Mathematical signal analysis: wavelets, Wigner distribution and a seismic application
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Chapter 4

A Framework for Multiresolution Analysis in Hilbert Spaces

In Section 3.2 we already introduced the DWT for functions in $L^2(\mathbb{R})$ by means of a multiresolution analysis. In this chapter we consider a functional analytical setting of an MRA for separable Hilbert spaces $H$. Using this framework necessary and sufficient conditions on operators on $H$ and functions in $H$ are derived such that they constitute wavelet bases in $H$.

Meyer already gave strong hints for a generalization of MRA in [62]. A more general concept, like we present here, was also investigated by Goodman, Lee and Tang in [36]. However they use a different approach to construct bases in $H$ using MRA.

This chapter is based on [75].

4.1 Frames and Riesz Systems

In the previous chapter we already mentioned the concept of frames and Riesz bases, related to the DWT in $L^2(\mathbb{R})$. Here we introduce these concept for arbitrary separable Hilbert spaces $H$ with inner product $(\cdot, \cdot)$. Furthermore, we come to some results that relate these concepts.

In the sequel we denote for a countable index set $\mathcal{D}$, the Hilbert space of all square summable functions from $\mathcal{D}$ into $C$ by $l^2(\mathcal{D})$ and its inner product by $(\cdot, \cdot)_{\mathcal{D}}$. The standard orthonormal basis in $l^2(\mathcal{D})$ is denoted by $\{e_j\}_{j \in \mathcal{D}}$, so $e_j(i) = \delta_{i,j}$ for $i, j \in \mathcal{D}$. The expression $l^2_0(\mathcal{D})$ indicates the linear span of the set $\{e_j \mid j \in \mathcal{D}\}$.
**Definition 4.1.1** A collection $\Omega = \{v_j| j \in \mathcal{D}\}$ in $H$ is called a frame with frame bounds $m_F$ and $M_F$, $0 < m_F \leq M_F$, if for all $x \in H$ the sequence $(x, v_j)_{j \in \mathcal{D}}$ belongs to $l^2(\mathcal{D})$ and

$$m_F\|x\|^2 \leq \sum_{j \in \mathcal{D}} |(x, v_j)|^2 \leq M_F\|x\|^2. \quad (4.1)$$

Note that the wavelet frame $(\psi, a_0, b_0)$ as introduced in Section 3.2 also satisfies this condition. Obviously, condition (3.16) equals (4.1) by taking

$$\mathcal{D} = \mathbb{Z}^2 \text{ and } v_{n,m} = \tau_{nb_0a_0^m}\mathcal{D}_{a_0^m}\psi,$$

for some wavelet $\psi \in L^2(\mathbb{R})$.

For $\{v_j| j \in \mathcal{D}\}$ a frame in $H$, define $S_F : H \to l^2(\mathcal{D})$ by

$$S_F x = \sum_{j \in \mathcal{D}} (x, v_j)e_j, \quad \forall x \in H. \quad (4.2)$$

According to Definition 4.1.1,

$$m_F\|x\|^2 \leq \|S_F x\|^2 \leq M_F\|x\|^2. \quad (4.3)$$

So $S_F$ is a bounded linear operator from $H$ into $l^2(\mathcal{D})$, such that $S_F^*S_F$ has a bounded inverse. The optimal constants $m_F$ and $M_F$ are given by

$$m_F = \|(S_F^*S_F)^{-1}\|^{-1} \text{ and } M_F = \|S_F^*S_F\|.$$

The operator $S_F$ is called the frame generator associated with the frame $\Omega$.

A straightforward computation shows that

$$S_F^*\alpha = \sum_{j \in \mathcal{D}} (\alpha, e_j)v_j, \quad \forall \alpha \in l^2(\mathcal{D}).$$

Hence, the adjoint frame generator $S_F^*$ is given by

$$S_F^*e_j = v_j.$$

The following lemma presents some auxiliary results on bounded operators. Using this lemma we are able to derive relations between the frame generator and its adjoint.

**Lemma 4.1.2** Let $A \in B(H_1, H_2)$, with $B(H_1, H_2)$ the space of all bounded linear operators from the Hilbert space $H_1$ to the Hilbert space $H_2$. Then, the following are equivalent

(i) There exists an operator $B \in B(H_2, H_1)$ such that $BA = I$,
There exists an $m > 0$ such that $\|Ax\| \geq m\|x\|$, for all $x \in H_1$.

The null space $\text{Ker}(A) = \{0\}$ and the range $\text{Ran}(A)$ is closed.

There exists an operator $C \in B(H_1, H_2)$ such that $A^*C = I$.

The range $\text{Ran}(A^*) = H_1$.

**Proof**

Assume (i) holds. Then

$$\|x\| = \|BAx\| \leq \|B\| \|Ax\|.$$  

So, property (ii) holds for $m = \|B\|$. If property (ii) holds, then $\text{Ker}(A) = \{0\}$, since $\|Ax\| = 0$ implies $\|x\| = 0$ using property (i). For proving that $\text{Ran}(A)$ is closed, we take a sequence $(x_n)_{n \in \mathbb{N}}$ in $H_1$ and a vector $y \in H_2$, such that

$$Ax_n \to y \ (n \to \infty).$$

Then

$$\|x_k - x_n\| \leq \frac{1}{m} \|Ax_k - Ax_n\| \to 0 \ (n \to \infty).$$

So, $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. For its limit $x \in H_1$ we have

$$Ax = \lim_{n \to \infty} Ax_n = y,$$

which yields that $\text{Ran}(A)$ is closed.

Obviously, (i) and (iv) are equivalent. If (i) holds, then then also (iv) holds for $C = B^*$ and conversely. Therefore, instead of proving property (iv), if property (iii) holds, we prove property (i). To do this, we consider the mapping $A_1$ from $H_1$ into $\text{Ran}(A)$, given by $A_1x = Ax$, for all $x \in H_1$. Since $\text{Ran}(A)$ is closed, we have that $A_1$ is a continuous linear bijection from the Hilbert space $H_1$ onto the Hilbert space $\text{Ran}(A)$. By the inverse mapping theorem, $A_1$ is invertible with bounded inverse $A_1^{-1} : \text{Ran}(A) \to H_1$. Now, define $B$ by $By = A_1^{-1}y_1$, for $y = y_1 + y_2$, with $y_1 \in \text{Ran}(A)$ and $y_2 \in \text{Ran}(A)^\perp$. Then $B \in B(H_2, H_1)$ and $BA = I$.

Next, we assume that (iv) holds. Then

$$H_1 = \text{Ran}(A^*C) \subset \text{Ran}(A^*) \subset H_1,$$

and therefore $\text{Ran}(A^*) = H_1$. The last thing we have to prove is that property (iv) holds, given property (v). This is shown as follows. Take $E$ as $A^*$ restricted to $\text{Ker}(A^*)^\perp$. Then $E$ is a continuous linear bijection from the Hilbert space $\text{Ker}(A^*)^\perp$ onto $H_1$. As before, the inverse mapping theorem yields that $E$ has a bounded inverse $E^{-1}$ from $H_1$ onto $\text{Ker}(A^*)^\perp$. By defining an operator $C$ by $Cx = E^{-1}x$, for all $x \in H_1$, property (iv) is established. \qed
It follows from the previous lemma, that $S_F$ is an injective bounded linear operator from $l^2(D)$ into $H$ with range, $\text{Ran}(S_F)$, closed in $H$. This is equivalent with the fact that $S_F^*S_F$ is a boundedly invertible operator. Also $S_F^*(S_F^*S_F)^{-1}$ is the right inverse of $S_F^*$ with minimal norm, which can be shown as follows. Take $P = S_F(S_F^*S_F)^{-1}S_F^*$, the orthoprojector from $l^2(D)$ onto $\text{Ran}(S_F)$. Let further $A$ be a right inverse of $S_F^*$. Then

$$
\|S_F(S_F^*S_F)^{-1}\| = \|P S_F^*(S_F^*S_F)^{-1}\| = \|PA\| \leq \|A\|.
$$

In addition to the frame elements $v_j$ we define $\tilde{v}_j$, $j \in D$, in $\text{Ran}(S_F^*)$ by

$$
\tilde{v}_j = (S_F^*S_F)^{-1}v_j = (S_F^*S_F)^{-1}S_F^*e_j.
$$

Then for all $x \in \text{Ran}(S_F^*)$

$$
x = S_F(S_F^*S_F)^{-1}x = S_F \sum_{j \in D} (S_F^*S_F)^{-1}x, e_j) e_j
$$

and

$$
x = (S_F^*S_F)^{-1}S_F x = (S_F^*S_F)^{-1}S_F \sum_{j \in D} (S_F x, e_j) e_j
$$

$$
= \sum_{j \in D} (x, S_F e_j)(S_F^*S_F)^{-1}S_F^*e_j = \sum_{j \in D} (x, v_j) \tilde{v}_j.
$$

Following these derivations and the equivalent properties as described in Lemma 4.1.2 we arrive at the following theorem.

