The birational geometry of the moduli space of curves
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Chapter 1

The geometry of the moduli space of curves of genus 23

1.1 Introduction

The problem of describing the birational geometry of the moduli space $\mathcal{M}_g$ of complex curves of genus $g$ has a long history. Severi already knew in 1915 that $\mathcal{M}_g$ is unirational for $g \leq 10$ (cf. [Sev]; see also [AC1] for a modern proof). In the same paper Severi conjectured that $\mathcal{M}_g$ is unirational for all genera $g$. Then for a long period this problem seemed intractable (Mumford writes in [Mu]. p.51: “Whether more $\mathcal{M}_g$’s, $g \geq 11$, are unirational or not is a very interesting problem, but one which looks very hard too, especially if $g$ is quite large”). The breakthrough came in the eighties when Eisenbud, Harris and Mumford proved that $\mathcal{M}_g$ is of general type as soon as $g \geq 24$ and that the Kodaira dimension of $\mathcal{M}_{23}$ is $\geq 1$ (see [HM], [EH3]). We note that $\mathcal{M}_g$ is rational for $g \leq 6$ (see [Dol] for problems concerning the rationality of various moduli spaces).

Severi’s proof of the unirationality of $\mathcal{M}_g$ for small $g$ was based on representing a general curve of genus $g$ as a plane curve of degree $d$ with $\delta$ nodes: this is possible when $d \geq 2g/3 + 2$. When the number of nodes is small, i.e. $\delta \leq (d+1)(d+2)/6$, the dominant map from the variety of plane curves of degree $d$ and genus $g$ to $\mathcal{M}_g$ yields a rational parametrization of the moduli space. The two conditions involving $d$ and $\delta$ can be satisfied only when $g \leq 10$. so Severi’s argument cannot be extended for other genera. However, using much more subtle ideas. Chang. Ran and Sernesi proved the unirationality of $\mathcal{M}_g$ for $g = 11, 12, 13$ (see [CR1], [Se1]), while for $g = 15, 16$ they proved that the Kodaira dimension is $-\infty$ (see [CR2,4] ). The remaining cases $g = 14$ and $17 \leq g \leq 23$ are still quite mysterious. Harris and Morrison conjectured in [HMo] that $\mathcal{M}_g$ is uniruled precisely when $g < 23$.

All these facts indicate that $\mathcal{M}_{23}$ is a very interesting transition case. Our main result is the following:

**Theorem 1.1** The Kodaira dimension of the moduli space of curves of genus 23 is $\geq 2$. 
We will also present some evidence for the hypothesis that the Kodaira dimension of $\mathcal{M}_{23}$ is actually equal to 2.

1.2 Multicanonical linear systems and the Kodaira dimension of $\mathcal{M}_g$

We study three multicanonical divisors on $\mathcal{M}_{23}$, which are (modulo some boundary components) of Brill-Noether type, and we conclude by looking at their relative position that $\kappa(\mathcal{M}_{23}) \geq 2$.

We review some notations. We shall denote by $\mathcal{M}_g$ and $\mathcal{C}_g$ the moduli spaces of stable and 1-pointed stable curves of genus $g$ over $\mathbb{C}$. If $C$ is a smooth algebraic curve of genus $g$, we consider for any $r$ and $d$, the scheme whose points are the $g^r_d$'s on $C$, that is,

$$G^r_d(C) = \{(L, V) : L \in \text{Pic}^d(C), V \subseteq H^0(C, L), \dim(V) = r + 1\}.$$  

(cf. [ACGH]) and denote the associated Brill-Noether locus in $\mathcal{M}_g$ by

$$\mathcal{M}'_{g,d} := \{[C] \in \mathcal{M}_g : G^r_d(C) \neq \emptyset\}.$$  

and by $\overline{\mathcal{M}}'_{g,d}$ its closure in $\overline{\mathcal{M}}_g$.

The distribution of linear series on algebraic curves is governed (to some extent) by the Brill-Noether number

$$\rho(g, r, d) := g - (r + 1)(g - d + r).$$  

The Brill-Noether Theorem asserts that when $\rho(g, r, d) \geq 0$ every curve of genus $g$ possesses a $g^r_d$, while when $\rho(g, r, d) < 0$ the general curve of genus $g$ has no $g^r_d$'s, hence in this case the Brill-Noether loci are proper subvarieties of $\mathcal{M}_g$. When $\rho(g, r, d) < 0$, the naive expectation that $-\rho(g, r, d)$ is the codimension of $\mathcal{M}'_{g,d}$ inside $\mathcal{M}_g$, is in general way off the mark, since there are plenty of examples of Brill-Noether loci of unexpected dimension (cf. [EH2]). However, we have Steffen's result in one direction (see [St]):

If $\rho(g, r, d) < 0$ then each component of $\mathcal{M}'_{g,d}$ has codimension at most $-\rho(g, r, d)$ in $\mathcal{M}_g$.

On the other hand, when the Brill-Noether number is not very negative, the Brill-Noether loci tend to behave nicely. Existence of components of $\mathcal{M}'_{g,d}$ of the expected dimension has been proved for a rather wide range, namely for those $g, r, d$ such that $\rho(g, r, d) < 0$, and

$$\rho(g, r, d) \geq \begin{cases} -g + r + 3 & \text{if } r \text{ is odd;} \\ -rg/(r + 2) + r + 3 & \text{if } r \text{ is even.} \end{cases}$$

We have a complete answer only when $\rho(g, r, d) = -1$. Eisenbud and Harris have proved in [EH2] that in this case $\mathcal{M}'_{g,d}$ has a unique divisorial component, and using the previously mentioned theorem of Steffen's, we obtain the following result:
If \( \rho(g, r, d) = -1 \), then \( \overline{\mathcal{M}}_{g,d} \) is an irreducible divisor of \( \overline{\mathcal{M}}_g \).

We will also need Edidin’s result (see [Ed2]) which says that for \( g \geq 12 \) and \( \rho(g, r, d) = -2 \), all components of \( \mathcal{M}_{g,d}^r \) have codimension 2. We can get codimension 1 Brill-Noether conditions only for the genera \( g \) for which \( g + 1 \) is composite. In that case we can write

\[
g + 1 = (r + 1)(s - 1), \quad s \geq 3
\]

and set \( d := rs - 1 \). Obviously \( \rho(g, r, d) = -1 \) and \( \overline{\mathcal{M}}_{g,d}^r \) is an irreducible divisor. Furthermore, its class has been computed (cf. [EH3]):

\[
[\overline{\mathcal{M}}_{g,d}] = c_{g,r,d} \left( (g + 3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g - i)\delta_i \right),
\]

where \( c_{g,r,d} \) is a positive rational number equal to \( 3\mu/(2g - 4) \), with \( \mu \) being the number of \( g_{ij}'s \) on a general pointed curve \( (C_0, q) \) of genus \( g - 2 \) with ramification sequence \((0, 1, 2, \ldots, 2)\) at \( q \). For \( g = 23 \) we have the following possibilities:

\[
(r, s, d) = (1, 13, 12), (11, 3, 32), (2, 9, 7), (7, 4, 24), (3, 7, 20), (5, 5, 24).
\]

It is immediate by Serre duality, that cases \((1, 13, 12)\) and \((11, 3, 32)\) yield the same divisor on \( \mathcal{M}_{23} \), namely the 12-gonal locus \( \mathcal{M}_{12}^{12} \); similarly, cases \((2, 9, 7)\) and \((7, 4, 24)\) yield the divisor \( \mathcal{M}_{17}^{17} \) of curves having a \( g_{17}^{17} \). While cases \((3, 7, 20)\) and \((5, 5, 24)\) give rise to \( \mathcal{M}_{20}^{20} \), the divisor of curves having a \( g_{20}^{20} \). Note that when the genus we are referring to is clear from the context, we write \( \mathcal{M}_g = \mathcal{M}_{g,d}^r \).

By comparing the classes of the Brill-Noether divisors to the class of the canonical divisor \( \mathcal{K}_{\overline{\mathcal{M}}_{g,reg}} = 13\lambda - 2\delta_0 - 3\delta_i - \cdots - 2\delta_{g/2} \); at least in the case when \( g + 1 \) is composite we can infer that

\[
\mathcal{K}_{\overline{\mathcal{M}}_{g,reg}} = a[\overline{\mathcal{M}}_{g,d}] + b\lambda + ( \text{positive combination of } \delta_0, \ldots, \delta_{g/2} ).
\]

where \( a \) is a positive rational number, while \( b > 0 \) as long as \( g \geq 24 \) but \( b = 0 \) for \( g = 23 \). As it is well-known that \( \lambda \) is big on \( \overline{\mathcal{M}}_g \), it follows that \( \mathcal{M}_g \) is of general type for \( g \geq 24 \) and that it has non-negative Kodaira dimension when \( g = 23 \). Specifically for \( g = 23 \), we get that there are positive integer constants \( m, m_1, m_2, m_3 \) such that

\[
mK = m_1[\overline{\mathcal{M}}_{12}] + E, \quad mK = m_2[\overline{\mathcal{M}}_{17}] + E, \quad mK = m_3[\overline{\mathcal{M}}_{20}] + E.
\]

where \( E \) is the same positive combination of \( \delta_1, \ldots, \delta_{11} \).

**Proposition 1.2.1 (Eisenbud-Harris, [EH3])** There exists a smooth curve of genus 23 that possesses a \( g_{12}^{12} \) but no \( g_{17}^{17} \). It follows that \( \kappa(\mathcal{M}_{23}) \geq 1 \).

Harris and Mumford proved (cf. [HM]) that \( \mathcal{M}_g \) has only canonical singularities for \( g \geq 4 \), hence \( H^0(\mathcal{M}_{g,reg}, nK) = H^0(\overline{\mathcal{M}}_g, nK) \) for each \( n \geq 0 \), with \( \overline{\mathcal{M}}_g \) a desingularization of \( \mathcal{M}_g \).
We already know that \( \dim(\text{Im} \phi_{mK}) \geq 1 \) where \( \phi_{mK} : \overline{\mathcal{M}}_{23} \rightarrow \mathbb{P}^r \) is the multicanonical map. \( m \) being as in (1.1). We will prove that \( \kappa(\mathcal{M}_{23}) \geq 2 \). Indeed, let us assume that \( \dim(\text{Im} \phi_{mK}) = 1 \). Denote by \( C := \text{Im} \phi_{mK} \) the Kodaira image of \( \overline{\mathcal{M}}_{23} \). We reach a contradiction by proving two things:

- \( \alpha \) The Brill-Noether divisors \( \mathcal{M}_{12}, \mathcal{M}_{17}^2 \) and \( \mathcal{M}_{20}^3 \) are mutually distinct.
- \( \beta \) There exist smooth curves of genus \( 23 \) which belong to exactly two of the Brill-Noether divisors from above.

