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Broadcast of classical information through a quantum channel

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Abstract. – We consider transmission of classical information through a quantum channel from one sender to several receivers. In contrast to quantum information, classical information carried by two non-orthogonal states of a quantum system can be cloned exactly. This restricts a naive extrapolation of the standard no-cloning theorem. The optimal cloning induces entanglement between clones which increases with non-orthogonality of the states. The capacity of the corresponding quantum channel is considered. As a byproduct of our results, we obtain that for an entangled system with an unknown state there is no transformation which removes the entanglement totally without affecting the corresponding marginal density matrices.

Introduction. – Quantum information theory considers the transmission, processing, and detection of information, taking into account the quantum nature of its carriers. In particular, this is the case when *classical information* is coded by states of a quantum system (quantum-classical information theory). If the transmitted states are mutually orthogonal, all the messages can be accurately extracted at the receiver with von Neumann's standard measurement. However, for many reasons the delivered states in the quantum communication system are non-orthogonal. For example, non-orthogonality can arise due to noise in the transmitter or energy loss in the channel. It can be connected also with the construction of the transmitter. An example can illustrate this case. The model with binary coherent-state signals $|\alpha\rangle$, $|- \alpha\rangle$ is widely used in optical communication [1]. They are nonorthogonal: $\langle \alpha | - \alpha \rangle = \exp[-2|\alpha|^2]$, but are used to encode the classical symbols 0, 1. Only in the classical case, namely if the energy is large ($|\alpha|^2 \gg 1$), they are nearly orthogonal. After receiving a system, an address makes a measurement, and decides what state was actually sent.

There has been a considerable interest in the quantum-classical information theory; there are important general results [2–4], and concrete realizations of the optimal schemes in practice [1]. However, all these results concern one-terminal channels, where there are one transmitter and one receiver. The theory of multi-terminal quantum channels attracted attention

only very recently [5,6]. In particular, the multi-access channels have been investigated, where several new questions arise due to the presence of several senders and only one receiver [5]. In the present work we consider a *broadcast* quantum channel for classical information transmission: The classical information from one source should be directed to two receivers (users), which are chosen equivalent for simplicity. Thus, the quantum states that represent this information should be *cloned*. The exact copying (cloning) of an unknown quantum state does not exist. This well-known theorem [7] has been recently generalized to mixed states [8]: The exact cloning is impossible (except an evident case) even entanglement between clones is allowed. The physical origin is the following: states from an arbitrary set cannot be distinguished from each other, therefore the complete information cannot be obtained. Otherwise, an obvious contradiction will arise: By cloning the system many times, its (unknown) state can be determined with arbitrary high precision, and the fundamental difference between orthogonality and non-orthogonality will effectively be surmounted. Thus *quantum information* cannot be cloned perfectly.

In this paper we wish to answer the related question: is it possible to clone [9] non-orthogonal quantum states which represent *classical information*? Evidently, if only classical information is transmitted, then it is reasonable to require *only* as distinguishable clones as possible. This is sufficient for the reliable broadcasting of classical information through a quantum channel. Many usual sources of information are broadcast: radio, TV, Internet, ... They are classical channels, but in future —with development of the technologies— quantum effects will be important too.

We consider the simplest binary quantum broadcast channel, and introduce a unitary cloning transformation. The entanglement introduced by this transformation is discussed. As a byproduct we get an interesting application for the physics of decoherence. The results are applied to a simple scheme of the broadcast communication, whose classical analog has practical applications.

Binary-state broadcast communication. – The binary-state communication is one of the most important schemes in the practice and theoretical considerations. It is simple enough, but at the same time contains many essential properties of more general cases.

The classical information is represented as a sequence of bits. We assume that in a long sequence the frequencies of 0 and 1 are the same, *i.e.*, the *a priori* probabilities are equal. This condition is nearly satisfied in many practical cases. A quantum coder generates the state $|0\rangle$ ($|1\rangle$) for the symbol 0 (1),

$$\langle 1|0\rangle = \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad (1)$$

where θ measures the degree of non-orthogonality. In the second step the quantum states should be cloned and broadcasted to the users A, B. Thus the whole scheme is the following:

$$\begin{aligned} \rho_{A,i} &= \text{tr}_B \sigma_i \mapsto A, \\ |i\rangle_A &\mapsto \hat{U}(|i\rangle_A |\text{blank}\rangle_B) = \sigma_i \mapsto \\ \rho_{B,i} &= \text{tr}_A \sigma_i \mapsto B, \end{aligned} \quad (2)$$

