = 0 , \quad 0 < x < l ,$$

in which

$$q(x, \omega) = \frac{a^2 \omega^2}{\omega_0^2(x) - \omega^2} .$$
We consider the case in which the stiffness increases exponentially for positive values of \( x \), and is given by

\[
\omega_0^2(x) = \exp(bx), \quad 0 < x < l.
\]  

(5.42)

As a result, the frequency range that is covered by the basilar membrane is

\[
1 < \omega < \exp(bx).
\]  

(5.43)

Again we denote the point of resonance by \( x_r \). As follows from (5.41) and (5.42), resonance takes place at \( x_r = 2b \ln \omega \). Let us compare (5.38) and (5.40) with each other beyond the point of resonance, that means between the ‘oval window’ and the point of resonance. In order to do that we first replace the point of resonance to the origin of the \( z \) axis. This is accomplished by putting \( z = x - x_r \). Consequently, (5.40) takes the shape

\[
\frac{d^2 p}{dz^2} + q(z)p = 0, \quad -x_r < z < l - x_r; \quad x_r = \frac{2}{b} \ln \omega,
\]  

(5.44)

with

\[
q(z) = \frac{a^2}{\exp(bz) - 1}.
\]  

(5.45)

Now we are in a position to formulate a test that is applicable when we want to decide on oscillating or non-oscillating behaviour of the pressure in that part of the membrane where the stiffness dominates. The test reads

**Test.** If in any part of the positive \( z \) axis, the function \( q(z) \) from eq. (5.44) obeys the inequality

\[
q(z) \geq \frac{k^2}{(z+1)^2}, \quad k^2 > \frac{1}{4},
\]

then the solution of (5.44) is oscillatory in that region. On the other hand, if

\[
q(z) \leq \frac{1}{4(z+1)^2}
\]  

(5.46)

in that part of the \( z \) axis, then solutions of (5.44) are non-oscillatory.

**Proof.** The first part of the test follows straightforwardly from Sturm’s theorem and the properties of the solution (5.39) of eq. (5.38). In order to elucidate the second part, we identify in Sturm’s theorem \( q_1(z) \) with \( q(z) \) as given by (5.45) and \( q_2(z) \) with \( 1/4(z+1)^2 \), respectively. Next we suppose that \( \bar{p}_1 \) oscillates in a part of the \( z \) axis beyond resonance. Then, as a result of the corollary, \( \bar{p}_2 \) oscillates in that region too.
Figure 5.3. Plot of the test for oscillating behaviour for the solution of equation (5.40) at the lowest frequency $\omega = 1$. The straight lines represent the left member of (5.47) at two parameter values of $b$. The slopes of these lines follow from the stiffness behaviour along the membrane. The curved lines are plots of the right member of (5.47) for different values of the parameter $a$. In the region where a straight line exceeds a curved one, the solution of (5.40) is not oscillatory.

However, this is in contradiction to (5.39), so that the supposition has been falsified. As a result, $p_1$ is not oscillating in the part under consideration.

This test leads to a graphical criterion that basically follows from (5.46). Let us insert (5.45) in (5.46). Then (5.46) reads

$$\frac{a^2}{\exp(bz) - 1} \leq \frac{1}{4(z + 1)^2},$$

and can be written as

$$bz \geq \ln\left(4a^2(z + 1)^2 + 1\right).$$

The numerical value of $a$ is rather large. Therefore, this condition can be approximated by

$$\frac{b}{2} z \geq \ln(2a) + \ln(z + 1).$$

(5.47)

When in any region between resonance and oval window this criterion has been satisfied, solutions of equation (5.40) are non-oscillatory.

Figure 5.3 is a plot of (5.47) for the lowest frequency from the range (5.43). This is the 'critical' case. In this case the distance from the point of resonance to the 'oval window' equals the length of the membrane. In the figure the ordinate is the length parameter from the point of resonance to the 'stapes'. Straight lines follow from the left member of the inequality (5.47). The steepest one corresponds to the parameter value $b = 3$. For the second one holds $b = 1.5$. Curved lines show the right hand side of the inequality. Each curve corresponds to a value of the parameter $a$. In a usual model it holds that $a^2 = 2p/mh$. This follows from
(5.12) and (5.13). Typical parameters are \( \rho = 1 \text{ g/cm}^3, \ m = 0.05 \text{ g/cm}^2 \) and \( h = 0.1 \text{ cm} \). For the upper curve holds \( a = 20 \). This is in accordance to the usual parameter values. The length \( l \) equals 3.5 cm. For the second curve holds \( a = 10 \) and the third one corresponds to \( a = 5 \).

