The birational geometry of the moduli space of curves
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Chapter 3

Divisors on moduli spaces of pointed curves

3.1 Introduction

For integers \( g \geq 3 \) and \( n \geq 1 \) we denote by \( \mathcal{M}_{g,n} \) the moduli space of complex \( n \)-pointed curves of genus \( g \) and by \( \overline{\mathcal{M}}_{g,n} \) its compactification, the moduli space of \( n \)-pointed stable curves. When \( n = 1 \) we will sometimes use the notation \( \tilde{C}_g = \overline{\mathcal{M}}_{g,1} \) for the universal (stable) curve of genus \( g \).

The loci \( \mathcal{M}_{g,d}^r \subseteq \mathcal{M}_g \) consisting of curves having a \( g_d^r \) turned out to be extremely useful for understanding the birational geometry of \( \mathcal{M}_g \). One can consider analogous Brill-Noether loci in \( \mathcal{M}_{g,n} \) defined as follows: if \( \alpha^1, \ldots, \alpha^n \) are Schubert indices of type \((r,d)\) (that is \( 0 \leq \alpha^i_1 \leq \ldots \leq \alpha^i_r \leq d - r \) for \( i = 1, \ldots, n \)), we consider the subvariety

\[
\mathcal{M}_{g,n}^r(\alpha^1, \ldots, \alpha^n) := \{[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n} : \exists l \in G_d^r(C) \text{ with } \alpha^i(p_i) \geq \alpha^i \text{ for all } i\}.
\]

The ‘strong Brill-Noether Theorem’ of Eisenbud and Harris (cf. Section 1.3) asserts that for a general \( n \)-pointed curve \((C, p_1, \ldots, p_n)\) of genus \( g \), the dimension of the variety

\[
G_d^r(C, (p_1, \alpha^1), \ldots, (p_n, \alpha^n)) = \{l \in G_d^r(C) : \alpha^i(p_i) \geq \alpha^i \text{ for } i = 1, \ldots, n\}
\]

is the adjusted Brill-Noether number \( \rho(g, r, d, \alpha^1, \ldots, \alpha^n) = \rho(g, r, d) - \sum_{i=1}^n \sum_{j=0}^r \alpha^i_j \).

When this number is \(-1\) one expects to find divisors on \( \mathcal{M}_{g,n} \). We will try to understand the geometry of such divisors when \( r = 2 \) and \( n \in \{1, 2\} \), that is, we will look at loci of 1 or 2-pointed curves having a \( g_2^2 \) with prescribed ramification at the marked points. We mention that the case \( r = n = 1 \) has already been treated in [Lo], but as it will turn out, computations are significantly more involved in the case of 2-dimensional linear series.

Experience shows that on \( \mathcal{M}_g \) the most interesting divisors defined in terms of linear series are those consisting of curves with certain \( g_d^r \)'s having ramification as ordinary as possible: in general the more ramification one imposes on a linear series, the higher the slope of the resulting divisor on \( \overline{\mathcal{M}}_g \) will be, hence it will be less relevant for understanding the birational geometry of \( \mathcal{M}_g \). It is natural to expect the same for Brill-Noether divisors.
on $\overline{\mathcal{M}}_{g,n}$, which means that we will be mainly interested in the case when the $\alpha$'s are minimal.

For an integer $g \equiv 1 \mod 3$ and $g \geq 4$, we set $d := (2g - 7)/3$, so that $\rho(g, 2, d) = 1$. There are two ways to get Brill-Noether divisors with minimal ramification on $\mathcal{M}_{g,1}$. Either we take $\alpha = (0, 1, 1)$ and then we consider the divisor of curves with a marked point that is a cusp, i.e.

$$CU := \mathcal{M}_{g,1}^2((0, 1, 1)) = \{(C, p) \in \mathcal{M}_{g,1} : \exists g_3^d \text{ on } C \text{ with a cusp at } p\}.$$ 

or we take $\alpha = (0, 0, 2)$ and then we get the divisor of curves with a marked point that is a hyperflex, i.e.

$$HF := \mathcal{M}_{g,1}^2((0, 0, 2)) = \{(C, p) \in \mathcal{M}_{g,1} : \exists g_4^d \text{ on } C \text{ with a hyperflex at } p\}.$$ 

We are going to compute the classes of the closures $\overline{CU}$ and $\overline{HF}$ in $\overline{\mathcal{M}}_{g,1}$.

On $\mathcal{M}_{g,2}$ there is only one way to get a Brill-Noether divisor by imposing minimal ramification, and that is by taking $\alpha^1 = \alpha^2 = (0, 0, 1)$ to obtain the divisor of curves with 2-marked points that are both flexes:

$$FL := \{(C, p_1, p_2) \in \mathcal{M}_{g,2} : \exists g_4^d \text{ on } C \text{ having flexes at } p_1 \text{ and } p_2\}.$$ 

We shall also compute the class $[\overline{FL}] \in \text{Pic}_{\overline{\mathcal{M}}_{g,2}}$.

Although I think that computations of divisor classes on $\overline{\mathcal{M}}_{g,n}$ are interesting in themselves because they enhance our understanding of families of (pointed) curves, the original motivation for studying the divisor $FL \subseteq \mathcal{M}_{g,2}$ was an attempt to prove that the moduli space $\mathcal{M}_{22,2}$ is of general type. It is proved in [Lo] that $\mathcal{M}_{22,n}$ is of general type for $n \geq 8$; since $\mathcal{M}_{23,n}$ is of general type for $n \geq 1$ and $\mathcal{M}_{21,n}$ is of non-negative Kodaira dimension for $n = 5$ and of general type for $n \geq 6$, it is natural to expect that the bound for genus 22 is some way off from being optimal. The divisor $\overline{FL}$ or some of its pullbacks to $\overline{\mathcal{M}}_{g,n}$ via the maps $\mathcal{M}_{g,n} \to \mathcal{M}_{g,2}$ forgetting some marked points, seemed the most likely candidate for being part of a multicanonical linear system, i.e. to have $K_{\mathcal{M}_{22,2}} \sim a \overline{FL} + (\text{effective divisor })$, for some $a \geq 0$. Unfortunately our calculation shows this not to be the case.

In Section 3.6 we compute the class of yet another divisor on the universal curve $\overline{\mathcal{M}}_{g,1}$. This time we consider a divisor which although is defined by a geometric condition in terms of linear series on curves, is different from the Brill-Noether divisors from Section 3.4, in the sense that it appears as the push-forward of a codimension 2 Brill-Noether locus in $\mathcal{M}_{g,2}$ under the map $\pi_2 : \mathcal{M}_{g,2} \to \overline{\mathcal{M}}_{g,1}$ forgetting the second point.

For an integer $d \geq 3$ we set $g := 2d - 4$. We define the following codimension 1 locus in the universal curve

$$TR := \{(C, p) \in \mathcal{M}_{g,1} : \exists f \in G_d^1(C), \exists x \in C - \{p\} \text{ such that } d_f^1(p) \geq 3 \text{ and } d_f^1(x) \geq 3\};$$

that is, $TR$ is the locus of 1-pointed curves $(C, p)$ for which there exists a degree $d$ map $f : C \to \mathbb{P}^1$ having triple ramification at the marked point $p$ and at some unmarked point.
\( x \in C, x \neq p \). Clearly, \( TR = \pi_2(M_{g,2,d}(0.2,(0.2))) \). Since \( \rho(q,1,d,(0.2),(0.2)) = -2 \)
the expected codimension of \( M_{g,2,d}(0.2,(0.2)) \) inside \( M_{g,2} \) is 2 and it is easy to see that this is also the actual codimension, hence \( TR \) is a divisor on \( M_{g,1} \). We shall compute the class \( [TR] \in \text{Pic}_{\mathbb{C}}(\mathcal{M}_{g,1}) \) of the closure of \( TR \) in \( \mathcal{M}_{g,1} \).

We close this chapter by proving in Section 3.7 the following

**Theorem 3.1** For \( g = 11, 12, 15 \) the Kodaira dimension of the universal curve \( C_g \) is \( -\infty \).

### 3.2 The Picard group of the moduli space of pointed curves

We review a few facts about the Picard groups of the moduli spaces \( \mathcal{M}_{g,n} \), when \( g \geq 3 \) and \( n \geq 1 \). The main references are [AC3] and [Lo] (but also [Mod], for a very comprehensible discussion on divisor classes on moduli stacks). All Picard groups we consider are with rational coefficients; in particular we have isomorphisms \( \text{Pic}_{\mathbb{C}}(\mathcal{M}_{g,n}) \cong \text{Pic}_{\mathbb{C}}(\mathcal{M}_{g,n}) \cong \mathbb{A}^2(\mathcal{M}_{g,n}) \). Here by \( \text{Pic}_{\mathbb{C}} \) we understand the Picard group of the moduli stack (functor). From now on we denote \( \text{Pic}_{\mathbb{C}}(\mathcal{M}_{g,n}) \) by \( \text{Pic}(\mathcal{M}_{g,n}) \).

For \( 1 \leq i \leq n \) let us denote by \( \pi_i : \mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,n-1} \) the morphism which forgets the \( i \)-th point. We denote by \( v_i \in \text{Pic}(\mathcal{M}_{g,n}) \) the class on the moduli stack which associates to every \( n \)-pointed family of curves \( (f : C \rightarrow B, \sigma_1, \ldots, \sigma_n : B \rightarrow C) \) the class of the line bundle \( \sigma_i^*(\omega_f) \) on \( B \), where \( \omega_f = \omega_{C/B} \) is the relative dualizing sheaf of \( f \).