**Theorem 4.1.3** Let $\Omega = \{v_j \mid j \in D\}$ be a collection in $H$ satisfying the right inequality in (4.1) and let $S_F$ be defined by (4.2). Then $\Omega$ is a frame if and only if the adjoint of the frame generator $S_F^*$ associated with $\Omega$ is surjective. If $\Omega$ is a frame, the collection $\{\tilde{v}_j \mid j \in D\}$, defined by $\tilde{v}_j = (S_F^*S_F)^{-1}S_F^*e_j$, is also a frame, which we call the frame dual to $\Omega$.

We observe that since $\{\tilde{v}_j \mid j \in D\}$ is a frame, there exists also an associated frame generator given by $S_F(S_F^*S_F)^{-1}$. Special cases of frames are exact frames and tight frames, which are defined as follows.

**Definition 4.1.4** An exact frame is a frame that is no longer a frame whenever any one of its elements is removed. A tight frame is a frame for which the frame bounds satisfy $m_F = M_F$.

It can be shown in a rather straightforward way that if $\Omega = \{v_j \mid j \in D\}$ is a tight frame for which $m_F = 1$ and $\|v_j\| = 1$ for all $j \in D$, then $\Omega$ is an orthonormal system in $H$ and conversely. A similar property also holds for Riesz systems, which are defined as follows.
Definition 4.1.5 The collection \( \Omega = \{ v_j \mid j \in D \} \) in \( H \) is called a Riesz system with Riesz bounds \( 0 < m_R \leq M_R \), if
\[
m_R \| \alpha \|^2_D \leq \| \sum_{j \in D} (\alpha, e_j) v_j \|^2 \leq M_R \| \alpha \|^2_D, \quad \forall \alpha \in l^2(D).
\] (4.4)

Obviously, a Riesz system with Riesz bounds \( m_R = M_R = 1 \) is an orthonormal system, which means that orthonormal systems constitute a special set of Riesz systems. Also to a Riesz system we associate an operator in a similar way as we have done for frames.

For \( \{v_j \mid j \in D\} \) a Riesz system, define \( S^{(0)}_R : l^2(D) \to H \) by
\[
S^{(0)}_R \alpha = \sum_{j \in D} \alpha(j) v_j, \quad \forall \alpha \in l^2(D).
\]

Hence, \( S^{(0)}_R e_j = v_j \). The operator \( S^{(0)}_R \) extends to a bounded linear operator \( S_R \) from \( l^2(D) \) into \( H \), called the Riesz system generator of \( \Omega \). This Riesz generator satisfies
\[
m_R \| \alpha \|^2 \leq \| S_R \alpha \|^2 \leq M_R \| \alpha \|^2.
\] (4.5)

In the same manner as we concluded for the frame generator, we conclude that \( S^*_R S_R \) is a boundedly invertible operator on \( l^2(D) \). Moreover, the left inverse of \( S_R \) with minimal norm is given by \( (S^*_R S_R)^{-1} S^*_R \). This can be verified in the same manner as for the right inverse of \( S^*_F \).

Beside the Riesz system \( \{v_j \mid j \in D\} \) we also define the elements \( \bar{v}_j, j \in D, \) in \( \text{Ran}(S_R) \) by \( \bar{v}_j = S_R(S^*_R S_R)^{-1} e_j \). Hence, \( S^*_R \bar{v}_j = e_j \) and consequently
\[
(\bar{v}_j, v_k) = (\bar{v}_j, S_R e_k) = (e_j, e_k) = \delta_{j,k}.
\]

Furthermore, for all \( x \in \text{Ran}(S_R) \) and for \( \alpha \in l^2(D) \) with \( x = S_R \alpha \), we have
\[
x = S_R \alpha = S_R(S^*_R S_R)^{-1} (S^*_R S_R) \alpha = S_R(S^*_R S_R)^{-1} S_R^* x = S_R \sum_{j \in D} ((S^*_R S_R)^{-1} S_R^* x, e_j) e_j
\]
\[
= \sum_{j \in D} (x, S_R(S^*_R S_R)^{-1} e_j) S_R e_j = \sum_{j \in D} (x, \bar{v}_j) v_j,
\]
and
\[
x = S_R(S^*_R S_R)^{-1} S_R^* x = S_R(S^*_R S_R)^{-1} \sum_{j \in D} (S_R^* x, e_j) e_j
\]
\[
= \sum_{j \in D} (x, S_R e_j) S_R(S^*_R S_R)^{-1} e_j = \sum_{j \in D} (x, v_j) \bar{v}_j.
\]

These results are summarized in the following theorem.
**Theorem 4.1.6** Let \( \Omega = \{ v_j \mid j \in D \} \) be a collection in \( H \). Then \( \Omega \) is a Riesz system if and only if there is a bounded linear injection \( S_R : l^2(D) \to H \) with closed range such that \( S_R e_j = v_j, j \in D \). If so, the collection \( \{ \tilde{v}_j \mid j \in D \} \), defined by \( \tilde{v}_j = S_R (S_R^* S_R)^{-1} e_j \), is the Riesz system dual to \( \Omega \).

Note that by definition a Riesz system is a linearly independent set. A special set of Riesz systems are Riesz bases, which are defined as follows.

**Definition 4.1.7** A Riesz system which is total is a Riesz basis.

Obviously, every Riesz system is a Riesz basis for the closure of its linear span. For a Riesz basis, the corresponding Riesz generator \( S_R \) is invertible. This yields immediately that the frame \( \{ v_j \mid j \in D \} \) is a Riesz basis if and only if \( S_R^* \) is invertible. It can be proved, see [7], that the concepts of exact frame and of Riesz basis are equivalent. So, an exact frame can also be seen as a frame for which \( S_R^* \) is invertible. Connections between frames and Riesz systems are given in the following theorem, which results from the previous considerations.

**Theorem 4.1.8** Let \( \Omega = \{ v_j \mid j \in D \} \) be any collection in \( H \), and define the operator \( S : l^2(D) \to H \) by \( S e_j = v_j, j \in D \). Then

(i) \( \Omega \) is a frame if and only if \( SS^* \) is a boundedly invertible operator on \( H \),

(ii) \( \Omega \) is a Riesz system if and only if \( S^* S \) is a boundedly invertible operator on \( l^2(D) \),

(iii) \( \Omega \) is a Riesz basis if and only if \( S \) is a boundedly invertible operator on \( l^2(D) \), i.e., if both \( SS^* \) and \( S^* S \) are boundedly invertible operators.

The relations between frames and Riesz systems as considered in this theorem are also depicted in Figure 4.1.