This suffices in order to prove Theorem 1.1: since \( \overline{\mathcal{M}}_{12}, \overline{\mathcal{M}}_{17}^2 \) and \( \overline{\mathcal{M}}_{20}^3 \) are part of different multicanonical divisors, they must be contained in different fibres of the multicanonical map \( \phi_{mK} \). Hence there exists different points \( x, y, z \in C \) such that

\[
\mathcal{M}_{12}^1 = \phi^{-1}(x) \cap \mathcal{M}_{23}, \quad \mathcal{M}_{17}^2 = \phi^{-1}(y) \cap \mathcal{M}_{23}, \quad \mathcal{M}_{20}^3 = \phi^{-1}(z) \cap \mathcal{M}_{23}.
\]

It follows that the set-theoretic intersection of any two of them will be contained in the base locus of \( mK_{\mathcal{M}_{23}} \). In particular:

\[
\text{supp}(\mathcal{M}_{12}^1) \cap \text{supp}(\mathcal{M}_{17}^2) = \text{supp}(\mathcal{M}_{17}^2) \cap \text{supp}(\mathcal{M}_{20}^3) = \text{supp}(\mathcal{M}_{20}^3) \cap \text{supp}(\mathcal{M}_{12}^1). \quad (1.2)
\]

and this contradicts \( \beta \). We complete the proof of \( \alpha \) and \( \beta \) is Section 1.5.

### 1.3 Deformation theory for \( g_d' \)'s and limit linear series

We recall a few things about the variety parametrising \( g_d' \)'s on the fibres of the universal curve (cf. [AC2]), and then we recap on the theory of limit linear series (cf. [EH1], [Mod]), which is our main technique for the study of \( \mathcal{M}_{23} \).

Given \( g, r, d \) and a point \([C] \in \mathcal{M}_g\), there is a connected neighbourhood \( U \) of \([C]\), a finite ramified covering \( h : \mathcal{M} \rightarrow U \), such that \( \mathcal{M} \) is a fine moduli space of curves (i.e. there exists \( \xi : \mathcal{C} \rightarrow \mathcal{M} \) a universal curve), and a proper variety over \( \mathcal{M} \):

\[
\pi : \mathcal{G}_d' \rightarrow \mathcal{M}
\]

which parametrizes classes of couples \([C, l]\), with \([C] \in \mathcal{M} \) and \( l \in \mathcal{G}_d'(C) \), where we have made the identification \( C = \xi^{-1}(l(C)) \).

Let \((C, l)\) be a point of \( \mathcal{G}_d' \) corresponding to a curve \( C \) and a linear series \( l = (\mathcal{L}, V) \), where \( \mathcal{L} \in \text{Pic}^d(C), V \subseteq H^0(C, \mathcal{L}) \), and \( \dim(V) = r + 1 \). By choosing a basis in \( V \), one has a morphism \( f : C \rightarrow \mathbb{P}^r \). The normal sheaf of \( f \) is defined through the exact sequence

\[
0 \rightarrow T_C \rightarrow f^*T_{\mathbb{P}^r} \rightarrow N_f \rightarrow 0. \quad (1.3)
\]

By dividing out the torsion of \( N_f \) one gets to the exact sequence

\[
0 \rightarrow \mathcal{K}_f \rightarrow N_f' \rightarrow N_f' \rightarrow 0. \quad (1.4)
\]

where the torsion sheaf \( \mathcal{K}_f \) (the cuspidal sheaf) is based at those points \( x \in C \) where \( df(x) = 0 \), and \( N_f' \) is locally free of rank \( r - 1 \). The tangent space \( T_{(C, l)}(\mathcal{G}_d') \) fits into an exact sequence (cf. [AC2]):

\[
0 \rightarrow \mathcal{C} \rightarrow \text{Hom}(V, V') \rightarrow H^0(C, N_f) \rightarrow T_{(C, l)}(\mathcal{G}_d') \rightarrow 0. \quad (1.5)
\]

from which we have that \( \dim T_{(C, l)}(\mathcal{G}_d') = 3g - 3 + p\rho(g, r, d) + h^2(C, N_f) \).
Proposition 1.3.1 Let $C$ be a curve and $l \in G^r_d(C)$ a base point free linear series. Then the variety $G^r_d$ is smooth and of dimension $3g - 3 - \rho(g, r, d)$ at the point $(C, l)$ if and only if $H^1(C, N_f) = 0.$

Remark: The condition $H^1(C, N_f) = 0$ is automatically satisfied for $r = 1$ as $N_f$ is a sheaf with finite support. Thus $G^r_d$ is smooth of dimension $2g + 2d - 5$. It follows that $G^r_d$ is birationally equivalent to the $d$-gonal locus $\mathcal{M}_d^r$ when $d < (g + 2)/2$.

In Chapter 2 we will be interested in the differential $(d\pi)_{C,l}: T_{(C,l)}(G^r_d) \to T_{C,l}(\mathcal{M}_g)$. Let $(C, l) \in G^r_d$ be a point such that $H^1(C, N_f) = 0$ and assume for simplicity that $l = (\mathcal{L}, V)$ is a complete, base point free linear series, that is, $V = H^0(C, \mathcal{L})$. By standard Kodaira-Spencer theory (see [AC2] or [Mod]) one has that \[ \text{Im}(d\pi)_{C,l} = \text{Im}\{\delta : H^0(C, N_f) \to H^1(C, T_C)\}, \] where $\delta$ is the coboundary of the cohomology sequence associated to (1.3). We thus get that $\text{rk}(d\pi)_{C,l} = 3g - 3 - h^1(C, f^*T_{\mathcal{L}})$). By pulling back to $C$ the Euler sequence \[ 0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{f+1} \to T_{\mathcal{L}} \to 0, \] we obtain that $H^1(C, f^*T_{\mathcal{L}}) \simeq (\ker(\mu_0)(C, \mathcal{L}))^\vee$, where \[ \mu_0(C, \mathcal{L}) : H^0(C, \mathcal{L}) \oplus H^0(C, K_C \otimes \mathcal{L}^\vee) \to H^0(C, K_C) \] is the Petri map. We obtain thus that the differential $(d\pi)_{C,l}$ has the expected rank $\min(3g - 3, 3g - 3 + \rho(g, r, d))$, if and only if the Petri map is of maximal rank (which means surjective when $\rho(g, r, d) < 0$).

It is convenient to have a description of the annihilator $(\text{Im}(d\pi)_{C,l})^\perp \subset H^0(C, 2K_C)$, where we have made the identification $T^\vee_{C,l}(\mathcal{M}_g) = H^0(C, 2K_C)$ via Serre duality. We introduce the Gaussian map (cf. [CGGH]) \[ \mu_1(C, \mathcal{L}) : \ker(\mu_0)(C, \mathcal{L}) \to H^0(C, 2K_C), \] as follows: let us consider the evaluation sequence corresponding to $(C, \mathcal{L})$ \[ 0 \to M_{\mathcal{L}} \to H^0(C, \mathcal{L}) \otimes \mathcal{O}_C \to \mathcal{L} \to 0. \] We restrict the $\mathcal{L}$-linear map $H^0(C, \mathcal{L}) \otimes \mathcal{O}_C \to \mathcal{O}_C \otimes \mathcal{L}$. \[ (s \otimes f) \mapsto df \otimes s, \] to the kernel $M_{\mathcal{L}}$ and get an $\mathcal{O}_C$-linear map $M_{\mathcal{L}} \to \mathcal{O}_C \otimes \mathcal{L}$. If we tensor this map with $\Omega_C \otimes \mathcal{L}^\vee$, then take global sections and finally use that $H^0(C, M_{\mathcal{L}} \otimes \Omega_C \otimes \mathcal{L}^\vee) \simeq \ker(\mu_0)(C, \mathcal{L})$, we get the map $\mu_1(C, \mathcal{L}) : \ker(\mu_0)(C, \mathcal{L}) \to H^0(C, 2K_C)$, which we can loosely refer to as the "derivative" of the Petri map.

The map $\mu_1(C, \mathcal{L})$ can be explicitly described: for an element $s_0 \otimes \eta_0 + \cdots + s_r \otimes \eta_r \in \ker(\mu_0)(C, \mathcal{L})$, with $s_i \in H^0(C, \mathcal{L})$ and $\eta_i \in H^0(C, \Omega_C \otimes \mathcal{L}^\vee)$, if we consider the meromorphic functions $f_i = s_i/s_0$ on $C$, we have that \[ \mu_1(C, \mathcal{L})(s_0 \otimes \eta_0 + \cdots + s_r \otimes \eta_r) = s_0(\eta_0 df_1 + \cdots + \eta_r df_r). \]
An easy calculation shows that

\[(\Im(d\pi)_{(C,L)})^\sim = \Im\mu_1(C, L) \subseteq H^0(C, 2K_C).\]

Limit linear series try to answer questions of the following kind: what happens to a family of \(g_c^d\)'s when a smooth curve specializes to a reducible curve? Limit linear series solve such problems for a class of reducible curves, those of compact type. A curve \(C\) is of compact type if its dual graph is a tree. A curve \(C\) is tree-like if, after deleting edges leading from a node to itself, the dual graph becomes a tree.

Let \(C\) be a smooth curve of genus \(g\) and \(l = (\mathcal{L}, V) \in G_d^g(C), \mathcal{L} \in \text{Pic}^d(C), V \subseteq H^0(C, \mathcal{L}),\) and \(\dim(V) = r + 1.\) Fix \(p \in C\) a point. By ordering the finite set \(\{\text{ord}_p(\sigma)\}_{\sigma \in V}\) one gets the vanishing sequence of \(l\) at \(p:\)

\[a_i'(p) : 0 \leq a_0'(p) < \ldots < a_i'(p) \leq d.\]

The ramification sequence of \(l\) at \(p\)

\[\alpha_i'(p) : 0 \leq \alpha_0'(p) \leq \ldots \leq \alpha_i'(p) \leq d - r\]

is defined as \(\alpha_i'(p) = a_i'(p) - i\) and the weight of \(l\) at \(p\) is

\[u_i'(p) = \sum_{t=0}^{r} \alpha_t'(p).\]

A Schubert index of type \((r, d)\) is a sequence of integers \(\beta : 0 \leq \beta_0 \leq \ldots \leq \beta_r \leq d - r.\) If \(\alpha\) and \(\beta\) are Schubert indices of type \((r, d)\) we write \(\alpha \leq \beta \iff \alpha_i \leq \beta_i, i = 0, \ldots, r.\) The point \(p\) is said to be a ramification point of \(l\) if \(u_i'(p) > 0.\) The linear series \(l\) is said to have a cusp at \(p\) if \(\alpha_i'(p) \geq 0, 1, \ldots, 1.\) For \(C\) a tree-like curve, \(p_1, \ldots, p_n \in C\) smooth points and \(\alpha^1, \ldots, \alpha^n\) Schubert indices of type \((r, d),\) we define

\[G_d^r(C, (p_1, \alpha^1), \ldots, (p_n, \alpha^n)) := \{l \in G_d^r(C) : \alpha_1'(p_1) \geq \alpha^1, \ldots, \alpha'(p_n) \geq \alpha^n\}.\]

This scheme can be realized naturally as a determinantal variety and its expected dimension is

\[\rho(g, r, d, \alpha^1, \ldots, \alpha^n) := \rho(g, r, d) - \sum_{i=1}^{n} \sum_{j=0} r \alpha_i^j.\]

If \(C\) is a curve of compact type, a crude limit \(g_d^r\) on \(C\) is a collection of ordinary linear series \(l = \{l_Y \in G_d^r(Y) : Y \subseteq C\text{ is a component}\},\) satisfying the following compatibility condition: if \(Y\) and \(Z\) are components of \(C\) with \(\{p\} = Y \cap Z,\) then

\[a_i^y(p) + a_k^z(p) \geq d.\]

If equality holds everywhere, we say that \(l\) is a refined limit \(g_d^r.\) The ‘honest’ linear series \(l_Y \in G_d^r(Y)\) is called the \(Y\)-aspect of the limit linear series \(l.\)
We will often use the additivity of the Brill-Noether number: if $C$ is a curve of compact type, for each component $Y \subseteq C$, let $q_1, \ldots, q_n$ be the points where $Y$ meets the other components of $C$. Then for any limit $g_d^Y$ on $C$ we have the following inequality:

$$\rho(g, r, d) \geq \sum_{Y \subseteq C} \rho(l_Y, \alpha^Y_1(q_1), \ldots, \alpha^Y_n(q_n)).$$

(1.6)

with equality if and only if $l$ is a refined limit linear series.

