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Insurance with multiple insurers: A game-theoretic approach

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Abstract: This paper studies the set of Pareto optimal insurance contracts and the core of an insurance game. Our setting allows multiple insurers with translation invariant preferences. We characterise the Pareto optimal contracts, which determines the shape of the indemnities. Closed-form and numerical solutions are found for various preferences that the insurance players might have. Determining associated premiums with any given optimal Pareto contract is another problem for which economic-based arguments are further discussed. We also explain how one may link the recent fast growing literature on risk-based optimality criteria to the Pareto optimality criterion and we show that the latter is much more general than the former one, which according to our knowledge, has not been pointed out by now. Further, we extend some of our results when model risk is included, i.e. there is some uncertainty with the risk model and/or the insurance players make decisions based on divergent beliefs about the underlying risk. These robust optimal contracts are investigated and we show how one may find robust and Pareto efficient contracts, which is a key decision-making problem under uncertainty.

Keywords and phrases: Risk management; Pareto optimal insurance; Cooperative game theory; Robust decision-making.

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1 Introduction

This paper studies the optimal insurance contract design, where contracts following from bargaining amongst multiple insurers. A *policyholder (PH)* seeks to insure a part of its risk in a market where multiple insurers are present. We assume that insurance contracts are *Pareto optimal*, implying that there is no alternative contract that is better for all parties and strictly better for at least one party. Moreover, the aggregate premium that the PH pays is shared amongst the insurers. We propose a cooperative game to determine a set of stable allocations that leads to a set of premiums. This set of stable allocations is obtained via the *core* of an appropriate cooperative game. The core is originally introduced for general cooperative games by Gillies (1953) and Scarf (1967).

Whereas most of the traditional literature on insurance contract design focuses on optimal contracts with given premium functions (Borch, 1960; Arrow, 1963), we assume that indemnities and premiums follow from bargaining amongst the insurers. This approach is in line with Raviv (1979), who focuses on Pareto optimal insurance contracts with expected utilities.[1] We assume that all agents are endowed with a specific class of translation invariant risk measures, of which dual utility (Yaari, 1987) is a canonical example. Maximising dual utility is equivalent to minimising a distortion risk measure, as introduced by Wang et al. (1997). Dual utilities are often used to represent the preferences of corporations as regulatory requirements are based on a distortion risk measure in the Swiss Solvency Test regulation for insurers (a conditional Value-at-Risk), and a popular insurance premium principle is the distortion premium principle (Wang, 1996).

In contrast to Raviv (1979) that imposes an expected value principle bound on the premiums, our approach proposes a two-stage process that separates the indemnities and premiums via Pareto optimality: Pareto optimality yields a particular shape of the indemnities, but not on the premiums. We are allowed to do this when the preferences are translation invariant, whereas Raviv (1979) focusses on expected utilities. We use a game-theoretic approach to determine a set of premiums. Game-theoretic approaches to optimal reinsurance contract design are not new in case of expected utility preferences. In particular, Baton and Lemaire (1981) investigate the core in reinsurance markets. Moreover, Suijs *et al.* (1998, 1999) study the core of insurance markets under the restriction that insurance contracts are proportional.

In this paper, we focus on risk sharing of insurance contracts, where there are multiple insurers

[1] Borch (1962) and DuMouchel (1968) study Pareto optimal risk sharing in a reinsurance markets. In reinsurance markets, the only constraint on risk sharing contracts is that all risk is redistributed. There do not appear constraints such as the non-negativity of premiums and the constraint is that the insurance coverage cannot be larger than the underlying loss.

that are willing to sell insurance to a PH. In insurance, there are constraints on indemnities that are used for a convex optimisation problem to determine Pareto optimal contracts. We allow for a very general class of insurance indemnities. We show that Pareto optimality provides us with a structure on the indemnity contracts, but not on the corresponding prices. For a Pareto optimal contracts, we study the core and anti-core of an appropriate cooperative game to select a range of premiums. The insurance contracts must be such that no subgroup of insurers has a joint incentive to stay in the market, while paying the other insurers their maximum joint welfare gain. The maximum joint welfare gain represents the maximum aggregate profit of this subgroup of insurers in a market without the other insurers. We show that this stability criterion leads us to allocations in the anti-core of a cooperative game. We show that the anti-core is non-empty and provide a closed-form expression in case the agents are endowed with dual utilities.

When the objectives are given by risk measures, Pareto optimality is also studied by Boonen *et al.* (2016a) and Cai *et al.* (2017). Boonen *et al.* (2016a) discard the non-negativity property of prices. Such constraints change the underlying method to solve Pareto optimal contracts, but we show that the outcomes are similar. Cai *et al.* (2017) focus on Pareto optimality of insurance arrangements, but with given premium functionals. The approach with given premium functionals has been popular in the related literature of optimal reinsurance contract design. These kind of assumptions facilitate obtaining explicit solutions of such optimal contracts, but it could be criticised for their ad-hoc mathematical representation. On the contrary, our approach aims to let the premium be part of the insurance contract to bargain for rather than imposing a rigid premium setting. We show that optimal reinsurance contracts are Pareto optimal, where optimal reinsurance contracts are obtained by optimising the preferences of one party under participation constraints of the other parties. There is a fast growing literature on optimal reinsurance contract design (see, e.g., Young, 1999; Balbás *et al.*, 2011; Chi, 2012; Asimit *et al.*, 2013; Cui *et al.*, 2013; Bernard *et al.*, 2015). We are the first to make a connection between Pareto optimality and optimal reinsurance contract design.

The uncertainty with choosing the right model, also known as *model risk*, is an important issue and cannot be ignored. The common practical issue is data scarcity, which represents a standard source for model error. Expert opinion is quite often another way of choosing a model believed to be the “best” possible one, which is limited to the past experience and individual perspectives about future outcomes. Another source of model error is given by proxy models, models that are socially accepted within a profession and widely spread in regulation standards. The underlying model is not unknowable, but unknown and the decision-maker is exposed to a higher level of uncertainty and robust decisions are desirable. Encapsulating the model error in the decision-making process may not be done in the same fashion amongst all decision-makers. That is, there might be divergent beliefs about the concurrent risk models that various decision-makers validate as plausible models. The classical theory of robust statistics (for example, see Huber, 1964; Huber and Ronchetti, 2009) helps the decision-maker to produce a robust model choice and this is usually possible if data scarcity is not present. Another disadvantage of this method when solving an optimisation problem is that the ultimate goal is to identify a robust optimal decision and not necessary a robust model choice. This is precisely why the robust optimisation becomes a standard method to resolve the issue of optimisation under uncertainty. Within the optimal insurance problem, some attempts appeared recently in Balbás *et al.* (2015) and Asimit *et al.* (2017a). Wilson (1968), Acciaio and Svindland (2009), Boonen (2016), and Ghossoub (2017) all study risk sharing where agents have heterogeneous beliefs, i.e. the model risk is refuted by all insurance players and there is one "true" and known model for each player.

This paper is set out as follows: Section 2 states the model set-up; Section 3 characterises the Pareto optimality set, while closed-form and numerical solutions are discussed in Section 4; Section 5 characterises a class of premiums using cooperative game theory for two classes of preferences; the link between the more recent literature on individual optimal insurance and the classical Pareto optimal concept is revealed in Section 6; Section 7 generalises our results to the case in which the model risk is no longer ignored; finally, Section 8 concludes the paper.

2 Model Set-up

Consider a probability space given by $(\Omega, \mathcal{F}, \mathbb{P})$ and then, for any $1 \leq p \leq \infty$, let $L^p(\mathbb{P})$ be the set of p-integrable random variables. Moreover, $L^p_+(\mathbb{P})$ is the set of non-negative and p-integrable random variables. The standard insurance usually assumes that there is one PH who wishes to insure its risk $X \in L^p_+(\mathbb{P})$ (a loss) on a given future reference period and let $N = \{1, \ldots, n\}$ be the set of available insurers that are willing to cover a part or possibly the entire risk. It is further assumed that the PH and each insurer have preferences ordered via risk measures on $L^p(\mathbb{P})$, denoted by ρ_{PH} and $\rho_i, i \in N$, respectively. That is, the functionals $\rho_{PH}, \rho_i : L^p(\mathbb{P}) \to \mathbb{R}$ are considered, where $i \in N$. Various risk measure properties appeared in the literature and the following are recalled in this paper for a generic risk measure $\rho : L^p(\mathbb{P}) \to \mathbb{R}$:

(P1) *Monotonicity*: If $Y \leq Z$ $\mathbb{P}$-almost surely, then $\rho(Y) \leq \rho(Z)$;

(P2) *Translation Invariance*: For any $m \in \mathbb{R}$, $\rho(Y+m) = \rho(Y) + m$;

(P3) *Positive homogeneity*: For any $\lambda \geq 0$, $\rho(\lambda Y) = \lambda\rho(Y)$;

(P4) *Subadditivity*: $\rho(Y+Z) \leq \rho(Y) + \rho(Z)$;

(P5) *Convexity*: For any $0 \leq \beta \leq 1$, $\rho\big(\beta Y + (1-\beta)Z\big) \leq \beta\rho(Y) + (1-\beta)\rho(Z)$;

(P6) *Strict convexity*: For any $0 < \beta < 1$, $\rho\big(\beta Y + (1-\beta)Z\big) < \beta\rho(Y) + (1-\beta)\rho(Z)$ provided that there exists no $a \in \mathbb{R}$ such that $Y - Z = a$ $\mathbb{P}$-almost surely;

(P7) *Comonotonic additivity*: If Y and Z are comonotonic, i.e. $\big(Y(\omega) - Y(\omega')\big)\big(Z(\omega) - Z(\omega')\big) \geq 0$ for any $\omega, \omega' \in \Omega$, then $\rho(Y+Z) = \rho(Y) + \rho(Z)$.

Throughout this paper, all risk measures are law-invariant[2] and satisfy (P2). Moreover, without loss of generality, we assume $\rho(0) = 0$. Recall that a risk measure that satisfies (P1)–(P4) is a *coherent risk measure*. Moreover, a *distortion risk measure* satisfies (P1), (P7), law-invariance and a continuity-type property (for details, see Wang *et al.*, 1997); these risk measures have been introduced in the insurance pricing context by Wang (1996), even though it appears earlier as a preference relation known as *dual utility* (see Yaari, 1987). Specifically, the mathematical formulation of a distortion risk measure is given by

$$\rho(Y) = \int_0^\infty g\big(S_Y(z)\big)\, dz - \int_{-\infty}^0 \big[1 - g\big(S_Y(z)\big)\big]\, dz, \tag{2.1}$$

for all $Y \in L^p(\mathbb{P})$, where $S_Y(\cdot) = \mathbb{P}(Y > \cdot)$ and $g : [0,1] \to [0,1]$ is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$ known as *distortion function*. Note that the integrals in (2.1) are assumed to be finite and distortion risk measures also satisfy (P2) and (P3). Distortion risk measures are characterised by Yaari (1987) as an alternative to expected utility. For distortion risk measures, the evaluation of a risk is linear in the pay-offs, but non-linear in the probabilities.

Throughout this paper, we assume that the PH seeks to share its risk $X \in L^p_+(\mathbb{P})$ with some insurers. The set of insurers that the PH is trading with is given by the set $S \subseteq N$. Each insurer accepts to cover $X_i \in L^p_+(\mathbb{P})$ such that $X_i \leq X$ in exchange of a premium $\pi_i \geq 0$, where $i \in S$ and as a result, the PH covers the remaining amount $X_{PH} = X - \sum_{i \in S} X_i$ and pays the total premium $\sum_{i \in S} \pi_i$.

[2] By definition, ρ is law-invariant if for any $Y, Z \in L^p(\mathbb{P})$ with $Y \overset{d}{=} Z$, then $\rho(Y) = \rho(Z)$.

Next, we explain the set of feasible insurance contracts, that include rationality constraints for all insurance agents. Each insurer aims to minimise its own risk, $\rho_i\big(X_i-\pi_i\big)$, under the rationality constraint $\rho_i\big(X_i-\pi_i\big)\le 0$, where $\rho_i(0)=0$ is the risk of insurer i before the transaction. Similarly, the PH aims to reduce its risk, $\rho_{PH}\Big(X_{PH}+\sum_{i\in S}\pi_i\Big)$, under its own rationality constraint $\rho_{PH}\Big(X_{PH}+\sum_{i\in S}\pi_i\Big)\le\rho_{PH}(X)$, where $\rho_{PH}(X)$ is the risk of the PH before the transfer is made. Therefore, we call a contract $\big(\underline{\boldsymbol{\pi}}^S,\underline{X}^S\big)$ *feasible* if

$$\rho_i\big(X_i-\pi_i\big)\le 0 \quad \text{for all} \quad i\in S \quad \text{and} \quad \rho_{PH}\Big(X_{PH}+\sum_{i\in S}\pi_i\Big)\le\rho_{PH}(X). \tag{2.2}$$

We write $\big(\underline{\boldsymbol{\pi}}^S,\underline{X}^S\big)\in\mathbb{R}_+^S\times\mathcal{A}_S(X)$, where

$$\mathcal{A}_S(X)=\Big\{\underline{X}^S:\sum_{i\in S\cup PH}X_i=X,\ \ X_i\in L_+^p(\mathbb{P})\text{ for all }i\in S\cup PH\Big\}.$$

Moreover, the risk profiles $\underline{X}^S\in\mathcal{A}_S(X)$ are called *risk allocations.*

The main purpose of the paper is to characterise the optimal insurance contract amongst all insurance players. The most common approach in the economic theory is the Pareto criterion, which is detailed in the next section.