where $i = 0, 1$, $\hat{U}$ is a cloning transformation, $|\text{blank}\rangle$ is the initial state of the “cloning machine”, tr_A , tr_B are the partial traces by the subspaces of A and B. In the third step the users recover the classical information by a measurement. Every positive-operator-valued measurement (POVM) $\{\Pi_0, \Pi_1\}$ ($\Pi_0 + \Pi_1 = 1$), corresponds to the concrete process of decision. Since the obtained states are non-orthogonal, there is some probability of error when distinguishing

between them. When transmitting classical information one should optimize the error in distinguishing between $\rho_{A,1}$ and $\rho_{A,0}$, as well as between $\rho_{B,1}$ and $\rho_{B,0}$. For simplicity we consider the case of equivalent users: $\rho_{A,i} = \rho_{B,i}$, $i = 0, 1$, which is enough to illustrate our main result. Because $|\text{blank}\rangle$ is only some known state under control, we shall choose $|\text{blank}\rangle = |0\rangle$. We also restrict ourselves to the consideration of unitary $\hat{U}$ in the corresponding four-dimensional Hilbert space. For the average error one gets (recall that *a priori* probabilities are the same)

$$P_e = \frac{1}{2}(p(0/1) + p(1/0)) = \frac{1}{2}(1 + \text{tr}(\rho\Pi_1)), \quad \rho = \rho_0 - \rho_1 \quad (3)$$

and $\text{tr}(\rho\Pi_1) = \sum_i (\rho\Pi_1)_{(i)}$ should be minimized (where $A_{(i)}$ is a corresponding eigenvalue of the matrix A). By the well-known matrix inequality [11] we have

$$(\rho\Pi_1)_{(i)} \geq \rho_{(i)} \max_i (\Pi_1)_{(i)}, \quad \text{if} \quad \rho_{(i)} \leq 0. \quad (4)$$

The minimum can be obtained by attaining this lower bound. Thus the optimal operators Π_0 , Π_1 are projectors, and the resulting formulas read [3]

$$\Pi_1^{(\text{opt})} = \sum_i \theta(-\rho_{(i)}) |\rho_{(i)}\rangle \langle \rho_{(i)}|, \quad P_e^{(\text{opt})} = \frac{1}{2} \left(1 + \sum_i \theta(-\rho_{(i)}) \rho_{(i)} \right). \quad (5)$$

For density matrices in a two-dimensional Hilbert space a simpler formula can be obtained

$$P_e^{(\text{opt})} = \frac{1}{2}(1 + \rho_{(\min)}), \quad (6)$$

where $\rho_{(\min)}$ is the minimal eigenvalue of ρ . The unitary operation $\hat{U}$ can be written as

$$\hat{U}(|\alpha\rangle_A |0\rangle_B) = a_\alpha |0\rangle_A |0\rangle_B + b_\alpha |0\rangle_A |\bar{0}\rangle_B + c_\alpha |\bar{0}\rangle_A |0\rangle_B + d_\alpha |\bar{0}\rangle_A |\bar{0}\rangle_B, \quad (7)$$

where $\alpha = 0, \bar{0}, 1$, and $\langle 0|\bar{0}\rangle = \delta_{0\bar{0}}$. Further we assume that all coefficients in (7) are real. For the conditions $\rho_{A,i} = \rho_{B,i}$ (equivalent users) it is enough to choose $b_0 = c_0$ and $b_1 = c_1$. Equation (6) can be written as

$$P_e = \frac{1}{2}(1 - \sqrt{\Lambda}), \quad \Lambda = (a_1 c_1 + b_1 d_1 - a_0 c_0 - b_0 d_0)^2 + a_1^2 + b_1^2 - a_0^2 - b_0^2, \quad (8)$$

and Λ should be maximized by the parameters taking into account the restrictions arising from the unitary property. After some calculations we get that for the optimal unitary transformations like

$$a_{(0)1} = \frac{1}{\sqrt{2}} \left(\pm \sin \frac{\theta}{2} + \cos \frac{\theta}{2} \cos \phi \right), \quad d_{(1)0} = -a_{(0)1}, \quad b_1 = b_0 = \frac{1}{\sqrt{2}} \cos \frac{\theta}{2} \sin \phi, \quad (9)$$