In the region where the straight line exceeds a curve, the solution of equation (5.40) is not oscillatory. From this figure we conclude that for the lowest frequencies, the pressure according to (5.40) tends to a non-oscillatory function near the oval window.

5.6 An overview

All results from the previous sections have been summarised in Fig. 5.4. The figure describes qualitatively the behaviour of Zwislocki’s equation at the membrane. The origin of the membrane axis (\( * \)) is the point of resonance. We assumed that the stiffness increases exponentially along the membrane. As a result, the stapes is found at the right of the point of resonance and the helicotrema at the left of this point.

Between the stapes and the point of resonance the stiffness dominates. In that region the pressure is undulating near the point of resonance. When the distance to the point of resonance increases, the pressure gradually loses this character. Ultimately, when the distance to the point of resonance is sufficiently large, the pressure has completely lost its undulating character.

If the rate of change of the stiffness is sufficiently large, then it is impossible that the solution of this equation contributes to a wavelike motion. This presumably occurs in the basal turn of a cochlear near the stapes. In that region the effective stiffness varies much faster than the usual exponential fit to this function suggests. Because the effective stiffness is proportional to the squared resonance frequency, Fig. 2.7 shows evidence for this.

<table>
<thead>
<tr>
<th>mass of the membrane dominates</th>
<th>stiffness dominates</th>
</tr>
</thead>
<tbody>
<tr>
<td>not undulating</td>
<td></td>
</tr>
<tr>
<td>*</td>
<td>undulating</td>
</tr>
<tr>
<td></td>
<td>not undulating</td>
</tr>
</tbody>
</table>

Figure 5.4. Qualitative behaviour of Zwislocki’s equation with respect to the point of resonance (\( * \)) when the stiffness increases exponentially.

At the left of the point of resonance the mass dominates. In that region the behaviour of the pressure according to Zwislocki’s equation is almost exponential. The present results are supported by arguments from asymptotic analysis and are given in Appendix 5-A.

5.7 Extension and responses

In section 4.4 we started with the boundary condition for the pressure in the shape (4.33). This condition is a first-order equation for the pressure and reads

\[
\frac{\partial \ln \bar{p}}{\partial \eta} = \frac{\mu}{A(\xi)}.
\]
The constant $\mu$ has been given by (4.7). The function $1/A(\xi)$ is proportional to the admittance of a point on the basilar membrane. This follows from the function of Green for an elastic bar or rod. The function reads

$$\frac{1}{A(\xi)} = \frac{1}{\xi - \xi_1} - \frac{1}{\xi - \xi_2}$$

First order poles are the points $\xi_1 = -1$ and $\xi_2 = +1$.

Here, the membrane coincides with the negative real axis of the complex $\xi = \xi + i\eta$ plane. Thus, when $\xi$ is negative, the point $(\xi, 0)$ belongs to the basilar membrane. The helicotrema coincides with the origin of this axis. Resonance takes place at the point $-1$. The stapes is found in the negative direction of this axis beyond the point of resonance. In this equation we assumed that the resonance frequency decreases linearly along the membrane.

In section 4.4 we proposed to introduce an additional term into this equation. The notion is that this term should model activity due to hair cell behaviour. The first proposal was to replace the constant $\mu$ by $\mu(1 + \gamma)$. The complex constant $\gamma$ resulted from the presence of an additional term in the equation of motion for the membrane. This term is a delayed pressure. The constant $\gamma$ has been defined by (4.32) and controls the magnitude of the additional pressure as well as its delay. A second proposal was to consider additional effects because of the presence of lateral stiffness (section 4.4.3). This resulted in an extension of the equation with a term in which the mean value of the admittance over a region $2c$ is the characteristic factor. The extended equation is (4.41) and reads

$$\frac{\partial \ln \bar{p}}{\partial \eta} = \mu \left( \frac{1}{A(\xi)} + \frac{\gamma}{2c} \int_{\xi-c}^{\xi+c} \frac{1}{A(x)} \, dx \right) .$$

(5.48)

The second term between the brackets multiplied by the pressure is the term that models the assumed activity. The number $\gamma$ determines the magnitude and the delay of this effect. The integral in this term is the cause of dipoles that occur in the equation for the pressure. The 'poles' of the dipoles are points at the real axis of the $\xi$ plane and have been given by (4.40). The quantity $2c$ is the distance between the poles of a dipole.