3 Characterisation of Pareto optimal contracts

It is an irrefutable fact that conflicting objectives arise amongst insurer players and therefore, a compromising mutually beneficiary solution is of interest, which is usually attained via Pareto optimality. By definition, for a given $S\subseteq N$, the contract $\big(\underline{\boldsymbol{\pi}}^S,\underline{X}^S\big)\in\mathbb{R}_+^S\times\mathcal{A}_S(X)$ is called *Pareto optimal* if it is feasible and there is no other feasible contract $\big(\underline{\tilde{\boldsymbol{\pi}}}^S,\underline{\tilde{X}}^S\big)\in\mathbb{R}_+^S\times\mathcal{A}_S(X)$ such that

$$\rho_i\big(\tilde{X}_i-\tilde{\pi}_i\big)\le\rho_i\big(X_i-\pi_i\big) \quad \text{for all} \quad i\in S \quad \text{and} \quad \rho_{PH}\big(\tilde{X}_{PH}+\sum_{i\in S}\tilde{\pi}_i\big)\le\rho_{PH}\big(X_{PH}+\sum_{i\in S}\pi_i\big),$$

with at least one strict inequality. Let us denote $\mathcal{P}_S$ as the set of all Pareto optimal contracts. The next theorem characterises the set of Pareto optimal contracts and is given as Theorem 3.1.

Theorem 3.1. If ρ_i satisfies (P2) for all $i\in S\cup PH$, then $\mathcal{P}_S=\mathcal{S}_S$, where

$$\mathcal{S}_S=\underset{\big(\underline{\boldsymbol{\pi}}^S,\underline{X}^S\big)\in\mathbb{R}_+^S\times\mathcal{A}_S(X)}{\arg\min}\ \sum_{i\in S\cup PH}\rho_i(X_i) \quad \text{s.t.} \quad \text{condition (2.2) holds.} \tag{3.1}$$

Proof. We first show by contradiction that $\mathcal{S}_S \subseteq \mathcal{P}_S$. Thus, there exists a contract $\left(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\right) \in \mathcal{S}_S$ that is not Pareto optimal. This means that there exists $\left(\underline{\tilde{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\right) \in \mathbb{R}_+^S \times \mathcal{A}_S(X)$ such that

$$\rho_i\big(\tilde{X}_i - \tilde{\pi}_i\big) \leq \rho_i\big(X_i^* - \pi_i^*\big), \text{ for all } i \in S \text{ and } \rho_{PH}\left(\tilde{X}_{PH} + \sum_{i \in S} \tilde{\pi}_i\right) \leq \rho_{PH}\left(X_{PH}^* + \sum_{i \in S} \pi_i^*\right)$$

with at least one strict inequality. Thus, since all risk preferences satisfy (P2),

$$\sum_{i \in S \cup PH} \rho_i\big(\tilde{X}_i\big) < \sum_{i \in S \cup PH} \rho_i\big(X_i^*\big), \tag{3.2}$$

which contradicts that $\left(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\right)$ solves (3.1).

Let us prove now $\mathcal{P}_S \subseteq \mathcal{S}_S$, once again, by contradiction. Thus, there exists a Pareto optimal contract $\left(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\right) \in \mathcal{P}_S$ that is not a solution of (3.1). This means that there exists $\left(\underline{\tilde{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\right) \in \mathbb{R}_+^S \times \mathcal{A}_S(X)$ such that (3.2) holds. For the sake of exposition, denote now

$$a_i\big(\underline{\boldsymbol{\pi}}^{S}, \underline{X}^{S}\big) = \rho_i\big(X_i - \pi_i\big) \text{ for all } i \in S \text{ and } a_{PH}\big(\underline{\boldsymbol{\pi}}^{S}, \underline{X}^{S}\big) = \rho_{PH}\left(X_{PH} + \sum_{i \in S} \pi_i\right).$$

Let $\underline{\hat{\boldsymbol{\pi}}}^{S} = \underline{\tilde{\boldsymbol{\pi}}}^{S} + \underline{\boldsymbol{\varepsilon}}^{S}$ with $\varepsilon_i = a_i\big(\underline{\tilde{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\big) - a_i\big(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\big)$ for all $i \in S$. Thus,

$$a_{PH}\big(\underline{\hat{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\big) = a_{PH}\big(\underline{\tilde{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\big) + \sum_{i \in S} \varepsilon_i < a_{PH}\big(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\big), \tag{3.3}$$

which is true due to (3.2). Further, for all $i \in N$, we have that

$$a_i\big(\underline{\hat{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\big) = a_i\big(\underline{\tilde{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\big) - \varepsilon_i = a_i\big(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\big). \tag{3.4}$$

It is not difficult to verify that $\big(\underline{\hat{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\big)$ is a feasible contract, i.e.

$$a_i\big(\underline{\hat{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\big) \leq 0 \text{ for all } i \in S \quad \text{and } a_{PH}\big(\underline{\hat{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\big) \leq \rho_{PH}\big(X_{PH}\big),$$

which are straightforward implications of relations (3.3), (3.4) and the rationality constraints for $\big(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\big)$. Note also that $\hat{\pi}_i \geq 0$, since $a_i\big(\underline{\hat{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\big) \leq 0$ is true for all $i \in S$. Finally, the feasibility of $\big(\underline{\hat{\boldsymbol{\pi}}}^{S}, \underline{\tilde{X}}^{S}\big)$ together with equations (3.3) and (3.4) imply that $\big(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\big) \notin \mathcal{P}_S$, which concludes the proof of $\mathcal{P}_S \subseteq \mathcal{S}_S$. Hence, $\mathcal{P}_S = \mathcal{S}_S$, and so the proof is now complete. $\square$

Theorem 3.1 implies that if the minimum in (3.1) does not exist, the Pareto optimal set is empty.

Recall that the rationality constraints imply that the PH does not pay any premium if the decision is to not transfer anything to the insurers. According to Theorem 3.1, (P2) yields to a particular structure on risk allocations if we focus on Pareto optimality, while the premiums may arbitrarily be chosen as long as they yield a feasible contract. For expected utilities that do not satisfy (P2), we are not always able to disentangle the risk allocations and the premiums in this way via Pareto optimality (see Raviv, 1979).

Theorem 3.1 still holds if the rationality constraints from (2.2) are replaced by more stringent transferability conditions such

$$\rho_i\big(X_i - \pi_i\big) \le M_i \quad \text{for all } i \in S \text{ and } \quad \rho_{PH}\big(X_{PH} + \sum_{i \in S} \pi_i\big) \le \rho_{PH}(X), \tag{3.5}$$

where $M_i \le 0, i \in S$. Therefore, the Pareto optimal set can be found by simply solving

$$\min_{\left(\underline{\boldsymbol{\pi}}^S, \underline{X}^S\right) \in \mathbb{R}_+^S \times \mathcal{A}_S(X)} \sum_{i \in S \cup PH} \rho_i\big(X_i\big) \quad \text{s.t.} \quad \text{condition (3.5) holds.}$$

Note that the premiums $\underline{\boldsymbol{\pi}}^S$ are not present in the above objective function, while the constraints are linear in the π_i's. Thus, the latter optimisation problem is equivalent to solving

$$\min_{\underline{X}^S \in \mathcal{A}_S(X)} \sum_{i \in S \cup PH} \rho_i\big(X_i\big) \quad \text{s.t.} \quad \sum_{i \in S \cup PH} \rho_i\big(X_i\big) \le \sum_{i \in S} M_i + \rho_{PH}(X), \tag{3.6}$$

where the premiums belong to a set that depends on the optimal solution $\underline{X}^S$. Interestingly, the standard set (when $M_i = 0$ for all $i \in S$) of Pareto contracts becomes much simpler and it is equivalent to solving

$$\min_{\underline{X}^S \in \mathcal{A}_S(X)} \sum_{i \in S \cup PH} \rho_i(X_i). \tag{3.7}$$

Note 3.2. Since the objective function appears as a constraint as well in (3.6), then solving (3.6) is the same as solving the unconstrained counterpart, i.e. (3.7), but one should check if the optimal objective value of (3.7) satisfies the inequality constraint; otherwise the set of feasible solutions of (3.6) is an empty set.

An usual assumption in risk transferring is to assume comonotonic risk allocations. More specifically, if we do not impose the retained risk X_{PH} by the PH to be non-decreasing in the total risk, then the PH would have an incentive to under-report their losses. On the other hand, if X_{PH} increases more rapidly than X, then the PH would have an incentive to create incremental losses. Similar arguments could be found if the insurance coverage X_i would not be non-decreasing in the total loss. This implies that we replace the feasible set $\mathcal{A}_S(X)$ in (3.7) by the following smaller set:

$$\mathcal{C}_S(X) = \Big\{ \underline{X}^S \in \mathcal{A}_S(X) : X_i, X_j \text{ are comonotonic for all } i, j \in S \cup PH \Big\}.$$

Note that risk allocations in $\mathcal{A}_S(X) \backslash \mathcal{C}_S(X)$ may be decreasing in the loss or have discontinuities and cut-off points, where the indemnity drops to zero after a certain loss level.[3] Recall that the insurance focused literature discusses many specific insurance contracts that are not elements of $\mathcal{C}_S(X)$; for example, marine breakdown insurance may expect a different risk behaviour, since the risk manager is not the direct beneficiary of the insurance contract. Despite these examples, the vast majority of the existing insurance contracts are elements of $\mathcal{C}_S$. A reasonable trade-off between practicality and generality would be to choose $\mathcal{C}_S$ as the actual feasible set. The following proposition shows that Pareto optimal contracts could then still be found.

Proposition 3.3. If ρ_i satisfy (P5) for all $i \in S \cup PH$, then (3.6) is solved by any solution of

$$\min_{\underline{X}^S \in \mathcal{C}_S(X)} \sum_{i \in S \cup PH} \rho_i\big(X_i\big) \quad \text{s.t.} \quad \sum_{i \in S \cup PH} \rho_i\big(X_i\big) \leq \sum_{i \in S} M_i + \rho_{PH}(X). \tag{3.8}$$

Moreover, if (3.6) has no solution, then (3.8) has no solution either.

Proof. Firstly, assume that (3.8) is solved by $\underline{X}^{*S} \in \mathcal{C}_S(X)$ and this solution does not solve (3.6). Thus, $\underline{X}^{*S}$ is feasible for (3.6) and in turn, there exists a feasible solution of (3.6), $\underline{\hat{X}}^S \in \mathcal{A}_S(X)$, such that

$$\sum_{i \in S \cup PH} \rho_i\big(\hat{X}_i\big) < \sum_{i \in S \cup PH} \rho_i\big(X_i^*\big).$$

Theorem 2.3 of Burgert and Rüschendorf (2006) shows that for law-invariant (as we implicitly assumed from the very beginning) preferences satisfying (P5), there exists $\underline{\tilde{X}}^S \in \mathcal{C}_S(X)$ such that $\rho_i\big(\tilde{X}_i\big) \leq \rho_i\big(\hat{X}_i\big)$ for all $i \in S \cup PH$. Recall that $\underline{\hat{X}}^S$ is feasible for (3.6) and therefore, $\underline{\tilde{X}}^S$ is feasible for (3.8) and improves its objective since

$$\sum_{i \in S \cup PH} \rho_i\big(\tilde{X}_i\big) \leq \sum_{i \in S \cup PH} \rho_i\big(\hat{X}_i\big) < \sum_{i \in S \cup PH} \rho_i\big(\tilde{X}_i^*\big) \leq \sum_{i \in S} M_i + \rho_{PH}(X),$$

[3]Such risk allocation is found in Bernard *et al.* (2015).

which contradicts our assumption that $\underline{X}^{*S}$ does not solve (3.6).

Secondly, assume that (3.6) has no solution, but (3.8) is solved by $\underline{X}^{*S} \in \mathcal{C}_S(X)$. Thus, $\underline{X}^{*S}$ is feasible for (3.6) and in turn, there exists a feasible solution to (3.6), $\hat{\underline{X}}^S \in \mathcal{A}_S(X)$, such that

$$\sum_{i\in S\cup PH} \rho_i\big(\hat{X}_i\big) < \sum_{i\in S\cup PH} \rho_i\big(X_i^*\big).$$

Once again, Theorem 2.3 of Burgert and Rüschendorf (2006) tells us that there exists $\tilde{\underline{X}}^S \in \mathcal{C}_S(X)$ such that $\rho_i\big(\tilde{X}_i\big) \le \rho_i\big(\hat{X}_i\big)$ for all $i \in S \cup PH$. It is not difficult to find that $\tilde{\underline{X}}^S$ is feasible for (3.8) and improves its objective, which contradicts our assumption. Hence, (3.8) has no solution whenever (3.6) has no solution. This concludes the proof. $\square$

The convexity is a crucial assumption in Proposition 3.3 and counterexamples are possible (for example, see Theorem 4.3 from Embrechts *et al.*, 2016); for numerical purposes, there is an advantage to know if the Pareto optimal contracts are comonotonic. The next theorem shows that all Pareto optimal contracts are comonotonic under some certain conditions.