For \( 0 \leq i \leq [g/2] \) and \( A \subseteq \{1, 2, \ldots, n\} \), we denote by \( \delta_{i,A} \) the irreducible divisor on \( \mathcal{M}_{g,n} \) whose general point is a reducible curve of two components, one of genus \( i \), the other of genus \( g-i \), meeting transversally at a point and such that the genus \( i \) component contains precisely the marked points corresponding to \( A \). The index set \( A \) is subject to the obvious conditions \( \text{card}(A) \geq 2 \) if \( i = 0 \) and \( 1 \leq A \) for \( i = g/2 \). We denote by \( \delta_{i,A} \) the class on the moduli stack associated to the divisor \( \Delta_{i,A} \). We also write \( \delta_{i,A} = \delta_{g-i,A} \), where \( A' := \{1, \ldots, n\} - A \), as well as \( \delta_i = \delta_{g-i,\emptyset} \). The key result is the following:

**Proposition 3.2.1 (Harer, Arbarello-Cornalba)** For \( g \geq 3 \) and \( n \geq 1 \) the group \( \text{Pic}(\mathcal{M}_{g,n}) \) is freely generated by the class of the Hodge line bundle \( \lambda \) and by the classes \( v_i \) for \( 1 \leq i \leq n \) and the boundary classes \( \delta_{i,A} \), where \( 0 \leq i \leq [g/2] \) and \( A \subseteq \{1, \ldots, n\} \).

We briefly discuss now the cases that are of interest to us, i.e. when \( n \in \{1, 2\} \). If \( n = 1 \) we denote by \( \pi : \mathcal{M}_{g,1} \rightarrow \mathcal{M}_g \) the natural projection. Clearly \( v = \omega = c_1(\omega_\tau) \): also

\[
\pi^*(\lambda) = \lambda, \quad \pi^*(\delta_0) = \delta_0 \quad \text{and} \quad \pi^*(\delta_i) = \delta_i + \delta_{g-i}, \quad \text{for } 1 \leq i \leq [g/2].
\]

If \( i = [g/2] \) we have that \( \pi^*(\delta_{g/2}) = \delta_{g/2} \).

When \( n = 2 \) we look at the maps \( \pi_i : \mathcal{M}_{g,2} \rightarrow \mathcal{M}_{g,1} \) forgetting one of the points. One has that \( v_i = c_1(\omega_\tau_1) + \delta_{0,\{1,2\}} \). As for the pullbacks, we will need the formulas (cf. [Lo])

\[
\pi_2^*(\lambda) = \lambda, \quad \pi_2^*(\delta_0) = \delta_0, \quad \pi_2^*(v) = c_1 - \delta_{0,\{1,2\}} \quad \text{and} \quad \pi_2^*(\delta_{i,A}) = \delta_{i,A} + \delta_{i,A,\{1,2\}}.
\]
3.3 Counting linear series on curves via Schubert calculus

In order to compute the class of Brill-Noether divisors on $\overline{M}_{g,n}$, one has to figure out the intersection numbers of such divisors with various curves in $\overline{M}_{g,n}$. Typically, we have to answer questions like:

Given $g, r, d$ and $\alpha^1, \ldots, \alpha^n$ Schubert indices of type $(r, d)$ such that $\mu(g, r, d, \alpha^1, \ldots, \alpha^n) = 0$, how many $g'_d$'s with prescribed ramification at the $s$ marked points does a general $s$-pointed curve of genus $g$ have? In other instances one has to compute the number of $g'_d$'s having prescribed ramification at unprescribed points.

In each case we will solve such problems using Schubert calculus: we let our curves degenerate to curves of compact type that are unions of $\mathbb{P}^1$'s and have elliptic tails and no other components. Computing the number of limit $g'_d$'s with special properties on such curves boils down to computations in the cohomology rings of certain Grassmannians.

We now outline the method. We use as references [GH] and [EH5], while [F] is a reference for general properties of Schubert cycles in Grassmannians. Let $C$ be an algebraic curve of genus $g$, let $p \in C$ be a point and $I = (\mathcal{L}, V) \in G^r_d(C)$. To this data we associate the following strictly decreasing flag in $H^0(C, \mathcal{L}/\mathcal{L}(-(d + 1)p)) \cong \mathcal{O}^{d+1}$:

$$\mathcal{F}(p) : H^0(C, \mathcal{L}/\mathcal{L}(-(d + 1)p)) = W_0 \supset W_1 \supset \ldots \supset W_d \supset W_{d-1} = 0,$$

where $W_i = H^0(C, \mathcal{L}(-ip)/\mathcal{L}(-(d + 1)p))$. If $\alpha = (\alpha_0, \ldots, \alpha_s)$ is a Schubert index of type $(r, d)$, the condition $\alpha'(p) \geq \alpha$ is equivalent with $V$ belonging to the Schubert cycle $\sigma_\alpha \subseteq G(r, H^0(C, \mathcal{L}/\mathcal{L}(-(d + 1)p))) = G(r, d)$ defined w.r.t. the flag $\mathcal{F}(p)$ (see Section 1.3 for our way of denoting Schubert cycles which basically coincides with that from [GriffHa] except that we write the indices in reversed order). For instance, $p$ is a ramification point of $l$ if and only if $V \in \sigma_{(0, \ldots, 0, 1)}$.

Let us now consider the special case $C = \mathbb{P}^1$. There is only one line bundle of degree $d$ on $\mathbb{P}^1$, namely $\mathcal{L} = \mathcal{O}(d)$. Making once and for all the identification $\mathcal{O}^{d+1} = H^0(\mathbb{P}^1, \mathcal{O}(d))$, the variety $G_0^r(\mathbb{P}^1)$ is just the Grassmannian $G(r, d)$ of projective $r$-planes in $\mathbb{P}^d = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(d)))$. For each point $p \in \mathbb{P}^1$ we have the flag $\mathcal{F}(p)$ with respect to which we can define Schubert cycles in $G(r, d)$.

Let $p_1, \ldots, p_s \in \mathbb{P}^1$ be distinct general points and $\alpha^1, \ldots, \alpha^s$ Schubert indices of type $(r, d)$. For each $1 \leq i \leq s$ we consider the Schubert cycle $\sigma_{\alpha^i} \subseteq G(r, d)$ defined in terms of $\mathcal{F}(p_i)$. It is proved in [EH5] that $\sigma_{\alpha^1} \ldots \sigma_{\alpha^s}$ are dimensionally transverse: every component of $\cap_{i=1}^s \sigma_{\alpha^i}$ has codimension

$$\sum_{i=1}^s \text{codim} (\sigma_{\alpha^i}, G(r, d)) = \sum_{i=1}^s \alpha^i,$$

and in particular $\cap_{i=1}^s \sigma_{\alpha^i} = \emptyset$ if and only if $\sigma_{\alpha^1} \ldots \sigma_{\alpha^s} = 0$ in $H^*(G(r, d), \mathbb{Z})$. This shows that there exists a $g'_d$ on $\mathbb{P}^1$ having ramification $\geq \alpha^i$ at $p_i$ for $i = 1, \ldots, s$ if and only if $\sigma_{\alpha^1} \ldots \sigma_{\alpha^s} \neq 0$. 

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Proposition 3.3.1 Let $\alpha_1, \ldots, \alpha^*_{s}$ be Schubert indices of type $(r, d)$ such that
\[
\sum_{i=1}^{s} \sum_{j=0}^{r} \alpha'_j + rg = (r + 1)(d - r),
\]
and $p_1, \ldots, p_g, x_1, \ldots, x_s \in \mathbb{P}^1$ distinct general points. Then, the variety of $g'_d$'s on $\mathbb{P}^1$ having cusps at the points $p_1, \ldots, p_g$ and ramification $\geq \alpha'$ at $x_i$ for $i = 1, \ldots, s$ is reduced, 0-dimensional and consists of $\sigma^g_{(0,1,\ldots,1)}\sigma_{\alpha'_1}\cdots\sigma_{\alpha^*}$ points.

Proof: This has been basically settled in [EH5]. To be precise, Eisenbud and Harris proved a similar statement for the Schubert cycles of the form $\sigma_{(0,\ldots,0,\alpha)}$ rather than $\sigma_{(0,\ldots,1)}$, but by duality we obtain the claimed statement from theirs. Just use that the dual of the cycle $\sigma_{(0,\ldots,1)}$ in $G(r, d)$ is the cycle $\sigma_{(0,\ldots,0,r)}$ in $G(d - r - 1, d)$.

We will repeatedly use the following formula (cf. [GH, page 269]): if $\alpha = (\alpha_0, \ldots, \alpha_r)$ is a Schubert index of type $(r, d)$ such that $\sum_{i=0}^{r} \alpha_i + rg = (r + 1)(d - r)$ we have that
\[
\sigma_{(\alpha_0,\ldots,\alpha_r)} \sigma^g_{(0,1,\ldots,1)} = g! \prod_{c<j} (\alpha_j - \alpha_i + j - i) \prod_{l=0}^{r} (g - d + i + \alpha_i + r)!.
\]

(3.1)

Another formula that we will find quite useful is the classical Plücker formula (see [Mod, page 257]): If $C$ is a smooth curve of genus $g$ and $l$ a $g^r_d$ on $C$, then
\[
\sum_{p \in C} w^l(p) = (r + 1)d + (r + 1)r(g - 1).
\]

(3.2)

where $w^l(p)$ is the weight of $p$ in $l$ (cf. Chapter 1).

3.4 Divisors on $\overline{M}_{g,1}$

In this section we compute the classes of the divisors $\overline{C'}$ and $\overline{H'}$ of curves with a marked points that is either a cusp or a hyperflex. We will use Theorem 4.1 from [EH2] which gives informations about the subspace of Pic($\overline{M}_{g,1}$) generated by the classes of the Brill-Noether divisors.

Fix $g \geq 3$ and let us denote by $\pi : \overline{M}_{g,1} \rightarrow \overline{M}_g$ the natural projection. Inside Pic($\overline{M}_{g,1}$) one can look at the subspace generated by the classes of all divisors $\overline{M}_{g,1}(\alpha)$, where $\alpha = (\alpha_0, \ldots, \alpha_r)$ is a Schubert index of type $(r, d)$ such that $\rho(g, r, d) - \sum_{l=0}^{r} \alpha_i = -1$. We note that the Brill-Noether loci we consider on $\overline{M}_{g,1}$ will have exactly one divisorial component (cf. [EH2] Theorem 1.2) and possibly some other lower dimensional components. It is not known whether the Brill-Noether divisors on $\overline{M}_{g,n}$ with $n \geq 2$ are irreducible or not.