In this chapter we conceived the fluid in a cochlear scala as a shallow layer. In section 5.1 we argued that in consequence of this, there is some reason to replace the normal derivative of the pressure at the membrane by a second derivative along the membrane. When the thickness of this layer equals $h$, it followed that the approximation reads

$$\frac{\partial \bar{p}}{\partial \eta} \approx h \frac{d^2 \bar{p}}{d\eta^2} ; \eta = 0 ,$$

at the real axis of the plane.
Let us insert this in (5.48). Then the approximation for the extended equation is

\[
\frac{d^2 \bar{p}}{d\xi^2} = a^2 \left( \frac{1}{A(\xi)} + \frac{\gamma}{2c} \int_{\xi-c}^{\xi+c} \frac{1}{A(x)} \, dx \right) \bar{p}
\]

(5.49)

with \( a^2 = \mu/h \) at the membrane axis of the plane. Now it is our aim to solve pressure \( \bar{p} \) from this equation in the region \(-2 < \xi < 0\) of the real axis. Therefore we subject the pressure to the boundary conditions \( \bar{p}(-2) = 1 \) and \( \bar{p}(0) = 0 \). The condition at the point \(-2\) is a prescribed pressure at the stapes. The vanishing pressure at the origin is the pressure at the helicotrema. Resonance takes place at the point \(-1\).

In order to apply numerical methods successfully, we shifted the singular points in the problem over a small distance to the lower half-plane. This models damping artificially. After that we applied a well-known numerical procedure.

We first divided the interval of integration in \( n \) equal steps with length \( \Delta = 2/n \). Points between successive intervals are the points \( \xi_i \); \( i = 0,1,...,n \). The pressure at a point \( \xi_i \) is denoted by \( \bar{p}_i \). The second derivative of the pressure at \( \xi_i \) has been approximated by

\[
\bar{p}_{\xi\xi}(\xi_i) \approx \frac{\bar{p}_{i-1} - 2\bar{p}_i + \bar{p}_{i+1}}{\Delta^2} ; \quad i = 1,2,...,n-1 .
\]

This approximation is accurate up to the order \( O(h^4) \). According to the present description the discrete approximation of problem (5.49) and the boundary conditions lead to a matrix equation of the kind \( A \bar{p} = b \). The matrix \( A \) is tridiagonal and of the order \((n-1) \times (n-1)\). Elements of the vector \( \bar{p} \) are \( \bar{p}_i \); \( i = 1,2,...,n-1 \). The vector \( b \) is known. Only its first element differs from zero and follows from the prescribed pressure at \( \xi = -1 \). We solved this system with a complex variant of the well-known tridiagonal algorithm (see for instance Press et al., 1986 or 1990).

From the numerical solution we determined the amplitude and phase characteristics. Plots of these quantities are shown in Fig. 5.5 and Fig. 5.6 for real values of \( \gamma \). The first row of Fig. 5.5 shows the amplitude and the phase of the pressure in absence of activity \( (\gamma = 0) \). It appears that the point of resonance has lost its original singular behaviour (compare for instance the present figure with Fig. 4.6).

At the left side of the point of resonance the pressure is a wave that travels to the right. In this region the stiffness dominates. Just after this point the phase of the pressure is approximately the same for all successive points of the membrane. This effect has been discussed qualitatively in section 5.5. In that region the mass is dominating.

The second row shows the same quantities of the pressure in the presence of activity. The parameter values that we used are a dipole distance \( 2c = 0.6 \) and \( \gamma = -0.3 \). It is as if the pressure near the point of resonance has been 'amplified'. However, the results from section 4.4 show that the notion of amplification near the point of resonance can be wrong, because in that section the pressure has been suppressed outside the region of resonance. Thus, it can be misleading to use the notion of amplification near resonance. Figure 5.6 is a detail of the pressure characteristic that has been shown in Fig. 5.5c.
In the model that we solved, the frequency-to-place map is linear. Therefore, the shape of the pressure in the frequency domain is similar to the pressure along the membrane. This offers the opportunity to derive responses for the pressure and the velocity in the frequency domain for an arbitrary point of the membrane.
The velocity that follows from the original membrane condition will be given in chapter 6 formula (6.22). In section 4.3 and 4.4 we already used this formula to find velocity characteristics.