Theorem 3.4. If ρ_i satisfy (P2) for all $i \in S \cup PH$, then every solution $\underline{X}^S$ to (3.7) is comonotonic if one of the following properties holds:

i) ρ_i satisfy (P6) for all $i \in S \cup PH$;

ii) ρ_i is a distortion risk measure with strictly concave distortion functions for all $i \in S \cup PH$; in addition, $X \in L_+^\infty(\mathbb{P})$ and $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic.[4]

Proof. Instead of (3.7), consider now the following auxiliary problem:

$$\min_{\underline{Y}^S\in\overline{\mathcal{A}}_S(X)} \sum_{i\in S\cup PH} \rho_i\big(Y_i\big),\ \overline{\mathcal{A}}_S(X) = \Big\{\underline{Y}^S : \sum_{i\in S\cup PH} Y_i = X,\ Y_i \in L^p\big(\mathbb{P}\big),\ i\in S\cup PH\Big\}. \tag{3.9}$$

Thus, we solve (3.7) for risks that are not necessarily non-negative. Then, if we find a solution of (3.9) that is feasible for the problem (3.7), then it solves (3.7) as well. Note that for the sake of simplicity, the superscript S is removed in the remaining part of the proof.

Part i) is first shown. Proposition 3.1 of Filipović and Svindland (2008) shows that there exists a unique allocation for (3.9) up to rebalancing the cash. In other words, if $\underline{Y}_1$ and $\underline{Y}_2$ solve (3.9) then $\underline{Y}_1 - \underline{Y}_2$ is a deterministic vector. Let $\big(h_i(X), i \in S \cup PH\big)$ be a solution of (3.9). Theorem 2.3

[4] By definition, $L_+^\infty(\mathbb{P})$ is the set of non-negative and bounded random variables, and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic if there is no $A \in \mathcal{F}$ such that for every $B \in \mathcal{F}$ with $\mathbb{P}(B) < \mathbb{P}(A)$ we have $\mathbb{P}(B) = 0$.

of Burgert and Rüschendorf (2006) implies that $\big(h_i(X), i \in S \cup PH\big)$ is comonotonic, implying the functions h_i to be non-decreasing. Thus, $h_i(X) + c_i := h_i(X) - h_i(0), i \in S \cup PH$ solves (3.9), since all risk measures satisfy (P2), Moreover, $h_i(X) + c_i, i \in S \cup PH$ is feasible to (3.8) and hence, it solves (3.8) as well. This concludes Part (i).

Consider now Part ii) and let $\underline{Y}$ be a non-comonotonic risk allocation. Theorem 3.1 of Carlier *et al.* (2012) says that there exists an allocation that strictly dominates in convex order $\underline{Y}$. Thus, there exists $\hat{\underline{Y}}$ such that $\mathbb{E}\big(\phi(\underline{Y})\big) > \mathbb{E}\big(\phi(\hat{\underline{Y}})\big)$ for every strictly convex ϕ and $\mathbb{E}\big(\hat{\phi}(\underline{Y})\big) \geq \mathbb{E}\big(\hat{\phi}(\hat{\underline{Y}})\big)$ for every convex $\hat{\phi}$ (for details, see Lemma 2.2 of Carlier *et al.*, 2012).

Strict concavity of a probability distortion function implies that ρ_i for all $i \in S \cup PH$ strictly preserve second order stochastic dominance (see Corollary 2 of Chew *et al.*, 1987) and in turn, $\rho_i\big(Y_i\big) > \rho_i\big(\hat{Y}_i\big)$ for all $i \in S \cup PH$. Hence, if $\underline{Y}$ is not comonotonic, then there exist $\hat{\underline{Y}}$ such that $\rho_i\big(Y_i\big) > \rho_i\big(\hat{Y}_i\big)$, $i \in S \cup PH$, implying that $\underline{Y}$ does not solve (3.9). Let $\big(h_i(X), i \in S \cup PH\big)$ be a comonotonic solution of (3.9). Note that $h_i(X) + c_i := h_i(X) - h_i(0)$ solves (3.9) due to (P2), which clearly solves (3.7) as well. Thus, $\underline{Y}$ does not solve (3.7) and hence, all solutions to (3.7) are comonotonic, concluding the proof of Part (ii). The proof is now complete. □

Theorem 3.4 provides us two conditions under which we know that Pareto optimal contracts are comonotonic under some fairly general conditions. Moreover, for finding Pareto optimal contracts, Theorem 3.4 helps us to justify a focus only on comonotonic contracts if one of the two conditions is satisfied. Finding Pareto optimal contracts is the topic of Section 4.

4 Finding the Pareto optimal contracts

We have shown in Section 3 that under some mild conditions, the Pareto contract set coincides with solving (3.7). This enables us to establish the optimal risk allocation, which is the main purpose of this section. Solving (3.7) is now investigated and sometimes, closed-form optimal solutions are possible in some particular settings, otherwise, numerical solutions are sought. In a nutshell, if (3.7) is solved in $\mathcal{C}_S$ and all risk measures are distortion risk measures or all risk measures are exponential utilities, then elegant closed-form optimal solutions are possible, as stated in Propositions 4.1 and 4.2. All other cases could be numerically solved and two examples are later provided for which efficient numerical methods are indicated.

4.1 Distortion risk measures

Let us assume that for all $i \in S \cup PH$, ρ_i satisfies (2.1) with distortion function g_i. It is not difficult to find that $g_S^*(\cdot) = \min\left\{g_i(\cdot), i \in S \cup PH\right\}$ is a proper distortion function and let ρ_S^* be the corresponding distortion risk measure. Closed-form solutions to (3.7) are possible under this setting and are stated in Proposition 4.1 (for details, see Section 4 of Boonen *et al.*, 2016b), where I_A is the indicator function that takes the value 1 if A is true and 0, otherwise.

Proposition 4.1. Let $X \in L_+^\infty(\mathbb{P})$ and ρ_i, $i \in S \cup PH$, be distortion risk measures as in (2.1). The risk allocation $\underline{X}^S$ solve

$$\min_{\underline{X}^S \in \mathcal{C}_S(X)} \sum_{i \in S \cup PH} \rho_i(X_i) \tag{4.1}$$

if and only if $X_i = f_i(X)$ with $f_i(0) = 0$ and

$$f_i'(\cdot) := I_{\left\{g_i\left(S_X(\cdot)\right) < g^*_{S\setminus i}\left(S_X(\cdot)\right)\right\}} + \lambda_i^S(\cdot) I_{\left\{g_i\left(S_X(\cdot)\right) = g^*_{S\setminus i}\left(S_X(\cdot)\right)\right\}}, \tag{4.2}$$

holds almost surely for all $i \in S$, where λ_i^S is a measurable and $[0,1]$-valued function such that

$$\sum_{i \in S : g_i\left(S_X(\cdot)\right) = g^*_{S\setminus i}\left(S_X(\cdot)\right)} \lambda_i^S(\cdot) \leq 1.$$

Proposition 4.1 provides explicit solutions for the optimal Pareto contracts that exhibits a layered indemnity schedule. Such layered risk allocations are the most known non-proportional contracts available on the insurance market, which gives even more evidence to support our model setting that confirms the long-time insurance risk culture (see Arrow, 1963; Venter, 1991). Deductibles appear in the optimal insurance literature with expected utilities, where objective is to maximize the utility of the PH and the premium is a given expectation principle (Arrow, 1963). However, without the presence of *ex-post* costs and when Pareto optimality is the criterion, deductible insurance is often not optimal with expected utilities (Raviv, 1979). Note that if $g_{PH}(\cdot) = g_i(\cdot)$ for all $i \in S$, then Proposition 4.1 shows that any comonotonic risk allocation $\underline{X}^S \in \mathcal{C}_S(X)$ solves (4.1). Also, if there is one agent that is risk neutral $g_i(s) = s$ for all $s \in [0,1]$, and the other agents are averse towards mean-preserving spreads (due to Yaari, 1987, this implies $g_i(s) \geq s$), then it is Pareto optimal when the risk-neutral agent bears all the risk X.

4.2 Exponential expected utilities

In this subsection, we briefly discuss another class of translation invariant preferences that is based on exponential utility. This is a popular expected utility function that could be expressed as a risk measure satisfying (P2). By definition, an exponential utility function is given by $u_i(z) = -\gamma_i \exp(-z/\gamma_i)$, where $\gamma_i > 0$ captures the risk tolerance and $z \in \mathbb{R}$ is interpreted as a gain. Since we focuss on a generic random loss variable Y, we aim to minimise $-\mathbb{E}[u_i(-Y)]$. Then, we have

$$-u_i^{-1}(\mathbb{E}[u_i(-(Y+c))]) = -u_i^{-1}(\mathbb{E}[u_i(-Y)]) + c,$$

for any constant $c \in \mathbb{R}$, where $u_i^{-1}(\cdot)$ in a strictly increasing function. Accordingly, we assume that the risk measures for insurance agents $i \in N \cup PH$ are given by

$$\rho_i(Y) = -u_i^{-1}(\mathbb{E}[u_i(-Y)]) = \gamma_k \ln \mathbb{E}\big[\exp\big(Y/\gamma_k\big)\big]. \tag{4.3}$$

This risk measure is also known as the entropic risk measure (see Barrieu and El Karoui, 2005). One of the key properties of (4.3) is (P2), but it satisfies (P1) and (P6) as well (see Filipović and Svindland, 2008).

Theorem 3.9 of Barrieu and El Karoui (2005) provides a solution to (4.1) for the two insurance agents case, but an extension to multiple agents is quite obvious and is stated in the following proposition.

Proposition 4.2. Let $X \in L_+^\infty(\mathbb{P})$ and ρ_i, $i \in S \cup PH$, be as in (4.3). The risk allocation $\underline{X}^S$ solve (4.1) if $X_i = \frac{\gamma_i}{\sum_{j \in S \cup PH} \gamma_j} X$ for any $i \in S$.

Proposition 4.2 shows that the optimal Pareto contracts exhibit a proportional indemnity schedule. In general risk sharing problems where we allow for negative risk allocations, affine contracts are Pareto optimal with exponential utilities (Borch, 1962). Moreover, zero-sum side-payments will not affect Pareto optimality. In other words, if we find a Pareto optimal contract, we can construct a set of Pareto optimal contracts by adding zero-sum side-payments. In our model, these side-payments are captured by the premiums $\underline{\pi}^S$.

4.3 Examples with numerical solutions

Numerical solutions to (3.7) are always possible if the total risk X is discrete with a finite sample space. This is the case if historical data are available, otherwise representative samples could be drawn from a parametric model that is either fitted on real data or is based on expert opinion experience. Solving a discretised version of (3.7) requires a careful look for efficient numerical methods. Two examples are further developed, which depend on the set of feasible solutions that could be either $\mathcal{A}_S$ or $\mathcal{C}_S$. Assume now that we have two insurers, i.e. $N = \{1, 2\}$, with risk preferences given by:

$$\rho_1(\cdot) := \mathbb{E}(\cdot) + a\Big(\mathbb{E}\big(\cdot\ -\mathbb{E}(\cdot)\big)^2\Big)^{1/2} \quad \text{and} \quad \rho_2(\cdot) := \mathbb{E}(\cdot) + b\Big(\mathbb{E}\big(\cdot -\mathbb{E}(\cdot)\big)_+^c\Big)^{1/c}$$

with $a \geq 0$, $0 \leq b \leq 1$, and $1 \leq c \leq \infty$. If $c = \infty$, the above is read as $\mathbb{E}(\cdot) + b\big(x_\cdot - \mathbb{E}(\cdot)\big)$. By definition, $x_+ = \max\{x, 0\}$ and $x_\cdot := \inf\big\{x \in \mathbb{R} : \mathbb{P}(\cdot \leq x) = 1\big\}$ represents the right-end point of the sample space of a random variable. Formally, ρ_1 is known as the *standard deviation risk measure*, while ρ_2 is a parametric class of non-comonotonic additive coherent risk measures that is introduced by Fischer (2003). Note that these two risk measures satisfy (P1)–(P5). Finally, the PH orders risk via a distortion risk measure with distortion function g_{PH}.