One distinguished Brill-Noether divisor on $\overline{M}_{g,1}$ is the locus of Weierstrass points
\[
W := \{ [C, p] \in \overline{M}_{g,1} : p \text{ is a Weierstrass point of } C \}.
\]

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Clearly $W = \mathcal{M}_{g,1}^+(0, g - 1)$, that is, the locus of those $(C, p)$ for which there is a $g^1_4$ with total ramification at $p$. The class of the closure $\overline{W}$ in $\overline{\mathcal{M}}_{g,1}$ has been computed (cf. Cuk):

$$[\overline{W}] = -\lambda + \frac{g(g+1)}{2} \nu - \sum_{i=1}^{g-1} \frac{(g-i)(g-i+1)}{2} \delta_i.$$

Another divisor class we consider on $\overline{\mathcal{M}}_{g,1}$ is

$$BN := (g+3)\lambda - \frac{g+1}{6} \delta_0 - \sum_{i=1}^{g-1} i(g-i) \delta_i.$$

Here, $BN$ is (modulo multiplication by a positive rational constant) the class of the pullback of any Brill-Noether divisor $\overline{\mathcal{M}}_{g,d}$ on $\overline{\mathcal{M}}_g$ when $\rho(g,r,d) = -1$. The class $BN$ is effective when there are Brill-Noether divisors on $\overline{\mathcal{M}}_g$ and this happens precisely when $g+1$ is composite. When $g+1$ is prime, it is not clear whether $BN$ is effective (as a matter of fact, the slope conjecture (see end of Chapter 1) predicts it is not). For $g+1$ prime (in particular we can then write $g = 2k - 2$), the effective divisor on $\overline{\mathcal{M}}_g$ having the largest slope known to this date, is the closure of the locus

$$E_k^1 := \{ [C] \in \mathcal{M}_g : \exists g^1_4 \text{ on } C \text{ such that } 2g^1_4 \text{ is special } \}.$$

The slope of $E_k^1$ is $(6k^2 + k - 6)/k(k-1)$ (cf. [EH3]). A general curve of genus $2k - 2$ has $(2k-2)!/(k!(k-1)!)$ linear systems $g^1_4$, and $E_k^1$ is the locus of curves $C$ for which the scheme $G_k^1(C)$ is nonreduced. This happens when two $g^1_4$'s come together, so that we get a $g_4^1$ with dim $2g^1_4 \geq 3$, or equivalently by Riemann-Roch, $2g^1_4$ is special.

We have the following remarkable result of Eisenbud and Harris (cf. [EH2]):

**Proposition 3.4.1** The Brill-Noether subspace in $\text{Pic}(\overline{\mathcal{M}}_{g,1})$ is two-dimensional, generated by the classes $[\overline{W}]$ and $BN$.

**Remark:** More generally, Logan has proved in [Lo] that for $n \geq 1$ the subspace of $\text{Pic}(\overline{\mathcal{M}}_{g,n})$ generated by the classes of the Brill-Noether divisors on $\overline{\mathcal{M}}_{g,n}$ has dimension $1 + \binom{n+1}{2}$.

We now compute the classes of the divisors $\overline{HF}$ and $\overline{CC}$.

**Theorem 3.2** Let $g \equiv 1 \text{ mod } 3$ be an integer $\geq 4$ and set $d := (2g + 7)/3$. We have the following relations in $\text{Pic}(\overline{\mathcal{M}}_{g,1})$:

1. $[\overline{HF}] = c (a \lambda + b \nu - \epsilon_0 \delta_0 - \sum_{i=1}^{g-1} i \delta_i)$

where

$$a = 2(g^3 + 7g^2 - 10g - 166), \quad b = 15g(g-2)(g-d+6), \quad \epsilon_0 = (g^2 + 9g + 23)(g-d+1).$$
\[\epsilon_r = (g - i)(5g^2 + 2ig^2 + 3ig + 45g - 110 - 74 - i) \quad \text{and} \]
\[c = 8(g - 3)!/( (g - d + 6)! (g - d + 3)! (g - d + 1)! ).\]

2. \([\overline{C\mathcal{U}}] = c'\left((g + 4) \lambda + g \psi - \frac{g^2}{6} \delta_0 - \sum_{i=1}^{g} (i + 1)(g - i) \delta_i \right).\]

where \(c' = 24(g - 2)!/( (g - d + 5)! (g - d + 3)! (g - d + 1)! ).\)

**Remark:** Let us try to understand the meaning of our formulas in the simplest case, \(g = 4, d = 5\). By substitution we get

\[\overline{HF} = -2\lambda + 20\psi - 12\delta_1 - 6\delta_2 - 2\delta_3.\]

On the other hand, on a curve \(C\) of genus 4 every \(g \lambda^4\) is of the form \(|K_C - x|\), for some \(x \in C\). The marked point \(p \in C\) is a hyperflex of \(|K_C - x|\) if and only if \(h^0(C, 4p + x) \geq 3\).

This implies that \(p\) is a Weierstrass point of \(C\) and \(x\) is one of the 2 points in the effective divisor \(K_C - 4p\). Therefore \(\overline{HF} = 2 \overline{W}\), and this can also be seen by comparing our formula to Cukierman’s (cf. [Cuk]):

\[\overline{W} = -\lambda + 10\psi + -6\delta_1 - 3\delta_2 - \delta_3.\]

As for the other divisor, when \(g = 4\), one finds that

\[CU = \{(C, p) \in C_4 : \text{there exists } x \in C \text{ such that } h^0(C, 2p + x) \geq 2 \}.\]

Our formula gives in this case \([\overline{C\mathcal{U}}] = 8\lambda + 4\psi - 6\delta_1 - 6\delta_2 - 4\delta_3\). Note that the class of the divisor \(CU\) on \(C_4\) already appears in [Fa] page 423 (that is the \(\{\lambda, \psi\}\) part in our formula).

**Proof:** 1) We start by computing \([\overline{HF}]\) which is technically a bit more difficult than computing \([\overline{C\mathcal{U}}]\).

Since \(\overline{HF} = \overline{M}_{g,1}^2 ((0, 0, 2))\) is a Brill-Noether divisor, by applying Prop. 4.1 it follows that there are rational constants \(\nu, \mu\) such that

\[\overline{HF} = \mu B N + \nu [\overline{W}].\]  \hspace{1cm} (3.3)

and we just have to determine the coefficients \(\mu\) and \(\nu\). We use the method of test curves, i.e. we intersects both sides of (3) with curves in \(\overline{M}_{g,1}\): we write down 1-dimensional families of 1-pointed curves of genus \(g\) and compute the degrees of \(\lambda, \psi\) and the \(\delta\)'s on that curve as well as the degree of \(\overline{HF}\). We need two test curves in \(\overline{M}_{g,1}\) which will provide two linear equations in \(\mu\) and \(\nu\). Since it is pretty difficult to write down explicit families of curves of genus \(g\) with smooth general member, most of the test curves we use, will be entirely contained in the boundary \(\overline{M}_{g,1} - \overline{M}_{g,1}\).

We obtain the first test curve as follows: Take a general curve \(B\) of genus \(g - 1\) and
a general 2-pointed elliptic curve \((E, 0, p)\). We get a 1-dimensional family in \(\Delta_1 \subseteq \mathcal{M}_g\) by identifying the fixed point \(0 \in E\) with a variable point \(q \in B\), the marked point being the fixed point \(p \in E\). Let us denote by \(\{X_q := E \cup_q B, p \in E\}_{q \in B}\) the resulting family.

The degrees of the generators of \(\text{Pic}(\mathcal{M}_g)\) on the family we have just constructed, are as follows:

\[
\deg(\lambda) = 0, \quad \deg(\nu) = 0, \quad \deg(\delta_i) = -\deg(K_B) = 4 - 2g.
\]

while \(\delta_0\) and \(\delta_i\) for \(2 \leq i \leq g - 1\) all vanish. Next we evaluate \(\deg(HF)\), that is the degree of the divisor \(HF\) on the curve in \(\mathcal{M}_g\) we have written down.

Let us take \((X_q = E \cup_q B, p \in E)\) a member of our family. Then \([X_q, p] \in HF\) if and only if there exists a (smoothable) limit \(g_2^d\) on \(X_q\), say \(l\), with vanishing \(\geq (0, 1, 4)\) at \(p\). Using the additivity of the Brill-Noether number we have that:

\[
-1 \geq \rho(l, \alpha^{l}(q)) \geq \rho(l_B, \alpha^{l_B}(q)) + \rho(l_E, \alpha^{l_E}(q), \alpha^{l_E}(p)).
\]

Since \(\rho(l_E, \alpha^{l_E}(q), \alpha^{l_E}(p)) \geq 0\) (if we assume \(p - q \in \text{Pic}^0(E)\) not to be a torsion class) and also \(\rho(l_B, \alpha^{l_B}(q)) \geq 1\) (because \([B] \in \mathcal{M}_g-1\) is general), it follows that \(\rho(l_B, \alpha^{l_B}(q)) = -1\) that \(\rho(l_E, \alpha^{l_E}(q), \alpha^{l_E}(p)) = 0\) and \(\alpha^{l_E}(p) = (0, 1, 4)\). By using Prop. 1.4.1 we have that \(d - 1 \leq a^{l_E}_i(q) + a^{l_E}_i(q) \leq d\) for \(i = 0.1, 2\) and there are two cases two consider:

1st case: \(a^{l_E}(q) = (d - 4, d - 2, d - 1)\) from which \(\alpha^{l_E}(q) = (1, 2, 4)\). By `The Regeneration Theorem' (cf. Chapter 1), all these linear series are smoothable. In order to compute the contribution to \(\deg(HF)\) in this case, we have to count how many points \(q \in B\) there are such that there is a \(g_2^{d-1}\) on \(B\) with ordinary ramification at \(q\).

