The birational geometry of the moduli space of curves
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Chapter 2

The geography of Brill-Noether loci in the moduli space of curves

2.1 Introduction

We start by explaining the meaning of the word 'geography' from the title of this chapter. Many papers have been published where people studied the geography of certain mathematical objects (e.g. surfaces), meaning that they looked at them from the point of view of a naturalist. Our understanding of the term 'geography' is rather different: we study the position of the Brill-Noether loci \( \mathcal{M}_{g,d} \) = \{ \([C] \in \mathcal{M}_g : C \text{ carries a } g^d \} \) on the 'map' of the moduli space of curves. We compare different Brill-Noether loci, look whether they meet transversally (or are in general relative position) inside \( \mathcal{M}_g \), or describe their position with respect to other distinguished loci in \( \mathcal{M}_g \) (e.g. loci of curves sitting on certain surfaces). In most cases we prove that two such loci in \( \mathcal{M}_g \) are as transversal (or intersect as properly) as possible, unless there are some obvious containment relations. The general philosophy is that there are no ways (except the obvious ones) to construct linear series on curves with specific properties.

This chapter consists of relatively independent sections. After Section 2.1 in which we set up the necessary techniques, we ask in Section 2.3 whether the only constraints on the possible \( g^d \)'s on a general \( k \)-gonal curve \( C \) of genus \( g \) are related to the \( g^k \) on \( C \) (as it is the case for hyperelliptic and trigonal curves). We prove that a general \( k \)-gonal curves \( C \) of genus \( g \), where \( k \) is rather high with respect to \( g \), has no other linear series with negative Brill-Noether number except \( g^k \) and \( \mathcal{K}_C - g^k \). In Section 2.4 we show that by imposing two distinct conditions on a curve \( C \) of genus \( g \) (the existence of a pencil \( g^k \) and of an embedding \( C \subseteq \mathbb{P}^r \) of degree \( d \), where \( r \geq 3 \) and \( \rho(g, r, d) = -1 \)), we bring down accordingly the number of moduli such curves depend on. Section 2.5 deals with the problem of computing the gonality of space curves: we show that for a wide range of \( d \) and \( g \) such that \( \rho(g, 3, d) < 0 \) one can find smooth curves \( C \subseteq \mathbb{P}^3 \) of degree \( d \) and genus \( g \) which fill up a component of the Hilbert scheme \( \text{Hilb}_{d,g,3} \) and for which \( \text{gon}(C) = \min(\{(g+3)/2, d-4\}) \); if \( d-4 < ((g+3)/2) \), every pencil computing the gonality is given by the planes through a 4-secant line to \( C \). Finally, in Section 2.6 we ask what
kind of surfaces can contain a Brill-Noether general curve.

2.2 Deformations of maps and smoothing of algebraic space curves

We review some facts about deformations of maps and smoothing of reducible, nodal curves in \( \mathbb{P}^r \). These techniques together with the theory of limit linear series already discussed in the previous chapter will be our main tools throughout this chapter. We start by describing the deformation theory of maps between (possibly singular) complex algebraic varieties. Our main reference is [Ran] (certain aspects of the theory are well treated in [Mod] as well).

Let \( f : X \to Y \) be a morphism between complex projective varieties. We denote by \( \text{Def}(X,f,Y) \) the space of first-order deformations of the map \( f \), while the space of first-order deformations of \( X \) (resp. \( Y \)) is denoted by \( \text{Def}(X) \) (resp. \( \text{Def}(Y) \)). The standard identification \( \text{Def}(X) = \text{Ext}^1(\Omega_X, \mathcal{O}_X) \) is obtained by associating to any first-order deformation \( \tilde{X} \) of \( X \) the class of the extension

\[
0 \to \mathcal{O}_X \to \Omega_{\tilde{X}} \to \mathcal{O}_X \to 0.
\]

The deformation space \( \text{Def}(X,f,Y) \) fits in the following exact sequence:

\[
\text{Hom}_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X) \to \text{Def}(X,f,Y) \to \text{Def}(X) \oplus \text{Def}(Y) \to \text{Ext}^1_{\mathcal{O}_Y}(\Omega_Y, \mathcal{O}_X). \tag{2.1}
\]

The second arrow is given by the natural forgetful maps, the space \( \text{Hom}_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X) = H^0(X, f^*T_Y) \) parametrizes first-order deformations of \( f : X \to Y \) when both \( X \) and \( Y \) are fixed, while for \( A,B \), respectively \( \mathcal{O}_X \) and \( \mathcal{O}_Y \)-modules, \( \text{Ext}^1_{\mathcal{O}_Y}(B,A) \) denotes the derived functor of \( \text{Hom}_{\mathcal{O}_Y}(B,A) = \text{Hom}_{\mathcal{O}_X}(f^*B, A) = \text{Hom}_{\mathcal{O}_X}(B, f_*A) \). Under reasonable assumptions (trivially satisfied when \( f \) is a finite map between nodal curves) one has that \( \text{Ext}^1_{\mathcal{O}_Y}(\Omega_Y, \mathcal{O}_X) = \text{Ext}^1(f^*\Omega_Y, \mathcal{O}_X) \). Using (2.1) it follows that when \( X \) is smooth and irreducible and \( Y \) is rigid (e.g. a product of projective spaces) \( \text{Def}(X,f,Y) = H^0(X, N_f) \), with \( N_f \) the normal sheaf of the map \( f \) (see Chapter 1 for the definition).

Next, we recount some basic facts about moduli spaces of maps from curves to projective varieties. For \( Y \) a smooth, projective variety and \( \beta \in H_2(Y, \mathbb{Z}) \), one can consider the Kontsevich moduli space \( \overline{\mathcal{M}}_g(Y, \beta) \) of stable maps \( f : C \to Y \) from reduced, connected, nodal curves of genus \( g \) to \( Y \), such that \( f_*([C]) = \beta \) (see [FP] for the construction of these moduli spaces). If \( f : C \to Y \) is a point of \( \overline{\mathcal{M}}_g(Y, \beta) \) with \( C \) smooth, \( \deg(f) = 1 \) and \( f \) has no cusps (i.e. it is an immersion), then by Riemann-Roch

\[
\chi(C, N_f) = \dim(Y) (1 - g) + 3g - 3 - \beta \cdot K_Y.
\]

Because \( T_f(\overline{\mathcal{M}}_g(Y, \beta)) = H^0(C, N_f) \), the number

\[
\dim(Y) (1 - g) + 3g - 3 - \beta \cdot K_Y
\]

is called the expected dimension of the Kontsevich moduli space. If there exists a point \( [f] \in \overline{\mathcal{M}}_g(Y, \beta) \), with \( C \) smooth, \( \deg(f) = 1 \) and \( H^1(C, N_f) = 0 \), then every class in \( H^0(C, N_f) \) is unobstructed. \( f \) is an immersion (cf. [AC1] Lemma 1.4) and \( \overline{\mathcal{M}}_g(Y, \beta) \) is
smooth of the expected dimension at the point \([f]\). An irreducible component of \(\overline{M}_g(Y, 3)\) which has the expected dimension and is generically smooth, is said to be regular.

We now describe a few smoothing techniques of algebraic curves in \(\mathbb{P}^r, r \geq 3\) (cf. [HH], [Se2]). Let \(X\) be a nodal curve in \(\mathbb{P}^r\), with \(p_a(X) = g, \text{deg}(X) = d\). We say that \(X\) is smoothable in \(\mathbb{P}^r\) if there exists a flat family of curves \(\{X_t\}\) in \(\mathbb{P}^r\) over a smooth and irreducible base, with the general fibre \(X_t\) smooth while the special fibre \(X_0\) is \(X\). In other words, if \(\text{Hilb}_{d,g,r}\) denotes the Hilbert scheme of curves in \(\mathbb{P}^r\) of degree \(d\) and (arithmetic) genus \(g\), then \(X\) is smoothable in \(\mathbb{P}^r\) if and only if the point \([X]\) belongs to a component of \(\text{Hilb}_{d,g,r}\) whose general member corresponds to a smooth curve.

For \(X \subseteq \mathbb{P}^r\) a nodal curve with normal sheaf \(N_X = N_{X/\mathbb{P}^r}\), one has the exact sequence

\[0 \to T_X \to T_{\mathbb{P}^r} \otimes \mathcal{O}_X \to N_X \to T_X^1 \to 0.\]

where \(T_X^1\) is the Lichtenbaum-Schlessinger cotangent sheaf based on \(\text{Sing}(X)\) and which describes deformations of the nodes of \(X\). The basic smoothing criterion is the following result of Hartshorne and Hirschowitz:

**Proposition 2.2.1** Let \(X \subseteq \mathbb{P}^r\) be a nodal curve. Assume \(H^1(X, N_X) = 0\) and that for each \(p \in \text{Sing}(X)\), the map \(H^0(X, N_X) \to H^0(T_X^1, p)\) is surjective (that is, non-zero). Then \(X\) is smoothable in \(\mathbb{P}^r\) and the Hilbert scheme is smooth of the expected dimension \(\chi(X, N_X) = (r + 1)d - (r - 3)(g - 1)\) at the point \([X]\).

We will be interested in smoothing curves \(X \subseteq \mathbb{P}^r\) which are unions of two curves \(C\) and \(E\) meeting quasi-transversally at a finite set \(\Delta\). For such a curve one has the Mayer-Vietoris sequence

\[0 \to \mathcal{O}_X \to \mathcal{O}_C \oplus \mathcal{O}_E \to \mathcal{O}_\Delta \to 0.\] (2.2)

as well as the exact sequences

\[0 \to \mathcal{O}_E(-\Delta) \to \mathcal{O}_X \to \mathcal{O}_C \to 0.\] (2.3)

and

\[0 \to \Omega_E \to \omega_X \to \Omega_C(\Delta) \to 0.\] (2.4)

where \(\omega_X\) is the dualizing sheaf of \(X\). We will also need the following results:

**Proposition 2.2.2** Let \(C \subseteq \mathbb{P}^r\) be a smooth curve with \(H^1(C, N_C) = 0\).

1. (Sernesi) Let \(H \subseteq \mathbb{P}^r\) be a hyperplane transversal to \(C\) and \(Q \subseteq H\) a smooth, irreducible, rational curve of degree \(r - 1\) meeting \(C\) quasi-transversally in \(\leq r + 2\) points. Then \(X = C \cup Q\) is smoothable and \(H^1(X, N_X) = 0\).

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2. (Ballico-Ellia) Let \( p_1, \ldots, p_{r-2} \in C \) be \( r-2 \) points in general linear position and let \( E \subseteq \mathbb{P}^r \) a smooth rational curve of degree \( r \) which meets \( C \) quasi-transversally at \( p_1, \ldots, p_{r-2} \). Then \( X = C \cup E \) is smoothable and \( H^1(X, N_X) = 0 \).

3. (Ballico-Ellia) Assume \( r = 3 \) and let \( L \subseteq \mathbb{P}^3 \) be a line meeting \( C \) quasi-transversally at \( k \leq 3 \) points. If \( k = 3 \) assume furthermore that not all tangent lines to \( C \) at the points in \( L \cap C \) lie in the same plane. Then \( X = C \cup L \) is smoothable and \( H^1(X, N_X) = 0 \).

2.3 Linear series on \( k \)-gonal curves

For a smooth curve \( C \) of genus \( g \) one defines the gonality sequence \( (d_1, d_2, \ldots, d_r, \ldots) \) by \( d_r := \min \{ d \in \mathbb{Z}_{\geq 1} : d \leq g - 1 \text{ and } d \text{ is a } g_r \text{ on } C \} \). This sequence is strictly increasing and clearly \( d_r \leq rd_1 \). The first term \( d_1 \) is just the gonality of \( C \), while obviously \( d_r = g + r \) for \( r \geq g \), so we will restrict ourselves to the first \( g - 1 \) terms of the sequence. The Brill-Noether Theorem tells us that \( d_r \leq \lceil (g-r+2)/(r+1) \rceil \) and we have optimality when \( C \) is a general curve of genus \( g \). The terms of the gonality sequence can be easily computed for various classes of curves (hyperelliptic, trigonal, smooth plane curves). In order to find the \( d_r \)’s for a curve \( C \), it suffices to look only at the set of linear series

\[
S(C) = \{ [D] : D \in \text{Div}(C), \deg(D) \leq g-1, h^0(D) \geq 2, h^1(D) \geq 2 \}.
\]

This is because any \( g_r \) with \( d \geq 2g-1 \) is non-special, hence \( r = d - g \), while for \( d \leq 2g-2 \) by interchanging if necessary \( g_r \) by \( K_C - g_r \), we land eventually in the range \( d \leq g - 1 \).