It has been proved in [EH1] that limit linear series arise indeed as limits of ordinary linear series on smooth curves. Suppose we are given a family $\pi : C \to B$ of genus $g$ curves, where $B = \text{Spec}(R)$ with $R$ a complete discrete valuation ring. Assume furthermore that $C$ is a smooth surface and that if $0, \eta$ denote the special and generic point of $B$ respectively, the central fibre $C_0$ is reduced and of compact type, while the generic geometric fibre $C_\eta$ is smooth and irreducible. If $l_\eta = (\mathcal{L}_\eta, V_\eta)$ is a $g_d^\eta$ on $C_\eta$, there is a canonical way to associate a crude limit series $l_0$ on $C_0$ which is the limit of $l_\eta$ in a natural way: for each component $Y$ of $C_0$, there exists a unique line bundle $\mathcal{L}^Y$ on $C$ such that

$$\mathcal{L}^Y_{\eta} = \mathcal{L}_{\eta} \text{ and } \deg_Z(\mathcal{L}^Y_{\eta}) = 0,$$

for any component $Z$ of $C_0$ with $Z \neq Y$. (This implies of course that $\deg_{l^Y}(\mathcal{L}^Y_{\eta}) = d$).

Define $V^Y = V_\eta \cap H^0(C, \mathcal{L}^Y) \subseteq H^0(C_0, \mathcal{L}_0)$. Clearly, $V^Y$ is a free $R$-module of rank $r + 1$. Moreover, the composite homomorphism

$$V^Y(0) \to (\pi_*, \mathcal{L}^Y)(0) \to H^0(C_0, \mathcal{L}^Y_{\eta}) \to H^0(Y, \mathcal{L}^Y_\eta)$$

is injective, hence $l_\eta = (\mathcal{L}^Y_{\eta}, V^Y(0))$ is an ordinary $g_d^\eta$ on $Y$. One proves that $l = \{l_\eta : Y \text{ component of } C_0\}$ is a limit linear series.

If $C$ is a reducible curve of compact type, $l$ a limit $g_d^\eta$ on $C$, we say that $l$ is smoothable if there exists $\pi : C \to B$ a family of curves with central fibre $C = C_0$ as above, and $(\mathcal{L}_\eta, V_\eta)$ a $g_d^\eta$ on the generic fibre $C_\eta$ whose limit on $C$ (in the sense previously described) is $l$.

**Remark:** If a stable curve of compact type $C$ has no limit $g_d^\eta$'s, then $[C] \not\in \mathcal{M}_{2, d}$. If there exists a smoothable limit $g_d^\eta$ on $C$, then $[C] \in \mathcal{M}_{2, d}$.

Now we explain a criterion due to Eisenbud and Harris (cf. [EH1]), which gives a sufficient condition for a limit $g_d^\eta$ to be smoothable. Let $l$ be a limit $g_d^\eta$ on a curve $C$ of compact type. Fix $Y \subseteq C$ a component, and $\{q_1, \ldots, q_s\} = Y \cap (C - Y)$. Let

$$\pi : \mathcal{Y} \to B, \quad \bar{q}_i : B \to \mathcal{Y}$$

be the versal deformation space of $(Y, q_1, \ldots, q_s)$. The base $B$ can be viewed as a small $(3g(Y) - 3 + s)$-dimensional polydisk. Using general theory one constructs a proper scheme over $B$.

$$\sigma : G_\mathcal{Y}(\mathcal{Y}/B; (\bar{q}_i, \alpha^Y(q_i))_{i=1}^s) \to B$$

whose fibre over each $b \in B$ is $\sigma^{-1}(b) = G_b^*(Y_b, (\bar{q}_i(b), \alpha^Y(q_i))_{i=1}^s)$. One says that $l$ is **dimensionally proper with respect to $Y$**, if the $Y$-aspect $l_Y$ is contained in some component $\mathcal{G}$ of $G_\mathcal{Y}(\mathcal{Y}/B; (\bar{q}_i, \alpha^Y(q_i))_{i=1}^s)$ of the expected dimension, i.e.

$$\dim \mathcal{G} = \dim B + \rho(l_Y, \alpha^Y(q_1), \ldots, \alpha^Y(q_s)).$$
One says that \( l \) is dimensionally proper, if it is dimensionally proper with respect to any component \( Y \subseteq C \). The 'Regeneration Theorem' (cf. [EH1]) states that every dimensionally proper limit linear series is smoothable.

The next result is a 'strong Brill-Noether Theorem', i.e. it not only asserts a Brill-Noether type statement, but also singles out the locus where the statement fails.

**Proposition 1.3.2 (Eisenbud-Harris)** Let \( C \) be a tree-like curve and for any component \( Y \subseteq C \), denote by \( q_1, \ldots, q_s \in Y \) the points where \( Y \) meets the other components of \( C \). Assume that for each \( Y \) the following conditions are satisfied:

a. If \( g(Y) = 1 \) then \( s = 1 \).

b. If \( g(Y) = 2 \) then \( s = 1 \) and \( q \) is not a Weierstrass point.

c. If \( g(Y) \geq 3 \) then \( (Y, q_1, \ldots, q_s) \) is a general \( s \)-pointed curve.

Then for \( p_1, \ldots, p_t \in C \) general points, \( \rho(l, \alpha^1(p_1), \ldots, \alpha^t(p_t)) \geq 0 \) for any limit linear series on \( C \).

Simple examples involving pointed elliptic curves show that the condition \( \rho(g, r, d) \geq \sum_{i=1}^{r} \alpha^i(p_i) \) does not guarantee the existence of a linear series \( l \in G_d^r(C) \) with prescribed ramification at general points \( p_1, p_2, \ldots, p_t \in C \). The appropriate condition in the pointed case can be given in terms of Schubert cycles. Let \( \alpha = (\alpha_0, \ldots, \alpha_r) \) be a Schubert index of type \( (r, d) \) and

\[
\mathbb{C}^{d+1} = W_0 \supset W_1 \supset \cdots \supset W_{d+1} = 0
\]

a decreasing flag of linear subspaces. We consider the Schubert cycle in the Grassmanian.

\[
\sigma_\alpha = \{ \Lambda \in G(r + 1, d + 1) : \dim(\Lambda \cap W_{\alpha_i}) \geq r + 1 - i, \ i = 0, \ldots, r \}.
\]

For a general \( t \)-pointed curve \( (C, p_1, \ldots, p_t) \) of genus \( g \), and \( \alpha^1, \ldots, \alpha^t \) Schubert indices of type \( (r, d) \), the necessary and sufficient condition that \( C \) has a \( g^d \) with ramification \( \alpha^i \) at \( p_i \) is that

\[
\sigma_{\alpha_1} \cdots \sigma_{\alpha_t} \cap \sigma_{(0,1,\ldots,1)}^g \neq 0 \text{ in } H^*(G(r + 1, d + 1), \mathbb{Z}). \tag{1.7}
\]

In the case \( t = 1 \) this condition can be made more explicit (cf. [EH3]): a general pointed curve \( (C, p) \) of genus \( g \) carries a \( g^d \) with ramification sequence \( (\alpha_0, \ldots, \alpha_r) \) at \( p \) if and only if

\[
\sum_{i=0}^{r} (\alpha_i + g - d + r) \leq g. \tag{1.8}
\]

where \( x_- = \max\{x, 0\} \). One can make the following simple but useful observation:
Proposition 1.3.3 Let \((C,p,q)\) be a general 2-pointed curve of genus \(g\) and \((\alpha_0, \ldots, \alpha_r)\) a Schubert index of type \((r,d)\). Then \(C\) has a \(g^r_d\) having ramification sequence \((\alpha_0, \ldots, \alpha_r)\) at \(p\) and a cusp at \(q\) if and only if
\[
\sum_{i=0}^{r} (\alpha_i + g + 1 - d + r)_{+} \leq g + 1.
\]

Proof: The condition for the existence of the \(g^r_d\) with ramification \(\alpha\) at \(p\) and a cusp at \(q\) is that \(\sigma_{\alpha} \cdot \sigma_{\alpha+1}^{r+1} \neq 0\) (cf. (1.7)). According to the Littlewood-Richardson rule (see [F]), this is equivalent with \(\sum_{i=0}^{r} (\alpha_i + g + 1 - d + r)_{+} \leq g + 1.\)

\[\square\]

1.4 A few consequences of limit linear series

We investigate the Brill-Noether theory of a 2-pointed elliptic curve (see also [EH4]), and we prove that \(\overline{\mathcal{M}}^l_{g,d} \cap \Delta_1\) is irreducible for \(\rho(g,r,d) = -1.\)

Proposition 1.4.1 Let \((E,p,q)\) be a two-pointed elliptic curve. Consider the sequences
\[a : a_0 < a_1 < \ldots a_r \leq d.\]
\[b : b_0 < b_1 < \ldots b_r \leq d.\]

1. For any linear series \(l = (\mathcal{L}, V) \in G^l_d(E)\) one has that \(\rho(l, \alpha^l(p), \alpha^l(q)) \geq -r.\) Furthermore, if \(\rho(l, \alpha^l(p), \alpha^l(q)) \leq -1.\) then \(p - q \in \text{Pic}^0(E)\) is a torsion class.

2. Assume that the sequences \(a\) and \(b\) satisfy the inequalities: \(d - 1 \leq a_i + b_{r-i} \leq d.\) \(i = 0, \ldots, r.\) Then there exists at most one linear series \(l \in G^l_d(E)\) such that \(\alpha^l(p) = a\) and \(\alpha^l(q) = b.\) Moreover, there exists exactly one such linear series \(l = (O_E(D), V)\) with \(D \in E^d,\) if and only if for each \(i = 0, \ldots, r\) the following is satisfied: if \(a_i + b_{r-i} = d,\) then \(D \sim a_i p + b_{r-i} q,\) and if \((a_i + 1) p + b_{r-i} q \sim D,\) then \(a_{i+1} = a_i + 1.\)

Proof: In order to prove 1. it is enough to notice that for dimensional reasons there must be sections \(\sigma_i \in V\) such that \(\text{div}(\sigma_i) \geq a^l(p) p + a^l_{r-i}(q) q.\) Therefore, \(a^l(p) + a^l_{r-i}(q) \leq d.\)

By adding up all these inequalities, we get that \(\rho(l, \alpha^l(p), \alpha^l(q)) \geq -r.\) Furthermore, \(\rho(l, \alpha^l(p), \alpha^l(q)) \leq -1\) precisely when for at least two values \(i < j\) we have equalities \(a_i + b_{r-i} = d,\) \(a_j + b_{r-j} = d,\) which means that there are sections \(\sigma_i, \sigma_j \in V\) such that \(\text{div}(\sigma_i) = a_i p + b_{r-i} q,\) \(\text{div}(\sigma_j) = a_j p + b_{r-j} q.\) By subtracting, we see that \(p - q \in \text{Pic}^0(E)\) is torsion. The second part of the Proposition is in fact Prop. 5.2 from [EH4].

\[\square\]

Proposition 1.4.2 Let \(g, r, d\) be such that \(\rho(g, r, d) = -1.\) Then the intersection \(\overline{\mathcal{M}}^l_{g,d} \cap \Delta_1\) is irreducible.