At this point, one might worry about the multiplicities with which we count such linear series. It turns out that all multiplicities we encounter during this proof are equal to 1. The reason is that if we denote by \((f : X \to B, \bar{p} : B \to \mathcal{X})\) the versal deformation space of \((X_q, p)\), then in a similar way to the proof of Lemma 3.4 from [EH2], one can show that the variety \(G^d_2(X/B, (\bar{p}, (0.0, 2)))\) of \(g_2^d\)'s with hyperflexes on 1-pointed curves nearby \((X_q, p)\), is transversal to our test curve.

By standard Brill-Noether theory, \(B\) possesses \(2(g - 1)!/(g - d + 2)!/(g - d + 3)!/(g - d + 4)!\) linear series \(g_2^{d-1}\). By Plücker's formula (3.2) each such \(g_2^{d-1}\) has \(3d + 6g - 15\) ramification points (all ordinary, since \(B\) is general). We thus get a contribution of

\[
\frac{2(3d + 6g - 15)(g - 1)!}{(g - d + 2)!/(g - d + 3)!/(g - d + 4)!}.
\]

2nd case: \(a^{l_E}(q) = (d - 5, d - 2, d)\), hence \(\alpha^{l_E}(q) = (0, 2, 5)\). Once again, all these linear series are smoothable and the contribution to \(\deg(HF)\) we get in this case, is the number of \(g_2^d\)'s on a general curve of genus \(g - 1\) with vanishing sequence \((0.2.5)\) at an unspecified point. We now determine this number.

Let the curve \(B\) degenerate in a 1-dimensional family \(\{B_t\}\) having smooth generic fibre and as special fibre, a curve of compact type \(B_0 := \mathbb{P}^1 \cup E_1 \cup \ldots \cup E_{g-1}\), where \(E_i\) are general elliptic curves. \(\{p_i\} = E_i \cap \mathbb{P}^1\) and \(p_1, \ldots, p_{g-1} \in \mathbb{P}^1\) are general points. We count the number of limit \(g_2^d\)'s on \(B_0\) with ramification \((0, 1, 3)\) at some unspecified point...
Note that in principle we could have $x = p_i$ for some $i$, that is, there exists $x_t \in B_t$ and a family of 2-dimensional linear series $l_t \in G^2_d(B_t, (x_t, (0, 1, 3)))$ for $t \neq 0$, such that $\lim_{t \to 0} x_t = p_i$. (i.e. the hyperflexes $x_t$ specialize to a node of the central fibre). In this case however, by making a finite base-change, blowing-up sufficiently often the nodes of $B_0$ and resolving the resulting singularities, we obtain a new generically smooth family $(B'_t, x'_t)$ and linear series $l'_t \in G^2_d(B'_t, (x'_t, (0, 1, 3)))$ for $t \neq 0$, such that no ramification point of $l'_t$ specializes to a node of $B'_0$. The central fibre $B'_0$ is derived from $B_0$ by inserting chains of $\mathbb{P}^1$’s at the nodes of $B_0$. Since this operation (explained in [EH1]) does not change the Brill-Noether theory of the central fibre, we may assume from the beginning that all ramification has been swerved away from the nodes.

Conversely, using semicontinuity of fibre dimension for the space of (limit) $g^2_d$’s with hyperflexes on curves nearby $B_0$, we find that all limit $g^2_d$ on $B_0$ with vanishing $\geq (0, 1, 4)$ at an unmarked point, are smoothable to every nearby curve in a way that maintains the hyperflex. This shows that the number of limit $g^2_d$’s on $B_0$ having a hyperflex, is the same as the number of (honest) $g^2_d$’s with a hyperflex on a general curve of genus $g - 1$.

Let $l$ be a limit $g^2_d$ on $B_0$ with ramification $\geq (0, 1, 3)$ at a smooth point $x$. In general, from the Plücker formula it follows that for any linear series $g^2_d$ on $\mathbb{P}^1$ and for any number of points $y_1, \ldots, y_m \in \mathbb{P}^1$, the inequality $\rho(l_{g^2_d}, \alpha(y_1) \ldots \alpha(y_m)) \geq 0$ holds. Using this observation and that on $B_0$ we have $\rho(l, \alpha^l(x)) = -1$, by additivity it follows that $x$ must be on one of the elliptic tails, say $x \in E_1$. Then we must have $\rho(l_{E_1}, \alpha^{l_{E_1}}(x), \alpha^{l_{E_1}}(p_i)) = -1$. $\rho(l_{E_1}, \alpha^{l_{E_1}}(p_i)) = 0$ for $2 \leq i \leq g - 1$ and $\rho(l_{E_1}, \alpha^{l_{E_1}}(p_1), \ldots, \alpha^{l_{E_1}}(p_{g-1})) = 0$. This means that the aspect $l_{E_1}$ has cusps at the points $p_2, \ldots, p_{g-1}$. As for the $E_1$-aspect of $l$ there are three possibilities:

- $a^{l_{E_1}}(p_1) = (d - 5, d - 2, d - 1)$, so $a^{l_{E_1}}(p_1) = (1, 2, 5)$. Then clearly $3p_1 + 2x \sim 5x$, so $3x \sim 3p_1$ on $E_1$. On $\mathbb{P}^1$ we have (after subtracting the base point $p_1$) a $g^2_d$ with ramification $(0, 0, 2)$ at $p_1$ and cusps at $p_2, \ldots, p_{g-2}$. According to Section 3.3 the number of such linear series is $\sigma(0, 0, 2) \sigma^{g-2}_{(0, 1, 1)}$ (the product is taken in $H^{top}(G(2, d - 1), \mathcal{Z})$). Since there are 8 choices for $x \in E_1$ and $x$ can lie on any of the $g - 1$ elliptic tails. using formula (3.1), we get a total contribution of

$$8(g - 1)^2 \sigma(0, 0, 2) \sigma^{g-2}_{(0, 1, 1)} = \frac{96(g - 1)!}{(g - d + 3)! (g - d + 2)! (g - d + 1)!}.$$  \hspace{1cm} (3.5)

- $a^{l_{E_1}}(p_1) = (d - 6, d - 2, d)$, so $a^{l_{E_1}}(p_1) = (0, 2, 6)$. Then $2x \sim 2p_1$ on $E_1$, and in this case we obtain a contribution of

$$3(g - 1) \sigma(0, 1, 4) \sigma^{g-2}_{(0, 1, 1)} = \frac{144(g - 1)!}{(g - d + 6)! (g - d + 2)! (g - d)!}.$$  \hspace{1cm} (3.6)
• $a^{d^2_1}(p_1) = (d - 5, d - 3, d)$, hence $a^{d^2_1}(p_1) = (0, 3, 5)$. Then $5x \sim 5p_1$ on $E_1$ and the contribution to $\deg(HF)$ is

$$24(g - 1) \sigma_{(0, 2, 3)} \sigma_{(0, 1, 1)}^{g - 2} = \frac{720(g - 1)!}{(g - d + 5)! (g - d + 3)! (g - d)!}.$$  (3.7)

By adding (3.4), (3.5), (3.6) and (3.7), we obtain that

$$\deg(HF) = \frac{16(g - 1)! (7g^2 + 48g - 184)}{(g - d + 6)! (g - d + 3)! (g - d + 2)!}.$$  (3.8)

Since from (3.3) we have that $\deg(HF) = 2(g - 1)(g - 2)\mu + g(g - 1)(g - 2)\nu$, the equation (3.8) provides one linear relation between $\mu$ and $\nu$.

In order to obtain a second relation, we use as test curve a general fibre of the map $\pi : \overline{M}_{g, 1} \to \overline{M}_g$. We fix $C$, a general curve of genus $g$ and let $p \in C$ vary. For this family of course $\deg(\nu) = 2g - 2$, while $\lambda$ and all the $\delta$'s vanish. We also need $\deg(HF)$ which is just the number of $g^2_\ell$'s on a general curve of genus $g$ having a hyperflex at an unspecified point.

To compute this number we let $C$ degenerate to $C_0 := \mathbb{P}^1 \cup E_1 \cup \ldots \cup E_g$, where $E_i$ are general elliptic curves. $\{p_1\} = E_i \cap \mathbb{P}^1$ and $p_1, \ldots, p_g \in \mathbb{P}^1$ are general points. We count limit $g^2_\ell$'s on $C_0$ with vanishing $\geq (0, 1, 4)$ at some point $x \in C_0$. As before, it turns out that all these $g^2_\ell$'s are smoothable and no two $g^2_\ell$ on smooth curves nearby $C_0$ coalesce. Plücker's formula forces the point $x$ to sit on an elliptic tail.

Take $\ell$ a limit $g^2_\ell$ on $C_0$ with a hyperflex at a point $x$ and assume that $x \in E_1$. It is straightforward to see that $a^{d^2_1}(x) = (0, 2, 4)$ and $a^{d^2_1}(p_1) = (d - 4, d - 2, d)$, from which $4p_1 \sim 4x$, which gives 15 choices for $x \in E_1$. On the spine $\mathbb{P}^1$ we have to count $g^2_\ell$'s with vanishing sequence $(0, 2, 4)$ at $p_1$ and cusps at $p_2, \ldots, p_g$. The number of such linear series is $\sigma_{(0, 1, 2)} \sigma_{(0, 1, 1)}^{g - 1}$, the product being computed in $H^{top}(G(2, d), \mathbb{Z})$. Since $x$ can sit on any of the tails $E_1, \ldots, E_g$, we get that

$$\deg(HF) = 15g \sigma_{(0, 1, 2)} \sigma_{(0, 1, 1)}^{g - 1} = \frac{240g!}{(g - d + 5)! (g - d + 3)! (g - d + 1)!}.$$  (3.9)

which immediately gives

$$\nu = \frac{240(g - 2)!}{(g + 1) (g - d + 5)! (g - d + 3)! (g - d + 1)!}. $$

By plugging in we obtain $\mu$ as well, hence $[HF]$ too.

2) In order to compute $[CT]$ we could use exactly the same test curves employed for the computation of $[HF]$, but there is a shorter way of doing things in this case. Thankfully, our results do not depend on which method we choose.