A characterization of Riesz systems which is used frequently in the sequel of this chapter is given in terms of a Gram matrix. For \( \Omega = \{ v_j \mid j \in D \} \) in \( H \), we define its Gram matrix \( \Gamma_\Omega \) by \( \Gamma_\Omega(i,j) = (v_j, v_i)_H, \ i, j \in D \). Since \( \Gamma_\Omega(i,j) = (S_R^* S_R e_j, e_i) \) we conclude \( \Gamma_\Omega \) is the matrix of \( S_R^* S_R \), yielding with (4.5) that \( \Omega \) is a Riesz system if and only if

\[
m_R I \leq \Gamma_\Omega \leq M_R I. \quad (4.6)
\]

**4.2 Multiresolution Analysis in Hilbert Spaces**

In Section 3.2.1 we introduced the concept of an MRA for \( L^2(\mathbb{R}) \) following the ideas of Mallat [60] and Meyer [62]. This definition can be extended in a canonical way to an MRA for \( L^2(\mathbb{R}^n) \), see e.g. [23, 62]. In this section we define an MRA for a separable Hilbert space \( H \) using mutually dependent unitary operators on \( H \).
**Definition 4.2.1** Let $A$ be an $(n \times n)$ matrix with integer entries and eigenvalues $\lambda_i$, $i = 1, \ldots, n$, such that $|\lambda_i| > 1$. Furthermore, let $H$ be a separable Hilbert space, $\phi \in H$ and $U_1, U_{2,1}, \ldots, U_{2,n}$ unitary operators on $H$, such that $U_{2,1}, \ldots, U_{2,n}$ mutually commute. Then $[\phi, U_1, U_{2,1}, \ldots, U_{2,n}]$ is an MRA in $H$ if

(i) $\{U_2^k \phi \mid k \in \mathbb{Z}^n\}$ is a Riesz system in $H$,

(ii) $\phi \in \text{clos span}\{U_1 U_2^k \phi \mid k \in \mathbb{Z}^n\}$,

(iii) $U_2^k U_1 = U_1 U_2^{Ak}$, for all $k \in \mathbb{Z}^n$,

with $U_2^k = U_{2,1}^{k_1} \cdots U_{2,n}^{k_n}$.

In the sequel $\phi$ is called the MRA generator.

To compare this definition with Definition 3.2.2 we construct a nested sequence of closed subspaces for $H$ by

$$V_j = \text{clos span} \{U_1^{-j} U_2^k \phi \mid k \in \mathbb{Z}^n\}.$$

Then we have

$$U_1(V_j) = V_{j-1}, \ U_2^k(V_j) = V_j, \ k \in \mathbb{Z}^n, \ \text{and} \ V_j \subset V_{j-1}.$$

For $n = 1, A = 2, U_1 = D^{-1}, U_{2,1} = T$ and $H = L^2(\mathbb{R})$, Definition 3.2.2 and Definition 4.2.1 are nearly the same. However in Definition 3.2.2 an orthonormal basis for $V_0$ was
constructed, which is a special case of a Riesz basis, and in the definition above we also did not introduce the conditions

\[ \text{clos } \bigcup_{j \in \mathbb{Z}} V_j = H \quad \text{and} \quad \bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \]

which occur in Definition 3.2.2. Whether or not inserting these conditions will not change any of the further derivations. Therefore these conditions have been omitted.

The concept of MRA in \( H \) is now used to construct Riesz systems in \( H \) of a special kind. We start this construction by defining a unique countable collection of closed subspaces \( W_j, j \in \mathbb{Z} \), like we have done as well in Section 3.2.1, namely by writing

\[ W_j = V_{j-1} \cap V_j^\perp, \]

for all \( j \in \mathbb{Z} \). Since \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) are unitary operators on \( H \), we have

\[
\begin{align*}
\mathcal{U}_1(W_j) &= \mathcal{U}_1(V_{j-1} \cap V_j^\perp) = \mathcal{U}_1(V_{j-1}) \cap \mathcal{U}_1(V_j^\perp) \\
&= \mathcal{U}_1(V_{j-1}) \cap (\mathcal{U}_1(V_j))^\perp = W_{j-1},
\end{align*}
\]

and similarly \( \mathcal{U}_{2,t}(W_j) = W_j, l = 1, \ldots, n \). Obviously, since \( W_j = \mathcal{U}_1^{-j}(W_0) \), each Riesz basis \( \Omega \) for \( W_0 \) yields the Riesz basis \( \mathcal{U}_1^{-j}(\Omega) \) for \( W_j, j \in \mathbb{Z} \), with the same Riesz bounds.

Following the same orthogonal decomposition as in Section 3.2.1 and adding Condition 4.7 we arrive at

\[ H = \bigoplus_{j=-\infty}^{\infty} W_j. \]

From this direct sum decomposition of \( H \) it follows immediately that \( \bigcup_{j \in \mathbb{Z}} \mathcal{U}_1^j(\Omega) \) is a Riesz basis for \( H \), if \( \Omega \) is a Riesz basis for \( W_0 \).

Now the idea is to construct Riesz systems in \( W_0 \) of the form \( \{ \mathcal{U}_k^l \psi \mid k \in \mathbb{Z}^n \} \). It will turn out that, for constructing a Riesz basis of this form in general more than one element \( \psi \) will be needed. Our aim here is to prove existence of an \( N \in \mathbb{N} \) and of a collection

\[ \{ \psi_1, \ldots, \psi_{N-1} \} \subset V_1, \]

such that

(a) \( (\psi_l, \mathcal{U}_2^k \phi) = 0, \ l = 1, \ldots, N - 1, \) for all \( k \in \mathbb{Z}^n \), i.e., \( \psi_l \in V_0^\perp \),

(b) \( \{ \mathcal{U}_k^l \psi_l \mid l = 1, \ldots, N - 1, \ k \in \mathbb{Z}^n \} \) is a Riesz basis for \( W_0 \).

Since \( V_{-1} = V_0 \oplus W_0 \) Condition (b) can be also written as
(b') \{U_2^k \psi_l | l = 0, \ldots, N-1, \, k \in \mathbb{Z}^n\} is a Riesz basis for \(V_{-1}\), with \(\psi_0 = \phi\).

Since \(V_0 \subset V_{-1}\) and \(W_0 \subset V_{-1}\) and since \(\{U_1 U_2^k \phi \mid k \in \mathbb{Z}^n\}\) is a Riesz basis in \(V_{-1}\), we come to

\[
\phi = \sum_{k \in \mathbb{Z}^n} p(k) U_1 U_2^k \phi, \quad \psi_l = \sum_{k \in \mathbb{Z}^n} q_l(k) U_1 U_2^k \phi, \quad l = 1, \ldots, N - 1,
\]

(4.8) (4.9)

where \(p \in \ell^2(\mathbb{Z}^n)\), known, and the \(q_l \in \ell^2(\mathbb{Z}^n)\), to be determined, are the generating sequences. For \(U_1 = D^{-1}\) and \(U_2 = T\), these sequences are called scaling sequences as we have seen already in Section 3.2.1. In the sequel the term scaling sequence will only be used for cases in which \(U_1 = D^{-1}\) has been chosen.

So the idea is to formulate constraints on the sequences \(q_l\), given the sequence \(p\), such that the Conditions (a) and (b') are satisfied. Therefore we reformulate these conditions in terms of the generating sequences.

Condition (a) can be put in a rather straightforward way in terms of the generating sequences by substituting (4.8) and (4.9) into this condition and using \(U_2^k U_1 = U_1 U_2^A k\). We get

\[
(\psi_l, U_2^k \phi) = (\tau_\phi \ast q_l, R^{A k} p)_{\mathbb{Z}^n},
\]

with \(\tau_\phi(k) = (\phi, U_2^k \phi), \quad (i, j, k) \in \mathbb{Z}^n\), and \(R^m = R_1^{m_1} \cdots R_n^{m_n}\) for \(m \in \mathbb{Z}^n\), a composition of bilateral shift operators on \(\ell^2(\mathbb{Z}^n)\), each one acting along a standard basis vector of \(\mathbb{Z}^n\). So Condition (a) is equivalent with

\[
(\tau_\phi \ast q_l, R^{A k} p)_{\mathbb{Z}^n} = 0, \quad \forall l \in \{1, \ldots, N-1\}, \quad \forall k \in \mathbb{Z}^n.
\]

(4.10)

In order to put Condition (b') in terms of the generating sequences we present the following lemma.