(where $0 \leq \phi \leq 2\pi$ is a free parameter) the error in distinguishing between ρ_1 and ρ_0 is the same as for $|1\rangle$ and $|0\rangle$:

$$P_e = \frac{1}{2}(1 - \sin \theta). \quad (10)$$

For the marginal density matrices we have

$$\rho_{(0)1} = \frac{1}{2}(1 \pm \sin \theta \cos \phi) |0\rangle \langle 0| + \frac{1}{2}(1 \mp \sin \theta \cos \phi) |\bar{0}\rangle \langle \bar{0}| \pm \frac{1}{2} \sin \theta \sin \phi (|0\rangle \langle \bar{0}| + |\bar{0}\rangle \langle 0|). \quad (11)$$

The optimal measurement can be written as

$$\Pi_{(0)1} = |\psi_{\pm}\rangle\langle\psi_{\pm}|, \quad |\psi_{\mp}\rangle = \frac{1}{\sqrt{2(1 \mp \cos \phi)}}((\mp 1 + \cos \phi)|0\rangle + \sin \phi|\bar{0}\rangle). \quad (12)$$

Two pure states in a Hilbert space of any dimension span only a two-dimensional subspace; hence any two non-orthogonal pure state can be cloned *perfectly* from the point of view of classical information. This is our main result. The possibility of cloning orthogonal states [7] can be employed to interpret this result [12]. Being initially prepared with non-orthogonal states, the classical information can first be extracted by the optimal measurement, and then broadcasted by a (classical) channel having orthogonal signal states. As we have discussed, this direct procedure cannot be realized in many cases, since it returns to the use of orthogonal signal states.

According to eq. (11) the clones are entangled. Since the clones are the subsystems of the pure system, quantum entropy of a marginal density matrix can be used as a measure of the corresponding entanglement [13]: $S = -\text{tr} \sigma \log_2 \sigma$. For pure state S is zero, and it is positive function for all other cases (S is maximized with the uniform distribution). Now for the entanglement of our clones we have

$$S(\rho_0) = S(\rho_1) = h(P_e), \quad h(x) = -x \log_2 x - (1-x) \log_2 (1-x). \quad (13)$$

We see that entanglement is *maximal* in the “worst” case when $\theta \mapsto \pi/2$. It can be viewed as some “cost” which arises from the possibility of the perfect cloning.

In the context of the obtained results, an interesting question comes to mind. Let us assume at the moment that the composite system with possible states σ_1, σ_2 (see eq. (2)) can be *decoherenced exactly*. In other words, it can be subjected to some quantum transformation after which the entanglement between clones is removed totally, but without affecting upon the marginal density matrices: $\sigma_i \rightarrow \rho_i \otimes \rho_i$. Such a process *contradicts* the laws of quantum theory, because if it exists, then it is possible to distinguish the non-orthogonal states $|0\rangle, |1\rangle$ with more success than it is given by eq. (6). Indeed, one can use *post-priori* results of the A-measurement as the *a priori* probabilities for the B-measurement. Continuing such a line of arguments, we get that non-orthogonal states can be distinguished without any error at all. Our particular result illustrates a more general thesis: A quantum system in unknown state cannot be decoherenced exactly [14]. The entanglement between clones prevents such a procedure, and conserves the self-consistent structure of the theory.

Applications to the special case. – In the last part of the paper we discuss the qualitative measures of information transmission through a particular broadcast channel. We consider only one possible, practically important scenario [10]: Two independent classical sources *simultaneously* communicate by the same generator of quantum states (coding machine) with the users A and B (correspondingly). We start with the classical theory [10], and apply it in our case.

Suppose that possible quantum states of the coding machine have *a priori* probabilities p_x . After action of a cloning transformation and a measurement the user A (B) obtains a classical “letter” y (z) with a probability p_y (p_z) (the role of the entanglement here will be discussed later):

$$p_y = \sum_x p_1(y/x)p_x, \quad p_z = \sum_x p_2(z/x)p_x, \quad (14)$$

where a noise is described by the following sets of conditional probabilities:

$$p_1(y/x), \quad p_2(z/x). \quad (15)$$

Let us denote the ensemble of states for the coding machine by X , and the messages for the user A (B) are in the ensemble Y (Z). Now for N_1 symbols of the source 1, and for N_2 symbols of the source 2, N states of the ensemble X are generated and transmitted to the users. The users must *separate* and recognize their messages because the user A (B) wants to have only messages from the source 1 (2). In this sense each source acts as a noise for another, because messages for different users are mixed. The problem is to find the maximal rates $R_1 = N_1/N$ and $R_2 = N_2/N$, where the reliable connection (the total probability of error $\rightarrow 0$ for $N \rightarrow \infty$) is still possible.