Proof: Let \(Y\) be an irreducible component of \(\overline{\mathcal{M}}^l_{g,d} \cap \Delta_1.\) Either \(Y \cap \text{Int} \Delta_1 \neq \emptyset,\) hence \(Y = Y \cap \text{Int} \Delta_1,\) or \(Y \subseteq \Delta_1 - \text{Int} \Delta_1.\) The second alternative never occurs. Indeed, if \(Y \subseteq \Delta_1 - \text{Int} \Delta_1,\) then since \(\text{codim} (Y, \overline{\mathcal{M}}_g) = 2,\) \(Y\) must be one of the irreducible components of \(\Delta_1 - \text{Int} \Delta_1.\) The components of \(\Delta_1 - \text{Int} \Delta_1\) correspond to curves with two nodes. We list these components (see [Ed1]):

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• For $1 \leq j \leq g - 2$, $\Delta_{1,j}$ is the closure of the locus in $\mathcal{M}_g$ whose general point corresponds to a chain composed of an elliptic curve, a curve of genus $g - j - 1$, and a curve of genus $j$.

• The component $\Delta_{0,1}$, whose general point corresponds to the union of a smooth elliptic curve and an irreducible nodal curve of genus $g - 2$.

• The component $\Delta_{0,g-1}$ whose general point corresponds to the union of a smooth curve of genus $g - 1$ and an irreducible rational curve.

As the general point of $\Delta_{1,j}$, $\Delta_{0,1}$ or $\Delta_{0,g-1}$ is a tree-like curve which satisfies the conditions of Prop.1.3.2 it follows that such a curve satisfies the 'strong' Brill-Noether Theorem. Hence $\Delta_{1,j} \not\subseteq \mathcal{M}_{g,d}$, $\Delta_{0,1} \not\subseteq \mathcal{M}_{g,d}$ and $\Delta_{0,g-1} \not\subseteq \mathcal{M}_{g,d}$, a contradiction. So, we are left with the first possibility: $Y = \overline{Y \cap \text{Int} \Delta_1}$. We are going to determine the general point $[C] \in Y \cap \text{Int} \Delta_1$. Let $X = C \cup E \cup g(C) = g - 1, E$ elliptic. $E \cap C = \{p\}$ such that $X$ carries a limit $\mathcal{g}_d^g$, say $l$. By the additivity of the Brill-Noether number, we have:

$$-1 = \rho(g, r, d) \geq \rho(l, C, p) + \rho(l, E, p).$$

Since $\rho(l, E, p) \geq 0$, it follows that $\rho(l, C, p) \leq -1$, so $\alpha^l_C(p) \geq r$. Let us denote by

$$\beta : C_{g-1} \times C_1 \rightarrow \text{Int} \Delta_1$$

the natural map given by $\beta([C, p], [E, q]) = [X := C \cup E / p \sim q]$. We claim that if we choose $X$ generically, then $\alpha^l_C(p) = 0$. If not, $p$ is a base point of $l_C$ and after removing the base point we get that $[C] \in \mathcal{M}_{g-1,d-1}$. Note that $\rho(g - 1, r, d - 1) = -2$, so $\dim \mathcal{M}_{g-1,d-1} = 3g - 8$ (cf. [Ed2]). If we denote by $\pi : C_{g-1} \rightarrow \mathcal{M}_{g-1}$ the morphism which 'forgets the point', we get that

$$\dim \beta^{-1}(\mathcal{M}_{g-1,d-1} \times C_1) = 3g - 6 < \dim Y,$$

a contradiction. Hence, for the generic $[X] \in Y$ we must have $\alpha^l_C(p) = 0$, so $\alpha^l_E(p) = d$. Since an elliptic curve cannot have a meromorphic function with a single pole, it follows that $\alpha^l_E(p) \leq d - 2$ and this implies $\alpha^l_C(p) \geq (0, 1, \ldots, 1)$, i.e. $l_C$ has a cusp at $p$. Thus, if we introduce the notation

$$C_{g-1,d}^E(0, 1, \ldots, 1) = \{(C, p) \in C_{g-1} : C^E_d(C, (0, 1, \ldots, 1)) \neq 0\},$$

then $Y \subseteq \beta(C_{g-1,d}^E(0, 1, \ldots, 1) \times C_1)$. On the other hand, it is known (cf. [EH2]) that $C_{g-1,d}^E(0, 1, \ldots, 1)$ is irreducible of dimension $3g - 6$ (that is, codimension 1 in $C_{g-1}$), so we must have $Y = \beta(C_{g-1,d}^E(0, 1, \ldots, 1) \times C_1)$, which not only proves that $\mathcal{M}_{g,d} \cap \Delta_1$ is irreducible, but also determines the intersection.

1.5 The Kodaira dimension of $\mathcal{M}_{23}$

In this section we prove that $\kappa(\mathcal{M}_{23}) \geq 2$ and we investigate closely the multicanonical linear systems on $\mathcal{M}_{23}$. We now describe the three multicanonical Brill-Noether divisors from Section 2.
1.5.1 The divisor $\overline{\mathcal{M}}_{12}^1$

There is a stratification of $\mathcal{M}_{23}$ given by gonality:

$$\mathcal{M}_{1}^2 \subseteq \mathcal{M}_{3}^1 \subseteq \ldots \subseteq \mathcal{M}_{12}^1 \subseteq \mathcal{M}_{23}.$$  

For $2 \leq d \leq g/2 + 1$ one knows that $\mathcal{M}_{1}^1 = \mathcal{M}_{g,k}$ is an irreducible variety of dimension $2g + 2d - 5$. The general point of $\mathcal{M}_{g,d}^1$ corresponds to a curve having a unique $g_d^1$.

1.5.2 The divisor $\overline{\mathcal{M}}_{17}^2$

The Severi variety $V_{d,g}$ of irreducible plane curves of degree $d$ and geometric genus $g$, where $0 \leq g \leq \left(\frac{d-1}{2}\right)$, is an irreducible subscheme of $\mathbb{P}^d(d+3)/2$ of dimension $3d + g - 1$ (cf. [H], [Mod]). Inside $V_{d,g}$ we consider the open dense subset $U_{d,g}$ of irreducible plane curves of degree $d$ having exactly $\delta = \left(\frac{d-1}{2}\right) - g$ nodes and no other singularities. There is a global normalization map

$$m : U_{d,g} \to \mathcal{M}_g, \ m([Y]) := [\hat{Y}], \text{ where } \hat{Y} \text{ is the normalization of } Y.$$  

When $d - 2 \leq g \leq \left(\frac{d-1}{2}\right), d \geq 5$, $U_{d,g}$ has the expected number of moduli, i.e.

$$\dim m(U_{d,g}) = \min(3g - 3, 3g - 3 + \rho(g, 2, d)) > 0.$$  

In our case we can summarize this as follows:

**Proposition 1.5.1** There is exactly one component of $G_{17}^2$ mapping dominantly to $\mathcal{M}_{17}^2$. The general element $(C, l) \in G_{17}^2$ corresponds to a curve $C$ of genus 23, together with a $g_{17}^1$ which provides a plane model for $C$ of degree 17 with 97 nodes.

1.5.3 The divisor $\overline{\mathcal{M}}_{20}^3$

Here we combine the result of Eisenbud and Harris (see [EH2]) about the uniqueness of divisorial components of $G_{d}^2$ when $\rho(g, r, d) = -1$, with Sernesi's (see [Sc2]) which asserts the existence of components of the Hilbert scheme $H_{d,g}$ parametrizing curves in $\mathbb{P}^3$ of degree $d$ and genus $g$ with the expected number of moduli, for $d - 3 \leq g \leq 3d - 18$, $d \geq 9$.

**Proposition 1.5.2** There is exactly one component of $G_{20}^3$ mapping dominantly to $\mathcal{M}_{20}^3$. The general point of this component corresponds to a pair $(C, l)$ where $C$ is a curve of genus 23 and $l$ is a very ample $g_{20}^3$.

We are going to prove that the Brill-Noether divisors $\overline{\mathcal{M}}_{12}^1, \overline{\mathcal{M}}_{17}^2$ and $\overline{\mathcal{M}}_{20}^3$ are mutually distinct.

**Theorem 1.2** There exists a smooth curve of genus 23 having a $g_{17}^2$, but no $g_{20}^3$'s. Equivalently, one has $\text{supp}(\mathcal{M}_{17}^2) \notin \text{supp}(\mathcal{M}_{20}^3)$.
Proof: It suffices to construct a reducible curve $X$ of compact type of genus 23, which has a smoothable limit $g_{20}^1$, but no limit $g_{20}^3$. If $|C| \in M_{20}$ is a nearby smoothing of $X$ which preserves the $g_{20}^1$, then $|C| \in M_{17} - M_{20}$. Let us consider the following curve:

![Curved diagram](image)

where $(C_1,p_1)$ and $(C_2,p_2)$ are general pointed curves of genus 11. $E$ is an elliptic curve, and $p_1 - p_2$ is a primitive 9-torsion point in Pic$^0(E)$.

**Step 1) There is no limit $g_{20}^3$ on $X$.** Assume that $l$ is a limit $g_{20}^3$ on $X$. By the additivity of the Brill-Noether number.

$$-1 \geq \rho(l_{c_1}, p_1) + \rho(l_{c_2}, p_2) + \rho(l_E, p_1, p_2).$$

Since $(C_i, p_i)$ are general points in $C_{11}$, it follows from Prop.1.3.2 that $\rho(l_{c_1}, p_i) \geq 0$, hence $\rho(l_E, p_1, p_2) \leq -1$. On the other hand $\rho(l_E, p_1, p_2) \geq -3$ from Prop.1.4.1.

Denote by $(a_0, a_1, a_2, a_3)$ the vanishing sequence of $l_E$ at $p_1$, and by $(b_0, b_1, b_2, b_3)$ that of $l_E$ at $p_2$. The condition (1.8) for a general pointed curve $[(C_i, p_i)] \in C_{11}$ to possess a $g_{20}^3$ with prescribed ramification at the point $p$, and the compatibility conditions between $l_{c_i}$ and $l_E$ at $p$, give that:

$$(14 - a_3)_+ + (13 - a_2)_+ + (12 - a_1)_+ + (11 - a_0)_+ \leq 11. \quad (1.9)$$

and

$$(14 - b_3)_+ + (13 - b_2)_+ + (12 - b_1)_+ + (11 - b_0)_+ \leq 11. \quad (1.10)$$

1st case: $\rho(l_E, p_1, p_2) = -3$. Then $a_i + b_i = 20$, for $i = 0, \ldots, 3$ and it immediately follows that $20(p_1 - p_2) \sim 0$ in Pic$^0(E)$, a contradiction.