**Lemma 4.2.2** Let \([\phi, U_1, U_2, 1, \ldots, U_2, n]\) be an MRA and let \(p\) be the generating sequence of \(\phi\). Then

\[
\{R^{A k} p \mid k \in \mathbb{Z}^n\}
\]

is a Riesz system in \(\ell^2(\mathbb{Z}^n)\). Furthermore, let \(\psi_0 = \phi\) and \(\psi_1, l = 1, \ldots, N - 1\), be in \(W_0\) with generating sequences \(q_l\). Then

\[
\{U_2^k \psi_l \mid l = 0, \ldots, N - 1, \, k \in \mathbb{Z}^n\}
\]

is a Riesz basis for \(V_{-1}\) if and only if

\[
\{R^{A k} q_l \mid l = 0, \ldots, N - 1, \, k \in \mathbb{Z}^n\},
\]

with \(q_0 = p\), is a Riesz basis for \(\ell^2(\mathbb{Z}^n)\).
Proof
We introduce the boundedly invertible operator $B : V_{-1} \to l^2(\mathbb{Z}^n)$ by

\[ Bf = \alpha \text{ if and only if } f = \sum_{k \in \mathbb{Z}^n} \alpha(k) \mathcal{U}_1 \mathcal{U}_2^k \phi. \]

Since $B \mathcal{U}_2^k = \mathcal{R}^A k B$, $k \in \mathbb{Z}^n$ and $B \phi = p$, applying $B$ on the Riesz system $\{ \mathcal{U}_2^k \phi \mid k \in \mathbb{Z}^n \}$ yields $\{ \mathcal{R}^A k p \mid k \in \mathbb{Z}^n \}$. This is also a Riesz system, since $B$ is a boundedly invertible operator. The second result follows immediately by observing that

\[ \{ \mathcal{R}^A k q_l \mid l = 0, \ldots, N - 1, k \in \mathbb{Z}^n \} = B(\{ \mathcal{U}_2^k \psi_l \mid l = 0, \ldots, N - 1, k \in \mathbb{Z}^n \}). \]

From this lemma it follows that if we can construct sequences $q_l \in l^2(\mathbb{Z}^n)$, such that

- $\langle \tau_\phi \circ q_l, \mathcal{R}^A k p \rangle_{\mathbb{Z}^n} = 0$, $\forall_{l \in \{1, \ldots, N - 1\}}$ $\forall_{k \in \mathbb{Z}^n}$,
- $\{ \mathcal{R}^A k q_l \mid l = 0, \ldots, N - 1, k \in \mathbb{Z}^n \}$ is a Riesz basis for $l^2(\mathbb{Z}^n)$, with $q_0 = p$,

then elements $\psi_l$, $l = 1, \ldots, N - 1$, in $V_{-1}$ can be constructed for which Conditions (a) and (b) are satisfied. This naturally leads us to the next item.

4.3 Riesz Systems Generated by Unitary Operators

In Section 4.1 we already considered necessary and sufficient conditions on a set $\Omega$ such that it is a Riesz system. Now, we will deal with sets $\Omega$ of a special kind, namely those sets that are generated by unitary operators acting on one or more elements in $H$. This is done, since such Riesz systems occurred in Lemma 4.2.2.

First we consider the case of a Riesz system generated by several unitary operators all acting on one element in $H$. So, our aim is to derive necessary and sufficient conditions on the tuple $[\mathcal{U}_1, \ldots, \mathcal{U}_n, \phi]$, such that $\{ \mathcal{U}^j \phi \mid j \in \mathbb{Z}^n \}$ is a Riesz system, with $\mathcal{U}^j = \mathcal{U}_1^{i_1} \cdots \mathcal{U}_n^{i_n}$. Besides, we compute its dual Riesz system. These derivations follow Section 4.1 with $D = \mathbb{Z}^n$ and with the Riesz generator $\mathcal{S}$ given by $\mathcal{S} \mathcal{E}_j = \mathcal{U}^j \phi$.

By Section 4.1, $\{ \mathcal{U}^j \phi \mid j \in \mathbb{Z}^n \}$ is a Riesz system if and only if its Gram matrix $\Gamma$, given by

\[ \Gamma(i, j) = \langle \mathcal{U}^j \phi, \mathcal{U}^i \phi \rangle = \langle \phi, \mathcal{U}^{i-j} \phi \rangle = \tau_\phi(i - j), \]

satisfies (4.6). Observing that $\mathcal{S}^* \mathcal{S}$ with matrix $\Gamma$ acts by convolution on $l^2(\mathbb{Z}^n)$,

\[ \mathcal{S}^* \mathcal{S} \alpha = \tau_\phi \ast \alpha, \]

we arrive at the following theorem.
Theorem 4.3.1 For commuting unitary operators $U_1, \ldots, U_n$ on $H$ and $\phi \in H$, the collection $\{U^j\phi \mid j \in \mathbb{Z}^n\}$ is a Riesz system if and only if the sequence $\tau_\phi$ defined by

$$\tau_\phi(j) = (\phi, U^j\phi), \; j \in \mathbb{Z}^n,$$

yields a boundedly invertible convolution operator on $l^2(\mathbb{Z}^n)$, i.e., if and only if

$$0 < \text{ess inf}_{z \in T^n} \hat{\tau}_\phi(z) \leq \text{ess sup}_{z \in T^n} \hat{\tau}_\phi(z) < \infty,$$

(4.11)

where $\hat{\tau}_\phi$ denotes the discrete Fourier transform of $\tau_\phi$

$$\hat{\tau}_\phi(z) = \sum_{j \in \mathbb{Z}^n} \tau_\phi(j) z^{-j}, \; z \in T^n,$$

with $z^j = z_1^{j_1} \cdots z_n^{j_n}, \; j \in \mathbb{Z}^n$, and $T^n$ the $n$-fold product of the unit circle with normalized Lebesgue measure $\mu$.

Note that from this theorem it follows that for $\tau_\phi \in l^1(\mathbb{Z}^n)$ the collection

$$\Omega = \{U^j\phi \mid j \in \mathbb{Z}^n\}$$

is a Riesz system if and only if $\hat{\tau}_\phi$ has no zeroes on $T^n$.

By definition, the dual Riesz system is given by $S(S^*S)^{-1}e_j$, where $Se_j = U^j\phi$ and $S^*S\alpha = \tau_\phi \ast \alpha$, for all $\alpha \in l^2(\mathbb{Z})$. This yields immediately that the dual Riesz system $\tilde{\Omega}$ of $\Omega$ is of the form

$$\tilde{\Omega} = \{U^j\tilde{\phi} \mid j \in \mathbb{Z}^n\}$$

with $\tilde{\phi}$ given by

$$\tilde{\phi} = \sum_{j \in \mathbb{Z}^n} \hat{\tau}_\phi(j) U^j\phi,$$

where $\hat{\tau}_\phi \ast \tau_\phi = \epsilon_0$.

Next we replace the vector $\phi \in H$ by a finite collection $\{\phi_1, \ldots, \phi_N\}$ and pose the same problem, namely under which conditions

$$\Omega_N = \{U^j\phi_l \mid l = 1, \ldots, N, \; j \in \mathbb{Z}^n\}$$

is a Riesz system. For this we take as index set $D = \{1, \ldots, N\} \times \mathbb{Z}^n$. Furthermore, we define the unitary operator $E_N$ from $l^2(D)$ into $L^2(T^n, C^N) = L^2(T^n) \otimes C^N$ by

$$(E_N e_{l,j})(z) = z^j e_l, \; \{e_1, \ldots, e_N\} \text{ the standard orthonormal basis in } C^N.$$ We see that $E_N$ is a Riesz system if and only if the Gram matrix $\Gamma$ with entries $(U^j\phi_l, U^k\phi_m)_{(m,j),(k,i)}$ represents a bounded invertible operator on $l^2(\mathbb{Z}^n)$. A straightforward computation shows

$$(E_N \Gamma e_{m,j})(z) = z^j \Gamma(z) e_m,$$
where \( \hat{\Gamma}(z) \in C^{N \times N} \) is defined a.e. by \( (\hat{\Gamma}(z))_{k,m} = \sum_{j \in \mathbb{Z}^n} (\phi_m, U^j \phi_k) z^{-j} \). Hence the Gram matrix \( \Gamma \) represents a boundedly invertible operator on \( l^2(T^n) \) if and only if the matrix valued function \( \hat{\Gamma} \) from \( T^n \) into \( C^{N \times N} \) satisfies
\[
\exists m, M > 0 \quad \forall z \in T^n \quad mI_N \leq \hat{\Gamma}(z) \leq MI_N \quad a.e.
\]
(4.12)

4.4 Riesz Bases in \( l^2(\mathbb{Z}^n) \)

In the previous section we derived necessary and sufficient conditions such that
\[
\Omega_N = \{ U^j \phi_l \mid l = 1, \ldots, N, \; j \in \mathbb{Z}^n \}
\]
is a Riesz system. According to Lemma 4.2.2 such conditions can also be derived in terms of the generating sequences. Therefore in this section we deal with the following problem. Let the sequence \( \gamma \) yield a boundedly invertible convolution operator on \( l^2(\mathbb{Z}^n) \) and let \( \beta_0 \in l^2(\mathbb{Z}^n) \). Find necessary and sufficient conditions on sequences \( \beta_l, \; l = 1, \ldots, N - 1 \), in \( l^2(\mathbb{Z}^n) \), and determine \( N \) such that

(i) \( (\gamma * \beta_l, \mathcal{R}^{Ak} \beta_0)_\mathbb{Z} = 0, \; \forall l \in \{1, \ldots, N-1\} \; \forall k \in \mathbb{Z}^n \),

(ii) \( \{ \mathcal{R}^{Ak} \beta_l \mid l = 0, \ldots, N - 1, \; k \in \mathbb{Z}^n \} \) is a Riesz basis for \( l^2(\mathbb{Z}^n) \).