As anticipated, the sample space of X is given by a finite set $\{x_k, 1 \leq k \leq \ell\}$ that without any loss of generality is assumed to be increasingly ordered, i.e. $x_1 \leq x_2 \leq \cdots \leq x_\ell$. We also assume that every outcome has equal probability to occur. Each loss outcome, x_k, is shared between the three insurance players and we have that $x_k = y_k + z_k + t_k$, where y_k and z_k are the risk portions for the first and second insurer, respectively, while t_k represents the PH's risk share. Therefore, $\rho_1\big(X_1\big) := \frac{1}{\ell}\mathbf{1}^T\mathbf{y} + a\frac{\|\mathbf{Q}\mathbf{y}\|}{\sqrt{\ell}}$, where $\|\cdot\|$ represents the *Euclidean distance*, $\mathbf{1}$ is a column vector of ones, $\mathbf{Q} = \mathbf{I} - \frac{1}{\ell}\mathbf{J}$ with $\mathbf{I}$ and $\mathbf{J}$ being the identity matrix and matrix of ones, respectively. Further,

$$\rho_2\big(X_2\big) := \frac{1}{\ell}\mathbf{1}^T\mathbf{z} + b\left(\frac{1}{\ell}\sum_{k=1}^{\ell}\Big(z_k - \frac{1}{\ell}\mathbf{1}^T\mathbf{z}\Big)_+^c\right)^{1/c}.$$

Let $t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(\ell)}$ be the order statistics sample. Thus, we have

$$\rho_{PH}\big(X_{PH}\big) := \sum_{k=1}^{\ell}\Big(g_{PH}\big((\ell-k+1)/\ell\big) - g_{PH}\big((\ell-k)/\ell\big)\Big)t_{(k)} = \sum_{k=1}^{\ell} d_k t_{(k)}, \tag{4.4}$$

where $d_k := g_{PH}\big((\ell-k+1)/\ell\big) - g_{PH}\big((\ell-k)/\ell\big)$.

Assume first that $\mathcal{C}_S$ is the set of feasible solutions. Hence, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{t}$ are increasingly ordered since $\mathbf{x}$ is ordered as well, which simplifies (4.4) and we have that $\rho_{PH}\big(X_{PH}\big) = \mathbf{d}^T\mathbf{t}$. The increasing ordering requires that $\mathbf{R}\mathbf{y} \leq \mathbf{0}$, $\mathbf{R}\mathbf{z} \leq \mathbf{0}$ and $\mathbf{R}\mathbf{t} \leq \mathbf{0}$, where $\mathbf{0}$ is a column vector of zeroes and $\mathbf{R}$ is an $\ell \times \ell$ matrix given by

$$\mathbf{R} := \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & \cdots & 0 & 0 \\ & \ddots & \ddots & & \\ & & & & \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}.$$

Recall that by convention, an equality/inequality between two vectors is understood componentwise. Now, (3.7) for this setting is equivalent to solving

$$\begin{aligned} \min_{\substack{(\mathbf{y},\mathbf{z},\mathbf{t})\in \\ \mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathbb{R}^\ell}} \quad & \left\{ \frac{1}{\ell}\mathbf{1}^T\mathbf{y} + \frac{a}{\sqrt{\ell}}\|\mathbf{Q}\mathbf{y}\| + \frac{1}{\ell}\mathbf{1}^T\mathbf{z} + b\left(\frac{1}{\ell}\sum_{k=1}^{\ell}\left(z_k - \frac{1}{\ell}\mathbf{1}^T\mathbf{z}\right)_+^c\right)^{1/c} + \mathbf{d}^T\mathbf{t}\right\} \\ \text{s.t.} \quad & \mathbf{R}\mathbf{y} \le \mathbf{0}, \mathbf{R}\mathbf{z} \le \mathbf{0}, \mathbf{R}\mathbf{t} \le \mathbf{0}, \mathbf{0} \le \mathbf{y}, \mathbf{0} \le \mathbf{z}, \mathbf{0} \le \mathbf{t}, \mathbf{y}+\mathbf{z}+\mathbf{t} = \mathbf{x}. \end{aligned} \tag{4.5}$$

For obvious computational reasons, the above is written as a *Second Order Cone Program(SOCP)*. Clearly, (4.5) is equivalent to solving

$$\begin{aligned} \min_{\substack{(\mathbf{y},\mathbf{z},\mathbf{t},u,\mathbf{v},w)\in \\ \mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}}} \quad & \left\{ \frac{1}{\ell}\mathbf{1}^T\mathbf{y} + \frac{a}{\sqrt{\ell}}u + \frac{1}{\ell}\mathbf{1}^T\mathbf{z} + b_1\left(\sum_{k=1}^{\ell} v_k^c\right)^{1/c} + \mathbf{d}^T\mathbf{t}\right\} \\ \text{s.t.} \quad & \|\mathbf{Q}\mathbf{y}\| \le u, \mathbf{z} - w\mathbf{1} \le \mathbf{v}, \mathbf{0} \le \mathbf{v}, \frac{1}{\ell}\mathbf{1}^T\mathbf{z} = w, \\ & \mathbf{R}\mathbf{y} \le \mathbf{0}, \mathbf{R}\mathbf{z} \le \mathbf{0}, \mathbf{R}\mathbf{t} \le \mathbf{0}, \mathbf{0} \le \mathbf{y}, \mathbf{0} \le \mathbf{z}, \mathbf{0} \le \mathbf{t}, \mathbf{y}+\mathbf{z}+\mathbf{t} = \mathbf{x} \end{aligned} \tag{4.6}$$

with $b_1 := b(1/\ell)^{1/c}$. The above formulation is almost written in an SOCP form and only the fourth term from the objective function requires more work. Recall that if $c \in \{1, \infty\}$, then (4.6) is directly SOCP-representable without any additional change, while the case in which $c = 2$ has a straightforward SOCP reformulation as follows:

$$\begin{aligned} \min_{\substack{(\mathbf{y},\mathbf{z},\mathbf{t},u,\mathbf{v},w,\varepsilon)\in \\ \mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}\times\mathbb{R}}} \quad & \left\{ \frac{1}{\ell}\mathbf{1}^T\mathbf{y} + \frac{a}{\sqrt{\ell}}u + \frac{1}{\ell}\mathbf{1}^T\mathbf{z} + b_1\varepsilon + \mathbf{d}^T\mathbf{t}\right\} \\ \text{s.t.} \quad & \|\mathbf{Q}\mathbf{y}\| \le u, \mathbf{z} - w\mathbf{1} \le \mathbf{v}, \mathbf{0} \le \mathbf{v}, \frac{1}{\ell}\mathbf{1}^T\mathbf{z} = w, \|\mathbf{v}\| \le \varepsilon, \\ & \mathbf{R}\mathbf{y} \le \mathbf{0}, \mathbf{R}\mathbf{z} \le \mathbf{0}, \mathbf{R}\mathbf{t} \le \mathbf{0}, \mathbf{0} \le \mathbf{y}, \mathbf{0} \le \mathbf{z}, \mathbf{0} \le \mathbf{t}, \mathbf{y}+\mathbf{z}+\mathbf{t} = \mathbf{x}. \end{aligned} \tag{4.7}$$

We now briefly discuss the same issue whenever $c = m/p$ with integers $m > p$, i.e. c is a rational number. This assumption is not restrictive in any sense, since the set of rational numbers is dense in $\mathbb{R}$. The main idea appeared in various ways in the literature (see for example, Ben-Tal and Nemirovski, 2001; Krokhmal and Soberanis, 2010) and is based on the so-called "tower of variables" construction. The representation is similar to the one displayed in (4.7) where the epigraph type constraint $\|\mathbf{v}\| \le \varepsilon$ is replaced by $\left(\sum_{k=1}^{\ell} v_k^c\right)^{1/c} \le \varepsilon$, which is indeed SOCP-representable (for details, see Morenko *et al.*, 2013). For example, if $c = 3$, then the constraint in question could be rewritten as follows:

$$\mathbf{1}^T\boldsymbol{\gamma} \le \varepsilon, \mathbf{0} \le \boldsymbol{\gamma}, \mathbf{0} \le \boldsymbol{\delta}, v_k^2 \le \varepsilon\delta_k, \delta_k^2 \le v_k\gamma_k, \ \text{ for all } \ 1 \le k \le \ell.$$

Thus, the SOCP reformulation of (4.6) when $c = 3$ is given by:

$$\begin{aligned}
&\min_{\substack{(\mathbf{y},\mathbf{z},\mathbf{t},u,\mathbf{v},w,\varepsilon,\boldsymbol{\gamma},\boldsymbol{\delta})\in\mathbb{R}^\ell\times\mathbb{R}^\ell\times\\ \mathbb{R}^\ell\times\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}^\ell}} \left\{\frac{1}{\ell}\mathbf{1}^T\mathbf{y} + \frac{a}{\sqrt{\ell}}u + \frac{1}{\ell}\mathbf{1}^T\mathbf{z} + b_1\varepsilon + \mathbf{d}^T\mathbf{t}\right\}\\
&\text{s.t.} \quad \|\mathbf{Q}\mathbf{y}\| \le u, \mathbf{z} - w\mathbf{1} \le \mathbf{v}, \mathbf{0} \le \mathbf{v}, \frac{1}{\ell}\mathbf{1}^T\mathbf{z} = w,\\
&\qquad \mathbf{1}^T\boldsymbol{\gamma} \le \varepsilon, \mathbf{0} \le \boldsymbol{\gamma}, \mathbf{0} \le \boldsymbol{\delta},\\
&\qquad \big\|\big(2v_k, \varepsilon - \delta_k\big)\big\| \le \varepsilon + \delta_k, \big\|\big(2\delta_k, v_k - \gamma_k\big)\big\| \le v_k + \gamma_k, \ \text{ for all } \ 1 \le k \le \ell,\\
&\qquad \mathbf{R}\mathbf{y} \le \mathbf{0}, \mathbf{R}\mathbf{z} \le \mathbf{0}, \mathbf{R}\mathbf{t} \le \mathbf{0}, \mathbf{0} \le \mathbf{y}, \mathbf{0} \le \mathbf{z}, \mathbf{0} \le \mathbf{t}, \mathbf{y} + \mathbf{z} + \mathbf{t} = \mathbf{x}.
\end{aligned} \tag{4.8}$$

Assume now that $\mathcal{A}_S$ is the set of feasible solutions. Since the vector $\mathbf{t}$ is not ordered anymore, the problem becomes more cumbersome. Even though a solution is possible, it could be computationally expensive (for details, see Asimit *et al.*, 2017b). The optimisation problem does not require a significant computational effort if $g_{PH}(x) = \min\big\{1, x/(1-\alpha)\big\}$, where $0 < \alpha < 1$, known in the literature as the *Conditional Value-at-Risk* (CVaR) at level α (see Rockafellar and Uryasev, 2000) or *Expected Shortfall* at level α (see Acerbi and Tasche, 2002). From the computational point of view, the CVaR formulation is more advantageous and is given by:

$$\text{CVaR}_\alpha(\cdot) := \inf_{\xi\in\mathbb{R}} \big\{\xi + \frac{1}{1-\alpha}\mathbb{E}(\cdot - \xi)_+\big\},$$

where $0 < \alpha < 1$. Note that the coherence of CVaRis shown by Pflug (2000). Consequently, if the PH orders risk via the CVaR, then optimising (3.7) under $\mathcal{A}_S$ is equivalent to solving the following

SOCP-type instance:

$$\min_{\substack{(\mathbf{y},\mathbf{z},\mathbf{t},u,\mathbf{v},w,\varepsilon,\xi,\mathbf{s})\in\\ \mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathbb{R}\times\mathbb{R}^\ell\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^\ell}} \left\{\frac{1}{\ell}\mathbf{1}^T\mathbf{y}+\frac{a}{\sqrt{\ell}}u+\frac{1}{\ell}\mathbf{1}^T\mathbf{z}+b_1\varepsilon+\xi+\frac{1}{n(1-\alpha)}\mathbf{1}^T\mathbf{s}\right\}$$
$$\text{s.t.}\quad \|\mathbf{Q}\mathbf{y}\|\le u, \mathbf{z}-w\mathbf{1}\le\mathbf{v}, \mathbf{0}\le\mathbf{v}, \frac{1}{\ell}\mathbf{1}^T\mathbf{z}=w, \|\mathbf{v}\|\le\varepsilon,$$
$$\mathbf{t}-\xi\mathbf{1}\le\mathbf{s}, \mathbf{0}\le\mathbf{s}, \mathbf{0}\le\mathbf{y}, \mathbf{0}\le\mathbf{z}, \mathbf{0}\le\mathbf{t}, \mathbf{y}+\mathbf{z}+\mathbf{t}=\mathbf{x},$$

where without loss of generality $c=2$ is assumed in the above.

Example 4.3. Assume that $a=b=1$, $c\in\{2,3\}$ and that the PH orders risk via $\text{CVaR}_{80\%}$. Random samples of size $1,000$ are drawn for the total risk from a Log-Normal distribution with parameters $(\mu,\sigma)=(10,1)$ and a two-parameter Pareto distributed with survival function $\big(1+\cdot/\lambda\big)^{-\gamma}$ such that the theoretical first two moments are matched with the Log-Normal distribution, i.e. $(\gamma,\lambda)=(4.78442,926.018)$. As before, the total risk samples are assumed to be increasingly ordered and $\mathcal{C}_N$ is the set of feasible solutions (our numerical investigations showed that the optimal contracts do not change if the optimisation is performed over $\mathcal{A}_N$).

The Pareto optimal solutions are plotted in Figure 1 and we notice a layered optimal risk contract. Figure 1 displays sensible results for a couple of reasons. Firstly, the Log-Normal distribution exhibits a moderately heavy tail, while the Pareto distribution has a very heavy tail and therefore, any insurer would absorb a higher amount of risk for the Pareto distribution. Secondly, the optimal indemnity of the second insurer is larger when the setting $c=3$ changes to $c=2$.