2nd case: $\rho(l_E, p_1, p_2) = -2$. We have two distinct possibilities here: i) $a_0 + b_3 = 20, a_1 + b_2 = 20, a_2 + b_1 = 20, a_3 + b_0 = 19$. Then it follows that $a_E(p_1) = (0, 9, 18, 19)$ and $a_E(p_2) = (0, 2, 11, 20)$, while according to (1.9), $a_3 \leq 15$. (because $\rho(l_{c_1}, p_1) \leq 1$), a contradiction. ii) $a_0 + b_3 = 20, a_1 + b_2 = 20, a_2 + b_1 = 19, a_3 + b_0 = 20$. Again, it follows that $a_3 = a_0 + 18 \geq 15$, a contradiction.

3rd case: $\rho(l_E, p_1, p_2) = -1$. Then $\rho(l_{c_1}, p_i) = 0$ and $l$ is a refined limit $g_{20}^3$. From (1.9) and (1.10) we must have: $a_E(p_i) \leq (11, 12, 13, 14)$. $i = 1, 2$. There are four possibilities: i) $a_0 + b_3 = a_1 + b_2 = 20, a_2 + b_1 = a_3 + b_0 = 19$. Then $a_i = a_0 + 9 \leq 12$, so $b_3 = 20 - a_0 \geq 17$, a contradiction. ii) $a_0 + b_3 = a_2 + b_1 = 20, a_2 + b_1 = a_3 + b_0 = 19$. Then $b_3 = 20 - a_0 \leq 14$, so $a_2 = a_0 + 9 \geq 15$, a contradiction. iii) $a_0 + b_3 = a_3 + b_0 = 20, a_1 + b_2 = a_2 + b_1 = 19$. Then $b_3 = 19 - a_0 \leq 14$, so $a_3 \geq a_0 + 9 \geq 15$, a contradiction. iv) $a_0 + b_3 = a_3 + b_0 = 18,
19. \( a_1 + b_2 = a_2 + b_1 = 20 \). Then \( b_3 = 19 - a_0 \leq 14 \), so \( a_2 \geq a_1 + 9 \geq 15 \). a contradiction again. We conclude that \( X \) has no limit \( g^3_{20} \).

**Step 2)** There exists a smoothable limit \( g^2_{17} \) on \( X \), hence \([X] \in \overline{\mathcal{M}}^2_{17}\). We construct a limit linear series \( l \) of type \( g^2_{17} \) on \( X \), aspect by aspect: on \( C_i \) take \( l_{C_i} \in G^2_{17}(C_i) \) such that \( d^{C_i}(p_i) = (4.9.13) \). Note that in this case \( \sum_{j=0}^r (a_j + g - d + r)_+ = g \), so (1.8) ensures the existence of such a \( g^2_{17} \). On \( E \) we take \( l_E = |V_E| \), where \( |V_E| \leq |4p_1 + 13p_2| = 4p_2 + 13p_1 \) is a \( g^2_{17} \) with vanishing sequence (4.8, 13) at \( p_i \). Prop.1.4.1 ensures the existence of such a linear series. In this way \( l \) is a refined limit \( g^2_{17} \) on \( X \) with \( \rho(l_{C_i}, p_i) = 0, \rho(l_E, p_1, p_2) = -1 \).

We prove that \( l \) is dimensionally proper. Let \( \pi_i : C_i \to \Delta_i, \tilde{p}_i : \Delta_i \to C_i \) be the versal deformation of \([C_i, p_i] \in C_{11} \), and \( \sigma_i : G^2_{17}(C_i/\Delta_i, (\tilde{p}_i, (4.8.11))) \to \Delta_i \) the projection.

Since being general is an open condition, we have that \( \sigma_i \) is surjective and dim \( \sigma_i^{-1}(t) = \rho(l_{C_i}, p_i) = 0 \). for each \( t \in \Delta_i \), therefore\[ \dim G^2_{17}(C_i/\Delta_i, (\tilde{p}_i, (4.8.11))) = \dim \Delta_i + \rho(l_{C_i}, p_i) = 31. \]

Next, let \( \pi : C \to \Delta \). \( \tilde{p}_1, \tilde{p}_2 : \Delta \to C \) be the versal deformation of \((E, p_1, p_2) \). We prove that\[ \dim G^2_{17}(C/\Delta, (\tilde{p}_i, (4.7.11))) = \dim \Delta + \rho(l_E, p_1, p_2) = 1. \]

This follows from Prop.1.4.1, since a 2-pointed elliptic curve \((E_i, \tilde{p}_1(t), \tilde{p}_2(t)) \) has at most one \( g^2_{17} \) with ramification (4.7, 11) at both \( \tilde{p}_1(t) \) and \( \tilde{p}_2(t) \), and exactly one when \( 9(\tilde{p}_1(t) - \tilde{p}_2(t)) \sim 0 \). Hence \( \text{Im} G^2_{17}(C/\Delta, (\tilde{p}_i, (4.7.11))) = \{ t \in \Delta : 9(\tilde{p}_1(t) - \tilde{p}_2(t)) \sim 0 \text{ in } \text{Pic}^0(E_i) \} \), which is a divisor on \( \Delta \), so the claim follows and \( l \) is a dimensionally proper \( g^2_{17} \).

A slight variation of the previous argument gives us:

**Proposition 1.5.3** We have \( \text{supp}(\overline{\mathcal{M}}^2_{17} \cap \Delta_1) \neq \text{supp}(\overline{\mathcal{M}}^3_{20} \cap \Delta_1) \).

**Proof:** We construct a curve \([Y] \in \Delta_1 \subseteq \overline{\mathcal{M}}_{23} \) which has a smoothable limit \( g^2_{17} \) but no limit \( g^3_{20} \). Let us consider the following:

\[
Y := C_1 \cup C_2 \cup E_1 \cup E_2,
\]

where \((C_2, p_2)\) is a general point of \( C_{11} \). \((C_1, p_1, x)\) is a general 2-pointed curve of genus 10. \((E_1, x)\) is general in \( C_1 \). \( E \) is an elliptic curve, and \( p_1 - p_2 \in \text{Pic}^0(E) \) is a primitive 9-torsion. In order to prove that \( Y \) has no limit \( g^3_{20} \), one just has to take into account that according to Prop.1.3.3, the condition for a general 1-pointed curve \((C, z)\) of genus \( g \) to have a \( g^2_d \) with ramification \( \alpha \) at \( z \) is the same with the condition for a general 2-pointed
curve \((D, x, y)\) of genus \(g - 1\) to have a \(g^2\) with ramification \(\alpha\) at \(x\) and a cusp at \(y\). Therefore we can repeat what we did in the proof of Theorem 1.2. Next, we construct \(l\), a smoothable limit \(g_{17}^2\) on \(Y\); take \(l_{C_1} \in G^2_{17}(C_2, (p_2, (4, 8, 11)))\), \(l_E = (V_E \subseteq \delta p_1 + 13p_2)\), with \(\alpha\) \((p_1) = (4, 7, 11)\), on \(E_1\) take \(l_{E_1} = 14x + 3x\), and finally on \(C_1\) take \(l_{C_1}\) such that \(\alpha l_{C_1}(p_1) = (4, 8, 11), \alpha l_{C_1}(x) = (0, 0, 1)\). Prop. 1.3.3 ensures the existence of \(l_{C_1}\). Clearly, \(l\) is a refined limit \(g_{17}^2\) and the proof that it is smoothable is all but identical to the one in the last part of Theorem 1.2.

The other cases are settled by the following:

**Theorem 1.3** There exists a smooth curve of genus 23 having a \(g_{12}\) but having no \(g_{17}^2\) nor \(g_{30}^2\). Equivalently, \(\text{supp}(M_{12}^2) \nsubseteq \text{supp}(M_{17}^2)\) and \(\text{supp}(M_{12}^2) \nsubseteq \text{supp}(M_{20}^2)\).

**Proof:** We take the curve considered in [EH3]:

\[
\begin{array}{ccc}
 & p_1 & p_2 \\
C_1 & & C_2 \\
& E & \\
\end{array}
\]

where \(C_1, p_i\) are general points of \(C_1, E\) is elliptic and \(p_1 - p_2 \in \text{Pic}^0(E)\) is a primitive 12-torsion. Clearly \(Y\) has a (smoothable) limit \(g_{12}^2\); on \(C_1\) take the pencil \(|12p_1|\); while on \(E\) take the pencil spanned by \(12p_1\) and \(12p_2\). It is proved in [EH3] that \(Y\) has no limit \(g_{17}^2\) and similarly one can prove that \(Y\) has no limit \(g_{30}^2\) either. We omit the details.\(\square\)

Now we are going to prove that equation (1.2)

\[
\text{supp}(M_{12}^2) \cap \text{supp}(M_{17}^2) = \text{supp}(M_{17}^1) \cap \text{supp}(M_{30}^3) = \text{supp}(M_{20}^3) \cap \text{supp}(M_{12}^1)
\]

is impossible, and as explained before, this will imply that \(\kappa(M_{23}) \geq 2\). The main step in this direction is the following:

**Proposition 1.5.4** There exists a stable curve of compact type of genus 23 which has a smoothable limit \(g_{30}^2\), a smoothable limit \(g_{15}^3\) (therefore also a \(g_{17}^3\)), but has generic gonality. that is, it does not have any limit \(g_{12}^2\).

**Intermezzo:** Before proceeding with the proof let us discuss a possible way to construct curves of genus 23 with such special Brill-Noether properties. Since we are looking for curves \(C\) of genus 23 with a \(g_{15}^3\), a possibility is to start with a (smooth) plane curve \(\Gamma \subseteq \mathbb{P}^2\) of degree \(d < 15\) and obtain \(C\) from \(\Gamma\) by several geometrical operations. We take \(\Gamma \subseteq \mathbb{P}^2\) smooth of degree \(d\) and pick general points \(p_i, q_i \in \Gamma\) for \(1 \leq i \leq \delta\).

Let us denote by \(\widehat{C}\) the curve obtained from \(\Gamma\) by identifying \(p_i\) and \(q_i\) and by \(\nu: \Gamma \rightarrow \widehat{C}\) the normalization map. Hence \(\nu(p_i) = \nu(q_i) = s_i\) for \(1 \leq i \leq \delta\). There exists a generalized \(g_{d-\delta}^2\) on the integral curve \(\widehat{C}\) which corresponds to a torsion-free rank one sheaf on \(\widehat{C}\).
Note that since \( \overline{C} \) is irreducible the variety of torsion-free rank one sheaves on \( \overline{C} \) is a compactification of \( \text{Pic}(\overline{C}) \). The generalized \( g_{d-2}^{2} \) is obtained from the (unique) \( g_{d}^{2} \) on \( \Gamma \) by adding the nodes \( s_{i} \) as base points, hence we have that \( \nu^{*}(g_{d-2}^{2}) = g_{d}^{2}(\sum_{i=1}^{s}(p_{i} + q_{i})) \).