We would like to determine the sequence \( (d_1, d_2, \ldots) \) for a general \( k \)-gonal curve of genus \( g \) when \( k < (g+2)/2 \) (i.e. \( \rho(g, 1, k) < 0 \)). Coppens and Martens (cf. [CM]) have investigated how the existence of a \( g_k \) on a curve \( C \) can be used to produce special linear series on \( C \) with negative Brill-Noether number (i.e. the ones you cannot expect to find on a general curve of genus \( g \)). Under certain numerical constraints, a general \( k \)-gonal curve \( C \) of genus \( g \) carries linear series \( g_k = (r-f)g_k + E \) (which we shall call Segre linear series, see the motivation below), where \( 0 < f \leq k - 2 \) and \( E \in \text{Div}(C), E \geq 0 \). For \( r = 2 \) one recovers a famous result of Beniamino Segre (see [AC1]): A general nonhyperelliptic \( k \)-gonal curve \( C \) of genus \( g \) has a linear series \( g_k = g_k + E \) with \( E \geq 0 \), when \( d \geq (g-k+2)/2 \). and which provides a plane model \( \Gamma \) of \( C \) with an ordinary \((d-k)\)-fold singularity \( p \) and nodes as other singularities. The original \( g_k \) can be retrieved by projecting \( \Gamma \) from \( p \). When \( r = 3 \) and \( k \geq 4 \), a general \( k \)-gonal curve of genus \( g \) has a linear series \( g_k = g_k + E \) when \( d \geq (2g+k+6)/3 \). Thus for a general \( [C] \in \mathcal{M}_{g,k}^1 \) one has that \( d_k(C) \leq \lfloor (g+k+3)/2 \rfloor \) and \( d_3(C) \leq \lfloor (2g+k+8)/3 \rfloor \) and we expect to have equality in the case when the Segre linear series have negative Brill-Noether number. This would certainly be the case if the following two expectations were true:

- \( \alpha \) For a general \( [C] \in \mathcal{M}_{g,k}^1 \), the Segre linear series are of minimal degree among those \( g_k = D \in S(C) \) for which \( D - g_k \neq 0 \). (This is known to be true at least when \( r = 2 \) and \( 2k > \lfloor (g+k+3)/2 \rfloor \), see [CKM] Proposition 1.1).
\[ 3) \text{If } g^* \in S(C) \text{ and } g^*_d - g^*_d = \emptyset, \text{ then } \rho(g, r, d) \geq 0. \text{ (This holds when } r = 1 \text{ (cf. [AC1]) and for } k \leq 4 \text{ (cf. [CM]).)} \]

We are going to prove that these expectations hold in the case when the curve \( C \) is of relatively high gonality (but still non-generic):

**Theorem 2.1** Let \( g \) and \( k \) be positive integers such that \(-3 \leq \rho(g, 1, k) < 0\). Assume furthermore that \( k \geq 6 \) when \( \rho(g, 1, k) = -3 \). Then a general \( k \)-gonal curve of genus \( g \) has no \( g^*_d \)'s with negative Brill-Noether number except \( g^*_k \) and \( |K_C - g^*_k| \). In other words, the \( k \)-gonal locus \( M^1_{g,k} \) is not contained in any other proper Brill-Noether locus \( M^r_{g,d} \).

**Remark:** A general \( k \)-gonal curve of genus \( g \) with \( \rho(g, 1, k) < 0 \) has a unique pencil \( g^*_k \) (cf. [AC1]) so there is no ambiguity when we speak of “the \( g^*_k \) of a general \( k \)-gonal curve”.

**Proof:** We will make use of the theory of limit linear series. In each case we construct \( k \)-gonal curves of compact type that do not possess any limit \( g^*_d \) with \( r \geq 2, d \leq g - 1 \) and \( \rho(g, r, d) < 0 \). Using the fact that the \( k \)-gonal locus \( M^1_{g,k} \) is irreducible we obtain the conclusion for a general \([C] \in M^1_{g,k} \). The case \( \rho(g, 1, k) = -1 \) (when \( M^1_{g,k} \) is an irreducible divisor in \( M_g \)) is settled using the curves constructed in the proof of Theorem 1.2. Since the proof goes along the same lines we skip the details.

Assume now that \( \rho(g, 1, k) = -2 \). Because any component of \( M^r_{g,d} \) has codimension \( \geq 3 \) when \( \rho(g, r, d) \leq -3 \) (cf. [Ed2]), it suffices to construct a \( k \)-gonal curve of genus \( g \) having no \( g^*_d \)'s when \( \rho(g, r, d) \in \{-1, -2\} \).

Let us consider the following curve of genus \( 2k \):

![Diagram](image)

where \((C_1, x)\) and \((C_2, y)\) are general pointed curves of genus \( k - 1 \). \( E, \) are elliptic curves and \( x - p \in \text{Pic}^0(E_1) \) is a primitive \( k \)-torsion as it is \( p - y \in \text{Pic}^0(E_2) \). It is straightforward to construct a limit \( g^*_d \) on \( X \): on \( C_1 \) take the pencil \( |kx| \), on \( C_2 \) take the pencil \( |ky| \), on \( E_1 \) the pencil \( \langle kx, kp \rangle \), spanned by \( kx \) and \( kp \), while on \( E_2 \) the pencil \( \langle kp, ky \rangle \).

Assume now that there is a limit \( g^*_d \) on \( X \), say \( l \), with \( r \geq 2, d \leq 2k - 1 \) and \( \rho(g, r, d) < 0 \).

From (1.6), we have that

\[ -1 \geq \rho(l_X) \geq \rho(l_{C_1}, x) + \rho(l_{C_2}, y) + \rho(l_{E_1}, x, p) + \rho(l_{E_2}, p, y). \]

Because of Prop. 1.3.2 one has \( \rho(l_{C_1}, x) \geq 0 \) and \( \rho(l_{C_2}, y) \geq 0 \). Moreover, we have that \( \rho(l_{E_1}, x, p) \geq -1 \) and \( \rho(l_{E_2}, p, y) \geq -1 \). Indeed, if say \( \rho(E_1, x, p) \leq -2 \), then by denoting
by \( \{a_0, \ldots, a_r\} \) the vanishing sequence of \( l_{E_1} \) at \( x \) and by \( \{b_0, \ldots, b_r\} \) that of \( l_{E_2} \) at \( p \), it would follow that for at least 3 indices \( i < j < k \) there are equalities \( a_i + b_{r-j} = a_j + b_{r-j} = a_k + b_{r-k} = d \), from which \( k (a_j - a_i) \) and \( k (a_k - a_i) \), hence \( d \geq a_k \geq a_i + 2k \geq 2k \), which is a contradiction since we assumed \( d < 2k - 1 \).

This implies that there are essentially two cases to consider:

1. \( \rho(l_{C_1}, x) = \rho(l_{C_2}, y) = 0, \rho(l_{E_1}, x, p) = -1, \rho(l_{E_2}, p, y) = 0. \)
2. \( \rho(l_{C_1}, x) = 0, \rho(l_{C_2}, y) = 1, \rho(l_{E_1}, x, p) = \rho(l_{E_2}, p, y) = -1. \)

In both cases \( l \) is a refined limit \( g_f^r \). The other possibilities can either be dismissed right away (when one of the adjusted Brill-Noether numbers is \( \geq 2 \)), or are equivalent to the cases just mentioned.

Let us first settle case 1. By using (1.8),

\[
\sum_{i=0}^{r} (k - 1 - a_i^{l_{E_1}}(x) + i) = \sum_{i=0}^{r} (k - 1 - a_i^{l_{E_1}}(x) + i) = k - 1. \text{ hence }
\]

\[
a_i^{l_{E_1}}(x) \leq k - 1 + i. \text{ and similarly } a_i^{l_{E_2}}(y) \leq k - 1 + i. \text{ for all } i = 0, \ldots, r. \tag{2.5}
\]

Since on \( E_2 \) we have inequalities \( a_i^{l_{E_2}}(p) + a_i^{l_{E_2}}(y) \geq d - 2 \), for all \( i \) (otherwise once again we would clash with the assumption \( d < 2k - 1 \)), we eventually obtain that

\[
a_i^{l_{E_1}}(p) \leq k + i + 1. \text{ for all } i = 0, \ldots, r. \tag{2.6}
\]

Since \( \rho(l_{E_1}, x, p) = -1 \), there must be indices \( i < j \) with \( a_i^{l_{E_1}}(x) + a_{r-i}^{l_{E_1}}(p) = a_j^{l_{E_1}}(x) + a_{r-j}^{l_{E_1}}(p) = d \), from where we get that \( a_j^{l_{E_1}}(x) - a_i^{l_{E_1}}(x) = k \). Then, because of (2.5) and (2.6) we can write

\[
d - k - r + i - 1 \leq d - a_i^{l_{E_1}}(p) = a_i^{l_{E_1}}(x) \leq j - 1.
\]

hence \( 2r + k \geq d \). Combine this with \( \rho(2k, r, d) \geq -2 \) to get that \( k \leq (r^2 + r + 2)/(r - 1) \). But we also have that \( 2k \geq r^2 + r + 1 \) (because \( \rho(2k, r, d) \geq -r \) and \( d \leq 2k - 1 \)). So all in all, we end up with \( r^3 - 2r^2 - 2r - 5 \leq 0 \), which can be possible only for \( r \leq 3 \). When \( r \in \{2, 3\} \), by plugging in one of the previous inequalities we have that \( 8 \leq g = 2k \leq 16 \). But these cases can be disposed of easily. First, notice that when \( g \notin \{10, 12, 16\} \), since \( g + 1 \) is prime, we have no codimension one Brill-Noether condition on \( M_g \). To treat one of the remaining cases when we do have a codimension one Brill-Noether locus, take for example \( g = 14 \) and \( r = 2 \), hence \( d = 11 \). In this case the inequalities (2.6) cannot be improved and this leads to a contradiction: since on \( E_1 \) we have that \( \rho(l_{E_1}, x, p) = -1 \), exactly two of the numbers \( a_i^{(k)}(x) + a_{r-i}^{(k)}(p) \) are equal to 11 while the remaining one is equal to 10. There are three cases and each can be dismissed swiftly.

We turn to case 2. Use again (1.8) to obtain that \( a_i^{l_{E_2}}(y) \leq k + r \). Moreover
\[ a_{r,e}^{l_{E_1}}(y) + a_0^{l_{E_2}}(p) \geq d - 1, \text{ hence } a_i^{l_{E_1}}(p) \leq k + i + 1, \text{ for all } i = 0, \ldots, r. \text{ hence we have obtained again (2.6). while the inequalities } a_i^{l_{E_1}}(x) \leq k - i + 1 \text{ for } i = 0, \ldots, r \text{ still hold, so the previous argument can be repeated here as well.} \]

We treat now the case \( \rho(g,1,k) = -3 \), that is \( g = 2k + 1 \), with \( k \geq 6 \). Note that when \( k = 5 \), Segre’s Theorem gives a \( g_3^2 = \left| g_3^1 + E \right| \), with \( E \geq 0 \), on a general 5-gonal curve of genus 11, i.e. the 5-gonal locus \( M_{11,5}^2 \) is contained in the Brill-Noether divisor \( M_{11,9}^2 \). The idea is the same, but the computations are a bit more cumbersome. We use the following curve:

\[ X := C_1 \cup C_2 \cup E_1 \cup E_2 \cup U. \]

where \((C_1,x)\) and \((C_2,y)\) are general pointed curves of genus \( k - 1 \). The curves \( E, E_1, E_2 \) are all elliptic, and the differences \( x - p_1 \in \text{Pic}^0(E_1), p_1 - p_2 \in \text{Pic}^0(E) \), and \( p_2 - y \in \text{Pic}^0(E_2) \) are all primitive \( k \)-torsions. Just as in the previous case, it is clear that \( X \) possesses a limit \( g_k^d \). Assume now by contradiction that there exists \( l \) a limit \( g_k^d \) on \( X \), with \( r \geq 2, d \leq g - 1 \) and \( \rho(g,r,d) < 0 \). There are many cases to consider, but it is clear that in order to maximize the chances for such a limit \( g_k^d \) to exist, the adjusted Brill-Noether numbers must be as evenly distributed and as close to 0 as possible: a very positive Brill-Noether number on one component, implies by (1.6) very negative Brill-Noether numbers on other components (2-pointed elliptic curves) and this immediately yields a contradiction. We will only treat one case the other being similar. Assume \( \rho(l_{C_1},x) = \rho(l_{C_2},y) = \rho(l_{E_1},x,p_1) = \rho(l_{E_2},p_2,y) = 0 \), and \( \rho(l_{E_1}.p_1,p_2) = -1 \). Then by (1.8) we have that \( a_r^{l_{E_1}}(x) \leq k + r - 1 \) and \( a_r^{l_{E_2}}(y) \leq k + r - 1 \). Since \( a_r^{l_{E_1}}(x) + a_0^{l_{E_2}}(p_1) \geq d - 2 \) and \( a_r^{l_{E_2}}(y) + a_0^{l_{E_2}}(p_2) \geq d - 2 \), we get that

\[ a_i^{l_{E_1}}(p_1) \leq i + k + 1 \text{ and } a_i^{l_{E_2}}(p_2) \leq i + k + 1 \text{, for } i = 0, \ldots, r. \]  \( (2.7) \)

As in the case \( \rho = -2 \), we can conclude from (2.7) that \( 2r + k + 2 \geq d \). This we combine with \( \rho(g,r,d) \geq -r \) to obtain that \( k \leq (r^2 + 3r + 2)/(r - 1) \). Also \( 2k \geq r^2 + r \) (just put together \( d \leq 2k \) and \( \rho(g,r,d) \geq -r \)), and in the end we get that \( r^2 - 7r - 4 < 0 \implies r \leq 4 \). The case \( r = 4 \) can be dismissed though right away, because then all inequalities we have written down become equalities, hence \( g = 21, d = 20 \), and \( \rho(g,r,d) = -4 \), contradiction since we assumed \( \rho(g,r,d) = -1 \). When \( r \leq 3 \) we have that \( k \leq 10 \). In these particular cases however, we can improve the inequalities (2.7) (which we have watered down to obtain an argument working for general \( r \)), and we easily reach a contradiction. \( \square \)

**Remark:** One can try to extend these results for more negative values of \( \rho(g,1,k) \). The cases \( \rho = -4 \) (resp. \( \rho = -5 \)) could be handled by slightly modifying the curves used for
treated the cases $\rho = -2$ (resp. $\rho = -3$): require that the points $x \in C_1$ and $y \in C_2$ are ordinary Weierstrass points instead of general points. We have checked that for $g \leq 23$ Theorem 2.1 still holds when $\rho \in \{-4, -5\}$. For instance we get that the general 10-gonal curve of genus 23 does not possess any $g'_v$'s with $v \geq 2$, $d \leq 22$ and negative Brill-Noether number. In these cases however, computations become horrendous therefore we think that limit linear series cannot provide the full answer to problem 3).