5 Stable premiums

The previous results have shown how to efficiently allocate the risk amongst all insurance players, but the premiums are yet to be set. For a Pareto optimal contract, the risk allocation $\underline{X}^N$ is fixed, but the premiums $\underline{\pi}^N$ that the insurers charge are not fixed, though $\underline{\pi}^N$ should satisfy the individual rationality constraints from (2.2). Due to Theorem 3.1, we have that

$$\sum_{i\in N\cup PH}\rho_i(X_i)\le\rho_{PH}(X),$$

with $\underline{X}^N$ satisfying (3.7). If the inequality above is strict, there are typically multiple ways to construct a premium vector that is individually rational for all insurance agents.

In this section, we notice that a vector of premiums could be perceived as an allocation of the

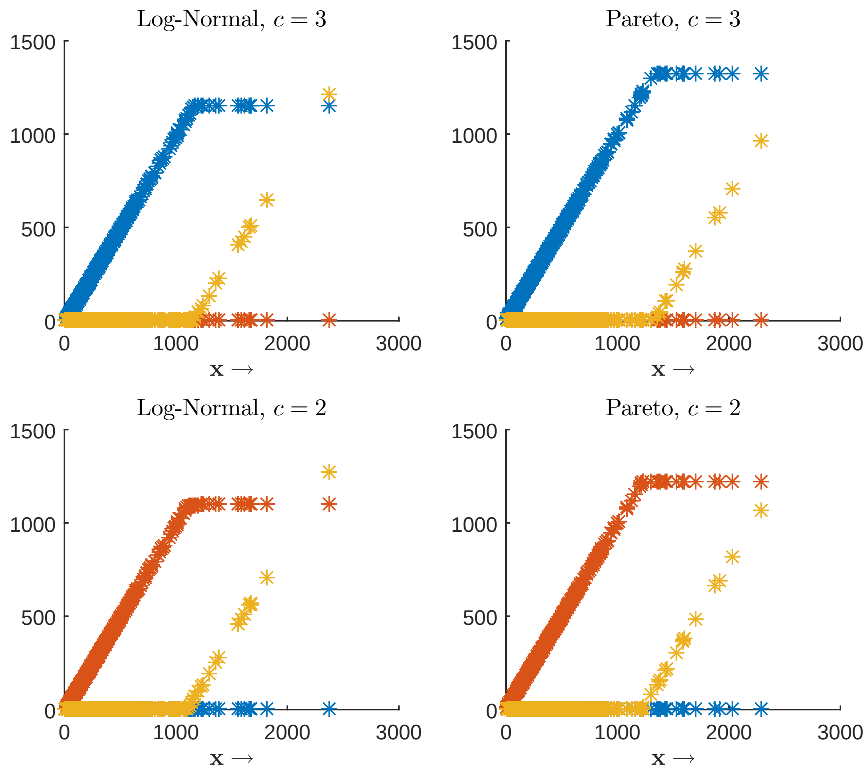

Figure 1: Pareto optimal risk allocations $(\mathbf{y}^*, \mathbf{z}^*, \mathbf{t}^*)$ that solve (4.8) and are respectively displayed in blue, red and yellow. The left and right panels show the Log-Normal and Pareto sample, respectively; the top and lower panels correspond to the $c = 3$ and $c = 2$ case, respectively.

welfare gains in the market to all insurers, which is the key ingredient for our approach to justify the premium charges. Specifically, we define a cooperative *Transferable Utility (TU)* game that assigns the maximum welfare gains to every subset of insurers. We call a premium vector *stable* if no subgroup of insurers has a joint incentive to stay in the market, while paying the other insurers their maximum joint welfare gain. It is shown that stable vectors of premiums constitute anti-core elements of a TU game.

As anticipated, we rely on cooperative game theory to determine prices. A TU game is given by $(N, \hat{v})$, where $\hat{v} : 2^N \rightarrow \mathbb{R}$ is a mapping that assigns to every subgroup of the set N a utility level[5]; by definition, an *allocation* is a vector $\underline{a}^N \in \mathbb{R}^N$ such that $\sum_{i \in N} a_i = \hat{v}(N)$. The core of TU games is originally introduced by Gillies (1953) and is given by

$$core(N, \hat{v}) = \Big\{\underline{a}^N \in \mathbb{R}^N : \sum_{i \in S} a_i \geq \hat{v}(S) \text{ for all } S \subset N, \sum_{i \in N} a_i = \hat{v}(N)\Big\}. \tag{5.1}$$

[5]Here, 2^N is the collection of all subgroups of N.

In other words, the core is the set of allocations such that no subgroup of agents has a joint incentive to not cooperate with the other agents. The core is one of the most important concepts in cooperative game theory (see, e.g., Scarf, 1967; Shapley and Shubik, 1969). It is generally accepted that, if the core is a nonempty set, then the allocation on which the players agree should be a core allocation. Formally, we have $a \in core(N, \hat{v})$ if, for each $S \subseteq N$, there exists no allocation $\hat{a} \in \mathbb{R}^S$ with $\sum_{i\in S} \hat{a}_i = \hat{v}(S)$ such that each insurance player $i \in S$ prefers the allocation $\hat{a}$ above a: $\hat{a}_i > a_i$ for all $i \in S$.

We further make use of the following two properties of a TU game $(N, \hat{v})$:

- *concavity*: $\hat{v}(S \cup \{i\}) - \hat{v}(S) \geq \hat{v}(T \cup \{i\}) - \hat{v}(T)$, for all $i \in N$ and $S \subset T \subseteq N\backslash\{i\}$;
- *additivity*: $\hat{v}(S) = \sum_{i\in S} \hat{v}(\{i\})$, for all $S \subseteq N$.

The remaining part of the section covers three settings. Firstly, a general setting is considered where all risk preferences are monotonic and translation invariant. Secondly, we assume that the risk preferences are given by distortion risk measures as defined in (2.1). Thirdly, we reflect on the setting where the PH strictly benefits in the insurance contract.

5.1 Stable premiums with risk preferences satisfying (P1) and (P2)

Let us assume that ρ_i satisfies (P2) for all $i \in N \cup PH$ and ρ_{PH} satisfies (P1) as well. Once a risk allocation is found via Pareto optimality as explained in Section 3 (for example, see (3.7)), another main problem is now to determine the premium vector $\underline{\boldsymbol{\pi}}^N \in \mathbb{R}^N_+$. We aim to use a TU game to model the pricing for transferring the risk to various insurers. This leads to a collection of premium vectors corresponding to the chosen risk allocation. At the moment, after identifying the optimal Pareto contract, the premiums are known to belong to a set that depends on the optimal solution $\underline{X}^S$ and a refinement is now sought.

The *welfare gain* for a given $S \subseteq N$ and $\underline{X}^S \in C_S(X)$ is as follows:

$$WG\big(S, \underline{X}^S\big) := \rho_{PH}(X) - \sum_{i\in S\cup PH} \rho_i\big(X_i\big).$$

Due to (P2), we can interpret the welfare gain as a monetary amount, i.e. $WG\big(S, \underline{X}^S\big)$ is the monetary amount that the insurers in S gain on aggregate by sharing the risk X via $\underline{X}^S$. Therefore, it is natural to study the following TU game that assigns the maximum welfare gains for the insurers in $S \subseteq N$:

$$v(S) = \max_{\underline{X}^S \in \mathcal{A}_S(X)} WG\Big(S, \underline{X}^S\Big) = \rho_{PH}(X) - \min_{\underline{X}^S \in \mathcal{A}_S(X)} \Big\{ \sum_{i \in S \cup PH} \rho_i\big(X_i\big) \Big\}, \tag{5.2}$$

for all $S \subseteq N$. By construction, we have $v(\emptyset) = 0$ and $0 \leq v(S) \geq v(T)$ for all $S \subset T \subseteq N$. This game assigns the maximum welfare gain that a subgroup may obtain by insuring the risk X altogether. It is further assumed that the minimum in (5.2) exists for all $S \subseteq N$.

Recall that an allocation is a vector $\underline{a}^N \in \mathbb{R}^N$ such that $\sum_{i \in N} a_i = v(N)$. Now, the premium charged by the i^{th} insurer for insuring the risk X_i is $\pi_i = \hat{\pi}_i(X_i, a_i)$, where

$$\hat{\pi}_i(X_i, a_i) := \rho_i(X_i) + a_i, \quad i \in N, \tag{5.3}$$

and $\underline{X}^N \in \mathcal{A}_N(X)$ is an optimal Pareto risk allocation. For any $\Big(\underline{\boldsymbol{\pi}}^N, \underline{X}^N\Big) \in \mathcal{P}_N$ such that $\pi_i = \hat{\pi}_i(X_i, a_i)$ for all $i \in N$, we get from (3.7) that

$$\sum_{i \in N} \hat{\pi}_i(X_i, a_i) = \sum_{i \in N} \rho_i(X_i) + v(N) = \rho_{PH}(X) - \rho_{PH}\big(X_{PH}\big).$$

Consequently, the PH is indifferent[6]. Moreover, if $a_i \geq 0$, then due to (P2) we have:

$$\rho_i(X_i - \hat{\pi}_i(X_i, a_i)) = -a_i \leq 0,$$

so that insurer i is individually rational. Moreover, this implies that we are allowed to interpret a_i are a welfare gain for insurer $i \in N$. Theorem 3.1 implies that $\Big(\underline{\boldsymbol{\pi}}^N, \underline{X}^N\Big) \in \mathcal{P}_N$ if $\underline{X}^N$ satisfies (3.7) and $\underline{\boldsymbol{\pi}}^N$ is as in (5.3) with $\underline{a}^N \geq 0$. Hence, an allocation $\underline{a}^N$ represents the welfare gains for the insurers in a Pareto optimal contract, where the aggregate welfare gains are given by $v(N)$.

We proceed with discussing the core of the game (N, v). First, we show that the core indeed leads to stability, and make an explicit link with the insurance contract $\big(\underline{\boldsymbol{\pi}}^N, \underline{X}^N\big)$.

Proposition 5.1. Assume that ρ_i is such that (P2) holds for all $i \in N \cup PH$, and let $\big(\underline{\boldsymbol{\pi}}^N, \underline{X}^N\big) \in \mathbb{R}_+^N \times \mathcal{A}_N(X)$ be Pareto optimal. Then, it holds that $\underline{a}^N \in core(N, v)$ for $\underline{a}^N$ solving $\pi_i = \hat{\pi}_i(X_i, a_i)$ for all $i \in N$ if and only if $\sum_{i \in N}(\pi_i - \rho_i(X_i)) = v(N)$ and for all $S \subset N$ there does not exist a

[6] This is relaxed in Section 5.3

$\left(\underline{\hat{\boldsymbol{\pi}}}^S, \underline{\hat{X}}^S\right) \in \mathbb{R}_+^S \times \mathcal{A}_S(X)$ such that

$$\rho_{PH}\Big(\hat{X}_{PH} + \sum_{i\in S}\hat{\pi}_i\Big) \leq \rho_{PH}\Big(X_{PH} + \sum_{i\in N}\pi_i\Big), \tag{5.4}$$

$$\rho_i(\hat{X}_i - \hat{\pi}_i) \leq \rho_i(X_i - \pi_i), i \in S, \tag{5.5}$$

with at least one inequality strict.

Proof. Let $\left(\underline{\boldsymbol{\pi}}^N, \underline{X}^N\right) \in \mathbb{R}_+^N \times \mathcal{A}_N(X)$ be Pareto optimal. Then, $\underline{a}^N \in core(N, v)$ is equivalent to $\sum_{i\in S} a_i \geq v(S)$ or, $\sum_{i\in S}(\pi_i - \rho_i(X_i)) \geq v(S)$ for all $S \subset N$, and $\sum_{i\in N} a_i = \sum_{i\in N}(\pi_i - \rho_i(X_i)) = v(N)$. Then, it is sufficient to show that for all $S \subset N$, $\sum_{i\in S}(\pi_i - \rho_i(X_i)) \geq v(S)$ is equivalent to the case that there does not exist a $\left(\underline{\hat{\boldsymbol{\pi}}}^S, \underline{\hat{X}}^S\right) \in \mathbb{R}_+^S \times \mathcal{A}_S(X)$ such that (5.4)-(5.5) hold with at least one inequality strict. Fix $S \subset N$. Since $\left(\underline{\hat{\boldsymbol{\pi}}}^S, \underline{\hat{X}}^S\right)$ is Pareto optimal and thus feasible, we have $\rho_i(X_i - \pi_i) \leq 0$. From this, and from the fact that the above preferences are linear in the π_i's, we get from similar arguments as in the proof of Theorem 3.1 that there does not exist a $\left(\underline{\hat{\boldsymbol{\pi}}}^S, \underline{\hat{X}}^S\right) \in \mathbb{R}_+^S \times \mathcal{A}_S(X)$ such that (5.4)-(5.5) hold with at least one inequality strict is equivalent to the case that there does not exist $\underline{\hat{X}}^S \in \mathcal{A}_S(X)$ such that

$$\sum_{i\in S\cup PH}\rho_i(\hat{X}_i) < \sum_{i\in S\cup PH}\rho_i(X_i) + \sum_{i\in N\setminus S}\pi_i.$$

Or, equivalently, for all $\underline{\hat{X}}^S \in \mathcal{A}_S(X)$ it holds

$$\sum_{i\in S\cup PH}\rho_i(\hat{X}_i) \geq \sum_{i\in S\cup PH}\rho_i(X_i) + \sum_{i\in N\setminus S}\pi_i,$$

so that

$$\min_{\underline{\hat{X}}^S\in\mathcal{A}_S(X)}\sum_{i\in S\cup PH}\rho_i(\hat{X}_i) \geq \sum_{i\in N\cup PH}\rho_i(X_i) + \sum_{i\in N\setminus S}(\pi_i - \rho_i(X_i)).$$

Since $\left(\underline{\boldsymbol{\pi}}^N, \underline{X}^N\right)$ is Pareto optimal, we have by Theorem 3.1 that

$$\sum_{i\in N\cup PH}\rho_i(X_i) = \min_{\underline{\hat{X}}^N\in\mathcal{A}_N(X)}\sum_{i\in N\cup PH}\rho_i(\hat{X}_i),$$

and hence this is equivalent to

$$v(N) - v(S) \geq \sum_{i \in N \backslash S} (\pi_i - \rho_i(X_i)).$$

Hence, since $v(N) = \sum_{i \in N}(\pi_i - \rho_i(X_i))$, this is equivalent to

$$v(S) \leq \sum_{i \in S}(\pi_i - \rho_i(X_i)).$$

This concludes the proof. $\square$

The next proposition shows that premium allocations via the core of our game is possible only under very restrictive conditions, i.e. additivity of the game (N, v). The additivity property is strong and it does not need to hold for the reinsurance problem in (5.2).