We inserted the frequency characteristics for the pressure according to the present approximation into this formula. Figure 5.7 shows the results for the amplitude and phase characteristics of the velocity and the corresponding impulse response. The first row shows the responses in absence of activity. In the second and third row the parameter \( \gamma \) equals \(-0.15\) and \(-0.3\), respectively. The dipole distance \(2c\) is 0.6. All responses are approximations to the velocity responses for almost all points of the membrane. Exceptions must be made for points near the ends of the membrane.

5.8 Conclusions

In this chapter the point of departure was similar to that of chapter 3. We started with a mathematical model for the pressure in the fluid that surrounds the basilar membrane. In the model the pressure at the membrane must obey an inhomogeneous boundary condition. Quite formally, this condition has the shape of an inhomogeneous radiation condition or, which is the same, an inhomogeneous boundary condition of the third kind. This condition includes both the pressure and its normal derivative. The inhomogeneous term follows from the stiffness of the membrane. In section 5.2 we approximated the derivative of the pressure by a term proportional to the second derivative of the pressure along the membrane. The constant of proportionality is the height of the cochlear scalae. It appeared to be impossible to prove the validity of this approximation near the point of resonance.

In spite of this mathematical imperfection, we introduced the approximation in the radiation condition. Then, the original first order equation for the pressure has been changed in a second order one. In mathematics it is common knowledge that the behaviour of the solution of an equation is determined by the singularities that are present in that equation. However, in consequence of the application of the approximation, we changed the order of the equation.
Figure 5.7. Responses for the velocity at an arbitrary point of the membrane according to (5.49) and (6.22) for real values of $\gamma$. The meaning of this constant follows from (4.29) and (4.30). Each row shows both the amplitude and the phase characteristic and the impulse response for the velocity at an arbitrary point of the membrane. The first row consists of responses in absence of activity. In the second one responses are shown in the presence of activity. In this row the magnitude of the activity parameter $\gamma$ is $-0.15$. The last row consists of responses in which $\gamma$ equals $-0.3$. In all examples the region in the frequency domain in which additional energy is supplied corresponds to a dipole distance $2c$ in the place domain and equals 0.6. One hundred units at the time axis are equal to the formal delay time. The units of $\ln|v|$ are Nepers. Units of $\arg(v)$ are radians.
Then we may expect that the solution of the new second order equation can differ considerably from the solution of the original first order boundary condition. Therefore, we studied properties of the second order equation from different points of view.

We expressed the solution of the second order equation in terms of an appropriate function of Green and a second term that models the input pressure at the stapes. As in chapter 3, the function of Green does not depend on time. The solution for the pressure has been introduced in the equation of motion for the deflection of the membrane. The result is an integral equation for the deflection of the membrane. The deflection has to obey this equation at every point of time. The equation is the counterpart of the ‘general’ equation (3.51) that has been given in section 3.5.3. The essential difference between the present and the ‘general’ equation is determined by the differences between the functions of Green that have been applied.

We proved that the time-domain representation in terms of the function of Green (5.17) is equivalent to the prototype of Zwislocki’s ‘long wave’ equation (Zwislocki, 1948, 1980; Siebert, 1974; De Boer 1980). Therefore, different considerations can lead to the same result. This last equation is appropriate to study the pressure at the membrane for fixed frequencies. The solution of this equation differs considerably from the solutions in the previous chapter.

In that chapter the point of resonance is the cause of a jump of the amplitude of the pressure combined with (almost) singular phase behaviour. From that phase behaviour follows that the point of resonance is a sink for the sound energy of vibrations with a frequency that equals the resonance frequency. In the present case both the amplitude and the phase of the pressure are continuous at resonance. Fig.5.6b shows the behaviour of the phase in the passive case. From this figure follows that the meaning of the point of resonance has been changed. Only before that point is there a flow of energy towards the point of resonance. Because the phase after the point of resonance is approximately constant, there is no flow of energy at all. This means in mathematical terms that a sink has been modified into a turning point. De Boer and MacKay (1980) were the first who noticed this aspect in a study on transport of energy in a simplified model of the cochlea. Recent observations of De Boer and Nuttall (personal communication) underline the existence of a sink at the point of resonance. These observations, combined with the impossibility to justify the approximations of this chapter near the point of resonance, point to the necessity to refine modelling near the point of resonance.