### 2.4 Existence of regular components of moduli spaces of maps to $\mathbb{P}^1 \times \mathbb{P}^r$

In this section we construct regular components of the moduli space $\overline{\mathcal{M}}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$ of stable maps $f : C \to \mathbb{P}^1 \times \mathbb{P}^r$ of bidegree $(k, d)$, in the case $k \geq r + 2$, $d \geq r \geq 3$, and $\rho(g, r, d) < 0$.

The spaces $\mathcal{M}_g(\mathbb{P}^r, d)$ (or the Hilbert schemes $\text{Hilb}_{d,g,r}$ of curves $C \subseteq \mathbb{P}^r$, deg($C$) = $d$, $p_a(C) = g$) have been the subject of much study in the past 20 years. For instance, in the case of curves in $\mathbb{P}^1$ one knows that for each $g$ there is $D(g) \in \mathbb{Z}$ such that for any $d \geq D(g)$, there exists a curve $C \subseteq \mathbb{P}^3$ of genus $g$ and degree $d$, with $H^1(C, N_{C/\mathbb{P}^3}(-2)) = 0$ (so also $H^1(C, N_{C/\mathbb{P}^3}) = 0$). The numbers $D(g)$ satisfy the estimate $\limsup D(g)g^{-2/3} \leq (9/8)^{1/3}$ (cf. [EllH]). Therefore, when (asymptotically) $d \geq g^{2/3}(9/8)^{1/3}$, there are regular components of $\mathcal{M}_g(\mathbb{P}^1, d)$ whose general points correspond to embeddings $C \hookrightarrow \mathbb{P}^3$. In the case $\rho(g, r, d) \geq 0$ there is a unique (regular) component of $\mathcal{M}_g(\mathbb{P}^r, d)$ which dominates $\mathcal{M}_g$ and whose general point corresponds to a non-degenerate map to $\mathbb{P}^r$ (i.e. the image is not contained in a hyperplane). This follows from the fact that $G^d_g(C)$ is irreducible for general $C$ when $\rho(g, r, d) = 1$ (see [ACGH]); when $\rho(g, r, d) = 0$ an extra monodromy argument is needed.

When the target space is $\mathbb{P}^1 \times \mathbb{P}^1$. Arbarello and Cornalba proved that any component of $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^1, (d, h))$, when $2 \leq g, d, h$, having general points corresponding to birational maps $C \to \mathbb{P}^1 \times \mathbb{P}^1$, is regular: as a matter of fact, it is not hard to see that there is exactly one such component. More generally, the methods from [AC1] can be used successfully in order to compute $\dim \mathcal{M}_g(Y, \beta)$ when $Y$ is a smooth surface: if $M \subseteq \mathcal{M}_g(Y, \beta)$ is a component of dimension $\geq g + 1$ and containing a point $[f : C \to Y]$ with $\deg(f) = 1$, then $M$ is regular. One uses here in an essential way the fact that the normal sheaf $N_f$ is of rank 1, hence the Clifford Theorem gives a straightforward condition for the vanishing of $H^1(C, N_f)$, so this techniques cannot be applied for handling moduli spaces of maps to higher dimensional target spaces $Y$.

Although we only treat the case of curves mapping into $\mathbb{P}^1 \times \mathbb{P}^r$ when $r \geq 3$, it will be clear that our methods can be also applied to study regular components of the moduli space of curves sitting on the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2$.

We start our study of moduli spaces of maps into $\mathbb{P}^1 \times \mathbb{P}^r$. Fix integers $g \geq 0$, $d \geq r \geq 3$ and $k \geq 2$, as well as $C$ a smooth curve of genus $g$ with maps $f_1 : C \to \mathbb{P}^1, f_2 : C \to \mathbb{P}^r$, such that $\deg(f_1) = k$, $\deg(f_2(C)) = d$ and $f_2$ is generically injective. Let us denote by $f : C \to \mathbb{P}^1 \times \mathbb{P}^r$ the product map.
There is a commutative diagram of exact sequences

\[
\begin{array}{cccc}
0 & \to & T_C & \to \ f^*(T_{\mathbb{P}^1 \times \mathbb{P}^r}) & \to \ N_f & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \to & T_C \otimes T_C & \to \ f_1^*(T_{\mathbb{P}^1}) \otimes f_2^*(T_{\mathbb{P}^r}) & \to \ N_{f_1} \otimes N_{f_2} & \to 0 \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0
\end{array}
\]

By taking cohomology in the last column, we see that the condition \( H^1(C, N_f) = 0 \) is equivalent with \( H^1(C, N_{f_1}) = 0 \) (trivial). \( H^1(C, N_{f_2}) = 0 \), and

\[
\text{Im}\{\delta_1 : H^0(C, N_{f_1}) \to H^1(C, T_C)\} + \text{Im}\{\delta_2 : H^0(C, N_{f_2}) \to H^1(C, T_C)\} = H^1(C, T_C),
\]

where the condition (2.8) is equivalent (cf. Chapter 1) with

\[
(d\pi_1)_{[f_1]}(T_{[f_1]}(\mathcal{M}_g(\mathbb{P}^1, k))) + (d\pi_2)_{[f_2]}(T_{[f_2]}(\mathcal{M}_g(\mathbb{P}^r, d))) = T_{[c]}(\mathcal{M}_g),
\]

(we assume that the curve \( C \) has no automorphisms, otherwise we work in the versal deformation space of \( C \); it makes no difference). The projections \( \pi_1 : \mathcal{M}_g(\mathbb{P}^1, k) \to \mathcal{M}_g \) and \( \pi_2 : \mathcal{M}_g(\mathbb{P}^r, d) \to \mathcal{M}_g \) are the natural forgetful maps. Slightly abusing the terminology, if \( C \) is a smooth curve and \( (l_1, l_2) \in G_k(C) \times G_k(C) \) is a pair of base point free linear series on \( C \), we say that \( (C, l_1, l_2) \) satisfies (2.9) if \( (C, f_1, f_2) \) satisfies (2.9), where \( f_1 \) and \( f_2 \) are maps associated to \( l_1 \) and \( l_2 \).

We prove the existence of regular components of \( \mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d)) \) inductively, using the following:

**Proposition 2.4.1** Fix positive integers \( g, r, d \) and \( k \) with \( d \geq r \geq 3, k \geq r + 2 \) and \( \rho(g, r, d) < 0 \). Let us assume that \( C \subseteq \mathbb{P}^r \) is a smooth nondegenerate curve of degree \( d \) and genus \( g \), such that \( h^1(C, N_C) = 0, h^0(C, \mathcal{O}_C(1)) = r + 1 \) and the Petri map

\[
\mu_0(C) = \mu_0(C, \mathcal{O}_C(1)) : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1)) \to H^0(C, K_C)
\]

is surjective. Assume furthermore that \( C \) possesses a simple base point free pencil \( \mathfrak{g}_k^1 \) say \( l \) such that \( \mathcal{O}_C(1)(-l) = 0 \) and \( C, l, \mathcal{O}_C(1) \) satisfies (2.9).

Then there exists a smooth nondegenerate curve \( Y \subseteq \mathbb{P}^r \) with \( g(Y) = g + r + 1 \), \( \deg(Y) = d + r \) and a simple base point free pencil \( l' \in G_k^1(Y) \), so that \( Y \) enjoys exactly the same properties: \( h^1(Y, N_Y) = 0, h^0(Y, \mathcal{O}_Y(1)) = r + 1 \). the Petri map \( \mu_0(Y) \) is surjective. \( \mathcal{O}_Y(1)(-l') = 0 \) and \( Y, l', \mathcal{O}_Y(1) \) satisfies (2.9).
Proof: We first construct a reducible $k$-gonal nodal curve $X \subseteq \mathbb{P}^r$, with $p_a(X) = g + r + 1$. $\deg(X) = d + r$, having all the required properties. Then we prove that $X$ can be smoothed in $\mathbb{P}^r$, preserving all properties we want.

Let $f_1 : C \to \mathbb{P}^1$ be the degree $k$ map corresponding to the pencil $l$. The covering $f_1$ is simple (i.e. over each branch point $\lambda \in \mathbb{P}^1$ there is only one ramification point $x \in f_1^{-1}(\lambda)$ and $e_x(f_1) = 2$), and hence the monodromy of $f_1$ is the full symmetric group. Then since $\mathcal{O}_C(1)^{-1}(-l) = \emptyset$, we have that for a general $\lambda \in \mathbb{P}^1$ the fibre $f_1^{-1}(\lambda) = p_1 + \cdots + p_k$ consists of $k$ distinct points in general linear position. Let $\Delta = \{p_0, \ldots, p_{r-2}\}$ be a subset of $f_1^{-1}(\lambda)$ and let $E \subseteq \mathbb{P}^r$ be a rational normal curve $\deg(E) = r$ passing through $p_1, \ldots, p_{r+2}$. Through any $r + 3$ points in general linear position in $\mathbb{P}^r$, there passes a unique rational normal curve, so we have picked $E$ out of a $1$-dimensional family of curves through the chosen points $p_1, \ldots, p_{r+2}$. Let $X := C \cup E$, with $C$ and $E$ meeting quasi-transversally at $\Delta$. Of course $\rho_0(X) = g + r + 1$ and $\deg(X) = d + r$. Note that $\rho(g, r, d) = \rho(g + r + 1, r, d + r)$.

We first prove that $[X] \in M^1_{g+r+1,k}$ (that is, $X$ is $k$-gonal), by constructing an admissible covering of degree $k$ having as domain a curve $X'$ stably equivalent to $X$. Let $X' := X \cup D_{r+3} \cup \ldots \cup D_k$, where $D_i \cong \mathbb{P}^1$ and $D_i \cap X = \{p_i\}$ for $i = r + 3, \ldots, k$. Take $Y := (\mathbb{P}^1)_{r+3} \cup (\mathbb{P}^1)_{k+1}$ a union of two lines identified at $\lambda$. We construct a degree $k$ admissible covering $f' : X' \to Y$ as follows: take $f'_c = f_1 : C \to (\mathbb{P}^1)_1$, $f'_r = f_2 : E \to (\mathbb{P}^1)_2$ a map of degree $r + 2$ sending the points $p_1, \ldots, p_{r+2}$ to $\lambda$, and finally $f'_{\Delta} : D_i \cong (\mathbb{P}^1)_k$ isomorphisms sending $p_i$ to $\lambda$. Clearly $f'$ is an admissible covering, so $X$ which is stably equivalent to $X'$ is a $k$-gonal curve.

Let us consider now the space $\mathcal{H}_{g+r+1,k}$ of Harris-Mumford admissible coverings of degree $k$ (cf. [HM]) and denote by $\pi_1 : \mathcal{H}_{g+r+1,k} \to M_{g+r+1}$ the natural projection which sends a covering to the stable model of its domain. If we assume that $\Aut(C) = \{Id_C\}$ (which we can safely do), then also $\Aut(f') = \{Id_{X'}\}$. $f'$ is a smooth point of $\mathcal{H}_{g+r+1,k}$. We compute the differential of the map $\pi_1$ at $[f']$. We notice that $T_{f'}(\mathcal{H}_{g+r+1,k}) = \Def(X, f', Y) = \Def(X, f, Y)$, where $f = f'_X : X \to Y$. The differential $(d\pi_1)|_{f'}$ is just the forgetful map $\Def(X, f, Y) \to \Def(X)$ and from the sequence (2.1) we get that $\Im(d\pi_1)|_{f'} = u_1^{-1}(\Im u_2)$, where $u_1 : \Def(X) \to \Ext^1(f^*\Omega_Y, \mathcal{O}_X)$ and $u_2 : \Def(Y) \to \Ext^1(f^*\Omega_X, \mathcal{O}_Y)$ are the dual maps of $u_2^* : H^0(X, \omega_X \otimes f^*\Omega_Y) \to H^0(Y, \omega_Y \otimes \Omega_X)$ and $u_2^* : H^0(X, \omega_X \otimes f^*\Omega_Y) \to H^0(Y, \omega_Y \otimes \Omega_X)$ (the last one induced by the trace map $tr : f_*\omega_X \to \omega_Y$). Starting with the exact sequence on $X$.

$$0 \to \Tors(\omega_X \otimes \Omega_X) \to \omega_X \otimes \Omega_X \to \Omega_X^2(\Delta) = \Omega_X^2(\Delta) \to 0,$$

we can write the following commutative diagram of sequences

$$
\begin{array}{ccc}
  & 0 & 0 & 0 \\
  & \downarrow & \downarrow & \downarrow \\
  0 & H^0(\Tors(\omega_X \otimes f^*\Omega_Y)) & H^0(\omega_X \otimes f^*\Omega_Y) & H^0(2K_E - R_1 + \Delta) \\
  \downarrow_{tr_{\Tors}} & \downarrow u_1 & \downarrow & \downarrow \\
  H^0(\Tors(\omega_X \otimes \Omega_X)) & H^0(\omega_X \otimes \Omega_X) & H^0(2K_E - R_2 + \Delta) \\
  & \downarrow_{tr_{\Tors}} & & \\
  & H^0(\omega_X \otimes \Omega_X) & H^0(2K_E + \Delta) \\
\end{array}
$$
where $R_1$ (resp. $R_2$) is the ramification divisor of the map $f_1$ (resp. $f_2$). Taking into account that $H^0(E, 2K_E - R_2 + \Delta) = 0$ and that $H^0(Y, \omega_Y \otimes \Omega_Y) = H^0(\text{Tors}(\omega_Y \otimes \Omega_Y))$, we obtain that

$$\text{Im}(d\pi_1)_f = (H^0(C, 2K_C - R_1 + \Delta) \supset \text{Ker}(u_{\Delta}^2)_{\text{tors}})^-.$$

where $(u_{\Delta}^2)_{\text{tors}} : H^0(\text{Tors}(\omega_X \otimes f^*\Omega_Y)) \to H^0(\text{Tors}(\omega_Y \otimes \Omega_Y))$ is the restriction of $u_{\Delta}^2$. The space $\text{Ker}(u_{\Delta}^2)_{\text{tors}}$ is just a hyperplane in $H^0(\text{Tors}(\omega_X \otimes f^*\Omega_Y)) \cong \mathbb{C}^{r+2}$.