Proposition 5.2. Assume that ρ_i is such that (P2) holds for all $i \in N \cup PH$ and ρ_{PH} satisfies (P1). Then, $core(N, v) \neq \emptyset$ if and only if the TU game (N, v) is additive.

The proof is delegated to the appendix and makes use of the proof of Theorem 5.4, that is stated later in this paper. Core elements yield insurance contracts in which all insurers benefit from having more insurers in the market. Proposition 5.2 shows that this only happens in a specific case. Moreover, if the core is non-empty, its unique element is given by $a_i = v(\{i\})$ for all $i \in N$, which is a well-known property of additive games. Hence, the core is either empty or single-valued.

Because the core is generally empty, the inequality constraints of the core $\sum_{i \in S} a_i \geq v(S)$ are strong. We now proceed with studying the set of *stable* allocations, that include "milder" constraints on $\sum_{i \in S} a_i$. We define the set $\mathcal{SA}(N, v)$ as follows:

$$\mathcal{SA}(N, v) = \Big\{ \underline{a}^N \in \mathbb{R}^N : \sum_{i \in S} a_i \geq v(N) - v(N \backslash S) \text{ for all } S \subset N, \sum_{i \in N} a_i = v(N) \Big\},$$

which has a very intuitive construction. The minimal allocation to a subgroup of insurers $S \subset N$ is determined as follows. The insurers in $N \backslash S$ get the maximum aggregate welfare gain, given by $v(N \backslash S)$; if this welfare gain is completely allocated to the insurers in $N \backslash S$, the insurers in S get at least the remaining, i.e. $v(N) - v(N \backslash S)$ which is non-negative. Later, in the proof of Theorem 5.4, we will formally show that $v(S) \geq v(N) - v(N \backslash S)$ for all $S \subseteq N$, so that the constraints in $\mathcal{SA}(N, v)$ are indeed milder than for the core: $core(N, v) \subset \mathcal{SA}(N, v)$.

We next show that the set $\mathcal{SA}(N,v)$ is the same as the anti-core of game (N,v), a statement that is true under a very general setting.

Proposition 5.3. If $v(S)$ is finite for all $S \subseteq N$, it holds that $\mathcal{SA}(N,v) = Acore(N,v)$, where

$$Acore(N,v) = \big\{ \underline{a}^N \in \mathbb{R}^N : \sum_{i\in S} a_i \le v(S) \text{ for all } S \subset N, \sum_{i\in N} a_i = v(N) \big\}. \tag{5.6}$$

Proof. Let $\underline{a}^N \in Acore(N,v)$. We get for every $S \subset N$ that

$$\sum_{i\in S} a_i = \sum_{j\in N} a_j - \sum_{j\in N\setminus S} a_j \ge v(N) - v(N\setminus S).$$

Thus, $\underline{a}^N \in \mathcal{SA}(N,v)$. Further, we get that if $\underline{a}^N \in \mathcal{SA}(N,v)$, then

$$\sum_{i\in S} a_i = \sum_{j\in N} a_j - \sum_{j\in N\setminus S} a_j \le v(N) - \big(v(N) - v(S)\big) = v(S)$$

for all $S \subset N$. Hence, $\underline{a}^N \in Acore(N,v)$. This concludes the proof. □

The elements $Acore(N,v)$ are the allocations of our interest. For a Pareto optimal contract, the set $Acore(N,v)$ helps us to construct a range of premiums that satisfy our stability conditions. The next theorem shows that this set is non-empty, which concludes that the premium allocation exercise is possible.

Theorem 5.4. Assume that ρ_i satisfies (P2) for all $i \in N \cup PH$ and ρ_{PH} satisfies (P1). If (5.2) is well-defined for all $S \subseteq N$, then $Acore(N,v) \neq \emptyset$.

Proof. For $S \subseteq N$, let e_S be the vector in $\mathbb{R}^N$ such that $e_S(i) = 1$ if $i \in S$, otherwise $e_S(i) = 0$. Bondareva (1963) and Shapley (1967) show that the core of a cooperative game (N,v) is non-empty if and only if (N,v) is balanced; a game (N,v) is called balanced when $\sum_{S\subseteq N} \lambda_S v(S) \le v(N)$ for all $\lambda_S \in [0,1]$ such that $\sum_{S\subseteq N} \lambda_S e_S = e_N$ (the Bondareva-Shapley Theorem). Since, in general, it holds that $Acore(N,v) = -core(N,-v)$ for any TU game (Monderer *et al.*, 1992), we have that $core(N,-v) \neq \emptyset$ is an equivalent formulation to our claim. Thus, the game $(N,-v)$ is balanced if

$$\sum_{S\subseteq N} \lambda_S v(S) \ge v(N) \quad \text{for all} \quad \lambda_S \in [0,1] \quad \text{such that} \quad \sum_{S\subseteq N} \lambda_S e_S = e_N.$$

Now, $\sum_{S\subseteq N} \lambda_S \geq 1$ and $\sum_{S:i\in S\subseteq N} \lambda_S = 1$ are clearly true and therefore

$$\begin{aligned}
\sum_{S\subseteq N} \lambda_S v(S) &= \rho_{PH}(X) \sum_{S\subseteq N} \lambda_S - \sum_{S\subseteq N} \lambda_S \min_{\underline{X}^S \in \mathcal{A}_S(X)} \Big\{ \sum_{i\in S} \rho_i\big(X_i\big) + \rho_{PH}\big(X_{PH}\big) \Big\} \\
&\geq \rho_{PH}(X) \sum_{S\subseteq N} \lambda_S - \min_{\underline{X}^N \in \mathcal{A}_N(X)} \sum_{S\subseteq N} \lambda_S \Big(\sum_{i\in S} \rho_i\big(X_i\big) + \rho_{PH}\big(X_{PH}\big) \Big) \\
&\geq \rho_{PH}(X) - \min_{\underline{X}^N \in \mathcal{A}_N(X)} \Big\{ \sum_{S\subseteq N} \lambda_S \sum_{i\in S} \rho_i\big(X_i\big) + \rho_{PH}\big(X_{PH}\big) \Big\} \\
&= \rho_{PH}(X) - \min_{\underline{X}^N \in \mathcal{A}_N(X)} \Big\{ \sum_{i\in N} \rho_i\big(X_i\big) \sum_{S:i\in S\subseteq N} \lambda_S + \rho_{PH}\big(X_{PH}\big) \Big\} \\
&= v(N).
\end{aligned}$$

Here, the second inequality follows from the fact that

$$\Big(\rho_{PH}(X) - \rho_{PH}\big(X_{PH}\big)\Big) \sum_{S\subseteq N} \lambda_S \geq \rho_{PH}(X) - \rho_{PH}\big(X_{PH}\big) \quad \text{for all} \quad \underline{X}^N \in \mathcal{A}_N(X),$$

which is due to the (P1) property of ρ_{PH} and the fact that $\sum_{S\subseteq N} \lambda_S \geq 1$ and $X_{PH} \leq X$ are true for any $\underline{X}^N \in \mathcal{A}_N(X)$. Therefore, $core(N, -v) \neq \emptyset$ due to the Bondareva-Shapley Theorem, which concludes the proof. □

5.2 Stable premiums with distorted risk preferences

We now assume more specific preferences than those considered in Section 5.1. This specific setting is interesting in the sense that a closed-form expression of the set of stable allocations, $\mathcal{SA}(N, v)$, is possible. That is, let us assume that ρ_i satisfies (2.1) with distortion function g_i for all $i \in S \cup PH$ and the feasible set of risk allocations is $C_N(X)$. It is well-known that distortion risk measures satisfy (P1) and (P2) (see Yaari, 1987). Then, it holds that:

$$WG\big(S, \underline{X}^S\big) = \sum_{i\in S} \Big(\rho_{PH}\big(X_i\big) - \rho_i\big(X_i\big)\Big),$$

for all $S \subseteq N$ and $\underline{X}^S \in C_S(X)$, since ρ_{PH} satisfies (P7) and the fact that the risk allocation is comonotonic. It is readily verified from Proposition 4.1 that

$$\sum_{i\in S\cup PH} \rho_i\big(X_i\big) = \rho^*_S(X),$$

whenever $\underline{X}^S$ is as in (4.2) with ρ^*_S being defined in Section 4.1. Then, we study the following TU game (N, v):

$$v(S) = \max_{\underline{X}^S \in \mathcal{C}_S(X)} WG\Big(S, \underline{X}^S\Big) = \rho_{PH}(X) - \rho^*_S(X), \text{ for all } S \subseteq N, \tag{5.7}$$

The next result shows that the TU game (N, v) has a special structure, namely is concave. Consequently, the set $\mathcal{SA}(N, v)$ could be characterised by its marginal vectors (Shapley, 1971) and for clarity, we state these results as Theorem 5.5. Recall that by definition, $\Pi(N)$ is the set of all permutations of N, while conv$\{\cdot\}$ represents the convex hull operator.

Theorem 5.5. Let ρ_i be distortion risk measures with distortion function g_i as in (2.1) for all $i \in N \cup PH$. Then, the TU game (N, v) from (5.7) is concave and $\mathcal{SA}(N, v) = \text{conv}\{m^\sigma : \sigma \in \Pi(N)\}$, where

$$m^\sigma_{\sigma(1)} = v\Big(\big\{\sigma(1)\big\}\Big) \quad \text{and} \quad m^\sigma_{\sigma(i)} = v\Big(\big\{\sigma(1), \ldots, \sigma(i)\big\}\Big) - v\Big(\big\{\sigma(1), \ldots, \sigma(i-1)\big\}\Big),\ i = 2, \ldots, n.$$

Proof. It holds for every $0 \le x \le 1$ that

$$\begin{aligned}
g^*_{S\cup\{i\}}(x) - g^*_S(x) &= \min\big\{g_j(x) : j \in S \cup \{i\} \cup PH\big\} - \min\{g_j(x) : j \in S \cup PH\big\} \\
&= \min\big\{g_i(x) - \min\{g_j(x) : j \in S \cup PH\}, 0\big\} \\
&\le \min\big\{g_i(x) - \min\{g_j(x) : j \in T \cup PH\}, 0\big\} \\
&= \min\big\{g_j(x) : j \in T \cup \{i\} \cup PH\big\} - \min\big\{g_j(x) : j \in T \cup PH\big\} \\
&= g^*_{T\cup\{i\}}(x) - g^*_T(x),
\end{aligned}$$

for all $i \in N$ and $S \subset T \subseteq N \backslash \{i\}$. Then, we derive

$$\begin{aligned}
&\rho^*_{T\cup\{i\}}(X) - \rho^*_T(X) - \rho^*_{S\cup\{i\}}(X) + \rho^*_S(X) \\
&\quad = \int_0^\infty g^*_{T\cup\{i\}}\big(S_X(z)\big)dz - \int_0^\infty g^*_T\big(S_X(z)\big)dz - \int_0^\infty g^*_{S\cup\{i\}}\big(S_X(z)\big)dz + \int_0^\infty g^*_S\big(S_X(z)\big)dz \\
&\quad = \int_0^\infty \Big(g^*_{T\cup\{i\}}\big(S_X(z)\big) - g^*_T\big(S_X(z)\big) - g^*_{S\cup\{i\}}\big(S_X(z)\big) + g^*_S\big(S_X(z)\big)\Big)dz,
\end{aligned}$$

which is clearly non-negative. Therefore, $v\big(S \cup \{i\}\big) - v(S) \ge v\big(T \cup \{i\}\big) - v(T)$ holds for all $i \in N$ and $S \subset T \subseteq N \backslash \{i\}$. Hence, the game (N, v) is concave. For concave games, the anti-core is given by $\operatorname{conv}\{m^\sigma : \sigma \in \Pi(N)\}$, due to Theorems 3 and 5 of Shapley (1971) and the fact that $Acore(N, v) = -core(N, -v)$. The proof is now complete. $\square$

Note 5.6. In Theorem 5.5, we provide a closed-form expression for $\mathcal{SA}(N, v)$. This set is however not necessarily single-valued. An open research question is still to characterise a single-valued allocation to determine premiums. A well-known allocation rule of TU games is the *Shapley value* (Shapley, 1953). A PH directly approaches the insurers to insure its risk in a Pareto optimal way. The Shapley value relies on a different approach and assumes that the PH first goes to randomly assigned insurer. This insurer selects with the PH a risk allocation via Pareto optimality, and the PH pays the aggregate premium such that it is indifferent. Then, the PH goes to the second randomly assigned insurer and selects the risk allocation via Pareto optimality for the first two insurers. The PH still pays the premium such that it is indifferent, the first insurer keeps its welfare gain and the second insurer gets the increase in the welfare gain. Continuing this procedure leads to an allocation; averaging this over all possible orderings of the insurers leads to the Shapley value. In case of distortion risk measures, we show that the corresponding TU game (N, v) is concave (see Theorem 5.5) and thus, the Shapley value constitutes an anti-core element (Shapley, 1971). Generally, we leave in this paper the question to an appropriate solution open for further research.