We fix a general elliptic curve $E$ and consider the map $j : \overline{M}_{g, 1} \to \overline{M}_{g+1}$ given by
\[ j([B, p]) := [B \cup_p E] \text{ (attaching an elliptic tail). Then } \overline{CU} = j^*(\overline{M}_{g-1, d}). \]  
As already pointed out in Chap. 1, we have
\[ [\overline{M}_{g-1, d}] = f((g + 4) \lambda - (g + 2) / 6 \delta_0 - \sum_{i=1}^{g(g+1)/2} i(g + 1 - i) \delta_i), \]
where \( f = 3a/(2g - 2) \), with \( a \) being the number of \( g \)'s on a 1-pointed curve of genus \( g - 1 \) and having ramification \((0, 1, 2)\) at the marked point. By degenerating the 1-pointed curve of genus \( g - 1 \) to \((\mathbb{P}^1 \cup E_1 \cup \ldots \cup E_g \cup p \subset \mathbb{P}^1)\), where the \( E_i \) are elliptic, \( \{p_i\} = \mathbb{P}^1 \cap E_i \) for \( i = 1, \ldots, g \) and \( p, p_1, \ldots, p_g \subset \mathbb{P}^1 \) are general points, we see that \( a = \sigma_{(0,1,2)} \sigma_{(0,1,1)}^{g-1} \in H^{top}(G(2, d), \mathbb{Z}) \), from which we obtain
\[ f = \frac{24(g - 2)!}{(g - d + 3)! (g - d + 3)! (g - 1)!}. \]
The pullback \( j^* \) acts on the generators of \( \text{Pic}(\overline{M}_{g+1}) \) as follows:
\[ j^*(\delta_0) = \delta_0, \quad j^*(\lambda) = \lambda, \quad j^*(\delta_i) = -u + \delta_{g-1} \text{ (by adjunction)}, \quad j^*(\delta_i) = \delta_{g-1} + \delta_{g-i-1} \quad \text{for} \quad i \geq 2. \]
We get immediately the stated formula for \( [\overline{CU}] \). □

### 3.5 A divisor on \( \overline{M}_{g, 2} \)

Using results from the previous section we compute the class of the divisor of curves with 2 marked points that are flexes of a 2-dimensional linear series on the curve:

**Theorem 3.3** Let \( g \equiv 1 \pmod{3} \) be an integer \( \geq 4 \) and set \( d := (2g + 7)/3 \). We have the following formula in \( \text{Pic}(\overline{M}_{g, 2}) \):
\[ [F L] = c^0(A \lambda + B (\psi_1 + \psi_2) - C \delta_0 - D \delta_0 \{1, 2\} - \sum_{i=1}^{g-1} a_i \delta_{i \{1\}} - \sum_{i=1}^{g-1} b_i \delta_{i \{2\}}), \]
where
\[ A = 6(g^3 + 9g^2 - 2g - 140), \quad B = 6(g + 1)(g + 11)(g - 2), \quad C = g^3 + 7g^2 - 10g - 76, \]
\[ D = 12g(g - 2)(g + 11), \quad b_i = 6(g - i)((g^2 + 4g)(i + 2) - (32i + 44)), \]
\[ a_i = 6(g^3(i + 1) - g^2(i^2 - 4i - 10) - g(4i^2 + 32i + 13) + (32i^2 - 22)) \quad \text{and} \]
\[ c^0 = 4(g - 3)!/((g - d + 6)! (g - d + 3)! (g - d + 1)!). \]

**Remark:** When \( g = 4, d = 5 \) we have that
\[ FL = [\{C, p_1, p_2\} \in \mathcal{M}_{4, 2} : \text{there exists } x \in C \text{ such that } h^0(C, x + 3p_i) \geq 2 \text{ for } i = 1, 2]. \]

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Our formula gives in this case

\[ \overline{FL} = 6\lambda - 15(\epsilon_1 + \epsilon_2) - \delta_0 - 24\delta_0(1,2) - 15(\delta_1(1)) + \delta_2(1,1) + \delta_1(1,2) - 18\delta_1(1,2) - 12\delta_2(1,2) - 6\delta_3(1,2). \]

**Proof of Theorem 3.3:** We determine the coefficients in the expression of \( \overline{FL} \) in three steps. First, we consider the map \( \pi_2 : M_{g,2} \to M_{g,1} \) which forgets the second point. We claim that

\[ (\pi_2)_*(\overline{FL} \cdot \delta_0(1,2)) = [\overline{HF}] - [C]U. \]  

This is almost obvious: If \( X = C \cdot q \cdot P^1 \cdot p_1 \cdot p_2 \), with \( p_1, p_2 \in P^1 \), is a general point in \( \overline{FL} \cap \Delta_{0(1,2)} \), then there exists \( l \), a limit \( g_i^l \) on \( X \) having flexes at \( p_1 \) and \( p_2 \). Using the additivity of the Brill-Noether number on \( X \) we have that such a linear series is refined. That \( \rho(l_{\pi_2}, \alpha^{l}(p_1), \alpha^{l}(p_2), \alpha^{l}(q)) = 0 \) and \( \rho(l_c, \alpha^{l}(q)) = -1 \). It follows that \( w^{l}(q) = 2 \) and this happens if either \( \alpha^{l}(q) = (0, 1, 4) \) and then \( [C, q] \in \overline{HF} \), or \( \alpha^{l}(q) = (0, 2, 3) \) (and then \( [C, q] \in \overline{CU} \)). On \( P^1 \) on the other hand, there is precisely one \( g_i^l \) with flexes at both \( p_1 \) and \( p_2 \) and vanishing \( \langle d - 4, d - 1, d \rangle \) (resp. \( \langle d - 3, d - 2, d \rangle \) at the point \( q \), so indeed \( (\pi_2)_*(\overline{FL} \cdot \delta_0(1,2)) = [\overline{HF}] + [C]U \). We use (3.10) together with Theorem 3.2 to determine a few coefficients in the expression of \( \overline{FL} \).

Let us write \( \overline{FL} = A + B (\epsilon_1 + \epsilon_2) - C \delta_0 - D \delta_0(1,2) - \sum_i g_i \delta_i(1) - \sum_i b_i \delta_i(2) \).

Since \( (\pi_2)_*(\lambda \cdot \delta_0(1,2)) = \lambda \), \( (\pi_2)_*(\delta_0(1,2)) = -\epsilon_i \cdot (\pi_2)_*(\delta_0 \cdot \delta_0(1,2)) = \delta_i(1) \) for \( 1 \leq i \leq g - 1 \), from (3.10) and Theorem 3.2 we obtain the coefficients \( A, C, D, D \) and \( b_i \) for \( 1 \leq i \leq g - 1 \).

In order to get the coefficients of \( \epsilon_1 \) and \( \epsilon_2 \), we use the following test curve: fix \( C \) a general curve of genus \( g \). When \( p_2 \) hits \( p_1 \), by blowing-up we insert \( p_1 \in C \) be a general fixed point and \( p_2 \in C \) a variable point. When \( p_2 \) hits \( p_1 \), by blowing-up we insert \( p_1 \) at \( C \) as a general point \( d = 6g - 7 \) ramification points different from \( p_1 \), thus we get that

\[ \deg(\overline{FL}) = (3d + 6g - 7)\sigma_{0(0,1)} \cdot \sigma_{0(1,1)}^2 = \frac{(3d + 6g - 7)g!}{(g - d + 5)! (g - d + 3)! (g - d + 2)!}. \]

We have in this way the relation \( 2gB - D = \deg(\overline{FL}) \), which allows us to determine \( B \).

We are left with the task of determining the coefficients \( a_i \), of \( \delta_i(1) \). for \( 1 \leq i \leq g - 1 \).

For this purpose, we use a new test curve. Take \( (B, q) \) a general 1-pointed curve of genus \( i \) and \( (C, q, p_2) \) a general 2-pointed curve of genus \( g - i \). We take \( X := B \cup_q C \) and consider as marked points a moving point \( p_1 \in B \) and the fixed point \( p_2 \in C \). For this family we have that

\[ \deg(\epsilon_1) = 2i - 1, \quad \deg(\epsilon_2) = 0, \quad \deg(\delta_{i-1}(1,2)) = 1, \quad \deg(\delta_i(1)) = -1. \]
while \( \lambda \) and the remaining \( \delta \)'s vanish. Thus we have the equation
\[
a_i = -(2i - 1)B + b_{g-i} + \deg(FL).
\] (3.11)
where the only unknown in the left-hand side is \( \deg(FL) \) which we now determine.

We have to solve the following problem: let \((Y, p)\) a general 1-pointed curve of genus \( g \) degenerating to \((X = B \cup_g C, p_2)\). There are \( \sigma_{(0,0,1)}, \sigma_{(0,1,1)}^g \) linear series \( g_2^g \) on \( Y \) having a flex at \( p \). Fix one of them. How many of its ramification points will end up on the genus \( i \) component \( B \) as \( Y \) specializes to \( X' \)?

To answer this we let \( X \) further degenerate to \( X' := E_1 \cup \ldots \cup E_g \), a string of \( g \) elliptic curves, the marked point, call it \( r_0 \) being on \( E_1 \). We assume that the component \( C \) of \( X \) degenerates to \( \cup_{j=1}^g E_j \), whereas \( B \) degenerates to \( \cup_{j=g-i+1}^g E_j \). If \( \{r_i\} = E_i \cap E_{i+1} \) assume \( r_i - r_{i-1} \in \text{Pic}^0(E_i) \) is not a torsion class. Pick 1 one of the limit \( g_2^g \)'s on \( X' \) that have a flex at \( r_0 \). Because of our assumptions, \( \rho(l_{E_i}, \alpha_{E_i}(r_{i-1}), \alpha_{E_i}(r_i)) = 0 \). For \( 1 \leq i \leq g \). By Plücker, the aspect \( l_{E_i} \) has \( 8 \) flexes which are smooth points of \( X' \), which means that there will be \( 8i \) flexes on the components \( E_{g-i+1} \ldots E_g \), hence finally.

\[
\deg(FL) = 8i \sigma_{(0,0,1)} \sigma_{(0,1,1)}^g = \frac{48ig!}{(g-d+5)! (g-d+3)! (g-d+2)!}.
\]

Substituting in (3.11) we have the coefficients \( a_i \) as well. We have determined all terms in the expression of \([FL]\).