**Remark:** Since $\Delta \in C_{r+2}$ was chosen generically in a fibre of the $g^k_1$ on $C$, it follows from Riemann-Roch that $h^0(C, 2K_C - R_1 + \Delta) = g - 2k + 3 + r = \text{codim}(\mathcal{M}_{g+r+1,k}, \mathcal{M}_{g+r+1})$. If $C$ has only finitely many $g^k_1$'s the fibre of the map $\pi_1 : \mathcal{M}_{g+r+1,k} \to \mathcal{M}_{g+r+1}$ over the point $[X]$ is $(r+1)$-dimensional: the fibre is basically the space of degree $r+1$ maps $f_2 : E \to \mathbb{P}^1$ such that $f_2(p_1) = \ldots = f_2(p_{r+2}) = \lambda$. Moreover, if we assume (in the case $g > 2k - 2$) that $[C]$ is a smooth point of the locus $\mathcal{M}_{g,k}$ (which happens precisely when $C$ has only one $g^k_1$ and $\text{dim}[2g^1] = 2$), then we have for the tangent cone

$$TC_{[X]}(\mathcal{M}_{g+r+1,k}) = \bigcup \{\text{Im}(d\pi_1)_z : z \in \pi_1^{-1}([X])\} = H^0(C, 2K_C - R_1 + \Delta)^-.$$

which shows that $[X]$ is a smooth point of the locus $\mathcal{M}_{g+r+1,k}$.

We compute now the differential

$$(d\pi_2)_f : T_{[X]}(\text{Hilb}_{d+r, g+r+1}) \to T_{[X]}(\mathcal{M}_{g+r+1}),$$

which is the same as the differential at the point $[X] \in \mathbb{P}^r$ of the projection $\pi_2 : \mathcal{M}_{g+r+1}(\mathbb{P}^r, d+r) \to \mathcal{M}_{g+r+1}$. We start by noticing that $X$ is smoothable in $\mathbb{P}^r$ and that $H^1(X, \mathcal{N}_X) = 0$ (apply Prop.2.2.2). We also have that $X$ is embedded in $\mathbb{P}^r$ by a complete linear system, that is $h^0(X, \mathcal{O}_X(1)) = r + 1$. Indeed, on one hand, since $X$ is nondegenerate, $h^0(X, \mathcal{O}_X(1)) > h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = r + 1$: on the other hand from the sequence (2.3) we have that $h^0(X, \mathcal{O}_X(1)) \leq h^0(C, \mathcal{O}_C(1)) = r + 1$.

If $X$ is embedded in $\mathbb{P}^r$ by a complete linear system, we know (cf. Section 1.3) that

$$\text{Im}(d\pi_2)_f = (\text{Im}\mu)(X)^-,$$

where $\mu(X) : \text{Ker}\mu_0(X) \to H^0(X, \omega_X \otimes \Omega_X)$ is the 'derivative' of the Petri map $\mu_0(X) : H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \omega_X(-1)) \to H^0(X, \omega_X)$. We compute the kernel of $\mu_0(X)$ and show that $\mu_0(X)$ is surjective in a way that resembles the proof of Prop.2.3 in [Se2].

From the sequence (2.4) we obtain $H^0(X, \omega_X) = H^0(C, K_C + \Delta)$, while from (2.3) we have that $H^0(X, \mathcal{O}_X(1)) = H^0(E, \mathcal{O}_E(1))$ (keeping in mind that $H^0(C, \mathcal{O}_C(1)(-\Delta)) = 0$, as $p_1, \ldots, p_{r+2}$ are in general linear position). Finally, using (2.4) again, we have that $H^0(X, \omega_X(-1)) = H^0(C, K_C(-1)+\Delta)$. Therefore we can write the following commutative
It follows that $\ker\mu_0(C) \subseteq \ker\mu_0(X)$. By using Corollary 1.6 from [CR], our assumptions $(\mu_0(C) \text{ surjective and } \text{card}(\Delta) \geq 4)$ enable us to conclude that $\mu_0(X)$ is surjective too. Then $\ker\mu_0(C) = \ker\mu_0(X)$ for dimension reasons, hence also $\operatorname{Im}_{\mu_1}(X) = \operatorname{Im}_{\mu_1}(C) \subseteq H^0(C, 2K_C) \subseteq H^0(X, \omega_X \otimes \Omega_X)$. We thus get that $\operatorname{Im}(d\pi_2)_{|X} = (\operatorname{Im}\mu_1(X))^{-} = (\operatorname{Im}\mu_1(C))^{-}.$

The assumption that $(C, f_1, f_2)$ satisfies (2.9) can be rewritten by passing to duals as

$$H^0(C, 2K_C - R_1)^{-} + (\operatorname{Im}_{\mu_1}(C))^{-} = H^1(C, T_C) \iff H^0(C, 2K_C - R_1) \cap \operatorname{Im}_{\mu_1}(C) = 0.$$

Then it follows that $\operatorname{Im}_{\mu_1}(X) \cap (H^0(C, 2K_C - R_1 + \Delta) = \text{Ker}((u_2')_{|\text{tor}})) = 0$, which is the same thing as

$$(d\pi_1)_{|f_1} (T_{f_1}([\overline{\mathcal{H}}_{g+r+1,k}]) + (d\pi_2)_{|X \rightarrow \mathbb{Z}^n} (T_{X \rightarrow \mathbb{Z}^n}([\overline{\mathcal{M}}_{g+r-1}((\mathbb{P}^{r}, d + r))]) = \operatorname{Ext}^1(\Omega_X, \mathcal{O}_X).$$

(2.11)

This means that the images of $\overline{\mathcal{H}}_{g+r+1,k}$ under the map $\pi_1$ and of $\overline{\mathcal{M}}_{g+r-1}((\mathbb{P}^{r}, d + r))$ under the map $\pi_2$, meet transversally at the point $[X] \in \overline{\mathcal{M}}_{g+r+1}$.

Claim: The curve $X$ can be smoothed in such a way that the $g_1$ and the very ample $g_{d+r}$ are preserved (while (2.11) is an open condition on $\overline{\mathcal{H}}_{g+r+1,k} \times \overline{\mathcal{M}}_{g+r+1}((\mathbb{P}^{r}, d + r))$).

Indeed, the tangent directions that fail to smooth at least one node of $X$ are those in $\bigcup_{i=1}^{r+2} H^0(\text{Tors}_{p_i}(\omega_X \otimes \Omega_X))^\perp$, whereas the tangent directions that preserve both the $g_1$ and the $g_{d+r}$ are those in

$$((\operatorname{Im}_{\mu_1}(C) + H^0(C, 2K_C - R_1 + \Delta)) \subseteq \ker(u_2')_{\text{tor}})^\perp.$$

Since obviously $H^0(\text{Tors}_{p_i}(\omega_X \otimes \Omega_X)) \not\subseteq \ker(u_2')_{\text{tor}}$ for $i = 1, \ldots, r + 2$, by moving in a suitable direction in the tangent space at $[f']$ of $\pi_1^{-1}\pi_2([\overline{\mathcal{M}}_{g+r+1}((\mathbb{P}^{r}, d + r))])$, we finally obtain a curve $Y \subseteq \mathbb{P}^{r}$ with $g(Y) = g + r + 1, \deg(Y) = d + r$ and satisfying all the required properties.

In order to use Prop.2.4.1 as the inductive step $(g, d) \mapsto (g + r + 1, d + r)$ in the construction of regular components of $\mathcal{M}_g((\mathbb{F}^1 \times \mathbb{P}^r, (k, d))$, we need curves $C \subseteq \mathbb{P}^r$ with all the properties listed in the statement of the Proposition (so that we can start the induction). We are able to construct such curves when $\rho(g, r, d) = -1$, i.e., when $\mathcal{M}_g^r$ is a divisor in $\mathcal{M}_g$. 40
Theorem 2.2 Let $r \geq 3, s \geq (2r + 1)/(r - 1)$ and $k \geq 3$ be integers such that 

$$(rs + s - 1)/2 \leq k \leq rs - r - 1.$$ 

Then for any integers $d, g$ such that $\rho(g, r, d) = -1$ and $g \geq (r + 1)(s - 1) - 1$ there exists a regular component of the moduli space of maps $\mathcal{M}_0(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$.

Remarks: 1. Theorem 2.2 actually provides regular components of the Hilbert scheme of curves of bidegree $(k, d)$ in $\mathbb{P}^1 \times \mathbb{P}^r$, where $k$ and $d$ are as above.

2. In the case $g = 23$ (extensively treated in Chapter 1), the theorem provides regular components of $\mathcal{M}_{23}(\mathbb{P}^1 \times \mathbb{P}^3, (k, 20))$ when $k \geq 8$.

Proof: We set $g_0 = (r + 1)(s - 1) - 1$ and $d_0 = rs - 1$. One checks that $\rho(g_0, r, d_0) = -1$ and any solution $(g, d)$ of the equation $\rho(g, r, d) = -1$ with $g \geq g_0$, can be obtained from $(g_0, d_0)$ by applying several times the transformation $(g, d) \to (g + r + 1, d + r)$. According to Prop.2.4.1 it suffices to construct a smooth curve $C \subset \mathbb{P}^r$ of genus $g_0$ and degree $d_0$, with $h^1(C, N_C) = 0, h^0(C, \mathcal{O}_C(1)) = r + 1$. having the Petri map $\mu_0(C)$ surjective, and also carrying a simple base point free pencil $g_k^1$ such that $2g_k^1$ is not-special and $|\mathcal{O}_C(1)|(-g_k^1) = 0$.

For such a triple $(C, g_k^1, \mathcal{O}_C(1))$, condition (2.9) also required in Prop.2.4.1 is immediately satisfied: if $f_1 : C \to \mathbb{P}^1$ is the map corresponding to $g_k^1$, we know (cf. Section 1.3) that $(d\pi_1)_{|f_1^*: (\mathcal{M}_{g_0}(\mathbb{P}^1, k))} = H^1(C, K_C - 2Z)^{\perp} = H^1(C, T_C)$, because $|2Z|$ is non-special. so (2.9) follows at once.

It is more convenient to replace the projection $\mathcal{M}_{g_0}(\mathbb{P}^1, k) \to \mathcal{M}_{g_0}$ by the surjective proper map $\pi : G_k^1 \to \mathcal{M}_{g_0}$, given by $\pi(C, l) = [C]$, where $l \in G_k^1(C)$. Of course $\pi$ does not exist quite as it stands, instead one should replace $\mathcal{M}_{g_0}$ by a finite cover over which the universal curve has a section, but we can safely ignore this minor nuisance. The map $\pi$ is surjective (and with connected fibres) because $\rho(g_0, 1, k) \geq r - 1$. Theorem 6.1 from [Se2] ensures the existence of an irreducible, smooth, open subset $U$ of $\mathcal{M}_{g_0}(\mathbb{P}^r, d_0)$. of the expected dimension, such that all points of $U$ correspond to embeddings of smooth curves $C \subset \mathbb{P}^r$, with $h^1(C, N_C) = 0, h^0(C, \mathcal{O}_C(1)) = r + 1$ and $\mu_0(C)$ surjective. Since $\mathcal{M}_{g_0, d_0}^r$ is irreducible (cf. Chapter 1), it follows that the natural projection $\pi_2 : U \to \mathcal{M}_{g_0, d_0}^r$ is dominant.

We now find a curve $C$ having the properties listed above. For a start, we notice that it is enough to find one curve $[C_0] \in \mathcal{M}_{g_0, d_0}^r$ possessing a complete, simple, base point free $g_k^1$ such that $2g_k^1$ is non-special, because then, by semicontinuity we get the same properties for a general point of $U$. To find one particular such curve we proceed as follows: take $C_0$ a general $(r + 1)$-gonal curve of genus $g_0$. These curves will have rather few moduli ($r + 1 < [(g + 3)/2]$) but we still have that $[C_0] \in \mathcal{M}_{g_0, d_0}^r$. Indeed, according to [CM] p. 348. we can construct a complete, birationally very ample $g_{d_0}^r = g_{r+1}^1 + F$ on $C_0$, where $F$ is an effective divisor on $C_0$ with $h^0(C_0, F) = 1$. Using Corollary 2.2.3 from [CM] we find that $C_0$ also possesses a complete, simple, base-point-free $g_k^1$ which is not composed with the $g_{r+1}^1$ computing gon$(C_0)$, and such that $2g_k^1$ is non-special. Since these are open conditions they will hold generically along a component of $G_k^1(C_0)$. Applying semicontinuity, for a general element $[C] \in \mathcal{M}_{g_0, d_0}^r$ (hence also for a general element $[C] \in U$) the
variety $G^i_k(C)$ will contain a component $A$ with general point $l \in A$ being simple, base point free and with $2l$ non-special.

We claim that there exists a pencil $l \in A$ having the properties listed above and moreover $O_C(1)(-l) = \emptyset$. Suppose not. Then if we denote by $V^{-1}_{d_0-k}(O_C(1))$ the variety of effective divisors of degree $d_0 - k$ on $C$ imposing $\leq r-1$ conditions on $O_C(1)$; we have that

$$\dim V^{-1}_{d_0-k}(O_C(1)) \geq \dim A \geq \rho(g_0, 1, k) \geq r - 1.$$  

the last inequality being the only point where we need the assumption $k \geq (rs + s - 1)/2$. Therefore $C \subseteq P^r$ has at least $\infty_{r-1}$ $(d_0 - k)$-secant $(r - 2)$-planes. Hence also at least $\infty_{r-1} r$-secant $(r - 2)$-planes (because $d_0 - k \geq r$). This last statement clearly contradicts the Uniform Position Theorem (see [ACGH], p. 112). All in all, the general point $[C] \in U$ enjoys all properties required to make Prop.2.4.1 work.