5.3 Welfare gains for the PH

In this section, the approach we propose hinges on the assumption that the PH is indifferent, which is now relaxed. Recall the maximum welfare gain $v(S) = \max_{\underline{X}^S \in \mathcal{A}_S(X)} WG\left(S, \underline{X}^S\right)$ from (5.2). Suppose that in any transaction, there is a profit for the PH that is proportional to $v(S)$. That is, the PH has an *ex-post* risk given by $\rho_{PH}(X) - \delta v(S)$ for $\delta \in [0, 1)$. Thus, in the market with all insurers in N, the PH gets the welfare gain $\delta v(N)$ and the remaining welfare gain, $(1 - \delta)v(N)$, is then allocated amongst the insurers in N.

Define $\hat{v}(S) = (1 - \delta)v(S), S \subseteq N$. For a vector $\underline{a}^N \in \mathbb{R}^N$ such that $\sum_{i \in N} a_i = \hat{v}(N)$ and $\hat{\pi}_i(X_i, a_i)$ as in (5.3), we get for any $\left(\underline{\boldsymbol{\pi}}^N, \underline{X}^N\right) \in \mathcal{P}_N$ with $\pi_i = \hat{\pi}_i(X_i, a_i)$ for all $i \in N$ that

$$\sum_{i \in N} \hat{\pi}_i(X_i, a_i) = \sum_{i \in N} \rho_i(X_i) + \hat{v}(N) = \rho_{PH}(X) - \rho_{PH}\left(X_{PH}\right) - \delta v(N).$$

Clearly,

$$core(N, \hat{v}) = (1 - \delta) \cdot core(N, v) := \{(1 - \delta)a : a \in core(N, v)\},$$

$$\mathcal{SA}(N,\hat{v}) = (1-\delta)\cdot\mathcal{SA}(N,v) := \{(1-\delta)a : a \in \mathcal{SA}(N,v)\}.$$

Hence, Propositions 5.2 and 5.3, Theorem 5.4 and Theorem 5.5 remain valid for this setting where (N,v) is replaced by $(N,\hat{v})$.

6 Optimal insurance and Pareto optimality

The current section aims to link the classical concept of Pareto optimality with the recent fast growing literature on optimal insurance/reinsurance. These two concepts have been independently investigated even though the ultimate aims are basically the same, i.e. finding efficient risk allocations. There is a vast literature on Pareto optimality and our focus has been on the risk sharing problem between a PH and one or more insurers willing to absorb part of the PH's risk. The Pareto solutions provides a "fair" allocation amongst all parties involved in the risk allocation exercise. In contrast, the optimal insurance/reinsurance problem is generically viewed as an optimisation problem from one risk bearer point of view. These two strands of research have not crossed yet in the literature and we try now to investigate whether the insurance/reinsurance problem leads to contracts that are Pareto optimal.

In the insurance/reinsurance problem framework, it is assumed that the PH or one insurer seeks to optimise its risk measure subject to rationality constraints, which in our case are given in (2.2). The mathematical formulations of the optimal insurance contract set from the PH and a generic insurer are

$$\mathcal{S}_S^{PH} = \underset{\left(\underline{\boldsymbol{\pi}}^S,\underline{X}^S\right)\in\mathbb{R}_+^S\times\mathcal{A}_S(X)}{\arg\min}\ \rho_{PH}\left(X_{PH}+\sum_{i\in S}\pi_i\right) \quad \text{s.t.} \quad \text{condition (2.2) holds} \tag{6.1}$$

and

$$\mathcal{S}_S^{i_0} = \underset{\left(\underline{\boldsymbol{\pi}}^S,\underline{X}^S\right)\in\mathbb{R}_+^S\times\mathcal{A}_S(X)}{\arg\min}\ \rho_{i_0}\left(X_{i_0}-\pi_{i_0}\right) \quad \text{s.t.} \quad \text{condition (2.2) holds,} \tag{6.2}$$

respectively, where $i_0 \in S$ and $S \subseteq N$ is a set of insurers. The next result shows that optimal insurance contracts via individual risk efficiency are Pareto optimal as well, which shows that the optimal insurance/reinsurance problem is well-posed. According to our knowledge, this link has not been discussed in the literature, even though Pareto optimality and individual risk optimisation are two related concepts.

Proposition 6.1. Let $i_0 \in S$ with $S \subseteq N$ and assume that ρ_i satisfies (P2) for all $i \in S \cup PH$. Then, $\mathcal{S}_S^{PH} \subseteq \mathcal{P}_S$ and $\mathcal{S}_S^{i_0} \subseteq \mathcal{P}_S$.

Proof. We show first that $\mathcal{S}_S^{PH} \subseteq \mathcal{P}_S$ and therefore, let $\left(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\right) \in \mathcal{S}_S^{PH}$. Keeping in mind that ρ_i satisfies (P2) and the fact that the objective function in (6.1) is continuous and increasing in each π_i for all $i \in S$, one may find that the insurers' rationality inequality constraints become equality constraints. Thus, $\left(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\right) \in \mathcal{S}_S$ and in turn, we get that $\left(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\right) \in \mathcal{P}_S$ due to Theorem 3.1, which justifies that $\mathcal{S}_S^{PH} \subseteq \mathcal{P}_S$.

Next, we show $\mathcal{S}_S^{i_0} \subseteq \mathcal{P}_S$ and therefore, let $\left(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\right) \in \mathcal{S}_S^{i_0}$. Recall that ρ_i satisfies (P2) and since the objective function in (6.2) is continuous and decreasing in π_{i_0}, then we may conclude that

$$\pi_{i_0}^* = \rho_{PH}(X) - \rho_{PH}\left(X_{PH}^*\right) - \sum_{i \in S\setminus\{i_0\}} \pi_i^*.$$

Consequently, $\left(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\right)$ must solve

$$\begin{aligned} &\min_{\left(\underline{\boldsymbol{\pi}}^{S}, \underline{X}^{S}\right) \in \mathbb{R}_+^S \times \mathcal{A}_S(X)} \rho_{i_0}\left(X_{i_0}\right) + \rho_{PH}\left(X_{PH}\right) + \sum_{i \in S\setminus\{i_0\}} \pi_i \\ &\text{s.t.}\ \ \rho_i\left(X_i\right) - \pi_i \le 0,\ \ i \in S\ \text{ and }\ \rho_{PH}\left(X_{PH}\right) + \sum_{i \in S} \pi_i \le \rho_{PH}(X). \end{aligned}$$

Similar arguments to those given in the proof of $\mathcal{S}_S^{PH} \subseteq \mathcal{P}_S$, one may show that $\left(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\right) \in \mathcal{S}_S$. Thus, $\left(\underline{\boldsymbol{\pi}}^{*S}, \underline{X}^{*S}\right) \in \mathcal{P}_S$ due to Theorem 3.1, which completes the proof. □

7 Model risk

Model risk has been ignored until this very moment and it has been assumed that all insurance players have known the "true" distribution of the total risk and this probabilistic model is the same amongst all players, i.e homogeneous beliefs are only considered. This convenient assumption may lead to decisions that are very sensitive to small changes in the chosen model, which affects the optimal decision. This problem could be tackled by using statistical tools, but the ultimate goal is to produce an optimal robust decision, which is the main objective of a robust optimisation. In addition, Asimit *et al.* (2017a) has shown the advantage of robust optimisation over the statistical methods and for these reasons, this section is focused on standard robust optimisation to deal with model risk. This source of risk could be viewed such that the "true" probability measure is not unknown, but it belongs to a set of possible available probability models that are based on statistical evidence, expert opinion or "good practice" in the sector.

The mathematical formulation of the model risk is such that the "true" probability measure for a given insurer or the PH is one amongst various available choices, $\{\mathbb{P}_j, j \in \mathcal{M}_i\}$ with $i \in N \cup PH$ $\mathcal{M}_i$ being an index set. clearly, these probability measures are defined on the same sample space As a result, $X, X_i \in L^p_+(\mathbb{P}_i)$ for all $i \in N \cup PH$ and $j \in \mathcal{M}_i$. We define $\rho_i(\cdot; \mathbb{P}_j)$ as the risk measure given the underlying probability measure $\mathbb{P}_j$ for $i \in N \cup PH$ and $j \in \mathcal{M}_i$. The rationality constraint for each insurer and PH is now given by

$$\rho_i\big(X_i - \pi_i; \mathbb{P}_{j_1}\big) \leq 0 \quad \text{and} \quad \rho_{PH}\left(X_{PH} + \sum_{k\in S} \pi_k; \mathbb{P}_{j_2}\right) \leq \rho_{PH}\big(X; \mathbb{P}_{j_2}\big) \tag{7.1}$$

for all $i \in S$, $j_1 \in \mathcal{M}_i$ and $j_2 \in \mathcal{M}_{PH}$. Therefore, a contract $\big(\underline{\boldsymbol{\pi}}^S, \underline{X}^S\big)$ is feasible and we write $\big(\underline{\boldsymbol{\pi}}^S, \underline{X}^S\big) \in \mathbb{R}^S_+ \times \mathcal{A}^{\mathcal{M}}_S(X)$, where

$$\mathcal{A}^{\mathcal{M}}_S(X) = \Big\{\underline{X}^S : \sum_{i\in S\cup PH} X_i = X, \;\; X_i \in L^p_+(\mathbb{P}_j) \text{ for all } i \in S \cup PH, \; j \in \mathcal{M}_i\Big\},$$

if (7.1) holds. Further, the contract $\big(\underline{\boldsymbol{\pi}}^S, \underline{X}^S\big) \in \mathbb{R}^S_+ \times \mathcal{A}^{\mathcal{M}}_S(X)$ is called *Pareto robust optimal* if there is no other feasible contract $\big(\underline{\tilde{\boldsymbol{\pi}}}^S, \underline{\tilde{X}}^S\big) \in \mathbb{R}^S_+ \times \mathcal{A}^{\mathcal{M}}_S(X)$ such that

$$\rho_i\big(\tilde{X}_i - \tilde{\pi}_i; \mathbb{P}_{j_1}\big) \leq \rho_i\big(X_i - \pi_i; \mathbb{P}_{j_1}\big) \text{ and } \rho_{PH}\left(\tilde{X}_{PH} + \sum_{k\in S} \tilde{\pi}_k; \mathbb{P}_{j_2}\right) \leq \rho_{PH}\left(X_{PH} + \sum_{k\in S} \pi_k; \mathbb{P}_{j_2}\right)$$

for all $i \in S$, $j_1 \in \mathcal{M}_i$ and $j_2 \in \mathcal{M}_{PH}$, with at least one strict inequality. Let us denote $\mathcal{P}^{\mathcal{M}}_S$ as the set of all Pareto robust optimal contracts.

The individual robust optimisation problems involve the well-known (in robust optimisation) *worst-case scenario* approach. The mathematical formulations for the optimal insurance contract set from the PH and a generic insurer under model risk are

$$\mathcal{S}^{PH,\mathcal{M}}_S = \underset{\big(\underline{\boldsymbol{\pi}}^S, \underline{X}^S\big)\in\mathbb{R}^S_+\times\mathcal{A}_S(X)}{\arg\min} \; \max_{j\in\mathcal{M}_{PH}} \left\{\rho_{PH}\big(X_{PH}; \mathbb{P}_j\big) + \sum_{k\in S} \pi_k\right\} \quad \text{s.t.} \quad (7.1) \quad \text{holds}$$

and

$$\mathcal{S}^{i_0,\mathcal{M}}_S = \underset{\big(\underline{\boldsymbol{\pi}}^S, \underline{X}^S\big)\in\mathbb{R}^S_+\times\mathcal{A}_S(X)}{\arg\min} \; \max_{j\in\mathcal{M}_{i_0}} \rho_{i_0}\big(X_{i_0} - \pi_{i_0}; \mathbb{P}_j\big) \quad \text{s.t.} \quad (7.1) \quad \text{holds},$$

respectively, where $i_0 \in S$ and $S \subseteq N$ is a set of insurers.