A possibility for this is to combine ‘near resonance’ behaviour in the sense of the previous chapter with results from ‘far from resonance’ behaviour from Appendix 5-A.

For low frequencies the point of resonance is far from the stapes. Then, the solution near the stapes tends to lose its undulating character. From this property we expect that the slope of the phase near those frequencies in characteristics is rather small. Because the formal delay time follows from this slope, this time must be rather short. The figures from the last section show that the formal delay is about half the delay time from the previous chapter.

We consider Zwislocki’s equation for the pressure as a rough approximation for the boundary condition at the membrane in models of the kind (3.15). An advantage of this notion is that in this case the pressure at the membrane can be solved independent of the original boundary value problem. Then, the mixed boundary condition at the membrane can be replaced by an explicit boundary condition for the pressure. This simplifies the boundary value problem considerably.

156
In section 5.7 we studied velocity responses of an arbitrary point on the membrane. We first extended the second order equation for the pressure with a term in which a possible influence of hair cell behaviour on the membrane can be incorporated. The way in which this has been done is similar to the proposals from section 4.4.2. We determined responses for both the pressure and the velocity for only real values of the parameter $\gamma$. The results show that the low frequency slope of the amplitude characteristics near the point of resonance has been enhanced. In addition to this, it holds that the range of the phase is not far from the range that follows from measurements. In his work on observations on cochlear mechanics Rhode (1978) gives styled response curves. It seems as if those curves have much in common with the characteristics from Fig. 5.8.

In the next chapter we will construct systems of coupled filters. One of the systems will be based on the impulse responses from section 5.7. The method to arrive at this is the subject of section 6.3.1. In section 6.3.3 an example will be given that rests on the results from section 5.7.

In section 6.3.4 the integral equation approach of section 6.3.5 will be used to derive spectral-temporal pattern of some basic sound stimuli.
Appendix 5-A. Asymptotic behaviour

1. Introduction

In this appendix we will investigate asymptotic properties of the solution of Zwislocki's equation. Our starting point is the equation (5.35)

\[
\frac{d^2 \bar{p}}{dz^2} + q(z) \bar{p} = 0 , \quad -\infty < z < +\infty .
\]

The function \( q(z) \) is given by (5.36) and reads

\[
q(z) = \left( c - \frac{a}{b} \right) e^z - \frac{c}{b} ,
\]

The origin of the \( z \) axis is the point of resonance. We shall study both near resonance and far from resonance behaviour of the solution of (5.35). Therefore we distinguish between the following two cases.

Case 1. Properties of the solution for \( z \to \pm \infty \).

Case 2. Properties of the solution near \( z = 0 \).

In section 2 we start with the first case. After that, attention is paid to what happens near resonance. This will be done in section 3 and in section 4 a summary will be given and some conclusions will be drawn.

2. Far from resonance

Consider (5.35) for large negative values of \( z \). The function \( q(z) \) is given by (5.36). Since in this case the term \( \exp(z) \) is very small, \( q(z) \) can be written as

\[
q(z) = -\frac{c^2}{e^z - 1}.
\]

When \( q(z) \) is replaced by the leading term, (5.35) takes the elementary shape

\[
\frac{d^2 p}{dz^2} - c^2 p = 0 .
\]

The solutions of this equation are exponential functions. Therefore, when \( z \to -\infty \) the behaviour of the pressure is

\[
\bar{p}(z) \approx \exp(\pm cz) .
\]
Next we turn our attention to the case $z \rightarrow +\infty$. In this case we can profit from a theorem of the theory of differential equations (Ledermann and Vajda, 1982).