Using results from [Ta], it is not difficult to show that the \( g_{d+1}^{2} \) on \( \overline{C} \) is smoothable, that is, \( \overline{C} \in \overline{\mathcal{M}}_{\text{pa}(\overline{C}),d-\delta}^{2} \). By solving the equations \( d + \delta = 15 \) and \( (d-1) + \delta = 23 \) we get \( d = 7 \) and \( \delta = 8 \), so we could start with a smooth plane septic \( \Gamma \subseteq \mathbb{P}^{2} \) and identify 8 pairs of general points \( p_{i}, q_{i} \in \Gamma, i = 1, \ldots, 8 \). The resulting curve \( \overline{C} \), of genus 23, will have a smoothable \( g_{8}^{2} \). Letting the points \( p_{i}, q_{i} \) come together in pairs, we obtain a curve of genus 23 with 8 cusps. From the point of view of the Stable Reduction Theorem (see [Mod]) this is the same as attaching 8 elliptic tails to \( \Gamma \) at the cusps.

**Proof** We shall consider the following stable curve \( X \) of genus 23:

\[
\begin{array}{c|c|c|c}
\Gamma & p_{1} & p_{2} & p_{8} \\
\hline
E_{1} & E_{2} & \cdots & E_{8} \\
\hline
X := \Gamma \cup E_{1} \cup \ldots \cup E_{8},
\end{array}
\]

where the \( E_{i} \)'s are elliptic curves, \( \Gamma \subseteq \mathbb{P}^{2} \) is a general smooth plane septic and the points of attachment \( \{p_{i}\} = \Gamma \cup E_{i} \) are general points of \( \Gamma \).

**Step 1** There is no limit \( g_{12}^{1} \) on \( X \). Assume that \( l \) is a limit \( g_{12}^{1} \) on \( X \). Since the elliptic curves \( E_{i} \) cannot have meromorphic functions with a single pole, we have that \( d^{E_{i}}(p_{i}) \leq (10,12) \), hence \( d^{E_{i}}(p_{i}) \geq (0,1) \), that is, \( l_{\Gamma} \) has a cusp at \( p_{i} \) for \( i = 1, \ldots, 8 \). We now prove that \( \Gamma \) has no \( g_{12}^{1} \)'s with cusps at the points \( p_{i} \).

First, we notice that \( \dim G_{12}^{1}(\Gamma) = \rho(15,1,12) = 7 \). Indeed, if we assume that \( \dim G_{12}^{1}(\Gamma) \geq 8 \), by applying Keem's Theorem (cf. [ACGH], p.200) we would get that \( \Gamma \) possesses a \( g_{1}^{1} \), which is impossible since \( \text{gon}(\Gamma) = 6 \). (In general, if \( Y \subseteq \mathbb{P}^{2} \) is a smooth plane curve, \( \deg(Y) = d \), then \( \text{gon}(Y) = d - 1 \). and the \( g_{d-1}^{0} \) computing the gonality is cut out by the lines passing through a point \( p \in Y \), see [ACGH].) Next, we define the variety

\[
\Sigma = \{(l,q_{1}, \ldots, q_{8}) \in G_{12}^{1}(\Gamma) \times \mathbb{P}^{8} : \alpha^{l}(q_{i}) \geq (0,1), i = 1, \ldots, 8 \}
\]

and denote by \( \pi_{1} : \Sigma \rightarrow G_{12}^{1}(\Gamma) \) and \( \pi_{2} : \Sigma \rightarrow \mathbb{P}^{8} \) the two projections. For any \( l \in G_{12}^{1}(\Gamma) \), the fibre \( \pi_{2}^{-1}(l) \) is finite, hence \( \dim \Sigma = \dim G_{12}^{1}(\Gamma) = 7 \), which shows that \( \pi_{2} \) cannot be surjective and this proves our claim.

**Step 2** There exists a smoothable limit \( g_{15}^{3} \) on \( X \), hence \( \{X\} \in \overline{\mathcal{M}}_{15}^{2} \). We construct \( l \), a limit \( g_{15}^{3} \) on \( X \) as follows: on \( \Gamma \) there is a (unique) \( g_{8}^{2} \) and we consider \( l_{\Gamma} = g_{8}^{2}(p_{1} + \cdots + p_{8}) \), i.e. the \( \Gamma \)-aspect \( l_{\Gamma} \) is obtained from the \( g_{8}^{2} \) by adding the base points \( p_{1}, \ldots, p_{8} \). Clearly \( d^{E_{i}}(p_{i}) = (1,2,3) \) for each \( i \). On \( E_{i} \) we take \( l_{E_{i}} = g_{3}^{3}(12p_{i}) \) for \( i = 1, \ldots, 8 \), where \( g_{3}^{3} \) is a complete linear series of the form \( 2p_{i} + x_{i} \), with \( x_{i} \in E_{i} - \{p_{i}\} \). Furthermore, \( d^{E_{i}}(p_{i}) = (12,13,14) \), so \( l = \{l_{\Gamma}, l_{E_{i}}\} \) is a refined limit \( g_{15}^{3} \) on \( X \). One sees that \( \rho(l_{E_{i}}, \alpha^{E_{i}}(p_{i})) = 1 \) for all \( i \), \( \rho(l_{\Gamma}, \alpha^{\Gamma}(p_{1}), \ldots, \alpha^{\Gamma}(p_{8})) = -15 \), and \( \rho(l) = -7 \). We now prove that \( l \) is dimensionally proper.
Let $\pi_i : \mathcal{C}_i \to \Delta_i$, $\hat{p}_i : \Delta_i \to \mathcal{C}_i$ be the versal deformation space of $(E_i, p_i)$, for $i = 1, \ldots, 8$. There is an obvious isomorphism over $\Delta_i$,

$$G^2_7(\mathcal{C}_i / \Delta_i, (\hat{p}_i, (12, 12, 12))) \simeq G^2_7(\mathcal{C}_i / \Delta_i, (\hat{p}_i, 0)).$$

If $\sigma_i : G^2_7(\mathcal{C}_i / \Delta_i, (\hat{p}_i, 0)) \to \Delta_i$ is the natural projection, then for each $t \in \Delta_i$, the fibre $\sigma_i^{-1}(t)$ is isomorphic to $\pi_i^{-1}(t)$, the isomorphism being given by

$$\pi_i^{-1}(t) \ni q \mapsto 2\hat{p}_i(t) + q \in G^2_7(\pi_i^{-1}(t)).$$

Thus, $G^2_7(\mathcal{C}_i / \Delta_i, (\hat{p}_i, 0))$ is a smooth irreducible surface, which shows that $l$ is dimensionally proper w.r.t. $E_i$. Next, let us consider $\pi : \mathcal{X} \to \Delta$, $\hat{p}_1, \ldots, \hat{p}_8 : \Delta \to \mathcal{X}$, the versal deformation of $(\Gamma, p_1, \ldots, p_8)$. We have to prove that

$$\dim G^2_7(\mathcal{X} / \Delta, (\hat{p}_i, (1.1.1))) = \dim \Delta + \rho l_{l_i}, \alpha^i(p_i)) = 35.$$ 

There is an isomorphism over $\Delta$,

$$G^2_7(\mathcal{X} / \Delta, (\hat{p}_i, (1.1.1))) \simeq G^2_7(\mathcal{X} / \Delta, (\hat{p}_i, 0)).$$

If $\pi_0 : \mathcal{C} \to \mathcal{M}$ is the versal deformation space of $\Gamma$, then we denote by $G^2_7 \to \mathcal{M}$ the scheme parametrizing $g^2_7$'s on curves of genus 15 `nearby' $\Gamma$ (See Section 1.3 for this notation). Clearly $G^2_7(\mathcal{X} / \Delta, (\hat{p}_i, 0)) \simeq G^2_7 \times \mathcal{M} \Delta$, so it suffices to prove that $G^2_7$ has the expected dimension at the point $(\Gamma, g^2_7)$. For this we use Prop.1.3.1. We have that $N_{l_{\mathcal{E}} \mathcal{F}} = \mathcal{O}_{\mathcal{E}}(7), K_{\mathcal{E}} = \mathcal{O}_{\mathcal{E}}(4)$, hence

$$H^1(\Gamma, \mathcal{N}_{l_{\mathcal{E}} \mathcal{F}}) \simeq H^0(\Gamma, \mathcal{O}(3)) \cap = 0,$$

so $l$ is dimensionally proper w.r.t. $\Gamma$ as well. We conclude that $l$ is smoothable.

**Step 3** There exists a smoothable limit $g^2_{20}$ on $X$, that is $[X] \in \mathcal{M}^3_{20}$. First we notice that there is an isomorphism $\Gamma \to G^2_6(\Gamma)$, given by

$$\Gamma \ni \rho \mapsto [g^2_{20} - \rho] \in G^2_6(\Gamma).$$

Consequently, there is a 2-dimensional family of $g^2_{12}$'s on $\Gamma$, of the form $g^2_{12} = g^2_{0} + h^\rho_{0} = 2g^2_{2} - p - q$, where $p, q \in \Gamma$. Pick $l_0 = l_0^p + l_0^q$ with $l_0^p, l_0^q \in G^1_7\{\Gamma\}$, a general $g^2_{12}$ of this type.

We construct $l$, a limit $g^2_{20}$ on $X$, as follows: the $\Gamma$-aspect is given by $l_{\mathcal{E}} = l_0(p_1 + \ldots + p_8)$, and because of the generality of the chosen $l_0$ we have that $\rho(l_{\mathcal{E}}, \alpha^{l_{\mathcal{E}}}(p_1), \ldots, \alpha^{l_{\mathcal{E}}}(p_8)) = -9$.

The $E_i$-aspect is given by $l_{E_i} = g^3_{1}(16p_i)$, where $g^3_{1} = 3p_i + x_i$, with $x_i \in E_i - \{p_i\}$, for $i = 1, \ldots, 8$. It is clear that $\rho(l_{E_i}, \alpha^{l_{E_i}}(p_i)) = 1$ and that $l' = \{l_{\mathcal{E}}, l_{E_i}\}$ is a refined limit $g^2_{20}$ on $X$.