Thus, the broadcast channel is defined by the ensembles X, Y, Z , the *a priori* probabilities p_x , and the conditional probabilities $p_1(y/x)$, $p_2(z/x)$. We assume additionally that the channel is *degraded*:

$$p_2(z/x) = \sum_y W(z/y)p_1(y/x), \quad (16)$$

where $W(z/y)$ is some conditional probability distribution. This means that the transmission scheme can be formally represented as

$$S \begin{array}{c} \xleftrightarrow{\quad} \\ \text{0-channel} \end{array} X \begin{array}{c} \xleftrightarrow{\quad} \\ \text{1-channel} \end{array} Y \begin{array}{c} \xleftrightarrow{\quad} \\ \text{W-channel} \end{array} Z, \quad (17)$$

where the noise in the W -channel is described by $W(z/y)$, and the 0-channel is introduced as some “trade-off” channel between R_1 and R_2 . The following result has been reviewed in [10]: The reliable connection between the sources and their addresses is possible if and only if

$$R_1 \leq I(X : Y/S), \quad R_2 \leq I(S : Z), \quad (18)$$

where

$$I(X : Y/S) = \sum_{x,y,s} p(x,y,s) \log_2 \frac{p(y/xs)}{p(y/s)}, \quad I(S : Z) = \sum_{s,z} p(s,z) \log_2 \frac{p(s/z)}{p(s)}. \quad (19)$$

The second quantity is the usual mutual information between the ensembles S and Z , and the first one is called mutual-conditional information (mc-information). The mutual information of two ensembles is the reduction of entropy of one ensemble if another one is observed. Mc-information has the same meaning but after realization of the conditional ensemble (S in our case). The physical meaning of (18) can be understood from eq. (17) as the generalization of Shannon’s theorem. R_2 is determined by the direct connection between S and Z , but for the determination of R_1 the ensemble S should be fixed. If 0-channel is absent then R_1 (R_2) is maximal (minimal), and if 0-channel is noise-free then the opposite case is realized: R_2 (R_1) is maximal (minimal).

We apply this theory to our problem: The states of coding machine are $|0\rangle$ and $|1\rangle$ with equal *a priori* probabilities, as a cloning transformation we use (9), and the users make the same measurement (12) for obtaining their classical messages. The resulting channel is degraded, and $W(z/y) = \delta_{zy}$. As we have seen the entanglement is introduced by the cloning transformation. Thus, the users are *dependent*, and the distributions (15) are the corresponding marginal distributions of $p(yz/x)$. Fortunately, the capacities of a degraded broadcast channel depend only on the distributions (15) [10]. On the other hand, no measurement of A (B) can influence the marginal probabilities obtained by B (A), in spite of the entanglement (no superluminal communication) [15].

We assume that 0-channel is memoryless, has equal *a priori* probabilities (it can be shown that this choice is optimal), and $p_0(0/1) = p_0(1/0) = \epsilon$. The final results are the following:

$$I(X : Y/S) = h((1 - P_e)\epsilon + P_e(1 - \epsilon)) - h(P_e), \quad I(S : Z) = 1 - h((1 - P_e)\epsilon + P_e(1 - \epsilon)). \quad (20)$$

It is enough to assume that $0 \leq \epsilon \leq 0.5$; if $\epsilon = 0(0.5)$, then $I(S : Z)$ ($I(X : Y/S)$) is maximal, and the opposite quantity is minimal. It is interesting that the maximal rates (capacities) depend only on P_e (10).

Conclusion. – We have considered the quantum broadcast channel, which illustrates the fact that non-orthogonal quantum states representing classical information can be cloned perfectly. This restricts naive applications of the standard no-cloning theorem in the quantum-classical information theory. The cloning introduces entanglement, which increases with indistinguishability of the initial states. As a byproduct of our main result, we obtain that an unknown quantum state cannot be decoherenced exactly. We compute the information capacities for the case, when two sources of information independently and simultaneously communicate by the broadcast channel. This scheme should have practical applications.

It must be noted that an approximate cloning of quantum information is considered by many authors (see, for example, [16]). As far as we know, the present paper is the first where broadcasting of classical information by quantum channels is investigated.

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