**Theorem.** If \( \lim_{z \rightarrow \infty} \int x |q(x)| \, dx < \infty \), then the solution of the equation

\[
\frac{d^2 \tilde{p}}{dz^2} + q(z) \tilde{p} = 0
\]

is

\[
\tilde{p}(z) = (A + Bz)(1 + o(1))
\]

The symbol \( o(1) \) means that any function tends to zero as \( z \) approaches the limiting value. A concise proof of this theorem is found in Cesari (1963). In our case it is not difficult to show that the condition of the theorem has been fulfilled. From (5-A.2) follows that for positive values of \( z \),

\[
c^2 z |q(z)| = c^2 \sum_{n=1}^{\infty} z e^{-nz}.
\]

Integration over \( z \) yields

\[
c^2 \int x |Q(x)| \, dx = -c^2 \sum_{n=1}^{\infty} \left( \frac{z}{n} + \frac{1}{n^2} \right) e^{-nz}.
\]

Each term of this series vanishes as \( z \) tends to infinity. Thus the condition of the theorem has been fulfilled. This theorem shows that for large positive values of \( z \) the solution of eq. (5-A.1) behaves as a linear function of place along the basilar membrane. This approximation can be written as

\[
\tilde{p}(z) \approx A + Bz,
\]

in which \( A \) and \( B \) are arbitrary constants. Because the stiffness function along the membrane is relatively simple, this behaviour is supported by a direct calculation. For positive values of \( z \), (5-A.2) can be written as

\[
q(z) = \frac{c^2}{e^z} \sum_{n=0}^{\infty} e^{-nz}.
\]

When \( z \) is sufficiently large, the right hand side of \( q(z) \) can be replaced by its leading term. This yields

\[
q(z) \approx \frac{c^2}{e^z}.
\]
and (5-A.1) takes the shape

$$\frac{d^2 p}{dz^2} + \frac{c^2}{e^z} p = 0 .$$

(5-A.6)

This equation can be reduced to a standard equation of Bessel (Abramowitz and Stegun, 1965, formula 9.1.54). In order to do that it is customary to define the new independent coordinate $u = \exp(x/2 + i\omega)$. Here we denote for the sake of simplicity the real part of $z$ by $x$. In order to return to the original $x$ axis we have, according to (5.34), to replace $x$ by $bx - 2\ln \omega$. However at this stage this is rather superfluous. Then it appears that the solution of (5-A.6) can be expressed in terms of zero order Bessel and Neumann functions that depend on the argument $u$. Here we shall use the notation

$$w = \frac{1}{u} |w| \exp(-i\omega)$$

so that the solution of (5-A.6) can be written as

$$p(z) = AJ_0(2w) + BY_0(2w),$$

in which $J_0(2w)$ and $Y_0(2w)$ are the Bessel and Neumann function of the zeroth order, respectively. $A$ and $B$ are constants. For large values of $x$, the magnitude of $w$ becomes so small that both the Bessel and the Neumann function can be replaced by the leading terms of the corresponding series. Because

$$J_0(r) \approx 1 \quad \text{and} \quad Y_0(r) \approx \frac{2}{\pi} \ln r ; r \to 0 ,$$

it is readily seen that

$$J_0(2w) \approx 1 \quad \text{and} \quad Y_0(2w) \approx \frac{2}{\pi} \ln 2w ; w \to 0 ,$$

from which follows that

$$Y_0(2w) \approx \frac{2}{\pi} \left( \ln 2 - \frac{x}{2} \right) ; x \to \infty .$$

The linear combination of both approximations supports (5-A.5). The conclusion that follows from (5-A.4) and (5-A.5) is: far from resonance the pressure tends to behave as a non-oscillating function.

In view of the summary of results in section 4 it is useful to rearrange both Bessel functions in terms of their complex counterparts, the Hankel functions. Then an equivalent shape of the solution for (5-A.6) is

$$\bar{p}(z) = AH_0^1(2w) + BH_0^2(2w) , \quad w = \exp\left( -\frac{x}{2} - i\omega \right) .$$

(5-A.7)
Here, $A$ and $B$ are undetermined constants. The Hankel functions are defined according to

$$H_0^{(1)}(2\bar{w}) = J_0(2\bar{w}) + iY_0(2\bar{w}),$$
$$H_0^{(2)}(2\bar{w}) = J_0(2\bar{w}) - iY_0(2\bar{w}).$$

The solution expressed in this shape is a convenient form to give an interpretation of the behaviour of the pressure in the region of the membrane where the stiffness dominates.