Remark: We discuss now the case \( g = 22 \), \( d = 17 \), which as we already pointed out, was the initial motivation for computing \([FL]\).

One tries to show that the Kodaira dimension of the moduli space \( \mathcal{M}_{22,2} \) is \( \geq 0 \) by exhibiting an explicit effective multicanonical divisor. Recall (cf. Chap. 1) that on \( \mathcal{M}_{23} \) the Brill-Noether divisor \( \mathcal{M}_{23,17}^2 \) of curves with a \( g_2^g \), is multicanonical (modulo a positive combination of the classes \( \delta_i \), for \( i \geq 1 \)). Therefore, it seemed possible that on \( \mathcal{M}_{22,2} \), the divisor \( F \mathcal{L} \) of 2-pointed curves with a \( g_2^g \); having flexes at both marked points, would be multicanonical as well.

The canonical class \( K_{M_{g,2}} \) can be computed easily (see also [Lo]): if \( \pi_2 : \mathcal{M}_{g,2} \to \mathcal{M}_{g,1} \) and \( \pi : \mathcal{M}_{g,1} \to \mathcal{M}_g \) are the natural maps, then \( K_{M_{g,2}} = \pi^*_2(K_{M_{g,1}}) + c_1(\omega_{M_{g,1}}) \) and \( K_{M_{g,1}} = \pi^*(K_{M_g}) + c_1 \), which gives.

\[
K_{\mathcal{M}_{g,2}} = 13\lambda + \nu - 2\delta_0 - 3(\delta_1 + \delta_{g-1}) - 2 \sum_{i=2}^{g-2} \delta_i,
\]
and

\[
K_{\mathcal{M}_{g,1}} = 13\lambda + \nu + 2\delta_0 - 2\delta_{0;2} - 3 \sum_A \delta_{1,A} - 2 \sum_{i \geq 2,A} \delta_{i,A}.
\]

To make computations easier to handle, we introduce the following notation: for \( D_1 \) and \( D_2 \) divisor classes on \( \mathcal{M}_{g,n} \), we write

\[
D_1 \geq_{\delta} D_2 \iff D_1 - D_2 \text{ is a non-negative combination of the classes } \delta_{i,A} \text{, where } i \geq 1.
\]

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Our Theorem 3.3 shows that in the case $g = 22$, we have that
\[
\frac{247}{2} \lambda + \frac{253}{2} (\epsilon_1 + \epsilon_2) - 242 \delta_{0,(1,2)} - \frac{229}{12} \delta_0 \geq 0,
\]
(the left-hand side is modulo boundary classes a rational multiple of $[FL]$). Other known effective divisor classes on $\overline{M}_{22,2}$ are the following:

- The class of the Weierstrass divisor $\text{Wei} := \pi_1^*(\overline{W}) - \pi_2^*(\overline{W})$, that is the closure of the locus of those $[C, p_1, p_2]$ for which either $p_1$ or $p_2$ is a Weierstrass point of $C$. One has that $[\text{Wei}] \leq \delta - 2 \lambda + 253 (\epsilon_1 + \epsilon_2) - 506 \delta_0$.

- The pullback of the divisor $E_{12}^1$ from $\overline{M}_{22}$, that is, the closure of the locus of those $[C, p_1, p_2]$ for which $C$ has a $g_1^2$ such that $\dim 2g_1^2 \geq 3$. One has that $[E_{12}^1] \leq \delta 870 \lambda - 132 \delta_0$.

- The closure of the locus $D := \{[C, p_1, p_2] \in \overline{M}_{22,2} : h^0(C, 11p_1 + 11p_2) \geq 2\}$. One has (cf. [Lo]) that $[D] \leq -\lambda + 66 (\epsilon_1 + \epsilon_2) - 253 \delta_{0,(1,2)}$.

One can then show that $[FL]$ is not expressible as a positive linear combination of $[\text{Wei}], [E_{12}^1]$ and $[D]$, so by knowing $[FL]$ we really extend the knowledge of the effective cone on $\text{Pic}(\overline{M}_{22,2})$.

On the other hand, one sees that $K_{\overline{M}_{22,2}}$ cannot be expressed as a positive combination of the four effective classes mentioned before. Although our computation of $[FL]$ provides a new effective divisor class on $\overline{M}_{22,2}$, this enlargement is not big enough to include the canonical class. In the spirit of the slope conjecture that predicts that on $\overline{M}_g$ the effective divisors of lowest slope are the Brill-Noether divisors, since $K_{\overline{M}_{22,2}}$ lies outside the Brill-Noether subspace in $\text{Pic}(\overline{M}_{22,2})$, we make the following:

**Conjecture 2** The Kodaira dimension of $\overline{M}_{22,2}$ (and hence that of $\overline{M}_{22,1}$) is $-\infty$.

### 3.6 The divisor of curves with two triple ramification points

In this section we compute the class of the divisor $\overline{TR}$ of 1-pointed curves that admit a map to $\mathbb{P}^1$ having both the marked point and some unspecified point as triple ramification points. Let us fix an integer $d \geq 3$ and we set $g := 2d - 4$. For a general 1-pointed curve $(C, p)$ of genus $g$ the variety of pencils $G_d^0(C)$ is an irreducible smooth surface. Among the $\infty^2$ pencils of degree $d$ there are finitely many $l \in G_d^0(C)$ for which $p$ is a triple ramification point, that is, $g_1^0(p) \geq 3$. Moreover, all linear series $l$ satisfying this condition are complete, base point free and all ramification apart from $p$ is ordinary. Imposing the condition that there exists a degree $d$ map $f : C \to \mathbb{P}^1$ with two triple ramification points, one of which is marked, we obtain a codimension 1 condition on $\overline{M}_{g,1}$. We have the following:
Theorem 3.4 Let \( d \geq 3 \) be an integer and set \( g := 2d - 4 \). We have the following relation in \( \text{Pic}(\overline{\mathcal{M}}_{g,1}) \):

\[
[TR] = m \left( a \lambda + b \psi - \sum_{i=0}^{g-1} c_i \delta_i \right),
\]

where

\[
\begin{align*}
a &= 2(18d^3 - 39d^2 - 120d + 290), \quad b = 8(6d^2 - 28d + 35)(2d - 4), \\
c_0 &= 6d^3 - 24d^2 + 13d + 30, \\
c_i &= 2(2d - 4 - i)(24d^2 + 9id^2 - 112d - 42id + 140 + 50i) \quad \text{for } i \geq 1 \quad \text{and} \\
m &= 6(2d - 6)!/(d! (d-3)!) .
\end{align*}
\]

Remark: In the simplest case \( d = 3, g = 2 \) our formula gives the relation

\[
[TR] = 80\lambda + 10\delta_0 - 120\lambda .
\]

where \( \lambda \) is a boundary class on \( \mathcal{M}_{2,1} \), namely \( \lambda = \delta_0/10 + \delta_1/5 \) (cf. [EH3]). For genus 2 one has the following interpretation for our divisor

\[
TR = \{ [C, p] \in \mathcal{M}_{2,1} : \exists x \in C : x \neq p, \text{ such that } 3p \sim 3x \}.
\]

During the proof of Theorem 3.4 we will need the result for the particular case \( g = 2 \), hence we will settle this case independently.

3.6.1 Counting pencils with two triple points

In order to determine the intersection multiplicities of \( TR \) and various test curves in \( \overline{\mathcal{M}}_{g,1} \) we will need certain enumerative results contained in the following result:

Proposition 3.6.1 1) Let \( (C, p, q) \) be a general 2-pointed curve of genus \( 2d - 6 \) with \( d \geq 3 \). The number of pencils \( g_d \) on \( C \) having triple points at both \( p \) and \( q \) is

\[
F(d) = (2d - 6)! \left( \frac{1}{(d-3)!} - \frac{1}{d!} \frac{1}{(d-6)!} \right).
\]

2) Let \( C \) be a general curve of genus \( 2d - 4 \) with \( d \geq 3 \). The number of pencils \( g_d \) on \( C \) having triple ramification at some distinct points \( x, y \in C \) is

\[
N(d) = \frac{48(6d^2 - 28d + 35)(2d - 4)!}{d! (d-3)!}.
\]

Remarks: 1. In the expression of \( F(d) \) we make the convention \( 1/n! = 0 \) for \( n < 0 \).
2. For \( d = 3 \) our formula gives \( N(3) = 80 \), that is, for a general curve \( C \) of genus 2 there are 160 = 2 · 80 pairs of points \( (x, y) \in C \times C \) such that \( 3x \sim 3y \). This can also be seen directly by considering the map \( \psi : C \times C \to \text{Pic}^0(C) \) given by \( \psi(x, y) = \mathcal{O}_C(3x - 3y) \). Then \( \psi^*(0) = \frac{1}{2} \int_{C \times C} \omega \wedge \omega = 2 \cdot 3^2 \cdot 3^2 = 162 \), where \( \omega \) is a
differential form representing $\theta$.

To get the answer to our enumerative question we have to subtract from 162 the contribution of the diagonal $\Delta \subseteq C \times C$. This excess intersection contribution is equal to 2 (cf. [Dj]), so in the end we get $160 = 162 - 2$ pairs of distinct points $(x, y) \in C \times C$ with $3x \sim 3y$.