We reformulate this into terms of the Hilbert space \( L^2(T^n) \).

Since we deal with a rather arbitrary matrix \( A \in \mathbb{Z}^{n \times n} \) we introduce the so-called Smith normal form of a matrix with integer entries, which is given in the following theorem. In [56] one can find a proof of this theorem for matrices over a ring of polynomials in one variable. This result generalizes immediately to the case of matrices over the ring of integers.

**Theorem 4.4.1 (Smith normal form)** Let \( A \in \mathbb{Z}^{n \times n} \). Then there are unimodular matrices \( U, V \in \mathbb{Z}^{n \times n} \), i.e., \( \det(U) = \det(V) = 1 \), and a diagonal matrix \( D \in \mathbb{Z}^{n \times n} \), such that
\[
A = UDV.
\]
(4.13)

This factorization is not unique.

In the sequel we use the notation \( L = |\det(D)| \).

It can be proved by some straightforward computations that the problem posed in the beginning of this section is equivalent with the following one. Give necessary and sufficient conditions on sequences \( \beta_l, \; l = 1, \ldots, N - 1 \), in \( l^2(\mathbb{Z}^n) \), and determine \( N \) such that

(i) \( (\gamma * \beta_l, \mathcal{R}^{Dk} \beta_0)_\mathbb{Z} = 0, \; \forall l \in \{1, \ldots, N-1\} \; \forall k \in \mathbb{Z}^n \),
(ii) \( \{ R^{DK} \beta_l \mid l = 0, \ldots, N - 1, k \in \mathbb{Z}^n \} \) is a Riesz basis for \( l^2(\mathbb{Z}^n) \),
with \( D \in \mathbb{Z}^{n \times n} \) a diagonal matrix involved in the Smith normal form of \( D \).

Let now \( d_i = D(i,i), i = 1, \ldots, n \). Define \( \omega_{di} = e^{2\pi i/d_i}, i = 1, \ldots, n \), and \( K_n \) the \( n \)-fold segment of all \( z \in T^n \) such that

\[
\arg(z_i) \in [0, 2\pi/d_i), i = 1, \ldots, n.
\]

We observe, that \( \{ e^k \mid k \in \mathbb{Z}^n \} \) is an orthonormal basis for \( L^2(T^n) \) and

\[
\{ \sqrt{L} z_1^{k_1 d_1} \ldots z_n^{k_n d_n} \mid k \in \mathbb{Z}^n \}
\]
is an orthonormal basis for \( L^2(K_n) \). So

\[
\{ \sqrt{L} z_1^{k_1 d_1} \ldots z_n^{k_n d_n} e_i \mid i = 1, \ldots, N, k \in \mathbb{Z}^n \}
\]
is an orthonormal basis for \( L^2(K_n, C^N) \), the Hilbert space consisting of all \( C^N \)-valued Euclidean square integrable functions on \( K_n \). In dealing with the above stated problem, we present some auxiliary results.

The proof of the following lemma is based on the fact, that the \( (n \times n) \) Fourier matrix \( F_n \) with entries

\[
F_n(i, j) = 1/\sqrt{n} \omega_{ij}^{ij}, i, j = 0, \ldots, n - 1,
\]
is unitary. Furthermore, we use the notation

\[
P_{\hat{\beta}, \hat{h}}(z) = \sum_{j_1=0}^{\lfloor d_1 \rfloor - 1} \cdots \sum_{j_n=0}^{\lfloor d_n \rfloor - 1} \hat{\beta}(\omega_{d_1}^{j_1} z_1, \ldots, \omega_{d_n}^{j_n} z_n) \hat{h}(\omega_{d_1}^{j_1} z_1, \ldots, \omega_{d_n}^{j_n} z_n).
\]

Lemma 4.4.2 Let \( g, h \in l^2(\mathbb{Z}^n) \). Then

\[
1/L \int_{T^n} P_{\hat{\beta}, \hat{h}}(z) z^m d\mu_n(z) = \begin{cases} (g, R^{DK} h) \mathbb{Z}^n & \text{if } m = Dk, k \in \mathbb{Z}^n, \\ 0 & \text{if } m \neq Dk, k \in \mathbb{Z}^n. \end{cases}
\]

Proof

We consider the following computation

\[
1/L \int_{T^n} P_{\hat{\beta}, \hat{h}}(z) z^m d\mu_n(z)
= 1/L \sum_{j_1=0}^{\lfloor d_1 \rfloor - 1} \cdots \sum_{j_n=0}^{\lfloor d_n \rfloor - 1} \omega_{d_1}^{-j_1 m_1} \cdots \omega_{d_n}^{-j_n m_n} \int_{T^n} \hat{\beta}(z) \hat{h}(z) z^{-m} d\mu_n(z)
= \begin{cases} \int_{T^n} \hat{\beta}(z) \hat{h}(z) z^{-m} d\mu_n(z) & \text{if } m = Dk, k \in \mathbb{Z}^n, \\ 0 & \text{if } m \neq Dk, k \in \mathbb{Z}^n. \end{cases}
\]
The proof is completed by observing that the $n$-dimensional discrete Fourier transform of $\mathcal{R}^{l}h$ is given by $z^{-l}h$. 

From this lemma we deduce the following result.

**Lemma 4.4.3** Let $g, h \in l^{2}(\mathbb{Z}^{n})$. Then for all $k \in \mathbb{Z}^{n}$

$$\int_{K_{n}} P_{g,h}(z)z_{1}^{k_{1}d_{1}} \cdots z_{n}^{k_{n}d_{n}}d\mu_{n}(z) = (g, \mathcal{R}^{Dk}h)_{\mathbb{Z}^{n}}.$$ 

**Proof**

Obviously, $P_{g,h}(z)z_{1}^{k_{1}d_{1}} \cdots z_{n}^{k_{n}d_{n}}$ remains unchanged if $z_{i}$ is replaced by $\omega_{\lambda_{i}}^{m_{i}}z_{i}$, \(i = 1, \ldots, n\), for $m \in \mathbb{Z}^{n}$ and so

$$\int_{K_{n}} P_{g,h}(z)z_{1}^{k_{1}d_{1}} \cdots z_{n}^{k_{n}d_{n}}d\mu_{n}(z) = \int_{K_{n}} P_{g,h}(z)z_{1}^{k_{1}d_{1}} \cdots z_{n}^{k_{n}d_{n}}d\mu_{n}(z),$$

with

$$K_{n}^{m} = [m_{1}\omega_{d_{1}}, (m_{1} + 1)\omega_{d_{1}}) \times \cdots \times [m_{n}\omega_{d_{n}}, (m_{n} + 1)\omega_{d_{n}}).$$

Consequently the result follows from Lemma 4.4.2. 

By Lemma 4.4.3, Condition (i) can be written as

$$\int_{K_{n}} P_{g,h}(z)z_{1}^{k_{1}d_{1}} \cdots z_{n}^{k_{n}d_{n}}d\mu_{n}(z) = 0.$$ 

Since this relation must hold for every $k \in \mathbb{Z}^{n}$ and since

$$\{\sqrt{L}z^{Dk} \mid k \in \mathbb{Z}^{n}\}$$

is an orthonormal basis for $L^{2}(K_{n})$, we get that

$$P_{g,h}(z) = 0 \text{ a.e. on } K_{n}, l = 1, \ldots, N - 1. \quad (4.15)$$

So, sequences $\beta_{l}$ that should satisfy Condition (i) are given in terms of their Fourier transforms that satisfy (4.15). Note, that for finite sequences $\beta_{0}$ and $\gamma$ Condition (4.15) only deals with polynomial function on the $n$-dimensional unit sphere.