3. **Near resonance**

Near the point of resonance, i.e. $z = 0$, the Laurent series of $q(z)$ which follows from (5-A.2) is

$$q(z) = \frac{c^2}{z} \left(1 - \frac{z}{2!} + \ldots\right).$$

Near $z = 0$, it is customary to replace $q(z)$ by its leading term

$$q(z) \approx \frac{c^2}{z}.$$

Then, (5-A.1) takes the shape

$$\frac{d^2 \bar{p}}{dz^2} + \frac{c^2}{z} \bar{p} = 0. \tag{5-A.8}$$

Eq. (5-A.8) is again a known example of an equation in which properties of Bessel functions determine the behaviour of the solution. In order to show this we introduce the new independent co-ordinate

$$x = \pm \sqrt{z}. \tag{5-A.9}$$

Next we consider the pressure $\bar{p}$ as a function of $x$ by putting

$$\bar{p}(x) = \frac{x}{2} u(x).$$

Then follows from (5-A.8) and (5-A.9) that $u(x)$ is the solution of the equation

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + \left(c^2 x^2 - 1\right)u = 0.$$

When $x$ is scaled with a factor $c$, this equation reduces to the standard form of Bessel's equation of the first order. Then the solution of the equation for $u(x)$ can be expressed in terms of Bessel and Neumann functions (or in terms of Hankel functions) that depend on the independent variable $cx$ and in which $x$ is given by (5-A.9).
To our purpose we write the solution as

$$p(z) = \sqrt{z} \left( AH^{(1)}(2c\sqrt{z}) + BH^{(2)}(2c\sqrt{z}) \right),$$

(5-A.10)

where $A$ and $B$ are undetermined constants and

$$H^{(1)}_1(2c\sqrt{z}) = J_1(2c\sqrt{z}) + iY_1(2c\sqrt{z}); \quad H^{(2)}_1(2c\sqrt{z}) = J_1(2c\sqrt{z}) - iY_1(2c\sqrt{z}).$$

Near the origin the Hankel functions can be approximated by

$$H^{(1)}_1(2c\sqrt{z}) \approx -\frac{i}{\pi c\sqrt{z}} \quad \text{and} \quad H^{(2)}_1(2c\sqrt{z}) \approx +\frac{i}{\pi c\sqrt{z}}$$

from which readily follows that

$$\lim_{z \to 0} \sqrt{z}H^{(1)}_1(2c\sqrt{z}) = -\frac{i}{\pi c} \quad \text{and} \quad \lim_{z \to 0} \sqrt{z}H^{(2)}_1(2c\sqrt{z}) = +\frac{i}{\pi c}.$$

These expressions show that the solution (5-A.10) at the origin - that means at the point of resonance - is free from singular behaviour. This is an important difference between the solution of the pressure according to Zwislocki’s equation and the solution of the original membrane equation that we studied in chapter 4. The weak singular behaviour that is present in the pressure that follows from the original membrane condition has been suppressed completely. It is impossible to consider this behaviour as an approximation of the pressure from the original membrane condition.

An important physical aspect follows from the argument of the Hankel functions. De Boer and McKay (1980) studied the transport of energy along the membrane in a simplified model of the cochlea. They noticed that due to the presence of the factor $2\sqrt{z}$ in the arguments of the Hankel functions, the solution (5-A.10) does not oscillate for negative values of $z$. In other words: in Zwislocki’s equation the point of resonance is a turning point.

4. Conclusions

The analysis from the preceding sections opens a way of describing properties of the membrane in terms of undulating behaviour. This behaviour has much in common with traditional travelling waves. We start with the situation near the point of resonance. Let us turn our attention to the solution (5-A.10). In the theory of Bessel functions, Hankel functions can be considered as the analogue of the familiar complex harmonic functions according to

$$H^{(1)}_1(z) \leftrightarrow e^{iz} \quad \text{and} \quad H^{(2)}_1(z) \leftrightarrow e^{-iz}.$$

In the description of travelling waves, complex exponentials play an important role. For instance, assume that we deal with complex vibrations proportional to $\exp(-i\omega t)$. Then the expression $\exp(-i(\omega t + kz))$ represents a travelling wave. When $k$ is positive, the wave travels
to the left. When \( k \) is negative, the wave travels in the opposite direction. The length of the wave is \( \lambda = 2\pi /|k| \). The number \( k \), the wave number, is the slope of the place-dependent part of the argument \( \omega t + kz \).