Remarks: 1. We could apply Prop.2.4.1 and get regular components of the moduli space $\mathcal{M}_g(P^1 \times P^1, (k, d))$ for lower values of $\rho(g, r, d)$ (and not only when $\rho(g, r, d) = -1$). If we knew that the $(r + 1)$-gonal locus $\mathcal{M}^1_{g, r-1}$ is contained in every component of $\mathcal{M}^r_{g, d}$ (or at least in a component of $\mathcal{M}^r_{g, d}$ with the expected number of moduli). No such result appears to be known at the moment (except in the case $\rho(g, r, d) = -1$).

2. Let us fix $g, k$ such that $\rho(g, 1, k) > 0$. One knows (cf. [ACGH]) that if $l \in G^i_k(C)$ is a complete, base point free pencil, then dim $T_l(G^i_k(C)) = \rho(g, 1, k) + h^1(C, 2l)$. Therefore if $A$ is a component of $G^1_k(C)$ such that dim $A = \rho(g, 1, k)$ and the general $l \in A$ is base point free such that $2l$ is special, then $A$ is nonreduced. We ask the following question: what is the dimension of the locus

$$M := \{[C] \in \mathcal{M}_g : \text{every component of } G^1_k(C) \text{ is nonreduced} \}?$$

A result of Coppens (cf. [Co4]) says that for a curve $C$, if the scheme $W^1_k(C)$ is reduced and of dimension $\rho(g, 1, k)$, then the scheme $W^1_{k+1}(C)$ is reduced too and of dimension $\rho(g, 1, k+1)$. Therefore it would make sense to determine dim $M$ when $\rho(g, 1, k) \in \{0, 1\}$ (depending on the parity of $g$). We suspect that $M$ depends on very few moduli. A suitable upper bound for dim $M$ would rule out the possibility of a component of $\mathcal{M}^r_{g, d}$ being contained in $M$ (we have the lower bound $3g - 3 + \rho(g, r, d)$ for all components of $\mathcal{M}^r_{g, d}$) and we could apply Prop.2.4.1 without having to resort to Corollary 2.2.3 from [CKM].

### 2.5 The gonality of space curves

#### 2.5.1 Preliminaries

The gonality of a curve is perhaps the second most natural invariant of a curve: it gives an indication of how far from being rational a curve is, in a way different from what the genus does. For $g \geq 3$ we consider the stratification of $\mathcal{M}_g$ given by gonality:

$$\mathcal{M}^1_{g, 2} \subseteq \mathcal{M}^1_{g, 3} \subseteq \ldots \subseteq \mathcal{M}^1_{g, k} \subseteq \ldots \subseteq \mathcal{M}_g.$$
where \( M_{g,k} = M_g \) for \( k \geq \lceil (g + 3)/2 \rceil \). The number \( \lceil (g + 3)/2 \rceil \) is thus the generic gonality for curves of genus \( g \). We want to study the relative position of the Brill-Noether loci \( M_{g,d} \) (with \( r \geq 3, \rho(g, r, d) < 0 \)) and the \( k \)-gonal loci \( M_{g,k} \) (where \( k < (g + 2)/2 \)). More precisely, we would like to know the gonality of a general point of \( M_{g,d} \). Since the geometry of the loci \( M_{g,d} \) is, as we already pointed out in Section 1.2, very messy (existence of many components, some unreduced and/or of unexpected dimension), we will content ourselves with computing \( \text{gon}(C) \) when \( [C] \) is a general point of a 'genuine' component of \( M_{g,d} \) (i.e. a component which is generically smooth, with general point corresponding to a curve with a very ample \( \mathcal{O}_C^d \)).

The same problem for \( r = 2 \) has already been solved by M. Coppens (cf. [Co3]):

**Proposition 2.5.1** Let \( \nu : C \to \Gamma \) be the normalization of a general, irreducible plane curve of degree \( d \) with \( \delta = g - (d - 1) \) nodes. Assume that \( 0 < \delta < (d^2 - 7d + 18)/2 \). Then \( \text{gon}(C) = d - 2 \).

**Remarks:**
1. The result says that there are no \( g_{d-1}^1 \)'s on \( C \). On the other hand a \( g_{d-2}^1 \) is given by the lines through a node of \( \Gamma \).
2. The condition \( \delta < (d^2 - 7d + 18)/2 \) is equivalent with \( \rho(g, 1, d - 3) < 0 \). This is the range in which the problem is non-trivial: if \( \rho(g, 1, d - 3) \geq 0 \), the Brill-Noether Theorem provides \( g_{d-3}^1 \)'s on \( C \).

For \( r \geq 3 \) we could expect a similar result. Let \( C \subseteq \mathbb{P}^e \) be a suitably general smooth curve of genus \( g \) and degree \( d \), with \( \rho(g, r, d) < 0 \). We can always assume that \( d \leq g - 1 \) (by duality \( g_d^1 \to K_{C - g_d^1} \) we can always land in this range). One can expect that a \( g_k^1 \) computing \( \text{gon}(C) \) is of the form \( g_d^1(-D) = \{E - D : E \in g_d^1, E \geq D \} \) for some effective divisor \( D \) on \( C \). Since the expected dimension of the variety of \( e \)-secant \((r - 2)\)-plane divisors

\[
\mathcal{V}^{r-1} = \{D \in C, \dim g_d^1(-D) \geq 1\}
\]

is \( 2r - 2 - e \) (cf. [ACGH]), we may ask whether \( C \) has finitely many \((2r - 2)\)-secant \((r - 2)\)-planes (and no \((2r - 1)\)-secant \((r - 2)\)-planes at all). This is known to be true for curves with general moduli. that is, when \( \rho(g, r, d) \geq 0 \) (cf. [Hirsch]): for instance a smooth curve \( C \subseteq \mathbb{P}^3 \) with general moduli has only finitely many 4-secant lines and no 5-secant lines. However, no such principle appears to be known for curves with special moduli.

**Definition:** We call the number \( \min(d - 2r + 2, \lceil (g + 3)/2 \rceil) \) the **expected gonality** of a smooth nondegenerate curve \( C \subseteq \mathbb{P}^e \) of degree \( d \) and genus \( g \).

The main result of this section is the following:

**Theorem** Let \( g \geq 5 \) and \( d \geq 8 \) be integers, \( g \) odd, \( d \) even, such that \( d^2 > 8g, 4d < 3g + 12, d^2 - 8g + 8 \) is not a square and either \( d \leq 18 \) or \( g < 4d - 31 \). If

\[
(d', g') \in \{(d, g), (d + 1, g + 1), (d + 1, g + 2), (d + 2, g + 3)\},
\]

then there exists a regular component of \( \text{Hilb}_{d' g' - 3} \) whose general point \([C']\) is a smooth curve such that \( \text{gon}(C') = \min(d' - 4, \lceil (g' + 3)/2 \rceil) \).

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One can approach this problem from a different angle: find recipes to compute the gonality of various classes of curves $C \subseteq \mathbb{P}^r$. Our knowledge in this respect is very scant: we know how to compute the gonality of extremal curves $C \subseteq \mathbb{P}^r$ (that is, curves attaining the Castelnuovo bound, see [ACGH]), and the gonality of complete intersections in $\mathbb{P}^3$ (cf. [Ba]): If $C \subseteq \mathbb{P}^3$ is a smooth complete intersection of type $(a,b)$ then $\text{gon}(C) = ab - 1$. 

where $l$ is the degree of a maximal linear divisor on $C$. Hence an effective divisor $D \subseteq C$ computing $\text{gon}(C)$ (that is $\deg(D) = \text{gon}(C)$ and $h^0(C, D) \geq 2$), is residual to a linear divisor of degree $l$ in a plane section of $C$. Of course, we know $\text{gon}(C)$ in a few other cases: It is a classical result that the gonality of a smooth plane $C$ curve of degree $d$ is $d - 1$ and every $g_{d-1}$ on $C$ is of the form $|\mathcal{O}_C(1)(-p)|$, where $p \in C$. If $C$ is a smooth curve of type $(a,b)$ on a smooth quadric surface in $\mathbb{P}^3$, then $\text{gon}(C) = \min(a,b)$, i.e. the gonality is computed by a ruling. One gets a similar result for a curve sitting on a Hirzebruch surface. Finally, in [Pa] there is a rather surprising lower bound for the gonality of a smooth curve $C \subseteq \mathbb{P}^r$ in terms of the Seshadri constant of $C$, which is an invariant measuring the positivity of $\mathcal{O}_{\mathbb{P}^r}(1)$ in a neighbourhood of $C$.

### 2.5.2 Linear systems on smooth quartic surfaces in $\mathbb{P}^3$

We recall a few basic facts about linear systems on $K3$ surfaces (cf. [SD]). Let $S$ be a smooth $K3$ surface. For an effective divisor $D \subseteq S$, we have $h^1(S, D) = h^0(D, \mathcal{O}_D) - 1$. If $C \subseteq S$ is an irreducible curve then $H^1(S, C) = 0$, and by Riemann-Roch we have that

$$\dim|C| = 1 + \frac{C^2}{2} = p_a(C).$$

In particular $C^2 \geq -2$ for every irreducible curve $C$.

**Proposition 2.5.2** Let $S$ be a K3 surface. We have the following equivalences:

1. $C^2 = -2 \iff \dim C = 0 \iff C$ is a smooth, rational curve.
2. $C^2 = 0 \iff \dim C = 1 \iff p_a(C) = 1$.

For a $K3$ surface one also has a 'strong Bertini' Theorem:

**Proposition 2.5.3** Let $\mathcal{L}$ be a line bundle on a $K3$ surface $S$. Then $\mathcal{L}$ has no base points outside its fixed components. Moreover, if $\text{deg} \mathcal{L} = 0$ then either

- $\mathcal{L}^2 > 0$, $h^1(S, \mathcal{L}) = 0$ and the general member of $|\mathcal{L}|$ is a smooth, irreducible curve of genus $\mathcal{L}^2/2 + 1$, or

- $\mathcal{L}^2 = 0$ and $\mathcal{L} = \mathcal{O}_S(kE)$, where $k \in \mathbb{Z}_{>1}$, $E \subseteq S$ is an irreducible curve with $p_a(E) = 1$. We have that $h^0(S, \mathcal{L}) = k + 1$, $h^1(S, \mathcal{L}) = k - 1$ and all divisors in $\mathcal{L}$ are of the form $E + \cdots + E_k$ with $E_i \sim E$.  

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We are interested in space curves sitting on K3 surfaces and the starting point is Mori’s Theorem (cf. [Mo]): if \( d > 0, g \geq 0 \), there is a smooth curve \( C \subseteq \mathbb{P}^3 \) of degree \( d \) and genus \( g \), lying on a smooth quartic surface \( S \), if and only if (1) \( g = d^2/8 + 1 \), or (2) \( g < d^2/8 \) and \( (d, g) \neq (5, 3) \). Moreover, we can choose \( S \) such that \( \text{Pic}(S) = \mathbb{Z}H = \mathbb{Z}(4/d)C \) in case (1) and such that \( \text{Pic}(S) = \mathbb{Z}H : \mathbb{Z}C \), with \( H^2 = 4, C^2 = 2g - 2 \) and \( H \cdot C = d \), in case (2). In each case \( H \) denotes a plane section of \( S \). Note that from the Hodge Index Theorem one has the necessary condition

\[(C \cdot H)^2 - H^2C^2 = d^2 - 8(g - 1) \geq 0.\]

We will repeatedly use the following observation:

**Proposition 2.5.4** Let \( S \subseteq \mathbb{P}^3 \) be a smooth quartic surface with a smooth curve \( C \subseteq S \) such that \( \text{Pic}(S) = \mathbb{Z}H : \mathbb{Z}C \) and assume that \( S \) has no \((-2)\) curves. For a divisor \( D \subseteq S \) we have that:

1. \( D \) is effective \( \iff D^2 \geq 0 \) and \( D \cdot H > 2 \).

2. If \( D^2 = 0 \) and \( D \cdot H > 2 \), then \( D = kE \), where \( E \) is an irreducible curve of genus 1 and \( h^0(S, D) = k + 1 \). \( h^1(S, D) = k - 1 \).

3. If \( D^2 > 0 \) and \( D \cdot H > 2 \), then the general element of \( D \) is smooth and irreducible.

**Remarks:**

a) The first part of Proposition 2.5.4 is based on the fact that if \( D \subseteq S \) is a curve with \( \deg(D) = D \cdot H \leq 2 \), then \( h^0(S, D) = 1 \), i.e. \( D \) is isolated. But every isolated curve is a \((-2)\) curve and we have assumed that there are no such curves on \( S \).

b) If \( S \subseteq \mathbb{P}^3 \) is a smooth quartic surface with Picard number 2 as above, \( S \) has no \((-2)\) curves when the equation

\[2m^2 + mnd + (g - 1)n^2 = -1.\]

has no solutions \( m, n \in \mathbb{Z} \). This is the case for instance when \( d \) is even and \( g \) is odd.

### 2.5.3 Brill-Noether special linear series on curves on K3 surfaces

The study of special linear series on curves lying on K3 surfaces began with Lazarsfeld’s proof of the Brill-Noether-Petri Theorem (cf. [La]). He noticed that there is no Brill-Noether type obstruction to embed a curve in a K3 surface: if \( C_0 \subseteq S \) is a smooth curve of genus \( g \geq 2 \) on a K3 surface such that \( \text{Pic}(S) = \mathbb{Z}C_0 \), then the general curve \( C \in C_0 \) satisfies the Brill-Noether-Petri Theorem, that is, for any line bundle \( A \) on \( C \), the Petri map \( \mu_0(C,A): H^0(C,A) \rightarrow H^0(C,K_C \cdot A^\vee) \rightarrow H^0(C,K_C) \) is injective. We mention that Petri’s Theorem implies (trivially) the Brill-Noether Theorem.