Condition (ii) can also be reformulated in terms of function on $T^{n}$, using (4.12). Therefore we introduce

$$B_{N} = \{\mathcal{R}^{Dk}\beta_{l} \mid l = 0, \ldots, N - 1, k \in \mathbb{Z}^{n}\}.$$
Since $B_N$ is generated by $N$ vectors $\beta_0, \ldots, \beta_{N-1}$ and the unitary operators $\mathcal{R}_1, \ldots, \mathcal{R}_N$, we may use result (4.12). This yields that $B_N$ is a Riesz system if and only if

$$mI_N \leq \hat{\mathbf{r}}(z) \leq M I_N \text{ a.e. } z \in \mathbb{T}^n,$$

with

$$\hat{\mathbf{r}}(z)_{k,m} = \sum_{l \in \mathbb{Z}^n} (\beta_m, \mathcal{R}^D l \beta_k) z^{-l}, \quad k, m = 0, \ldots, N - 1. \quad (4.16)$$

This result can also be put in terms of the Fourier transforms of $\beta_l$. Therefore we derive the relation

$$\hat{\mathbf{r}}(z_1, \ldots, z_n)_{k,m} = 1/L \mathbf{p}_{\beta_m, \beta_k}(z), \quad k, m = 0, \ldots, N - 1, \quad (4.17)$$

using Lemma 4.4.3, and the fact that

$$\{\sqrt{L} z^{d_1} \ldots z^{d_n} | k \in \mathbb{Z}^n\}$$

is an orthonormal basis for $L^2(K_n)$. Define $\hat{\mathbf{F}}_d(z) = \hat{\mathbf{r}}_d(z_1, \ldots, z_n) = \hat{\mathbf{r}}(z_1^{d_1}, \ldots, z_n^{d_n})$. Then we arrive at the following theorem by combining (4.12) and (4.17).

**Theorem 4.4.4** Let $N \in \mathbb{N}$ be fixed and $\{\beta_0, \ldots, \beta_{N-1}\}$ be a subset of $l^2(\mathbb{Z}^n)$. Let further $D$ be an $(n \times n)$ diagonal matrix with integer entries. Then the collection

$$B_N = \{\mathcal{R}^{Dl} \beta_l | l = 0, \ldots, N - 1, k \in \mathbb{Z}^n\}$$

is a Riesz system if and only if the $(N \times N)$ matrix valued function $z \mapsto \hat{\mathbf{F}}_d(z)$, $z \in K_n$, with entries

$$\hat{\mathbf{F}}_d(z)_{k,m} = 1/L \mathbf{p}_{\beta_m, \beta_k}(z), \quad k, m = 0, \ldots, N - 1,$$

admits real positive constants $m$ and $M$, such that

$$m I_N \leq \hat{\mathbf{F}}_d(z) \leq M I_N \text{ a.e. } z \in K_n. \quad (4.18)$$

So, Theorem 4.4.4 presents necessary and sufficient conditions on $\beta_0, \ldots, \beta_{N-1}$, so that $B_N$ is a Riesz system. We proceed by searching for similar conditions on the Fourier transforms of $\beta_0, \ldots, \beta_{N-1}$, such that $B_N$ is a Riesz basis for $l^2(\mathbb{Z}^n)$. As a starting point for deriving such conditions we present a corollary of the preceding theorem.

**Corollary 4.4.5** If $B_N$ is a Riesz system, then $N \leq L$.

**Proof**

Define the $(L \times N)$ matrix valued function $z \mapsto \hat{\mathbf{M}}(z)$, $z \in K_n$, with entries

$$\hat{\mathbf{M}}(z)_{r,l} = L^{-1/2} \beta_l^{(\pi(r))}(z_1, \ldots, \omega^{(\pi(r))}(n) z_n), \quad (4.19)$$
\[ l = 0, \ldots, N-1, \quad r = 0, \ldots, L-1, \text{ where } \pi \text{ is an arbitrary bijection from the collection } \{0, 1, \ldots, L-1\} \text{ onto } \]
\[ \{r \in \mathbb{Z}^n \mid 0 \leq r_i \leq |d_i| - 1, i = 1, \ldots, n\}. \]

Since
\[ \hat{\Gamma}_d(z) = \hat{M}^*(z)\hat{M}(z), \quad z \in K_n, \]
\(\hat{\Gamma}_d\) is invertible a.e. if and only if \(\hat{M}\) is injective a.e. If therefore \(\hat{\Gamma}_d\) satisfies (4.18), i.e., \(\hat{\Gamma}_d(z)\) is invertible for almost all \(z \in K_d\), then \(N \leq L\). \(\square\)

For deriving conditions on \(\hat{\beta}_0, \ldots, \hat{\beta}_{N-1}\), such that \(B_N\) is a Riesz basis for \(l^2(\mathbb{Z}^n)\) we consider the special case \(N = L\) and we assume that \(\hat{\Gamma}_d\) satisfies (4.18). Now the proof of Corollary 4.4.5 yields that \(\hat{\Gamma}_d\) is invertible a.e. if and only if \(\hat{M}\), as introduced in the above proof, is invertible a.e. So (4.18) is equivalent with \(\hat{M}\) being invertible a.e. on \(K_n\). Furthermore, let \(\mathcal{G}\) and \(\mathcal{M}\) denote bounded linear operators on \(L^2(K_n; C^L)\) corresponding to \(\hat{\Gamma}_d\) and \(\hat{M}\), respectively, i.e.,
\[ (\mathcal{G}\eta)(z) = \hat{\Gamma}_d(z)\eta(z) \quad \text{and} \quad (\mathcal{M}\eta)(z) = \hat{M}(z)\eta(z) \quad \text{a.e. } z \in K_n, \]
for all \(\eta \in L^2(K_n; C^L)\). Then \(\mathcal{G} = \mathcal{M}^*\mathcal{M}\) and \(\mathcal{M}^{-1} = \mathcal{G}^{-1}\mathcal{M}^*\). Thus \(\mathcal{M}\) is a boundedly invertible operator, since \(\mathcal{G}\) is a boundedly invertible operator and
\[ (\mathcal{M}^{-1}\eta)(z) = \hat{M}(z)^{-1}\eta(z). \]

These considerations are used to give conditions on \(\hat{\beta}_0, \ldots, \hat{\beta}_{N-1}\), such that \(B_N\) is a Riesz basis for \(l^2(\mathbb{Z}^n)\).

**Theorem 4.4.6** Let \(N \in \mathbb{N}\) be fixed and \(D = \text{diag}(d_1, \ldots, d_n)\) with \(d_i \in \mathbb{Z}\). Let further \(\{\beta_0, \ldots, \beta_{N-1}\}\) be a subset of \(l^2(\mathbb{Z}^n)\), such that the collection
\[ B_N = \{R^Dk\beta_l \mid l = 0, \ldots, N-1, \quad k \in \mathbb{Z}^n\} \]
is a Riesz system. Then this collection is a Riesz basis if and only if \(N = L\).

**Proof**
Take \(N = L\) and let \(B_L\) be a Riesz system in \(l^2(\mathbb{Z}^n)\). Besides, let \(\hat{M}\) be defined as in (4.19) and let \(\mathcal{M}\) be associated with \(\hat{M}\) by (4.20). Then \(\hat{M}(z)\) is invertible for almost all \(z \in K_n\) and \(\mathcal{M}\) is invertible, since \(\hat{\Gamma}_d\) satisfies (4.18). Define \(\tilde{\epsilon}_{l,k} \in L^2(K_n; C^L)\) for \(l = 0, \ldots, L-1\), and \(k \in \mathbb{Z}^n\) by
\[ \tilde{\epsilon}_{l,k}(z) = \sqrt{L}z^Dk\epsilon_l, \quad z \in K_n. \]
Furthermore, introduce $\mathcal{V}_D : l^2(\mathbb{Z}^n) \to L^2(K_n; \mathbb{C}^L)$ by

$$(\mathcal{V}_D h)(z) = \left( \hat{h}(\omega_{d_1}^{(\pi(0))(1)} z_1, \ldots, \omega_{d_n}^{(\pi(0))(n)} z_n), \ldots, \hat{h}(\omega_{d_1}^{(\pi(L-1))(1)} z_1, \ldots, \omega_{d_n}^{(\pi(L-1))(n)} z_n) \right)$$

where $\pi$ is an arbitrary bijection from the set $\{0, 1, \ldots, L - 1\}$ onto the collection

$E = \{r \in \mathbb{Z}^n \mid 0 \leq r_i \leq |d_i| - 1, i = 1, \ldots, n\}$.