When the length of the wave becomes extremely large, the magnitude of \( k \) tends to zero. In that case successive points of the wave move in almost the same phase.

In our case, in which Hankel functions determine the properties of wave behaviour, a similar description is applicable. Let us first write both Hankel functions in the polar shape.

\[
H_1^{(1)}(2c\sqrt{z}) = R_1(z)e^{i\theta_1(z)} \quad \text{and} \quad H_1^{(2)}(2c\sqrt{z}) = R_1(z)e^{-i\theta_1(z)},
\]

where

\[
R_1(z) = \sqrt{J_1^2(2c\sqrt{z}) + Y_1^2(2c\sqrt{z})} \quad \text{and} \quad \theta_1(z) = \arctan \frac{Y_1(2c\sqrt{z})}{J_1(2c\sqrt{z})}.
\]

Figure 5.8 shows a plot of \( R_1(z) \) and \( \theta_1(z) \). The near resonance solution for complex time behaviour according to \( \exp(-i\omega t) \) can be written as

\[
p(z, t) \sim \text{Re} \left( A\sqrt{z}R_1(z)e^{-i(\omega t - \theta_1(z))} + B\sqrt{z}R_1(z)e^{-i(\omega t + \theta_1(z))} \right),
\]

The slope of \( \theta_1(z) \) is positive for \( z > 0 \). Therefore, the first term between the brackets is a travelling wave to the right. The second term describes a wave that travels in the opposite direction.

Near resonance the behaviour of the waves changes because the slope of \( \theta_1(z) \) tends to zero. A direct calculation shows that this slope can be written as

\[
\frac{d\theta_1(z)}{dz} = \frac{1}{\pi z R_1^2(z)}.
\]
Figure 5.9. Modulus $R_0$ and argument $\theta_0$ of the Hankel function $H_1^0$. In this figure the argument of the Hankel function reads $2\exp(-z/2)$.

Because $R_i(z)$ rapidly tends towards infinity in its limiting value, both waves lose their undulating character at resonance.

The asymptotic behaviour far from resonance for positive values of $z$ has been discussed in section 2. There it appeared that the solution (5-A.7) for equation (5-A.6) can be composed of Bessel and Neumann functions or Hankel functions of the zero-th order, which depend on the argument $\bar{w} = \exp(-z/2-i0)$. In view of the relations for Hankel functions depending on complex conjugate arguments (Abramowitz and Stegun, 1965), the solution can be written as

$$\overline{H}_0^{(2)}(2\exp(-z/2)) = R_0(z)e^{i\theta_0(z)} \quad \text{and} \quad \overline{H}_0^{(1)}(2\exp(-z/2)) = R_0(z)e^{-i\theta_0(z)},$$

in which

$$R_0(z) = \sqrt{J_0^2(2\exp(-z/2)) + Y_0^2(2\exp(-z/2))} \quad \text{and} \quad \theta_0(z) = \arctan \frac{Y_0(2\exp(-z/2))}{J_0(2\exp(-z/2))}.$$

Here, $z$ is a point at the positive real axis. Figure 5.9 shows plots of $R_0$ and $\theta_0$.

The far-from-resonance behaviour, which corresponds to the travelling wave towards the point of resonance, is

$$p(z,t) = \Re\left(R_0(z)e^{-i(\omega t - \theta_0(z))}\right),$$

The slope of $-\theta_0(z)$ is positive. When $z$ grows to infinity, $2\exp(-z/2)$ tends to zero. For small values of $2\exp(-z/2)$ both $J_0$ and $Y_0$ can be replaced by the first terms of the
corresponding series expansions. Then, the argument $\theta_0(z)$ is simplified considerably and reads

$$\theta_0(z) = -\arctan \frac{z}{2},$$

from which immediately follows that in its limiting value the slope of $-\theta_0(z)$ tends to be a constant. Again in this region the wave has lost its undulating character. The same arguments hold for a wave in the opposite direction.

Let us summarise the behaviour of the pressure that follows from (5-A.1). Between the point of resonance and the helicotrema the pressure is not undulating. The behaviour of the pressure at the other side of the point of resonance is quite different. Near the stapes the behaviour of the pressure tends to be a not-undulating function. Near the point of resonance the pressure undulates. This behaviour corresponds to the qualitative description from section 5.6.