The general philosophy when studying linear series on a K3-section \( C \subseteq S \) of genus \( g \geq 2 \) is that the type of a Brill-Noether special \( g^r_d \) often does not depend on \( C \) but only on its linear equivalence class in \( S \), i.e. a \( g^r_d \) on \( C \) with \( \rho(g, r, d) < 0 \) is expected to propagate to all smooth curves \( C' \in C \). This expectation, in such generality, is perhaps
a bit too optimistic, but it was proved to be true for the Clifford index of a curve (see [GL]): for $C \subseteq S$ a smooth $K3$-section of genus $g \geq 2$, one has that $\text{Cliff}(C'') = \text{Cliff}(C)$ for every smooth curve $C'' \in C'$. Furthermore, if $\text{Cliff}(C') < \lfloor (g-1)/2 \rfloor$ (the generic value of the Clifford index), then there exists a line bundle $L$ on $S$ such that for all smooth $C'' \in C'$ the restriction $L|_{C''}$ computes $\text{Cliff}(C')$. Recall that the Clifford index of a curve $C$ of genus $g$ is defined as

$$\text{Cliff}(C) := \min \{ \text{Cliff}(D) : D \in \text{Div}(C) \land h^0(D) \geq 2, h^1(D) \geq 2 \}.$$ 

where for a divisor $D$ on $C$, we have $\text{Cliff}(D) = \deg(D) - 2(h^0(D) - 1)$. Note that in the definition of $\text{Cliff}(C)$ the condition $h^1(D) \geq 2$ can be replaced with $\deg(D) \leq g - 1$.

Another invariant of a curve is the Clifford dimension of $C$ defined as

$$\text{Cliff-dim}(C) := \min \{ r \geq 1 : \exists \mathfrak{g}^r_d \text{ on } C \text{ with } d \leq g - 1 \text{, such that } d - 2r = \text{Cliff}(C) \}.$$ 

Curves with Clifford dimension $\geq 2$ are rare; smooth plane curves are precisely the curves of Clifford dimension 2, while curves of Clifford dimension 3 occur only in genus 10 as complete intersections of two cubic surfaces in $\mathbb{P}^4$.

Harris and Mumford during their work in [HM] conjectured that the gonality of a $K3$-section should stay constant in a linear system: if $C \subseteq S$ carries an exceptional $\mathfrak{g}^1_d$, then every smooth $C'' \in C'$ carries an equally exceptional $\mathfrak{g}^1_d$. This conjecture was later disproved by Donagi and Morrison (cf. [DMo]) who showed that the gonality can vary in a linear system: Consider the following situation: let $\pi : S \rightarrow \mathbb{P}^2$ be a $K3$ surface, double cover of $\mathbb{P}^2$ branched along a smooth sextic and let $L = \pi^*\mathcal{O}_{\mathbb{P}^2}(3)$. The genus of a smooth $C \in L$ is 10. The general $C' \in |L|$ carries a very ample $\mathfrak{g}^2_d$, hence $\text{gon}(C) = 5$. On the other hand, any curve in the codimension 1 linear system $\pi^*H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ is bielliptic, therefore has gonality 4. Under reasonable assumptions this turns out to be the only counterexample to the Harris-Mumford conjecture. Ciliberto and Pareschi proved (in [CilP]) that if $C \subseteq S$ is such that $C$ is base-point-free and ample, then either $\text{gon}(C') = \text{gon}(C)$ for all smooth $C'' \in C'$, or $(S, C)$ are as in the previous counterexample. Although $\text{gon}(C)$ can drop as $C$ varies in a linear system, base point free $\mathfrak{g}^1_d$'s on $K3$-sections do propagate:

**Proposition 2.5.5 (Donagi-Morrison)** Let $S$ be a $K3$ surface, $C \subseteq S$ a smooth, non-hyperelliptic curve and $Z$ a complete, base point free $\mathfrak{g}^1_d$ on $C$ such that $p(g, 1, d) < 0$. Then there is an effective divisor $D \subseteq S$ such that:

- $h^0(S, D) \geq 2, h^0(S, C - D) \geq 2, \deg(D_C) \leq g - 1$.
- $\text{Cliff}(C', D_C) \leq \text{Cliff}(C', Z)$, for any smooth $C'' \in C'$.
- There is $Z_0 \in Z$, consisting of distinct points such that $Z_0 \subseteq D \cap C$. 

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2.5.4 The gonality of curves on quartic surfaces

For a wide range of \( d \) and \( g \) we construct curves \( C \subseteq \mathbb{P}^3 \) of degree \( d \) and genus \( g \) having the expected gonality. We start with the case when \( g \) is odd and \( d \) is even when we can realize our curves as sections of smooth quartic surfaces.

**Theorem 2.3** Let \( g \geq 5, d \geq 8 \) be integers, \( g \) odd, \( d \) even, such that \( d^2 > 8g \), \( 4d < 3g+12 \) and \( d^2 - 8g + 8 \) is not a square. Then there exists a smooth curve \( C \subseteq \mathbb{P}^3 \) of degree \( d \) and genus \( g \) such that \( \text{gon}(C) = \min(d-4, [(g+3)/2]) \). If \( \text{gon}(C) = d-4 < [(g+3)/2] \), every \( g^4 \) computing the gonality is given by the planes through a 4-secant line to \( C \). Moreover, \( C \) has only finitely many 4-secant lines, finitely many tangential trisecants and no 3-secant lines.

**Proof:** By Mori’s Theorem, for such \( d \) and \( g \), there exists a smooth quartic surface \( S \subseteq \mathbb{P}^3 \) and \( C \subseteq S \) a smooth curve of degree \( d \) and genus \( g \) such that \( \text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \), where \( H \) is a plane section. The conditions \( d \) and \( g \) are subject to, ensure that \( S \) does not contain \((-2)\) curves or genus 1 curves (the existence of a curve with self-intersection 0 would imply that \( d^2 - 8g + 8 \) is a square).

We prove first that \( \text{Cliff}(C) = 1 \). It suffices to show that \( C \subseteq S \) is an ample divisor, because then by using Prop.3.3 from [ClifP] we obtain that either \( \text{Cliff}(C) = 1 \) or \( C \) is a smooth plane sextic, \( g = 10 \) and \((S, C)\) are as in Donagi-Morrison’s example (then \( \text{Cliff}(C) = 2 \)). The latter case obviously does not happen.

We prove that \( C \cdot D > 0 \) for any effective divisor \( D \subseteq S \). Let \( D \sim mH + nC \), with \( m, n \in \mathbb{Z} \), such a divisor. Then \( D^2 = 4m^2 + 2mnd + n^2(2g-2) \geq 0 \) and \( D \cdot H = 4m + dn > 2 \). The case \( m \leq 0, n \leq 0 \) is impossible, while the case \( m \geq 0, n \geq 0 \) is trivial. Let us assume \( m > 0, n < 0 \). Then \( D \cdot C = md + n(2g-2) > -2n(d^2/8 - g + 1) + d/2 > 0 \), because \( d^2/8 > g \). In the remaining case \( m < 0, n > 0 \) we have that \( nD \cdot C \geq -mD \cdot H > 0 \), so \( C \) is ample by Nakai-Moishezon.

Our assumptions imply that \( d \leq g-1 \), so \( \mathcal{O}_C(1) \) is among the line bundles from which \( \text{Cliff}(C) \) is computed. We get thus the following estimate on the gonality of \( C \):

\[
\text{gon}(C) = \text{Cliff}(C) + 2 \leq \text{Cliff}(C, H_C) + 2 = d - 4.
\]

which yields \( \text{gon}(C) \leq \min(d-4, [(g+3)/2]) \).

Assume now that \( \text{gon}(C) < [(g+3)/2] \). We will then show that \( \text{gon}(C) = d - 4 \). Let \( |Z| \) be a complete, base point free pencil computing \( \text{gon}(C) \). By applying Prop.2.5.5., there exists an effective divisor \( D \subseteq S \) satisfying

\[
h^0(S, D) \geq 2, h^0(S, C-D) \geq 2, \deg(D_C) \leq g-1, \text{gon}(C) = \text{Cliff}(D_C) + 2 \text{ and } Z \subseteq D \cap C.
\]

We consider the exact cohomology sequence:

\[
0 \to H^0(S, D-C) \to H^0(S, D) \to H^0(C, D_C) \to H^1(S, D-C).
\]

Since \( C-D \) is effective and \( \sim 0 \), one sees that \( D-C \) cannot be effective, so \( H^0(S, D-C) = 0 \). The surface \( S \) does not contain \((-2)\) curves, so \( C-D \) has no fixed components; the
equation \((C - D)^2 = 0\) has no solutions, therefore \((C - D)^2 > 0\) and the general element of \(C - D\) is smooth and irreducible. Then it follows that \(H^1(S, D-C) = H^1(S, C-D)^\vee = 0\). Thus \(H^0(S, D) = H^0(C, D_C)\) and
\[
gon(C) = 2 + \Cliff(D_C) = 2 + D \cdot C - 2 \dim D = D \cdot C - D^2.
\]

We consider the following family of effective divisors
\[
\mathcal{A} := \{ D \in \text{Div}(S) : h^0(S, D) \geq 2, h^0(S, C-D) \geq 2, C \cdot D \leq g-1\}.
\]
and since we already know that \(d-4 \geq \gon(C) \geq \alpha\), where \(\alpha = \min\{ D \cdot C - C^2 : D \in \mathcal{A}\}\), we are done if we prove that \(\alpha \geq d-4\). Take \(D \in \mathcal{A}\) such that \(D \sim mH + nC. m, n \in \mathbb{Z}\). The conditions \(D^2 > 0, D \cdot C \leq g-1\) and \(2 < D \cdot H < d-2\) (use Prop.2.5.4 for the last inequality) can be rewritten as
\[
2m^2 + mnd + n^2(g-1) > 0 \quad \text{(i)}, \quad 2 < 4m + nd < d-2 \quad \text{(ii)}, \quad md + (2n-1)|g-1| \leq 0 \quad \text{(iii)}.
\]

We have to prove that for any \(D \in \mathcal{A}\) the following inequality holds
\[
f(m, n) = D \cdot C - D^2 = -4m^2 + m(d - 2nd) + (n - n^2)(2g - 2) \geq f(1, 0) = d - 4.
\]

We solve this standard calculus problem. Denote by \(a := (d + \sqrt{d^2 - 8g + 8})/4\) and \(b := (d - \sqrt{d^2 - 8g + 8})/4\). We dispose first of the case \(n < 0\). Assuming \(n < 0\), from (i) we have that either \(m < -bn\) or \(m > -an\). If \(m < -bn\) from (ii) we obtain that \(2 < n(d - 4b) < 0\), because \(n < 0\) and \(d - 4b > \sqrt{d^2 - 8g + 8} > 0\), so we have reached a contradiction.

We assume now that \(n < 0\) and \(m > -an\). From (iii) we get that \(m \leq (g-1)(1-2n)/d\). If \(-an > (g-1)(1-2n)/d\) we are done because there is no \(m \in \mathbb{Z}\) satisfying (i), (ii) and (iii), while in the other case for any \(D \in \mathcal{A}\) with \(D \sim mH + nC\), one has the inequalities
\[
f(m, n) > f(-an, n) = (2g - 2 - ad)n = \frac{(d^2 - 8g + 8) + d\sqrt{d^2 - 8g + 8}}{4}(-n) > d - 4.
\]

unless \(n = -1\) and \(d^2 - 8g < 8\) (which forces \(d^2 - 8g = 4\)). In this last case we obtain \(m \geq (d + 4)/4\) so \(f(m, -1) \geq f((d + 4)/4, -1) > d - 4\).

The case \(n > 0\) can be treated rather similarly. From (i) we get that either \(m < -an\) or \(m > -bn\). The first case can be dismissed immediately. When \(m > -bn\) we use that for any \(D \in \mathcal{A}\) with \(D \sim mH + nC\).
\[
f(m, n) \geq \min\{f(-(g-1)(2n-1)/d, n), \max\{f(-bn, n), f((2 - nd)/4, n)\}\}.
\]

Elementary manipulations give that
\[
f(-(g-1)(2n-1)/d, n) = (g-1)/2 \left[ (2n-1)^2(d^2 - 8g + 8)/d^2 + 1 \right] \geq d - 4
\]
(use that \(d^2 > 8g\) and \(d \leq g - 1\). Note that we have equality if and only if \(n = 1, m = -1\) and \(d = g - 1\). This possibility is compatible with the other conditions only for \(g \in 48\).
Furthermore, \( f(-bn, n) = n(2g - 2 - bd) \geq 2g - 2 - bd \) and \( 2g - 2 - bd \geq d - 4 \). When this does not happen we proceed as follows: if \( \sqrt{d^2 - 8g + 8} > d - 4 \) then if \( n = 1 \) we have that \( m > -b > -1 \), that is \( m \geq 0 \), but this contradicts \( (i) \). When \( n \geq 2 \), we have \( f((2 - nd)/4, n) = \left\lfloor (d^2 - 8g + 8)(n^2 - n) + (2d - 4)\right\rfloor /4 \geq d - 4 \). Finally, the remaining possibility \( 4 > \sqrt{d^2 - 8g + 8} \) can be disposed of easily by an ad-hoc argument (our assumptions force in this case \( d^2 - 8g = 4 \)).

All this leaves us with the case \( n = 0 \). when \( f(m, 0) = -4m^2 + md \). Clearly \( f(m, 0) \geq f(1, 0) \) for all \( m \) complying with \((i),(ii)\) and \((iii)\).

Thus we proved that \( \text{gon}(C) = d - 4 \). We have equality \( D - C - D^2 = d - 4 \) where \( D \in A \). if and only if \( D = H \) or in the case \( d = g - 1 \). \( g \in \{11,13,15\} \) also when \( D = C - H \). It is easy to show that if \( d = g - 1 \) then \( K_C = 2H_C \). therefore we can always assume that the divisor on \( S \) cutting a \( g_{d-4}^1 \) on \( C \) is the plane section of \( S \). Since \( Z \subseteq H \cap C \), if we denote by \( \Delta \) the residual divisor of \( Z \) in \( H \cap C \). we have that \( h^0(C, H \cap C - \Delta) = 2 \). so \( \Delta \) spans a line and \( Z \) is given by the planes through the 4-secant line \( \langle \Delta \rangle \). This shows that every pencil computing \( \text{gon}(C) \) is given by the planes through a 4-secant line.