**Proof:** 1) We let $(C, p, q)$ degenerate to the following 2-pointed curve of compact type $\{C_0 := \mathbb{P}^1 \cup E_1 \cup \ldots \cup E_{2d-6}, p_0, q_0\}$, where $E_i$ are general elliptic curves, $\{p_i\} = E_i \cap \mathbb{P}^1$ and $p_1, \ldots, p_{2d-6}, p_0, q_0 \in \mathbb{P}^1$ are general points. We have to count the number of limit $g_d'$'s on $C_0$ having triple ramification at $p_0$ and $q_0$. This is the same as the number of $g_d'$'s on $\mathbb{P}^1$ having cusps (i.e., ordinary ramification) at $p_1, \ldots, p_{2d-6}$ and triple ramification at $p_0$ and $q_0$. By Prop.3.3.1 this number is $\sigma_{10.21}^2 \sigma_{(0.1)}^{2d-6}$ (in $H^{top}(G(1, d), \mathbb{Z})$). This product can be computed using formula (v) at the bottom of page 273 in [F] and one has that

$$\sigma_{10.21}^2 \sigma_{(0.1)}^{2d-6} = (2d - 6)! \left( \frac{1}{(d - 3)!} - \frac{1}{d! (d - 6)!} \right).$$

2) Once more, we let $C$ degenerate to $C_0 = \mathbb{P}^1 \cup E_1 \cup \ldots \cup E_{2d-1}$, where $E_i$ are general elliptic curves, $\{p_i\} = \mathbb{P}^1 \cap E_i$ and $p_1, \ldots, p_{2d-1} \in \mathbb{P}^1$ are general points. We count limit $g_d'$'s on $C_0$ with vanishing $\geq (0, 3)$ at two distinct points $x, y \in C_0$. Let $l$ be such a limit $g_d'$. By a standard argument we have already outlined before, we can assume that both $x$ and $y$ are smooth points of $C_0$ and by the additivity of the Brill-Noether number we obtain that $x, y$ must lie on the tails $E_i$. Since $E_i$ are general, we can assume that $j(E_i) \neq 0$ (that is, none of the $E_i$'s is the Fermat cubic), hence there can be no $g_d'$ on $E_i$ with three triple points. There are two cases:

a) There are $1 \leq i < j \leq 2d - 4$ such that $x \in E_i$ and $y \in E_j$. Then $d^{E_i}(p_i) = d^{E_j}(p_j) = (d - 3, d)$, hence $3x \sim 3p_i$ on $E_i$ and $3y \sim 3p_j$ on $E_j$. There are 8 choices for $x \in E_i$, 8 choices for $y \in E_j$ and $(2d - 4)$ choices for the tails $E_i$ and $E_j$ containing the triple points. On $\mathbb{P}^1$ we count $g_d'$'s with cusps at $\{p_1, \ldots, p_{2d-1}\} - \{p_i, p_j\}$ and triple points at $p_i$ and $p_j$. This number is again $\sigma_{10.21}^2 \sigma_{(0.1)}^{2d-6}$, so we get in this case a contribution of

$$64 \binom{2d - 4}{2} \sigma_{10.21}^2 \sigma_{(0.1)}^{2d-6} = 32(2d - 4)! \left( \frac{1}{(d - 3)!} - \frac{1}{d! (d - 6)!} \right).$$

b) There is $1 \leq i \leq 2d - 4$ such that $x, y \in E_i$. We distinguish between two cases here:

$b_{1})$ $d^{E_i}(p_i) = (d - 3, d - 1)$. On $\mathbb{P}^1$ we count $g_d'$'s with cusps at $p_1, \ldots, p_{2d-1}$ and this number is $\sigma_{10,1}^{d-4}$ (in $H^{top}(G(1, d - 1), \mathbb{Z})$). On $E_i$ we have to compute the number of $g_d'$'s having triple ramification at some unspecified points $x, y \in E_i - \{p_i\}$ and which also have simple ramification at $p_i$. Let us denote $(E_i, p_i) = (E, p)$. If we regard $p \in E$ as the origin of $E$, then the translation $(x, y) \rightarrow (y - x, -x)$ establishes a bijection between the set of pairs $(x, y) \in E \times E - \Delta, x \neq p \neq y$ such that there is a $g_d'$ in which $x, y, p$ appear with multiplicities $3, 3$ and 2 respectively, and the set of pairs $(u, v) \in E \times E - \Delta$ with $u \neq p \neq v$ such that there is a $g_3'$ in which $u, v, p$ appear with multiplicities $3, 2$ and 3 respectively. The latter set has obviously cardinality 16, hence the number of pencils $g_d'$

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we are counting is $8 = 16/2$. All in all we have a contribution of

$$8(2d - 4) \sigma_{(0,1)}^{2d-4} = \frac{8(2d - 4) (2d - 4)!}{(d - 2)! (d - 1)!}. \quad (3.14)$$

$b_2) \ a^\varepsilon(p_i) = (d - 4, d)$. This time, on $\mathbb{P}^1$ we look at $\mathfrak{g}_i$,s with cusps at $\{p_1, \ldots, p_{2d-4}\} \setminus \{p_i\}$ and a 4-fold point at $p_i$. Their number is $\sigma_{(0,3)}^5 \sigma_{(0,1)}^{2d-5}$ (in $H^{top}(\mathbb{G}(1, d), \mathbb{Z})$). On $E_i$ we shall compute the number of $\mathfrak{g}_i$,s for which there are distinct points $x, y \in E_i - \{p_i\}$ such that $p, x, y$ appear with multiplicities 4, 3, and 3 respectively. Again, for simplicity we denote $(E_i, p_i) = (E, p)$ and we proceed as follows: We consider $\Sigma$ the closure in $E \times E$ of the locus

$$\{(u, v) \in E \times E - \Delta : \exists l \in G^1(E) \text{ such that } a^l_1(p) = 4, a^l_1(u) \geq 3, a^l_1(v) \geq 2\}.$$

The class of the curve $\Sigma$ can be computed readily. If $F_i$ denotes the numerical equivalence class of a fibre of the projection $\pi_i : E \times E \to E$, for $i = 1, 2$, then

$$\Sigma \sim 10F_1 + 5F_2 - 2\Delta. \quad (3.15)$$

The coefficients in this expression are determined by intersecting $\Sigma$ with $\Delta$ and the fibres of $\pi_i$. One has that $\Sigma \cap \Delta = \{(x, x) \in E \times E : x \neq p, 4p \sim 4x\}$ and $\Sigma \cap \pi_2^{-1}(p) = \{(y, p) \in E \times E : y \neq p, 3p \sim 3y\}$. It is easy to check that these intersections are transversal, hence $\Sigma : \Delta = 15, \Sigma : F_2 = 8$ whereas obviously $\Sigma : F_1 = 3$ and these relations yield (3.15).

The number of pencils $l \subseteq \{p\}$ having two extra triple points will then be equal to $1/2 \#(\text{ramification points of } \pi_2 : \Sigma \to E) = \Sigma^2/2 = 20$. We have obtained in this case a contribution of

$$20(2d - 4) \sigma_{(0,3)}^5 \sigma_{(0,1)}^{2d-5} = \frac{20(2d - 4)!}{(d - 4)! d!}. \quad (3.16)$$

Adding together (3.13), (3.14) and (3.16), we obtain the stated number $N(d)$. \hfill $\square$

### 3.6.2 A divisor class on $\overline{M}_{2,1}$

Here we compute the class of $\overline{M}$ when $g = 2$. We have the following:

**Proposition 3.6.2** Let us consider the divisor

$$TR = \{[C, p] \in \overline{M}_{2,1} : \exists x \in C - \{p\} \text{ such that } 3x \sim 3p\}.$$ 

Then $[\overline{M}] = 80\Upsilon + 10\delta_0 - 120\lambda$.

**Proof:** There are a few ways to compute $[\overline{M}]$. One is to consider the map $j : \overline{M}_{2,1} \to \overline{M}_4$ given by $j([B, p]) := [B \cup_p C_0]$, where $(C_0, p)$ is a general 1-pointed curve of genus 2. On $\overline{M}_4$ we have the divisor of curves with an abnormal Weierstrass point, that is,

$$D := \{[C] \in \overline{M}_4 : \exists x \in C \text{ such that } h^0(C, 3x) \geq 2\}.$$
On knows (cf. [Di]) that
\[
[D] = 264\lambda - 30\delta_0 - 96\delta_1 - 128\delta_2. \tag{3.17}
\]

We claim that \(j^*(D) = TR + 16 W\). Indeed, let \([B, p] \in \mathcal{M}_{2,1}\) be such that \(j([B, p]) \in D\). Then there is a limit \(g^*_1\) on \(X = B \cup C_0\), say \(l\), which has a point of total ramification at some \(x \in X\). There are two cases depending on whether \(x\) lies on \(C_0\) or on \(B\).

If \(x \in B\), then \(d^{e_0}(p) = (0, 3)\), hence \(l^{e_0} = 3p\) (and there is a single choice for \(l_{C_0}\)). While on \(B\) we have the linear equivalence \(3p \sim 3x\). Hence \([B, p] \in TR\).

If \(x \in C_0\), then \(d^{e_0}(p) = (1, 3)\), i.e. \(p \in B\) is a Weierstrass point and \(l_{B} = p + 2p\). On \(C_0\) we have \(d^{e_0}(p) = (0, 2)\) and \(d^{e_0}(x) = (0, 3)\), so there exists \(y \in C_0 - \{p, x\}\) such that \(3x \sim 2p + y\). To compute the number of such points \(x \in C_0 = C\), we intersect the curves \(f_1(C)\) and \(f_2(C)\) inside \(\text{Pic}^3(C)\), where \(f_1 : C \to \text{Pic}^3(C)\) and \(f_2 : C \to \text{Pic}^3(C)\) are given by \(f_1(t) = \mathcal{O}_C(3t)\) and \(f_2(t) = \mathcal{O}_C(2p + t)\) respectively. Clearly \([f_1(C)] = 9\theta\) and \([f_2(C)] = 6\theta\), hence \(f_1(C) \cdot f_2(C) = 18\). However, we have to discard from this intersection the point \(\mathcal{O}_C(3p)\) at which the condition \(x \neq p\) is no longer satisfied. At this point the curves \(f_i(C)\) have a common tangent line and using Lemma 6.2 of [Di] one gets that the intersection multiplicity at \(\mathcal{O}_C(3p)\) is actually 2. The answer to our enumerative problem is thus 16 = 18 - 2.