With this definition

$$||\mathcal{V}_D h||^2_{L^2(K_n; \mathbb{C}^L)} = \int_{K_n} ||(\mathcal{V}_D h)(z)||^2_{\mathbb{C}^L} d\mu_n(z) = \int_{K_n} P_{h,z}(z) d\mu_n(z) = ||h||^2_{L^2(\mathbb{Z}^n)},$$

for all $h \in l^2(\mathbb{Z}^n)$, so that $\mathcal{V}_D$ is an isometry. Define $h_{l,k} \in l^2$ by

$$\hat{h}_{l,k}(z) = \left\{ \begin{array}{ll} \sqrt{L} z_1^{d_1} \ldots z_n^{d_n} & \text{if } (\omega_{d_1}^{-l_1} z_1, \ldots, \omega_{d_n}^{-l_n} z_n) \in K_n, \\ 0 & \text{otherwise}, \end{array} \right.$$ 

with $l \in E, k \in \mathbb{Z}^n$. Then $\{h_{l,k} \mid l \in E, k \in \mathbb{Z}^n\}$ is an orthonormal basis for $l^2(\mathbb{Z}^n)$. This yields $\mathcal{V}_D h_{\pi(l),k} = \hat{e}_{l,k}$ and so the operator $\mathcal{V}_D$ is unitary. Applying $\mathcal{V}_D$ on $\mathcal{R}^{Dk} \beta_l$ now yields

$$\mathcal{V}_D(\mathcal{R}^{Dk} \beta_l)(z) = z^{-Dk} \left( \hat{\beta}_l(\omega_{d_1}^{(\pi(0))(1)} z_1, \ldots, \omega_{d_n}^{(\pi(0))(n)} z_n), \ldots, \hat{\beta}_l(\omega_{d_1}^{(\pi(L-1))(1)} z_1, \ldots, \omega_{d_n}^{(\pi(L-1))(n)} z_n) \right) = \sqrt{L} z^{-Dk} \hat{M}(z) \hat{e}_l = \hat{M}(z) \hat{e}_{l,-k}(z) = (\hat{M} \hat{e}_{l,-k})(z),$$

for all $l = 0, \ldots, L - 1$. So $\mathcal{R}^{Dk} \beta_l = (\mathcal{V}_D)^* \hat{M} \hat{e}_{l,-k}$.

Since $\{\hat{e}_{l,k} \mid l = 0, \ldots, L - 1, k \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(K_n; \mathbb{C}^L)$ and $(\mathcal{V}_D)^* \hat{M}$ is boundedly invertible, it follows that $B_N$ is a Riesz basis for $l^2(\mathbb{Z}^n)$.

For proving the converse, we assume $B_L$ to be a Riesz basis for $l^2(\mathbb{Z}^n)$. Then $\hat{M}$ has to be invertible, since $\mathcal{R}^{Dk} \beta_l = \mathcal{V}_D M \hat{e}_{l,k}$. It follows that the matrix valued function $\hat{M}$ has to be invertible a.e. on $K_n$, and thus $N = L$. \[\square\]
From the preceding theorem we conclude that for satisfying Condition (ii) we have to search for sequences $\beta_1, \ldots, \beta_{|\det A|-1}$, given $\beta_0$, such that (4.18) holds.

Although we are not dealing with the concept of frames in $l^2(\mathbb{Z}^n)$ in this section, similar results can now be given in rather straightforward way such that

$$B_N = \{R^D \beta_l | l = 0, \ldots, N - 1, k \in \mathbb{Z}^n\}$$

is a frame. We can write

$$R^D \beta_l = V^*_D M \tilde{e}_{l,k} = V^*_D M U e_{l,k},$$

with $U : L^2(\{0, \ldots, N-1\} \times \mathbb{Z}^n) \rightarrow L^2(K_n; \mathbb{C}^N)$ the unitary operator given by $U e_{l,k} = \tilde{e}_{l,k}$. So $S_F = U^* M^* V_D$ is the frame generator of $B_N$ if $B_N$ is a frame. Now Theorem 4.1.8 immediately yields the following theorem.

**Theorem 4.4.7** Let $N \in \mathbb{N}$ be fixed and $\{\beta_0, \ldots, \beta_{N-1}\}$ be a subset of $l^2(\mathbb{Z}^n)$. Then the collection

$$\{R^D \beta_l | l = 0, \ldots, N - 1, k \in \mathbb{Z}^n\}$$

is a frame if and only if for the $(L \times L)$ matrix valued function $z \mapsto \tilde{M}(z)\tilde{M}^*(z)$, $z \in K_n$, with $\tilde{M}$ defined as in (4.19) there exists real positive constants $m_F$ and $M_F$, such that

$$m_F I_L \leq \tilde{M}(z)\tilde{M}^*(z) \leq M_F I_L \text{ a.e. } z \in K_n. \quad (4.21)$$

So, we presented necessary and sufficient conditions on $\hat{\beta}_0, \ldots, \hat{\beta}_{N-1}$, so that $B_N$ is a frame in a similar way as in Theorem 4.4.4. Finally, we also present a corollary of Theorem 4.4.7, analogous to Corollary 4.4.5.

**Corollary 4.4.8** If $B_N$ is a frame, then $N \geq L$.

### 4.5 MRA and Riesz Bases in Hilbert Spaces

In Section 4.2 we used the concept of MRA to construct Riesz bases of the form

$$\{U_1^l U_2^k \psi_l | l = 1, \ldots, N - 1, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$$

for the separable Hilbert space $H$. We showed that the vectors $\psi_l$ were uniquely determined by (4.9) and that their generating sequences $q_l$ had to be determined such that

- $(\tau_\phi * q_l, R^A q_0)_{\mathbb{Z}^n} = 0, \quad \forall l \in \{1, \ldots, N-1\}, \quad \forall k \in \mathbb{Z}^n$,

- $\{R^A q_l | l = 0, \ldots, N - 1, k \in \mathbb{Z}^n\}$ is a Riesz basis for $l^2(\mathbb{Z}^n)$, with $q_0 = p$, the generating sequence of the MRA generator $\phi$.  


By taking $\gamma = \tau_\phi$ and $\beta_l = q_l$, $l = 0, \ldots, N-1$, in (4.15), Theorem 4.4.4 and Theorem 4.4.6, we arrive at the following theorem on the construction of Riesz bases in Hilbert spaces using MRA.

**Theorem 4.5.1** Given a sequence $q_0 \in l^2(\mathbb{Z}^n)$. Then the following two problems are equivalent.

**Problem 1:** Construct sequences $q_l$, $l = 1, \ldots, |\det A| - 1$, in $l^2(\mathbb{Z}^n)$ such that

1.1 $$(\tau_\phi * q_l, \mathcal{R}^A q_0)_{\mathbb{Z}^n} = 0, \quad \forall l \in \{1, \ldots, |\det A| - 1\} \quad \forall k \in \mathbb{Z}^n,$$

1.2 $\{\mathcal{R}^A q_l | l = 1, \ldots, |\det A| - 1, k \in \mathbb{Z}^n\}$ is a Riesz basis for $l^2(\mathbb{Z}^n)$.

**Problem 2:** Construct $\hat{q}_l$, $l = 1, \ldots, |\det A| - 1$, in $L^2(\mathbb{T}^n)$, the n-dimensional discrete Fourier transforms of $q_l \in l^2(\mathbb{Z}^n)$, such that

2.1 $P_{r,\phi,\phi_0}(z) = 0$ a.e. on $K_n$, $l = 1, \ldots, |\det A| - 1$, with $P$ as defined in (4.14).

2.2 The matrix-valued function $z \mapsto \hat{M}(z)$, $z \in K_n$, with entries

$$(\hat{M}(z))_{r,l} = |\det A|^{-1/2} \hat{q}_l(\omega_{d_1}^{(r)}(1) z_1, \ldots, \omega_{d_n}^{(r)}(n) z_n),$$

$l, r = 0, \ldots, |\det A| - 1$, where $\pi$ is an arbitrary enumeration from

$$\{0, 1, \ldots, |\det A| - 1\}$$

onto

$$\{m \in \mathbb{Z}^n | 0 \leq m_i \leq |d_i| - 1, i = 1, \ldots, n\}$$

and with $d_i = D(i, i)$ as in Theorem 4.4.1, is invertible for almost all $z \in K_n$.