There are a few ways to see that \( C \) has only finitely many 4-secant lines. The shortest is to invoke Theorem 3.1 from [CilP]; since \( \text{gon}(C') = d - 4 \) is constant as \( C' \) varies in \( |C| \), for the general smooth curve \( C' \in C \) one has \( \text{dim} W^1_{d-4}(C') = 0 \). Thus \( C \) has only finitely many 4-secant lines and no 5-secant lines. Note that the last part of this assertion can also be seen directly using Bezout’s Theorem: if \( L \subseteq \mathbb{P}^3 \) were a 5-secant line to \( C \), then \( L \subseteq S \), but \( S \) contains no lines. Finally, \( C \) has only finitely many tangential trisecants because \( C \) is nondegenerate and we can apply a result from [Kaji].

**Remarks:**

1. One can find quartic surfaces \( S \subseteq \mathbb{P}^3 \) containing a smooth curve \( C \) of degree \( d \) and genus \( g \) in the case \( g = d^2/8 + 1 \) (which is outside the range Theorem 2.3 deals with). Then \( d = 4m, g = 2m^2 + 1 \) with \( m \geq 1 \) and \( C \) is a complete intersection of type \( (4, m) \). For such a curve, \( \text{gon}(C) = d - l \), where \( l \) is the degree of a maximal linear divisor on \( C \) (cf. [Ba]). If \( S \) is picked sufficiently general so that it contains no lines, by Bezout, \( C \) cannot have 5-secant lines so \( \text{gon}(C) = d - 4 \) in this case too.

2. Mori’s Theorem can be extended to curves sitting on K3 surfaces which are embedded in higher dimensional projective spaces: for \( r \geq 3, d > 0, g \geq 0 \) such that \( g < d^2/(4r - 4) \) and \( (d, g) \neq (2r - 1, r) \), there exists a K3 surface \( S \subseteq \mathbb{P}^r \) of degree \( 2r - 2 \) containing a smooth curve \( C \) of degree \( d \) and genus \( g \) and such that \( \text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \), where \( H \) is a hyperplane section of \( S \) (see [Kn]). It seems very likely (although I have only checked several particular cases) that under the same conditions (i.e. \( S \) contains no genus 0 or genus 1 curves) the analogue of Theorem 2.3 still holds. that is \( \text{gon}(C) = \min(\{(g + 3)/2, d - 2r + 2\}) \).

We want to find out when the curves constructed in Theorem 2.3 correspond to ‘good points’ of \( \text{Hilb}_{d,g,3} \). We have the following:

**Proposition 2.5.6** Let \( C \subseteq S \subseteq \mathbb{P}^3 \) be a smooth curve sitting on a quartic surface such that \( \text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \) with \( H \) being a plane section and assume furthermore that \( S \) contains no \((-2)\) curves. Then \( H^1(C, N_{C/S}) = 0 \) if and only if \( d \leq 18 \) or \( g < 4d - 31 \).
Proof: We use the exact sequence
\[ 0 \rightarrow N_{C/S} \rightarrow N_{C/T_3} \rightarrow N_{S/T_3} \otimes \mathcal{O}_C \rightarrow 0. \] (2.13)
where \( N_{S/T_3} \otimes \mathcal{O}_C = \mathcal{O}_C(4) \) and \( N_{C/S} = K_C \). We claim that there is an isomorphism 
\( H^1(C, N_{C/T_3}) = H^1(C, \mathcal{O}_C(4)) \). Suppose this is not the case. Then the injective map 
\( H^1(C, K_C) \rightarrow H^1(C, N_{C/T_3}) \) provides a splitting of the sequence (2.13) and by using 
Proposition 3.25 from [Mod] we obtain that \( C \) is a complete intersection with \( S \). This is clearly a contradiction.

We have isomorphisms 
\( H^1(C, 4H_C) = H^2(S, 4H-C) = H^0(S, C-4H)^* \). According to 
Prop.2.5.4 the divisor \( C-4H \) is effective if and only if \( (C-4H)^2 \geq 0 \) and \( (C-4H) \cdot H > 2 \), 
from which the conclusion follows. \( \square \)

We need to determine the gonality of nodal curves not of compact type and which 
consist of two components (like those appearing in Prop.2.2.2). The following result is 
intuitively clear if one uses admissible coverings:

**Proposition 2.5.7** Let \( C = C_1 \cup_\Delta C_2 \) be a quasi-transversal union of two smooth curves 
\( C_1 \) and \( C_2 \) meeting at a finite set \( \Delta \). Denote by \( g_1 = g(C_1), g_2 = g(C_2), \delta = \text{card}(\Delta) \). Let 
us assume that \( C_1 \) has only finitely many pencils \( g_1^\delta \), where \( \delta \leq d \) and that the points of 
\( \Delta \) do not occur in the same fibre of one of these pencils. Then \( \text{gon}(C) \geq d+1 \). Moreover 
if \( \text{gon}(C) = d+1 \) then either (1) \( C_2 \) is rational and there is a degree \( d \) map \( f_1 : C_1 \rightarrow \mathbb{P}^1 \) 
and a degree 1 map \( f_2 : C_2 \rightarrow \mathbb{P}^1 \) such that \( f_1 \circ f_2 = f_2 \circ f_1 \), or (2) there is a \( g_{d+1}^\delta \) on \( C_1 \) 
containing \( \Delta \) in a fibre.

Proof: For the proof we use Section 2 of [EH1] (the one which works for nodal curves not 
necessarily of compact type). We briefly reviewed this in Chapter 1 (see also [Est] for a 
clear account on limit linear series on (general) reducible nodal curves). Let us assume 
that \( C \) is \( k \)-gonal, that is, a limit of smooth \( k \)-gonal curves. Then there exists a family 
of curves \( \pi : \mathcal{C} \rightarrow B \), with \( B = \text{Spec}(R) \), \( R \) being a discrete valuation ring, such that the 
central fibre \( C_0 \) is \( C \), the generic fibre \( C_\eta \) is smooth \( (\eta \in B \) is the generic point) and there 
is a \( g_{\eta}^\delta \) on \( C_\eta \), which as in Chapter 1 we denote by \( l_\eta = (\mathcal{L}_\eta, V_\eta) \), where \( V_\eta \subseteq \pi_* \mathcal{L}_\eta \) is a 
vector bundle of rank 2. To the family of pencils \( l_\eta \) we can associate a limit linear series 
on \( C \) as follows (cf. [Est]): there are unique line bundles \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) on \( C \) such that:

1. \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are extensions of \( \mathcal{L}_\eta \); \( \mathcal{L}_i \cdot C_\eta = \mathcal{L}_\eta \), for \( i = 1, 2 \).

2. If \( V_{\mathcal{L}_i} := V_\eta \cap \pi_* \mathcal{L}_i \subseteq \pi_* \mathcal{L}_i \), then the map \( V_{\mathcal{L}_1}(0) \rightarrow H^0(C_1, \mathcal{L}_1(0) \cdot C_1) \) is injective and 
the map \( V_{\mathcal{L}_2}(0) \rightarrow H^0(C_2, \mathcal{L}_2(0) \cdot C_2) \) is \( \neq 0 \). Similarly, \( V_{\mathcal{L}_1}(0) \rightarrow H^0(C_1, \mathcal{L}_2(0) \cdot C_1) \) is 
injective and \( V_{\mathcal{L}_2}(0) \rightarrow H^0(C_1, \mathcal{L}_2(0) \cdot C_1) \) is \( \neq 0 \).

Note that in the case of curves of compact type, it was possible to get for each component 
of the special fibre one extension of \( \mathcal{L}_\eta \) whose restriction had degree \( k \) on the chosen 
component and degree 0 on all the other components of the special fibre; obviously we 
cannot expect something like this for arbitrary nodal curves. We also point out that
even in the case when \( C \) is of compact type the extensions \( L_1 \) and \( L_2 \) may differ from Eisenbud-Harris’ extensions; this happens when there is some ramification at the nodes of \( C \).

Let us denote by \( l \geq 0 \) the unique integer such that \( L_1 = L_2(lC_2) \) and by \( d_1 = \deg_{C_1}(L_1, C_1), d_2 = \deg_{C_2}(L_2, C_2) \). Then \( d_1 + d_2 = k + l\delta \).

We show first that \( k \geq d + 1 \). Suppose \( k = d \). Then \( d_1 = d, d_2 = l\delta \geq 1 \) (because \( h^0(C_2, L_2(0, C_2)) \geq 2 \). and then \( (L_1(0) C_1, V_C(0)) \) is one of the finitely many \( g^1 \)'s on \( C_1 \). From \( 2 \), we have that \( 1 \leq h^0(C_1, L_2(0, C_1)) \leq h^0(C_1, L(0) C_1 - \Delta) \), that is. \( \Delta \) is contained in a fibre of a \( g^1 \) on \( C_1 \), a contradiction.

Assume now \( k = d + 1 \). There are two cases to consider: (i). \( d_1 = d, d_2 = l\delta + 1 \) which forces \( l = 0 \) (if \( l \geq 1 \) once again \( \Delta \) would be entirely contained in a fibre of a \( g^1 \) on \( C_1 \)). hence \( L_2 = L_1 \), so we have only one line bundle on \( C \) which gives a degree \( d + 1 \map \) \( C \cup \Delta C_2 \to \mathbb{P}^1 \), which is case \( (1) \) of Prop.2.5.7. (ii). \( d_1 = d+1, d_2 = l\delta \). Again, the condition \( h^0(C_2, L_2(0, C_2)) \geq 2 \) gives \( l \geq 1 \), hence \( 1 \leq h^0(C_1, L_2(0, C_1)) \leq h^0(C_1, L(0) C_1 - \Delta) \), which yields case \( (2) \) of Prop.2.5.7.

Theorem 2.3 provides space curves of expected gonality when \( d \) is even and \( g \) is odd. Naturally, we would like to have such curves when \( d \) and \( g \) have other parities as well. We will achieve this by attaching to a ‘good’ curve of expected gonality, either a 2 or 3-secant line or a 4-secant conic.

**Theorem 2.4** Let \( g \geq 5. d \geq 8 \) be integers with \( g \) odd and \( d \) even, such that \( d^2 > 8g, 4d < 3g + 12, d^2 - 8g + 8 \) is not a square and either \( d \leq 18 \) or \( g < 4d - 31 \). If

\[
(d', g') \in \{(d, g), (d + 1, g + 1), (d + 1, g + 2), (d + 2, g + 3)\},
\]

then there exists a regular component of \( \text{Hilb}^d_{g^1,3} \) with general point \( [C'] \) a smooth curve such that \( \text{gon}(C') = \min(d' - 4, [(g' + 3)/2]) \).

**Proof:** For \( d \) and \( g \) as in the statement we know by Theorem 2.3 and Prop.2.5.6 that there exists a smooth, nondegenerate curve \( C \subseteq \mathbb{P}^3 \) of degree \( d \) and genus \( g \), with \( \text{gon}(C) = \min(d - 4, [(g + 3)/2]) \) and \( H^1(C, N_{C/\mathbb{P}^3}) = 0 \). We can also assume that \( C \) sits on a smooth quartic surface \( S \) and \( \text{Pic}(S) = \mathbb{Z}H \cong \mathbb{Z}C \). Moreover, in the case \( d - 4 < [(g + 3)/2] \) the curve \( C \) has only finitely many \( g^1 \)'s, all given by planes through a 4-secant line.

**i)** Let us settle first the case \( (d', g') = (d + 1, g + 1) \). Take \( p, q \in C \) general points.
\( L = \overline{pq} \subseteq \mathbb{P}^3 \) and \( X := C \cup L \). By Prop.2.2.2 \( X \) is smoothable and \( H^1(X, N_{X/\mathbb{P}^3}) = 0 \). If \( d - 4 < [(g + 3)/2] \), then since \( C \) has only finitely many \( g^1 \)'s by applying Prop.2.5.7 we get that \( \text{gon}(X) = d - 3 \). In the case \( d - 4 \geq [(g + 3)/2] \) we just notice that \( \text{gon}(X) \geq \text{gon}(C) = [(g' + 3)/2] \).

**ii)** Next, we tackle the case \( (d', g') = (d + 1, g + 2) \). Assume first that \( d - 4 < [(g+3)/2] \iff d' - 4 < [(g'+3)/2] \). Let \( p \in C \) be a general point. The image of the projection \( \pi_p : C \to \mathbb{P}^2 \) from \( p \) is a plane curve of degree \( d - 1 \) having only nodes as singularities, which means that \( C \) has no stationary trisecants through \( p \) (i.e. trisecants \( pqq' \) such that \( T_q(C) \) and \( T_{q'}(C) \) meet). because a stationary trisecant would correspond to a tacnode of \( \pi_p(C) \).
Pick \( L \) one of the \( \binom{d-1}{2} - g \) trisecants through \( p \) and consider \( X := C \cup L \). The conditions required by Prop.2.2.2 (part 3) being satisfied, \( X \) is smoothable and \( H^1(X, N_X) = 0 \). To conclude that \( \text{gon}(X) = d - 3 \), we have to show that there is no \( g_{d-1}^1 \) on \( C \) containing \( L \cap C \) in a fibre. A line in \( \mathbb{P}^1 \) (hence also a 4-secant line to \( C \)) can meet only finitely many trisecants. Indeed, assuming that \( m \subseteq \mathbb{P}^3 \) is a line meeting infinitely many trisecants, by considering the correspondence

\[
T = \{(p, t) \in C \times m : \exists l \text{ a trisecant to } C \text{ passing through } p \text{ and } t\},
\]

the projection \( \pi_2 : T \to m \) yields a \( g_3^1 \) on \( T \), hence \( C \) is trigonal as well, a contradiction. Since \( p \) and \( L \) have been chosen generally we may assume that \( L \) does not meet any of the 4-secant lines.