In order to prove that $l'$ is dimensionally proper, we first notice that $l'$ is dimensionally proper w.r.t. the elliptic tails $E_i$. We now prove that $l'$ is dimensionally proper w.r.t. $\Gamma$. As in the previous step, we consider $\pi : \mathcal{X} \to \Delta$, $\hat{p}_1, \ldots, \hat{p}_8 : \Delta \to \mathcal{X}$, the versal deformation
of $(\Gamma, p_1, \ldots, p_8)$ and $\pi_0 : \mathcal{C} \to \mathcal{M}$, the versal deformation space of $\Gamma$. There is an isomorphism over $\Delta$

$$G^3_{20}(\mathcal{X}/\Delta, (\tilde{p}_1, \alpha^{ir}(p_1), \ldots, (\tilde{p}_8, \alpha^{ir}(p_8))) \simeq G^3_{12}(\mathcal{C}/\mathcal{M}) \times_{\mathcal{M}} \Delta.$$ 

It suffices to prove that $G^3_{20} = G^3_{12}(\mathcal{C}/\mathcal{M})$ has a component of the expected dimension passing through $(\Gamma, l_0)$. In this way, the genus 23 problem is turned into a deformation theoretic problem in genus 15. Denote as usual by $\sigma : G^3_{12} \to \mathcal{M}$ the natural projection. According to Prop.1.3.1 it will be enough to exhibit an element $(C, l) \in G^3_{20}$, sitting in the same component as $(\Gamma, l_0)$, such that the linear system $l$ is base point free and simple, and if $\phi_1 : C \to \mathbb{P}^3$ is the map induced by $l$, then $H^1(C, N_{\phi_1}) = 0$. Certainly we cannot take $C$ to be a smooth plane septic because in this case $H^1(C, N_{\phi_1}) \neq 0$, as one can easily see. Instead, we consider the 6-gonal locus in a neighbourhood of the point $[\Gamma] \in \mathcal{M}_{15}$, or equivalently, the 6-gonal locus in $\mathcal{M}$, the versal deformation space of $\Gamma$. One has the projection $G^3_6 \to \mathcal{M}$ and the scheme $G^3_6$ is smooth (and irreducible) of dimension $37 (= 2g + 2d - 5; g = 15, d = 6)$. We denote by

$$\mu : G^3_6 \times_{\mathcal{M}} G^3_6 \to \mathcal{M}. \quad \mu([C, l, l']) = [C].$$

There is a stratification of $\mathcal{M}$ given by the number of pencils: for $i \geq 0$ we define,

$$\mathcal{M}(i)^0 := \{[C] \in \mathcal{M} : C \text{ possesses } i \text{ mutually independent, base-point-free } g_6\text{'s} \}.$$ 

and $\mathcal{M}(i) : = \mathcal{M}(i)^0$. The strata $\mathcal{M}(i)^0$ are constructible subsets of $\mathcal{M}$, the first stratum $\mathcal{M}(1) = \text{Im} (G^3_6)$ is just the 6-gonal locus; the stratum $\mathcal{M}(2)$ is irreducible and $\dim \mathcal{M}(2) = g + 4d - 7 = 32$ (cf. [AC1]). We denote by $\mathcal{M}_{\text{sept}} := m(U_{7,15}) \cap \mathcal{M}$ the closure of the locus of smooth plane septics in $\mathcal{M}$, and by $\mathcal{M}_{\text{oct}} := m(U_{8,15}) \cap \mathcal{M}$ the closure of the locus of curves which are normalizations of plane octics with 6 nodes. Since the Severi varieties $U_{7,15}$ and $U_{8,15}$ are irreducible, so are the loci $\mathcal{M}_{\text{sept}}$ and $\mathcal{M}_{\text{oct}}$. Furthermore $\dim \mathcal{M}_{\text{sept}} = 27$ and $\dim \mathcal{M}_{\text{oct}} = 30$. We prove that $\mathcal{M}_{\text{sept}} \subseteq \mathcal{M}_{\text{oct}}$. Indeed, let us pick $Y \subseteq \mathbb{P}^2$ a smooth plane septic, and $L \subseteq \mathbb{P}^2$ a general line, $L \cdot Y = p_1 + \cdots + p_7$. Denote $Z := C \cup L$, $\deg (Z) = 8, p_6(Z) = 21$. We consider the node $p_7$ unassigned, while $p_1, \ldots, p_6$ are assigned. By using [Ta] Theorem 2.13, there exists a flat family of plane curves $\pi : Z \to B$ and a point $0 \in B$, such that $Z_0 = \pi^{-1}(0) = Z$, while for $0 \neq b \in B$, the fibre $Z_b$ is an irreducible octic with nodes $p_1(b), \ldots, p_6(b)$, and such that $p_i(0) \to p_i$, when $b \to 0$, for $i = 1, \ldots, 6$. If $Z' \to B$ is the family resulting by normalizing the surface $Z$, and $\eta : Z'' \to B$ is the stable family associated to the semistable family $Z' \to B$, then we get that $\eta^{-1}(0) = Y$, while $\eta^{-1}(b)$ is the normalization of $Z_b$ for $b \neq 0$. This proves our contention.

Since $\mathcal{M}_{\text{oct}}$ is irreducible there is a component $A$ of $G^3_6 \times_{\mathcal{M}} G^3_6$, such that $\mu(A) \supseteq \mathcal{M}_{\text{oct}}$. The general point of $A$ corresponds to a curve $C$ and two base-point-free pencils $l', l'' \in G^3_6(C)$ such that if $f' : C \to \mathbb{P}^1$ and $f'' : C \to \mathbb{P}^1$ are the corresponding morphisms, then

$$\sigma = (f', f'') : C \to \mathbb{P}^1 \times \mathbb{P}^1$$

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is birational. Since \( \Gamma \in \mu(\mathcal{A}) \) we can assume that \( [\Gamma, l_0, l_0'] \in \mathcal{A} \). As a matter of fact, we can start the construction of a limit \( g_{12}^1 \) on the genus 23 curve \( X = G \cup E_1 \cup \ldots \cup E_6 \), by taking any pair \((l_0, l_0') \in G_{12}^1(\Gamma) \times G_{12}^1(\Gamma)\), such that \( \dim l_0 - l_0' = 3 \), the argument does not change.

We denote by \( \eta : \mathcal{A} \to \mathcal{G}_{12}^1 \) the map given by \( \eta(C, l, l') := (C, l - l') \). The fact that \( \eta \) maps to \( \mathcal{G}_{12}^1 \) follows from the base-point-free-pencil-trick.

We are going to show that given a general point \([C] \in \mathcal{M}_{\text{gen}}, \) and \([C, l, l'] \in \mu^{-1}([C])\), the condition \( H^1(C, N_{o_1}) = 0 \) is satisfied, hence \( \mathcal{G}_{12}^1 \) is smooth of the expected dimension at the point \((C, l, l')\). This will prove the existence of a component of \( \mathcal{G}_{12}^1 \) passing through \((\Gamma, l_0)\) and having the expected dimension. We take \( \overline{\mathcal{C}} \subseteq \mathbb{P}^2 \), a general point of \( \mathcal{C}_{8,15} \), with nodes \( p_1, \ldots, p_6 \in \mathbb{P}^2 \) in general position. Theorem 3.2 from [AC1] ensures that there exists a plane octic having 6 prescribed nodes in general position. Let \( \nu : \mathcal{C} \to \overline{\mathcal{C}} \) be the normalization map, \( \nu^{-1}(p_i) = q_i + q_i'' \) for \( i = 1, \ldots, 6 \). Choose two nodes, say \( p_1 \) and \( p_2 \), and denote by \( g_i = H - q_i - q_i'' \) and \( h_i = H - q_i + q_i'' \), the linear series obtained by projecting \( \overline{\mathcal{C}} \) from \( p_1 \) and \( p_2 \) respectively. Here \( H \in \nu^*O_{\mathbb{P}^2}(1) \) is an arbitrary line section of \( \mathcal{C} \). The morphism induced by \((g_i, h_i)\) is denoted by \( \phi : \mathcal{C} \to \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \phi_1 = s \circ \phi : \mathcal{C} \to \mathbb{P}^4 \), with \( s : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3 \) the Segre embedding. There is an exact sequence over \( \mathcal{C} \)

\[
0 \to N_{o_1} \to N_{o_1} \to \phi^*N_{\mathbb{P}^1 \times \mathbb{P}^1} \to 0. \tag{1.11}
\]

We can argue as in [AC2] p.473, that for a general \((C, g_i, h_i)\) with \([C] \in \mathcal{M}_{\text{net}}, \) we have \( h^1(C, N_{o_1}) = 0 \). Indeed, let us denote by \( A_0 \) the open set of \( \mathcal{A} \) corresponding to points \((X, l, l')\) such that \( \chi : X \to \mathbb{P}^1 \times \mathbb{P}^1 \), the morphism associated to the pair of pencils \((l, l')\) is birational, and by \( U \subseteq A_0 \) the variety of those points \((X, l, l') \in A_0 \) such that \( H^1(X, N_{\chi}) \neq 0 \). Define

\[
\mathcal{V} := \{ x = (X, l, l', \mathcal{F}, \mathcal{F}') : (X, l, l') \in U, \mathcal{F} \text{ is a frame for } l, \mathcal{F}' \text{ is a frame for } l' \}.
\]

We may assume that for a generic \( x \in U \), the corresponding pencils \( l \) and \( l' \) are base-point-free. Suppose that \( U \) has a component of dimension \( \alpha \). For any \( x \in \mathcal{V} \),

\[
T_x(V) \subseteq H^0(X, N_{\chi}), \quad \text{and} \quad \dim T_x(V) \geq \alpha + 2 \dim \text{PGL}(2) = \alpha + 6.
\]

If \( K_{\chi} \) is the cuspidal sheaf of \( \chi \) and \( N_{\chi}' = N_{\chi}/K_{\chi} \), then according to [AC1] Lemma 1.4, for a general point \( x \in \mathcal{V} \) one has that,

\[
T_x(V) \cap H^0(X, K_{\chi}') = 0.
\]

from which it follows that \( \alpha \leq g - 6 \). If not, one would have that \( h^0(X, N_{\chi}') \geq g + 1 \) and therefore by Clifford's Theorem, \( h^1(X, N_{\chi}') = h^1(X, N_{\chi}) = 0 \), which contradicts the definition of \( U \). Since clearly \( \dim \mathcal{M}_{\text{net}} > g - 6 \), we can assume that \( h^1(C, N_{o_1}) = 0 \), for the general \([C] \in \mathcal{M}_{\text{net}} \). Therefore, by taking cohomology in (1.11), we get that

\[
H^1(C, N_{o_1}) = H^1(C, O_C(2)).
\]
where $O_C(1) = \mathcal{O}_{\mathbb{P}^3}(1)$. By Serre duality,

$$H^1(C, \mathcal{O}_C(2)) = 0 \iff K_C - 2g_1 - 2h_1 = \emptyset.$$  

(1.12)

Since $K_C = 5H - \sum_{i=1}^6 (q'_i + q''_i)$, equation (1.12) becomes

$$H + q'_1 + q''_1 + q'_2 + q''_2 - \sum_{i=3}^6 (q'_i + q''_i) = \emptyset.$$  

(1.13)

If $L = m_1 \ell_2 \subseteq \mathbb{P}^2$, we can write $\nu^*(L) = q'_1 + q''_1 + q'_2 + q''_2 + x + y + z + t$. and (1.13) is rewritten as

$$2H - x - y - z - t - \sum_{i=3}^6 (q'_i + q''_i) = \emptyset.$$  

So, one has to show that there are no conics passing through the nodes $p_3, p_4, p_5$ and $p_6$ and also through the points in $L \cdot C = 2p_1 - 2p_2$. Because $C \in U_{9,15}$ is general we may assume that $x, y, z$ and $t$ are distinct, smooth points of $C$. Indeed, if the divisor $x + y + z + t$ on $C$ does not consist of distinct points, or one of its points is a node, we obtain that $C$ has intersection number 8 with the line $L$ at 5 points or less. But according to [DH], the locus in the Severi variety

$$\{[X] \in U_{d,g} : X \text{ has total intersection number } m + 3 \text{ with a line at } m \text{ points} \}$$

is a divisor on $U_{d,g}$, so we may assume that $C$ lies outside this divisor. Now, if $x, y, z$ and $t$ are distinct and smooth points of $C$, a conic satisfying (1.13) would necessarily be a degenerate one, and one gets a contradiction with the assumption that the nodes $p_3, p_4, p_5$ and $p_6$ of $C$ are in general position.