We have proved that \(j^*(D) = TR + 16 W\). From this and from (3.17) we get the expression for \([TR]\) if we take into account that \(j^*(\delta_0) = \delta_0\), \(j^*(\delta_1) = \delta_1\), \(j^*(\delta_2) = -\psi\) and \(j^*(\lambda) = \lambda = \delta_0/10 + \delta_1/5\).  \(\square\)

### 3.6.3 The class of the divisor \(TR\)

We now compute the class of the divisor \(TR\) in \(\text{Pic}(\mathcal{M}_{g,1})\):

**Proof of Theorem 3.4:** By Prop. 3.2.1 there are rational constants \(A, B, C_0, \ldots, C_{g-1}\) such that the following relation holds:

\[
[TR] = A \lambda + B \psi - \sum_{i=0}^{g-1} C_i \delta_i.
\]

We first consider the map \(\phi : \mathcal{M}_{0,g-1} \to \mathcal{M}_{g,1}\) obtained by associating to a \((g + 1)\)-pointed curve \((R, p_0, \ldots, p_g)\) of genus 0 the 1-pointed curve \((C, p_0)\), with \(C = R \cup E_1 \cup \ldots \cup E_g\) and \(\{p_i\} = E_i \cap \mathbb{F}^1\) for \(1 \leq i \leq g\), where \(E_i\) are general elliptic curves. We show that \(\phi(\mathcal{M}_{0,g-1}) \cap \overline{TR} = \emptyset\). Then by using Lemma 4.2 from [EH2] we obtain relations between the coefficients of \(\overline{TR}\): for \(i \geq 1\),

\[
C_i = \frac{(g - i)(g - i - 1)}{g(g - 1)} B + \frac{i(g - i)}{g - 1} C_{g-1}.
\]

Note that Lemma 4.2 as stated in [EH2] is not applicable to the divisor \(\overline{TR}\), but a brief inspection of its proof shows that its conclusions are valid for any divisor on \(\mathcal{M}_{g,1}\) whose support is disjoint from \(\text{Im}(\phi)\), hence for \(\overline{TR}\) too.
The proof that $\text{Im}(\sigma) \cap T\mathcal{R} = \emptyset$ is an immediate application of Prop.1.4.1 together with Plücker's formula (3.2).

Next, we take the map $j : \mathcal{M}_{2,1} \to \mathcal{M}_{g,1}$ sending a 1-pointed curve $(B.q)$ of genus 2 to $(X = B \cup_q C_0.p)$, where $(C_0,p,q)$ is a general 2-pointed curve of genus $g-2$. The pull-back $j^*$ acts on the generators of the Picard group as follows: $j^*(\lambda) = \lambda$, $j^*(\upsilon) = 0$, $j^*(\delta_0) = \delta_0$, $j^*(\delta_{g-2}) = -\upsilon$, $j^*(\delta_{g-1}) = \delta_1$ and $j^*(\delta_i) = 0$ for $1 \leq i \leq g-3$. Since on $\mathcal{M}_{2,1}$ we also have the relation $\delta_1 = 5\lambda - \delta_0/2$, we obtain

$$j^*(\langle T\mathcal{R} \rangle) = (A - 5C_{g-1}) \lambda + (C_{g-1}/2 - C_0)\delta_0 + C_{g-2}\upsilon.$$  (3.18)

We now compute $j^*(\overline{T\mathcal{R}})$. Let us take $[B.q] \in j^*(\overline{T\mathcal{R}})$. Then there exists $l$, a limit $g_d$ on $X = B \cup_q C_0$ and a point $x \in X$ such that $a_1^l(x) \geq 3$ and $a_1^l(p) \geq 3$.

If $x \in C_0$, then $\rho(l_d.a_1^l(q)) = -1$ and we get that $q \in B$ is a Weierstrass point. The multiplicity with which $\overline{W}$ appears in $j^*(\overline{T\mathcal{R}})$ is the number of $g_d$'s on $C_0$ in which $p,q$ and an unspecified point $x \neq p,q$ appear with multiplicities 3, 2 and 3 respectively. By Schubert calculus this number is

$$n_1 = 8(2d - 6) \sigma^2_{(0,2)} \sigma^{2d-6}_{(0,1)} = 8(2d - 6)(2d-6)! \left( \frac{1}{(d-3)!^2} - \frac{1}{d!(d-6)!} \right).$$

If $x \in B$, then we have the linear equivalence $3q \sim 3r$. Then there exists $l$, a limit $g_d$ on $X = B \cup_q C_0$ and a point $x \in X$ such that $a_1^l(x) \geq 3$ and $a_1^l(p) \geq 3$.

We have thus obtained that $j^*(\overline{T\mathcal{R}}) = n_1 \overline{W} + n_2 \overline{T\mathcal{R}_2}$, which according to (3.18) provides three new relations between $A$ and the $C_i$'s.

Finally we determine the coefficient $B$. It is enough to intersect $\overline{T\mathcal{R}}$ with a general fibre of the map $\pi : \mathcal{M}_{g,1} \to \mathcal{M}_g$ and to divide the intersection number by $2g - 2$. The intersection number is twice the number of $g_d$'s on a general curve $C$ of genus $2d - 4$ having two points of triple ramification. By Prop.3.6.1 this number is $2\overline{N}(d)$, hence we obtain that $B = \overline{N}(d)/(2d - 5)$. 

\[
3.7 \quad \text{The Kodaira dimension of the universal curve}
\]

At the beginning of Chapter 1 we recounted attempts by various people to understand the birational geometry of $\mathcal{M}_g$, in particular to compute its Kodaira dimension. Similar questions can be asked about the moduli spaces $\mathcal{M}_{g,n}$. Obviously the problem is non-trivial only for $g \leq 23$: since for $g \geq 24$ the moduli space $\mathcal{M}_g$ is of general type, the spaces $\mathcal{M}_{g,n}$ with $n \geq 1$ will be of general type too.

The case $g = 23$ turns out to be quite easy too: since the relative dualizing sheaf of the map $\pi : \overline{\mathcal{M}}_{g,n} \to \mathcal{M}_{23}$ is big one only needs the effectiveness of $K_{\overline{\mathcal{M}}_{g,n}}$ to conclude that $K_{\overline{\mathcal{M}}_{g,n}} = c + \pi^*(K_{\overline{\mathcal{M}}_{23}})$ is big too, hence $\mathcal{M}_{23,n}$ is of general type for all $n \geq 1$. In the case $4 \leq g \leq 22$, Logan computed a number $f(g)$ such that for all $n \geq f(g)$ the
moduli space $\mathcal{M}_{g,n}$ is of general type (see [Lo]).

We shall content ourselves with the case $n = 1$ which we approach from a different angle: it is known that for $g \leq 16$, $g \neq 14$, the Kodaira dimension of $\mathcal{M}_g$ is $-\infty$. Is there a similar result for the universal curve $\mathcal{C}_g = \mathcal{M}_{g,1}$ in this range?

The problem is almost trivial for $g \leq 10$: the universal curve is unirational for these genera. To see this, one can easily adapt Severi’s argument about the unirationality of $\mathcal{M}_g$. This is also remarked in [Lo]. For most remaining cases we have the following:

**Theorem 1** For $g = 11, 12, 15$ the Kodaira dimension of the universal curve $\mathcal{C}_g$ is $-\infty$.

**Proof**: We assume that $\kappa(\mathcal{C}_g) \geq 0$, i.e., some multiple of the canonical divisor $K_{\mathcal{C}_g}$ is effective. We are going to reach a contradiction with some estimates for the slope $s_g$ of $\mathcal{M}_g$ (see end of Chapter 1 for the definition of $s_g$).

We denote as usual by $\pi : \overline{\mathcal{C}}_g \to \mathcal{M}_g$ the natural projection and we have seen that

$$K_{\mathcal{C}_g} = 13\lambda + \psi - 3(\delta_1 + \delta_{g-1}) - 2 \sum_{i=2}^{g-2} \delta_i.$$

Assume there exists $m > 0$ such that $mK_{\mathcal{C}_g}$ is effective. We consider the divisor of Weierstrass points $\overline{\mathcal{W}} \subset \overline{\mathcal{C}}_g$, whose class, we recall, is

$$[\overline{\mathcal{W}}] = -\lambda + g(g+1)/2 \psi - \sum_{i=1}^{g-1} (g-i)(g-i+1)/2 \delta_i.$$

Clearly $mK_{\mathcal{C}_g}$ cannot contain $\overline{\mathcal{W}}$ with arbitrarily high multiplicity. In fact, it suffices to choose $a \in \mathbb{Z}_{\geq 1}$ such that $ag(g+1)/2 > m$ and then $a\overline{\mathcal{W}} \not\subseteq mK_{\mathcal{C}_g}$. Indeed, otherwise the divisor $mK_{\mathcal{C}_g} - a\overline{\mathcal{W}}$ would have negative degree on the fibres $\pi^{-1}[C]$, where $[C] \in \mathcal{M}_g$ is arbitrary and this is impossible.

After choosing such an $a$, we consider the push-forward $D := \pi_* (mK_{\mathcal{C}_g} - a\overline{\mathcal{W}})$, which is an effective divisor on $\overline{\mathcal{M}}_g$. In particular, we have that $s_D \geq s_g$, where by $s_D$ we denote the slope of $D$. Since

$$[D] = ma \pi_* (K_{\mathcal{C}_g} : [\overline{\mathcal{W}}]) \geq \delta (13g^3 + 6g^2 - 9g - 2)\lambda - 1/2 g(g+1)(4g-3)\delta,$$

we obtain that

$$s_D = \frac{2(13g^3 + 6g^2 - 9g + 2)}{g(g+1)(4g-3)}.$$

On the other hand, it is known that $s_{11} = 7(= 6 + 12/(g+1))$, $s_{12} \geq 41/6 = 6.83...$ (cf. [Tan]) and $s_{15} \geq 6.667$ (cf. [CR4]). The values of $s_D$ are $6.61...$ for $g = 11$, $6.61...$ for $g = 12$ and $6.59...$ for $g = 15$, hence in each of these cases we have found an effective divisor on $\overline{\mathcal{M}}_g$ having slope $\geq s_g$, which is a contradiction.

We note that there is also a bound $s_{16} \geq 6.56$ (cf. [CR4]), but for this genus we have $s_D = 6.58...$, so we cannot conclude that $\kappa(\mathcal{C}_{16}) = -\infty$. \qed