In the remaining case \( d - 4 \geq [(g + 3)/2] \) we apply Theorem 2.3 to obtain a smooth curve \( C_1 \subseteq \mathbb{P}^3 \) of degree \( d \) and genus \( g + 2 \) such that \( \text{gon}(C_1) = (g + 5)/2 \) and \( H^1(C_1, N_{C_1}) = 0 \). We take \( X_1 := C_1 \cup L_1 \) with \( L_1 \) being a general 1-secant line to \( C_1 \). Then \( X_1 \) is smoothable and \( \text{gon}(X_1) = \text{gon}(C_1) = (g + 5)/2 \).

iii) Finally, we turn to the case \( (d', g') = (d + 2, g + 3) \). Take \( H \subseteq \mathbb{P}^4 \) a general plane meeting \( C \) in \( d \) distinct points in general linear position and pick 4 of them: \( p_1, p_2, p_3, p_4 \in C \cap H \). Choose \( Q \subseteq H \) a general conic such that \( Q \cap C = \{p_1, p_2, p_3, p_4\} \). Prop.2.2.2 ensures that \( X := C \cup Q \) is smoothable and \( H^1(X, N_X) = 0 \).

Assume first that \( d' - 4 \leq [(g' + 3)/2] \). We claim that \( \text{gon}(X) \geq \text{gon}(C) + 2 \). According to Prop. 2.5.7 the opposite could happen only in 2 cases: \( a \) there exists a \( g_{d-3}^1 \) on \( C \), say \( Z \) such that \( |Z|(-p_1 - p_2 - p_3 - p_4) \neq \emptyset \). \( b \) there exists a degree \( d - 4 \) map \( f : C \to \mathbb{P}^1 \) and a degree 1 map \( f' : Q \to \mathbb{P}^1 \) such that \( f(p_1) = f'(p_i) \) for \( i = 1, \ldots, 4 \).

Assume that \( a \) does happen. We denote by \( U = \{D \in C_1 : |O_C(1)(-D)| \neq \emptyset\} \) the irreducible 3-fold of divisors of degree 4 spanning a plane and also consider the correspondence

\[
\Sigma = \{(l, D) \in G_d^1(C) \times U : l(-D) \neq \emptyset\}.
\]

with the projection \( \pi_1 : \Sigma \to G_d^1(C) \). We have that \( \dim \Sigma \geq 3 \). There are two possibilities: \( a_1 \) There is \( l \in \pi_1(\Sigma) \) such that \( |O_C(1)(-l)| = \emptyset \). Then \( \pi_1^{-1}(l) \) is finite hence \( \dim G_d^1(C) \geq 3 \). By using the theory of excess linear series (cf. [ACGH]) we get that \( \dim G_{d-3}^1(C) \geq 1 \), a contradiction. \( a_2 \) For all \( l \in \pi_1(\Sigma) \) we have that \( |O_C(1)(-l)| \neq \emptyset \) and then \( C \) has \( \infty^2 \) trisecants. But a nondegenerate curve in \( \mathbb{P}^4 \) can have at most \( \infty^4 \) trisecants, so by picking the plane section \( H \cap C \) generally we can assume that \( a_2 \) does not happen either.

We now rule out case \( b \). Suppose that \( b \) does happen and denote by \( L \subseteq \mathbb{P}^1 \) the 4-secant line corresponding to \( f \). Let \( \{p\} = L \cap H \) and pick \( l \subseteq H \) a general line. As \( Q \) was a general conic through \( p_1, \ldots, p_4 \) we may assume that \( p \notin Q \). The map \( f' : Q \to l \) is (up to a projective isomorphism of \( l \) the projection from a point \( q \in Q \), while \( f(p_i) = p_i \cap l \) for \( i = 1, \ldots, 4 \). By Steiner's Theorem from classical projective geometry, the condition \( (f(p_1)f(p_2)f(p_3)f(p_4)) = (f'(q_1)f'(p_2)f'(p_3)f'(p_4)) \) is equivalent with \( p_1, p_2, p_3, p_4, p \) and \( q \) being on a conic, a contradiction since \( p \notin Q \).

Finally, when \( d' - 4 > [(g' + 3)/2] \), we have to show that \( \text{gon}(X) \geq \text{gon}(C) + 1 \). We
note that \( \dim G_{g-3/2}^1(C) = 1 \) (for any curve one has the inequality \( \dim G_{\text{even}}^1 \leq 1 \)). By taking \( H \in (\mathbb{P}^3)^t \) general enough, we obtain that \( p_1, \ldots, p_4 \) do not occur in the same fibre of a \( g_{g-3/2}^1 \).

As an application of all these results we give a totally different proof of the most difficult part of Chapter 1, namely Prop. 1.5.4:

**Theorem 2.5** The Kodaira dimension of \( \mathcal{M}_{23} \) is \( \geq 2 \).

**Proof:** We apply Theorem 2.4 when \((d, g) = (18, 23)\). There exists a curve \( C \subseteq \mathbb{P}^3 \) of degree 18 and genus 23 such that \( \text{gon}(C) = 13 \) (generic). Hence \([C] \in \mathcal{M}^2_{23,17} \cap \mathcal{M}^3_{23,20}\) but \([C] \notin \mathcal{M}^1_{23,12}\), which basically proves 3) of Chapter 1. \( \square \)

### 2.6 Miscellany

In this section we gather several facts about the relative position of certain loci in \( \mathcal{M}_g \). The Brill-Noether Theorem asserts that the general curve of genus \( g \) has no linear series with negative \( \rho \). However, it is notoriously difficult to find smooth Brill-Noether general curves. We discuss whether various geometrically defined subvarieties of \( \mathcal{M}_g \) (e.g. loci of curves which lie on certain surfaces, or admit irrational involutions) might possess Brill-Noether general curves.

Let us look first what kind of surfaces can a Brill-Noether general curve lie on. Lazarsfeld proved that a general \( K3 \) surface (with Picard number 1) contains Brill-Noether-Petri general curves. It seems pretty hard to obtain such results for other classes of surfaces. We have the following observation:

**Proposition 2.6.1** 1. A general surface \( S \subseteq \mathbb{P}^3 \) of degree \( d \geq 5 \) does not contain non-degenerate Brill-Noether general curves.

2. A smooth curve of genus \( g \geq 29 \) with generic gonality \( [(g + 3)/2] \) cannot lie on an Enriques surface.

**Proof:** By the Noether-Lefschetz Theorem, if \( S \) is a general surface of degree \( d \geq 5 \), \( \text{Pic}(S) = \mathbb{Z}H \), with \( H \) being a plane section. Hence any curve \( C \subseteq S \) is a complete intersection. For \( C \sim mH \) with \( m \geq 2 \), we have that \( 2g(C) - 2 = md(m + d - 4) \), from which clearly \( \rho(g, 3, md) < 0 \). So \( C \) is not Brill-Noether general.

Suppose now that \( C \subseteq S \) is a smooth curve of genus \( g \), sitting on an Enriques surface. There exists \( 2E \) an elliptic pencil on \( S \) such that \( C \cdot E \leq \sqrt{2g-2} \) (cf. [CD] Corollary 2.7.1). In the exact sequence

\[ 0 \longrightarrow H^0(S, 2E - C) \longrightarrow H^0(S, 2E) \longrightarrow H^0(C, 2E_C) \]

we have that \( h^0(S, 2E) \geq 2 \) and \( H^0(S, 2E - C) = 0 \) (because \( (2E - C) \cdot E < 0 \)). Therefore \( C \) carries a \( g_{2E}^1 \). Since for \( g \geq 29 \) we have that \( 2\sqrt{2g-2} \leq (g + 1)/2 \), the curve \( C \) does not have generic gonality. \( \square \)

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We would like to know whether curves having an irrational involution can be Brill-Noether general. We will restrict ourselves to double covers, although these considerations can be carried out for coverings of arbitrary degree. We have the following results:

**Proposition 2.6.2** 1. For \( g \geq 1 \), the general point of the locus \( \{ C \} \in \mathcal{M}_{2g-1} : g \sigma : C \to X, \deg(\sigma) = 2 \), \( g(X) = g \) is Brill-Noether general.

2. For odd \( g \geq 1 \), the general point of the locus \( \{ C \} \in \mathcal{M}_{2g-1} : g \sigma : C \to X \), étale, with \( \deg(\sigma) = 2 \), \( g(X) = g \) is of generic gonality \( g + 1 \).

**Proof:** 1. Using limit linear series we find a Brill-Noether general curve \( C \) of compact type and genus \( 2g \), having a map of degree 2 onto a curve \( X \) of compact type and genus \( g \). Take \( (A, \mu) \), a general pointed curve of genus \( g \), and \( R \) a smooth rational curve. Consider \( X := A \cup_p R \), which is of genus \( g \). Let \( (C_1, p_1) \) and \( (C_2, p_2) \) be two copies of \( (A, p) \) and \( (E, x, y) \) a 2-pointed elliptic curve such that \( x - y \notin \text{Pic}^0(E) \) is not torsion. We construct a curve of compact type of genus \( 2g + 1 \) by taking \( C := C_1 \cup_{p_1 - x} E \cup_{y, p_2} C_2 \). It is straightforward to construct a degree 2 map \( \sigma : C \to X \): take \( \sigma(C, p_1) = (A, p) \) and \( \sigma : E \to R \), the double covering given by the linear system \( x + y \) on \( E \). This shows that \( C \) is a limit of smooth curves of genus \( 2g + 1 \) having a double cover with 4 branch points. The claim that \( C \) is Brill-Noether general (i.e., it does not admit any limit linear series with negative Brill-Noether number) is a byproduct of Propositions 1.3.2 and 1.4.1.

2. The idea is the same, to construct an unramified double cover \( \sigma : C \to X \), with \( X \) and \( C \) of compact type, \( g(C) = 2g - 1 \), \( g(X) = g \) and \( C \) having no \( g_x \)'s. This time we take \( X := A \cup_p E \), with \( (A, p) \) a general curve of genus \( g - 1 \) and \( E \) an elliptic curve, and \( C := C_1 \cup_{p_1} E' \cup_{p_2} C_2 \), where \( (C_1, p_1) \) are just copies of \( (A, p) \). \( E' \) is a copy of \( E \) and \( p_1 - p_2 \in \text{Pic}^0(E') \). We obtain an étale double cover \( \sigma : C \to X \), by mapping \( C_1 \) and \( C_2 \) to \( A \), \( E' \) to \( E \), such that \( \sigma(p_1) = \sigma(p_2) = p \). The proof that \( C \) has no limit \( g_x \)'s is similar to a few other such proofs in this thesis, so we omit it. Note that this construction in the unramified case can also be found in [Ber].

In the previous proposition, the restriction to odd \( g \) in the unramified case seems to be rather artificial. Although we believe that for a sufficiently large even \( g \), there are Brill-Noether general curves of genus \( 2g - 1 \) mapping \( 2:1 \) to a curve of genus \( g \), for \( g = 6 \) we have the rather surprising result which we regard as a one-off:

**Proposition 2.6.3** If \( \sigma : \tilde{C} \to C \) is an étale double cover with \( g(\tilde{C}) = 11, g(C) = 6 \), then \( \tilde{C} \) is 6-gonal (whereas the generic gonality on \( \mathcal{M}_{11} \) is \( 7 \)).

**Proof:** Let us consider the moduli space \( \mathcal{R}_6 \) of pairs \((C, \eta)\), where \( C \) is a smooth curve of genus 6 and \( \eta \in \text{Pic}^0(C) \). We denote by \( \phi : \mathcal{R}_6 \to \mathcal{M}_{11} \) the map given by \( \phi(C, \eta) := [\tilde{C}] \), where \( \sigma : \tilde{C} \to C \) is the étale double cover corresponding to \( \eta \), i.e. \( \sigma^*\mathcal{O}_C = \mathcal{O}_{\tilde{C}} \cdot \eta \).

The key observation is the following result of Verra (cf. [Ve]): for a general point \((C, \eta) \in \mathcal{R}_6\), there exists an Enriques surface \( S \) such that \( C \subseteq S, C^2 = 10 \) (hence \( \text{dim} \ C = 5 \)). \( C \) is very ample and \( \eta = K_S \mid C \). Moreover, if \((C, \eta) \in \mathcal{R}_6\) is general, the Enriques surface \( S \) can be chosen generally too in the 10-dimensional moduli space of Enriques surfaces. By general theory (cf. [CD]) \( S \) contains 10 curves of genus 1, \( E_1, \ldots, E_{10} \), such
that $E_i : E_j = 1 - \delta_{ij}$ and $C \sim 1/3(E_1 + \cdots + E_{10})$. Let $\pi : X \to S$ be the $K3$ cover of $S$. $C = \pi^{-1}(C)$ and $F_i = \pi^{-1}(E_i)$. with $3C \sim F_1 + \cdots + F_{10}$. For some $1 \leq i \leq 10$, consider the exact sequence

$$0 \longrightarrow H^0(X, F_i - C) \longrightarrow H^0(X, F_i) \longrightarrow H^0(\tilde{C}, F_i, C).$$

Certainly $H^0(\tilde{C}, F_i, C) = 0$ and although $E_i$ might be isolated on $S$, when we pass to the $K3$ surface $X$, we get that $h^0(X, F_i) = 2$, so $F_i$ gives a pencil on $\tilde{C}$ of degree $6(= F_i \cdot \tilde{C} = 2E_i \cdot C)$. This shows that $\phi(\mathcal{R}_6) \subseteq M_{11,6}$.

**Remark:** In a similar way, we can show that any smooth curve lying on the $K3$-cover of a general Enriques surface cannot have maximal gonality, so it is Brill-Noether special.