Extensions of Theorem 3.1 in the presence of model risk are now given in Proposition 7.1. Its proof is omitted, since is similar to the arguments given in the "if" part proof of Theorem 3.1 and the proof of Proposition 6.1.

Proposition 7.1. Let $S \subseteq N$ and assume that ρ_i and ρ_{PH} satisfy (P2) for all $i \in S$. Then,

i) $\mathcal{S}_S^{+\mathcal{M}} \subseteq \mathcal{P}_S^{\mathcal{M}}$ if $|\mathcal{M}_j| = |\mathcal{M}_{PH}|$ for all $j \in S$, where $|\cdot|$ represents the cardinality of a set,

ii) $\mathcal{S}_S^{PH,\mathcal{M}} \subseteq \mathcal{S}_S^{\mathcal{M}}$ and $\mathcal{S}_S^{i_0,\mathcal{M}} \subseteq \mathcal{S}_S^{\mathcal{M}'}$,

where

$$\begin{aligned}
\mathcal{S}_S^{+\mathcal{M}} = & \underset{\left(\underline{\boldsymbol{\pi}}^S, \underline{X}^S\right) \in \mathbb{R}_+^S \times \mathcal{A}_S^{\mathcal{M}}(X)}{\arg\min} \sum_{i \in S \cup PH} \sum_{j \in \mathcal{M}_i} \rho_i\big(X_i; \mathbb{P}_j\big) \\
& \text{s.t. } \rho_i\big(X_i - \pi_i; \mathbb{P}_{j_1}\big) \leq 0, \;\; \rho_{PH}\left(X_{PH} + \sum_{k \in S} \pi_k; \mathbb{P}_{j_2}\right) \leq \rho_{PH}\big(X; \mathbb{P}_{j_2}\big) \\
& \text{for all } i \in S, j_1 \in \mathcal{M}_i \text{ and } j_2 \in \mathcal{M}_{PH},
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_S^{\mathcal{M}} = & \underset{\left(\underline{\boldsymbol{\pi}}^S, \underline{X}^S\right) \in \mathbb{R}_+^S \times \mathcal{A}_S^{\mathcal{M}}(X)}{\arg\min} \sum_{i \in S \cup PH} \max_{j \in \mathcal{M}_i} \rho_i\big(X_i; \mathbb{P}_j\big) \\
& \text{s.t. } \rho_i\big(X_i - \pi_i; \mathbb{P}_{j_1}\big) \leq 0, \;\; \rho_{PH}\big(X_{PH} + \sum_{k \in S} \pi_k; \mathbb{P}_{j_2}\big) \leq \rho_{PH}\big(X; \mathbb{P}_{j_2}\big) \\
& \text{for all } i \in S, j_1 \in \mathcal{M}_i \text{ and } j_2 \in \mathcal{M}_{PH},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{S}_S^{\mathcal{M}'} = & \underset{\left(\underline{\boldsymbol{\pi}}^S, \underline{X}^S\right) \in \mathbb{R}_+^S \times \mathcal{A}_S^{\mathcal{M}}(X)}{\arg\min} \sum_{i \in S} \max_{j \in \mathcal{M}_i} \rho_i\big(X_i; \mathbb{P}_j\big) + \max_{j \in \mathcal{M}_{PH}} \Big\{\rho_{PH}\big(X_{PH}; \mathbb{P}_j\big) - \rho_{PH}\big(X; \mathbb{P}_j\big)\Big\} \\
& \text{s.t. } \rho_i\big(X_i - \pi_i; \mathbb{P}_{j_1}\big) \leq 0, \;\; \rho_{PH}\big(X_{PH} + \sum_{k \in S} \pi_k; \mathbb{P}_{j_2}\big) \leq \rho_{PH}\big(X; \mathbb{P}_{j_2}\big) \\
& \text{for all } i \in S, j_1 \in \mathcal{M}_i \text{ and } j_2 \in \mathcal{M}_{PH}.
\end{aligned}$$

Proposition 7.1 tells us how to find some Pareto robust optimal contracts, namely by finding $\mathcal{S}_S^{+\mathcal{M}}$. Finding all Pareto robust optimal contracts is a much more difficult problem and involves standard multi-objective optimisation methods such as the weighted sum scalarisation (for example, see Miettinen, 1999; Ehrgott, 2005) when all risk measures are convex. It is well-known that the worst-case robust optimisation problems may not lead to Pareto robust contracts if there are multiple solutions. Interesting discussions that link the multi-objective optimisation with the worst-case robust optimisation are provided in Ehrgott *et al.* (2014) and Ide and Schöbel (2016), but $\mathcal{S}_S^{\mathcal{M}}$ represents the most natural worst-case robust optimisation formulation when the optimisation is viewed from a joint point of view. It is still challenging to find elements of $\mathcal{S}_S^{\mathcal{M}} \bigcap \mathcal{P}_S^{\mathcal{M}}$ and $\mathcal{S}_S^{\mathcal{M}} \bigcap \mathcal{P}_S^{\mathcal{M}'}$, but specific applications require different practical solutions (for example, see Iancu and Trichakis, 2014; Asimit *et al.*, 2017a). The last main result of the section shows a situation in which finding solutions for $\mathcal{S}_S^{\mathcal{M}} \bigcap \mathcal{P}_S^{\mathcal{M}}$ and $\mathcal{S}_S^{\mathcal{M}} \bigcap \mathcal{P}_S^{\mathcal{M}'}$ is possible. We next provide an extension of Theorem 5.1 of Asimit *et al.* (2017a), which allows us to explain our point, but the proof is left to the reader since it is similar to the proof of Theorem 5.1 of Asimit *et al.* (2017a).

Proposition 7.2. Let $(\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*)$ be any optimal solution of the following problem

$$\min_{\substack{(\mathbf{x}_1,\ldots,\mathbf{x}_n)\in \\ \mathbb{R}^\ell\times\cdots\times\mathbb{R}^\ell}} \sum_{i=1}^{n} \max_{j\in\mathcal{M}_i} \left\{\mathbf{c}_{ij}^T\mathbf{x}_i + d_{ij}\right\} \quad \text{s.t.} \quad \mathbf{A}_i\mathbf{x}_i \le \mathbf{b}_i,\ 1\le i\le n,\ \mathbf{x}_1+\cdots+\mathbf{x}_n = \mathbf{x}, \tag{7.2}$$

with known $\mathbf{A}_i$, $\mathbf{b}_i$, $\mathbf{c}_{ij}$ and $\mathbf{x}$ matrices and column vectors of appropriate dimensions and known scalars d_{ij}. Moreover, consider the following optimisation problem:

$$\min_{\substack{(\mathbf{y}_1,\ldots,\mathbf{y}_n)\in \\ \mathbb{R}^\ell\times\cdots\times\mathbb{R}^\ell}} \sum_{i=1}^{n} \sum_{j\in\mathcal{M}_i} \mathbf{c}_{ij}^T\mathbf{y}_i \quad \text{s.t. } \mathbf{A}_i(\mathbf{x}_i^*+\mathbf{y}_i) \le \mathbf{b}_i,\ \mathbf{c}_{ij}^T\mathbf{y}_i \le 0,\ 1\le i\le n, j\in\mathcal{M}_i,\ \mathbf{y}_1+\cdots+\mathbf{y}_n = \mathbf{0}. \tag{7.3}$$

i) If the optimal objective value in (7.3) is zero, then $(\mathbf{x}_1^*, \ldots, \mathbf{x}_n^*)$ is Pareto robust efficient in the sense that there is no other $(\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_n) \in \mathbb{R}^\ell\times\cdots\times\mathbb{R}^\ell$ feasible in (7.2) such that $\mathbf{c}_{ij}^T\mathbf{x}_i^* \ge \mathbf{c}_{ij}^T\tilde{\mathbf{x}}_i$ for all $1\le i\le n$, $j\in\mathcal{M}_i$ and at least one inequality is strict.

ii) If the optimal objective value in (7.3) is negative, then $(\mathbf{x}_1^*+\mathbf{y}_1^*, \ldots, \mathbf{x}_n^*+\mathbf{y}_n^*)$ solves (7.2) and is Pareto robust efficient, where $(\mathbf{y}_1^*, \ldots, \mathbf{y}_n^*)$ is an optimal solution of (7.3).

Note 7.3. One may apply Proposition 7.2 for finding solutions for $\mathcal{S}_S^{\mathcal{M}} \bigcap \mathcal{P}_S^{\mathcal{M}}$ and $\mathcal{S}_S^{\mathcal{M}} \bigcap \mathcal{P}_S^{\mathcal{M}'}$ in a particular setting where the state space is finite, i.e. $\Omega = \{x_1, \ldots, x_\ell\}$. If ρ is a distortion risk measure as defined in (2.1), then $\rho(X;\mathbb{P}) = \boldsymbol{d}^T\mathbf{x}$ provided that $\mathbf{x}$ is increasingly ordered, where

$$d_k = g\Big(1-\sum_{s=1}^{k-1}p_s\Big) - g\Big(1-\sum_{s=1}^{k}p_s\Big) \quad \text{and} \quad \mathbb{P}(X=x_k)=p_k \quad \text{for all} \quad 1\le k\le \ell$$

for further details, see Dhaene *et al.* (2012). Clearly, if ρ_i satisfy (2.1) for all $i\in S\cup PH$ and the admissible set of $\underline{X}^S$ is $\mathcal{C}_S(X)$, then the optimisation problems with the set of optimal solutions given by $\mathcal{S}_S^{\mathcal{M}} \bigcap \mathcal{P}_S^{\mathcal{M}}$ and $\mathcal{S}_S^{\mathcal{M}'} \bigcap \mathcal{P}_S^{\mathcal{M}}$ have only linear terms and become a special case of problem (7.2) in Proposition 7.2.

A final conclusion could be drawn after all of these findings. The individual robust solutions could be written in various ways, but $\mathcal{S}^{\mathcal{M}}$ is the most natural formulation. Proposition 7.2 shows that linear formulations have the advantage of finding optimal Pareto contracts that solve $\mathcal{S}^{\mathcal{M}}$ or $\mathcal{S}^{\mathcal{M}'}$, which is the ultimate goal of our analysis. Finding optimal Pareto contracts for non-linear instances is an open problem that remains to investigate in the future.

8 Conclusions

This paper provides micro-economic theory for optimal insurance contract design with translation invariant risk measures. Whereas traditional actuarial papers on optimal insurance focus on optimising future utility of one specific party, we study Pareto optimality and the anti-core of an appropriate game. The set of Pareto optimal contracts is characterised, and we find that Pareto optimality leads to a structure on the indemnity functions, and the premiums need guarantee individual rationality. This allows us to disentangle the indemnity functions and premiums, where the set of premiums that can be chosen with Pareto optimality is not necessarily single-valued. Further, we propose to select premiums in the anti-core of an appropriate cooperative game.

The optimal reinsurance contract design has been investigated in the last decade, but independent of the vast classical Pareto optimality literature; we show that these two concepts are very much related and in fact, optimal reinsurance contracts are in fact Pareto optimal. This is an interesting result that shows why the optimal reinsurance contract design provides valuable information even though focuses on a small subset of the set of Pareto optimal contracts. The final part of the paper discusses the model risk issue and we manage to explain how a robust and optimal Pareto optimal contract could be found.

Existing literature on Pareto optimality of insurance contracts with expected utility preferences focuses on the effect of costs. When there are *ex-post* costs, insurance policies with an upper limit are not part of the solution. When these costs are variable, deductibles appear in Pareto optimal contracts (see Raviv, 1979; Blazenko, 1985; Spaeter and Roger, 1997). Aase (2017) extends this result of Pareto optimal deductibles to the case of *ex-post* quasi-costs: the costs include a fixed cost each time a claim is made. In this paper, we neglect costs, and find that Pareto optimal deductibles may exist when distortion risk measures are used. We believe that a study of costs on Pareto optimal contracts with risk measures would be interesting to investigate in the coming future.

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A Proof of Proposition 5.2

It is well-known that additive games have a non-empty core and therefore, we only need to show the "only if" part of the proof. Now, suppose $core(N, v) \neq \emptyset$. The Bondareva-Shapley Theorem (see the proof of Theorem 5.4) states that $core(N, v)$ is non-empty if and only if the game (N, v) is balanced. Take the balanced collection $\lambda_{\hat{S}} = 1$ if $|\hat{S}| = 1$ and $\lambda_{\hat{S}} = 0$ otherwise. Let $S \subseteq N$ and since (N, v) is balanced, then it holds that $\sum_{i \in S} v(\{i\}) \leq v(S)$. Moreover, since $(N, -v)$ is balanced as well (see Theorem 5.4), we have $\sum_{i \in S} v(\{i\}) \geq v(S)$ and hence, $\sum_{i \in S} v(\{i\}) = v(S)$. The latter holds for all $S \subseteq N$ and consequently, the TU game (N, v) is additive, which concludes our proof.