**Remark:** We have a nice geometric characterization of some of the strata $\mathcal{M}_i$. First, by using Zariski's Main Theorem for the birational projection $\mathcal{G}_6 \to \mathcal{M}(1)$, one sees that $[C] \in \mathcal{M}(1)_{\text{sing}}$ if and only if either $[C] \in \mathcal{M}(2)_{\text{sing}}$ or $C$ possesses a $g_2^1$ such that $	ext{dim } 2g_2 \geq 3$. In the latter case, the $g_2^1$ is a specialization of 2 different $g_6^1$'s in some family of curves, hence $\mathcal{M}(2) = \mathcal{M}(1)_{\text{sing}} \cup \mathcal{G}_2$. As a matter of fact, Coppens has proved that for $4 \leq k \leq [(g + 1)/2]$ and $8 \leq g \leq (k - 1)^2$, there exists a $k$-gonal curve of genus $g$ carrying exactly 2 linear series $g_2^1$'s. Thus, the general point of $\mathcal{M}(2)$ corresponds to a curve $C$ of genus 15, having exactly 2 base-point-free $g_2^1$'s. Furthermore, using Coppens' classification of curves having many pencils computing the gonality (see [Co1]), we have that $\mathcal{M}(6) = \mathcal{M}_{\text{act}}$ and $\mathcal{M}(i) = \mathcal{M}_{\text{ext}}$, for each $i \geq 7$.

Now we are in a position to complete the proof of Theorem 1.1:  

**Proof of Theorem 1.1** According to (1.2), it will suffice to prove that there exists a smooth curve $[X] \in \mathcal{M}_{23}$ which carries a $g_{10}^1$, a $g_{12}^1$, but has no $g_{12}^1$'s. In the proof of Prop.1.5.4 we constructed a stable curve of compact type $[X] \in \overline{\mathcal{M}}_{23}^\text{I}$ such that $[X] \in \overline{\mathcal{M}}_{17}^\text{I} \cap \overline{\mathcal{M}}_{20}^\text{I}$, but $[X] \notin \mathcal{M}_{12}^\text{I}$. If we prove that $[X] \in \mathcal{M}_{17}^\text{I} \cap \mathcal{M}_{20}^\text{I}$, that is, there are smoothings of $X$ which preserve both the $g_{17}^1$ and the $g_{20}^1$, we are done. One can write $\mathcal{M}_{17}^\text{I} \cap \mathcal{M}_{20}^\text{I} = Y_1 \cup \ldots \cup Y_i$.
where \( Y_i \) are irreducible codimension 2 subvarieties of \( \overline{M}_{23} \). Assume that \([X] \in Y_1\). If \( Y_1 \cap \overline{M}_{23} \neq \emptyset \), then \([X] \in Y_1 \cap \overline{M}_{23} \subseteq \overline{M}_{17}^2 \cap \overline{M}_{20}^3\), and the conclusion follows. So we may assume that \( Y_1 \subseteq \overline{M}_{23} - \overline{M}_{23} \). Because \([X] \in \Delta_1 - \bigcup_{i 
eq 1} \Delta_i\), we must have \( Y \subseteq \Delta_1 \). It follows that \( \overline{M}_{17}^2 \cap \Delta_1 \) and \( \overline{M}_{20}^3 \cap \Delta_1 \) have \( Y_1 \) as a common component. According to Prop. 1.4.2, both intersections \( \overline{M}_{17}^2 \cap \Delta_1 \) and \( \overline{M}_{20}^3 \cap \Delta_1 \) are irreducible. Hence \( \overline{M}_{17}^2 \cap \Delta_1 = \overline{M}_{20}^3 \cap \Delta_1 = Y_1 \), which contradicts Prop. 1.5.3. Theorem 1.1 now follows. 

\[ \square \]

### 1.6 The slope conjecture and \( \mathcal{M}_{23} \)

In this final section we explain how the slope conjecture in the context of \( \mathcal{M}_{23} \) implies that \( \kappa(\mathcal{M}_{23}) = 2 \), and then we present evidence for this.

The slope of \( \mathcal{M}_g \) is defined as \( s_g := \inf \{a \in \mathbb{R}_{>0} : a \lambda - \delta \neq 0\} \), where \( \delta = \delta_1 + \delta_2 + \cdots + \delta_{g/2}, \lambda \in \text{Pic}(\mathcal{M}_g) \otimes \mathbb{R} \). Since \( \lambda \) is big, it follows that \( s_g < \infty \). If \( \mathcal{E} \) is the cone of effective divisors in \( \text{Div}(\mathcal{M}_g) \otimes \mathbb{R} \), we define the slope function \( s : \mathcal{E} \to \mathbb{R} \) by the formula

\[
s_D := \inf \{a/b : a, b > 0 \text{ such that } \exists c_i \geq 0 \text{ with } [D] = a \lambda - b \delta - \sum_{i=0}^{g/2} c_i \delta_i \}.
\]

for an effective divisor \( D \) on \( \mathcal{M}_g \). Clearly \( s_g \leq s_D \) for any \( D \in \mathcal{E} \). When \( g + 1 \) is composite, we obtain the estimate \( s_g \leq 6 + 12/(g + 1) \) by using the Brill-Noether divisors \( \mathcal{M}_{g,d}^q \), with \( \rho(g, r, d) = -1 \).

**Conjecture 1 ([HMo])** We have that \( s_g \geq 6 + 12/(g + 1) \) for each \( g \geq 3 \), with equality when \( g + 1 \) is composite.

Harris and Morrison also stated (in a somewhat vague form) that for composite \( g + 1 \), the Brill-Noether divisors not only minimize the slope among all effective divisors, but they also single out those irreducible \( D \in \mathcal{E} \) with \( s_D = s_g \).

The slope conjecture has been proved for \( 3 \leq g \leq 11 \), \( g \neq 10 \) (cf. [HMo], [CR3.4], [Tan]). A strong form of the conjecture holds for \( g = 3 \) and \( g = 5 \): on \( \mathcal{M}_3 \) the only irreducible divisor of slope \( s_3 = 9 \) is the hyperelliptic divisor, while on \( \mathcal{M}_5 \) the only irreducible divisor of slope \( s_5 = 8 \) is the trigonal divisor (cf. [HMo]). Conjecture 1 would imply that \( \kappa(\mathcal{M}_g) = -\infty \) for all \( g \leq 22 \). For \( g = 23 \), we rewrite (1.1) as

\[
nK_{\mathcal{M}_{23}} = \frac{n}{c_{23,r,d}} [\mathcal{M}_{g,d}^r] + 8n \delta_1 + \sum_{i=2}^{11} \frac{(i(23 - i) - 4)}{2} n \delta_i \quad (n \geq 1). \tag{1.14}
\]

(see Section 1.2 for the coefficients \( c_{g,r,d} \)). As Harris and Morrison suggest, we can ask the question whether the class of any \( D \in \mathcal{E} \) with \( s_D = s_g \) is (modulo a sum of boundary components \( \Delta_i \)) proportional to \([\mathcal{M}_{23,d}^r] \), and whether the sections defining (multiples of) \( \mathcal{M}_{23,d}^r \) form a maximal algebraically independent subset of the canonical ring \( R(\mathcal{M}_{23}) \).
If so, it would mean that the boundary divisor $Sn\delta_i + (1/2) \sum_{i=1}^{12} p(i) (23 - i) - 4\delta_i$ is a fixed part of $nK_{M_{23}}$. Moreover, using our independence result for the three Brill-Noether divisors, it would follow that $h^0(M_{23}, nK_{M_{23}})$ grows quadratically in $n$, for $n$ sufficiently high and sufficiently divisible, hence $\kappa(M_{23}) = 2$. We would also have that $\Sigma \cap M_{23} = M_{12} \cap M_{17} \cap M_{20}$, with $\Sigma$ the common base locus of all the linear systems $nK_{M_{23}}$.

Evidence for these facts is of various sorts: first, one knows (cf. [Tan], [CR3]) that $nK_{M_{17}}$ has a large fixed part in the boundary: for each $n \geq 1$, every divisor in $nK_{M_{23}}$ must contain $\Delta_i$ with multiplicity $16n$ when $i = 1, 19n$ when $i = 2$, and $(21 - i)n$ for $i = 3, \ldots, 9$ or 11. The results for $\Delta_1$ and $\Delta_2$ are optimal since these multiplicities coincide with those in (1.14). Note that $|\Delta_i| = 2\delta_i$.

Next, one can show that certain geometric loci in $M_{23}$ which are contained in all three Brill-Noether divisors, are contained in $\Sigma$ as well. The method is based on the trivial observation that for a family $f : X \to B$ of stable curves of genus 23 with smooth general member, if $B.K_{M_{23}} < 0$ (or equivalently, slope(X/B) = $\delta_n/\lambda_B > 13/2$), then $\varphi(B) \subseteq \Sigma$, where $\varphi : B \to \overline{M}_{23}, \varphi(b) = [X_b]$ is the associated moduli map. We have that:

- One can fill up the $d$-gonal locus $M^d_i$ with families $f : X \to B$ of stable curves of genus $g$ such that slope(X/B) = $8 + 1/g$ in the hyperelliptic case, $> 7 + 6/g$ in the trigonal and $> 6 + 12/(g + 1)$ in the tetragonal case (cf. [Sta]). For $g = 23$ it follows that $M^d_1 \subseteq \Sigma$. Note that this result is not optimal if we believe the slope conjecture since we know that $M^d_1 \subseteq M_{17} \cap M_{12} \cap M_{20}$. (The inclusion $M^d_1 \subseteq M_{20}$ is a particular case of a result from [CM].)

- We take a pencil of nodal plane curves of degree $d$ with $f$ assigned nodes in general position such that $(d - 1) - f = 23$, and with $b$ base points, where $4f + b = d^2$. After blowing-up the base points, we have a pencil $Y \to \mathbb{P}^1$ with fibre $[Y] \in M^2_4$. For this pencil $\lambda_B = \chi(O_Y) + 23 - 1 = 23$ and $\delta_B = c_2(Y) + 88 = 91 + b + f$. The condition $\delta_B/\lambda_B > 13/2$ is satisfied precisely when $d \leq 10$, hence taking into account that such pencils fill up $M^2_4$, we obtain that $M^2_{21} \subseteq \Sigma$. Note that $M^2_{21} \subseteq M^2_4$, and as mentioned above, the 8-gonal locus is contained in the intersection of the Brill-Noether divisors.

- In a similar fashion we can prove that $M_{23} \subseteq [2]$, the locus of curves of genus 23 which are double coverings of curves of genus $\gamma$ is contained in $\Sigma$ for $\gamma \leq 5$.

The fact that the slopes of other divisors on $\overline{M}_{23}$ (or on $\overline{M}_g$ for arbitrary $g$) consisting of curves with special geometric characterization, are larger than $6 + 12/(g + 1)$, lends further support to the slope hypothesis. In another paper we will compute the class of other divisors on $\overline{M}_{23}$: the closure in $\overline{M}_{23}$ of the locus

$$\{[C] \in M_{23} : C \text{ possesses a } g^1_{14} \text{ with two different triple points}\}$$

and the closure of the locus

$$\{[C] \in M_{23} : C \text{ has a } g^2_{18} \text{ with a } 5\text{-fold point, i.e. } \exists D \in C^{(5)} \text{ such that } g^2_{18}(-D) = g^1_{14}\}$$

In each case we will show that the slope estimate holds.