Possible solutions to these problems are given in [37, 62] and [74].

To illustrate how to deal with Theorem 4.5.1 we consider an example of an MRA, which we already mentioned in Section 3.2.1, namely an MRA for $L^2(\mathbb{R})$ using Riesz systems. It will turn out that in this example, Problem 2 is not hard to solve. Moreover, for this example Conditions 2.1 and 2.2 will reduce to conditions, which are described thoroughly in the literature [12, 23, 62].

**Example 4.5.2** This example deals with an MRA for $L^2(\mathbb{R})$ as introduced in Section 3.2.1. However, here we take an MRA generator $\phi \in L^2(\mathbb{R})$, such that

$$\{T^k \phi | k \in \mathbb{Z}\}$$
is a Riesz system. So, according to Definition 4.2.1 we take $H = L^2(\mathbb{R})$, $U_1 = \mathcal{D}$ and $U_{2,1} = \mathcal{T}$. Obviously, Condition (iii) in Definition 4.2.1 holds for $A = 2$. As MRA generator we take $\phi = \phi_m$, the cardinal B-spline of order $m \geq 1$, which is defined by

$$
\phi_m = \begin{cases} 
\chi_{[0,1]}, & m = 1, \\
\phi_1 * \phi_{m-1}, & m \geq 2.
\end{cases}
$$

(4.22)

In Figure 4.2, $\phi_2$ and $\phi_4$ have been depicted.

For cardinal B-splines we have the following properties

1. $\text{supp } \phi_m = [0, m]$,
2. $\sum_{k \in \mathbb{Z}} \phi_m(x - k) = 1 \ \forall x \in \mathbb{R}$,
3. $\phi_m(m/2 - x) = \phi_m(m/2 + x) \ \forall x \in \mathbb{R}$,
4. $\{T^k \phi_m \mid k \in \mathbb{Z}\}$ is a Riesz system,
5. $\phi_m \in \text{clos span } \{D_a T^k \phi_m \mid k \in \mathbb{Z}\} \ \forall a \in \mathbb{N} \setminus \{0\}$.

For a proof of these and other properties of cardinal B-splines we refer to [12, 88].
For solving Problem 2, i.e., to search for a sequence \( q_1 \) whose Fourier transform satisfies Conditions 2.1 and 2.2, we have to determine \( \hat{p} \) and \( \hat{\phi} \).

By \( p_m \) we denote the generating sequence of \( \phi_m \). To give an expression for \( p_m \), we derive
\[
\phi_m = \phi_1 * \phi_{m-1} = \left( \sum_{k \in \mathbb{Z}} p_1(k) DT^k \phi_1 \right) * \left( \sum_{l \in \mathbb{Z}} p_{m-1}(l) DT^l \phi_{m-1} \right) = \sum_{k,l \in \mathbb{Z}} p_1(k)p_{m-1}(l) \left( DT^k \phi_1 * DT^l \phi_{m-1} \right)
\]
\[
= \frac{1}{\sqrt{2}} \sum_{k,l \in \mathbb{Z}} p_1(k)p_{m-1}(l) DT^{k+l} \phi_1 * \phi_{m-1}
\]
\[
= \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} \left( p_1 * p_{m-1} \right)(k) DT^k \phi_m.
\]

Recursively we get \( \hat{p}_m(z) = 2^{1/2-m}(\hat{p}_1(z))^m \). For \( \phi_1 \) we find in a straightforward way
\[
p_1(z) = (1 + z^{-1})/\sqrt{2},
\]
which yields
\[
\hat{p}_m(z) = 2^{1/2-m}(1 + z^{-1})^m = 2^{1/2-m} \sum_{k=0}^{m} \binom{m}{k} z^{-k}.
\]

So, the generating sequence \( p_m \) of \( \phi_m \) is given by
\[
p_m(k) = \begin{cases} 2^{1/2-m}(\binom{m}{k}), & k = 0, \ldots, m, \\ 0, & \text{otherwise.} \end{cases}
\]

Using Property 3 of cardinal B-splines we derive
\[
\tau_{\phi_m}(k) = \int_{\mathbb{R}} \phi_m(x) \phi_m(x - k) dx = \int_{\mathbb{R}} \phi_m(x) \phi_m(M + k - x) dx = (\phi_m * \phi_m)(m + k) = \phi_{2m}(m + k).
\]

Its Fourier transform is given by
\[
\hat{\tau}_{\phi_m}(z) = \sum_{k=-m}^{m} \phi_{2m}(m + k) z^{-k} = \frac{z^{1-m}}{(2m-1)!} E_{2m-1}(z),
\]
with \( E_{2m-1} \) the Euler-Frobenius polynomial of order \( 2m - 1 \), see [12]. We observe that \( \hat{\tau}_{\phi_m}(1) = 1 \), which follows from Property 2 of cardinal B-splines. Besides, as a property of Euler-Frobenius polynomials we have \( E_{2m-1}(z) \neq 0 \), \( z \in \mathbb{T} \), for all \( m \in \mathbb{N} \setminus \{0\} \). These
two considerations yield that \( \hat{r}_{\phi_m} \) satisfies (4.11). So, indeed \( \{ T^k \phi_m \mid k \in \mathbb{Z} \} \) is a Riesz system for all \( m \in \mathbb{N} \setminus \{0\} \).

According to Problem 2, we have to search for a \( q_m \in L^2(\mathbb{Z}) \), such that

- \( \hat{r}_{\phi_m}(z) \hat{q}_m(z) \hat{p}_m(z) + \hat{r}_{\phi_m}(-z) \hat{q}_m(-z) \hat{p}_m(-z) = 0 \) a.e. \( z \in T \),
- \( \hat{M}(z) = \begin{pmatrix} \hat{p}_m(z) & \hat{q}_m(z) \\ \hat{p}_m(-z) & \hat{q}_m(-z) \end{pmatrix} \) is invertible for almost all \( z \in T \).

It can be verified that the first condition holds for

\[
\hat{q}_m(z) = z^{2k+1} \hat{r}_{\phi_m}(-z) \hat{p}_m(-z),
\]

\( k \in \mathbb{Z} \). With this choice for \( \hat{q}_m \) we compute

\[
|\det(\hat{M}(z))| = |\hat{p}_m(z)\hat{q}_m(-z) - \hat{p}_m(-z)\hat{q}_m(z)|
\]

\[
= (\hat{r}_{\phi_m}(z)|\hat{p}_m(z)|^2 + \hat{r}_{\phi_m}(-z)|\hat{p}_m(-z)|^2)
\]

\[
= \hat{r}_{\phi_m}(z)|\hat{p}_m(z)|^2 + \hat{r}_{\phi_m}(-z)|\hat{p}_m(-z)|^2 \text{ a.e. } z \in T.
\]

We already observed that

\[
\text{ess inf}_{z \in T} \hat{r}_{\phi_m}(z) \geq m_r,
\]

for a certain positive constant \( m_r \). Furthermore, we derive

\[
|\hat{p}_m(z)|^2 + |\hat{p}_m(-z)|^2 = \sum_{k,l} (1 + (-1)^l) p_m(k)p_m(k-l)z^{-l}
\]

\[
= 2 \sum_{k,l} p_m(k)(\mathcal{R}^{2l}p_m)(k)z^{-2l}
\]

\[
= 2 \hat{\Gamma}(z^2),
\]

with \( \hat{\Gamma} \) as in (4.16). So,

\[
|\hat{p}_m(z)|^2 + |\hat{p}_m(z)|^2 \geq m_p \text{ a.e. } z \in T,
\]

for a certain positive constant \( m_p \). Together these results yield

\[
|\det(\hat{M}(z))| \geq m_rm_p > 0.
\]

Concluding, \( q_m \) and its corresponding wavelet function \( \psi_m \) can be obtained from the coefficients of the polynomial

\[
\hat{q}_m(z) = \frac{(-2z)^{1-m}}{\sqrt{2}(2m-1)!} (1 + z)^m E_{2m-1}(-z).
\]