Mathematical signal analysis: wavelets, Wigner distribution and a seismic application
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Chapter 5

The FRFT and Affine Transformations in the Wigner Plane

This chapter provides a classification of all unitary operators that act as affine transformations in the multi-dimensional Wigner plane. Moreover, a representation formula is given that encloses all these operators.

The problem of finding these operators is inspired by studying the fractional Fourier transform. This operator, which is introduced in the first section of this chapter, turns out to be acting as a rotation in the Wigner plane. Using a group theoretical approach we arrive at a classification of all linear transformations in the Wigner plane that correspond to unitary operators. This classification is used to come to a representation formula for the corresponding operators on $L^2(\mathbb{R}^n)$. This is done in the second section of this chapter.

This chapter is mainly based on [63].

5.1 The Fractional Fourier Transform

The fractional Fourier transform (FRFT) was introduced by Namias in [68] as a Fourier transform of fractional order. This was done starting from fractional powers of the eigenvalues of the Fourier transform and their corresponding eigenvalues. With this formalism he derived in a heuristic manner an integral representation of this operator. In [53, 61], McBride and Kerr provided a rigorous mathematical framework in which the formal work of Namias could be situated. We discuss this mathematical framework and Namias formal work in the first part
of this section.

Recently, the FRFT turned out to be an interesting transformation for time-frequency signal processing and optical engineering. The growing interest for the FRFT is the consequence of a series of papers that deal with the relation of the FRFT to time-frequency representations of a signal, like the Wigner distribution, see e.g. [4, 67, 77, 78]. This relation is discussed in the second part of this section.

5.1.1 Definition and Properties

We start with the definition of the FRFT for functions in $L^2(\mathbb{R})$.

**Definition 5.1.1** Take $f \in L^2(\mathbb{R})$. Its fractional Fourier transform of order $\alpha \in (-\pi, \pi]$ is given by

$$\mathcal{F}_\alpha[f](x) = \frac{C_\alpha}{\sqrt{2\pi} |\sin \alpha|} \int_{\mathbb{R}} f(u) e^{i((u^2+x^2)\cdot(\cot \alpha)/2 - ux \csc \alpha)} du,$$  

for $0 < |\alpha| < \pi$, with

$$C_\alpha = e^{i\left(\frac{\pi}{2} \text{sgn} \alpha - \alpha / 2\right)}. \quad (5.2)$$

Furthermore, for $\alpha = 0$ and $\alpha = \pi$ the FRFT is defined by

$$\mathcal{F}_0[f](x) = f(x) \text{ and } \mathcal{F}_\pi[f](x) = f(-x).$$

For $\alpha \notin (\pi, \pi]$ the FRFT is defined by periodicity $\mathcal{F}_{\alpha+2\pi} = \mathcal{F}_\alpha$.

Particularly, we have from this definition

$$\mathcal{F}_{\pi/2} = \mathcal{F} \text{ and } \mathcal{F}_{n\pi/2} = \mathcal{F}^n \forall n \in \mathbb{Z},$$

with $\mathcal{F}$ the Fourier transform on $L^2(\mathbb{R})$.

The factor $C_\alpha$ in (5.2) is chosen to guarantee that $\mathcal{F}_\alpha$ is continuous in $\alpha$ and that $\mathcal{F}_\alpha$ is properly normalized. Indeed, it can be shown that

$$\lim_{\beta \to \alpha} \| \mathcal{F}_\beta f - \mathcal{F}_\alpha f \|_2 = 0,$$  

for all $f \in L^2(\mathbb{R})$ and for this particular choice of $C_\alpha$.

This result is obtained by combining two properties of the FRFT. The first property of the FRFT is known as the index law, i.e.,

$$\mathcal{F}_\alpha \mathcal{F}_\beta f = \mathcal{F}_{\alpha + \beta} f,$$  

(5.4)
for all $\alpha, \beta \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$. A rigorous proof of this property for functions in the Schwartz space $S(\mathbb{R})$ is given in [61]. Consequently, this result can be extended to functions in $L^2(\mathbb{R})$ since $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

The second property we need for proving the continuity of $\mathcal{F}_\alpha$ is the continuity of the FRFT either in $\alpha = 0$ or $\alpha = \pi$. In [53], it is proven that

$$\lim_{\alpha \to 0} \|\mathcal{F}_\alpha f - f\|_2 = 0,$$

for all $f \in L^2(\mathbb{R})$. Result (5.3) can now be obtained in a straightforward way by combining (5.4) and (5.5). We observe, that (5.3) also holds for other choices of $C_\alpha$, see e.g. [4].

Considering again (5.4) we have in particular

$$\mathcal{F}_\alpha \mathcal{F}_{-\alpha} = \mathcal{I} \text{ and } \mathcal{F}_{-\alpha} \mathcal{F}_\alpha = \mathcal{I}.$$

It follows that the inverse of $\mathcal{F}_\alpha$ is given by $\mathcal{F}_{-\alpha}$, for all $\alpha \in \mathbb{R}$.

For $t \in \mathbb{R}$, we introduce the unitary operator $C_t$ on $L^2(\mathbb{R})$ by

$$C_t[f](x) = e^{itx^2/2} f(x). \quad (5.6)$$

Obviously, $C_t$ multiplies a given function $f \in L^2(\mathbb{R})$ with a quadratic chirp, i.e., a Fourier mode with a quadratic argument. Using this chirp multiplication and the dilation operator $D$ as defined in (3.2), we can write $\mathcal{F}_\alpha$, $\alpha \in (-\pi, \pi)$, also as

$$\mathcal{F}_\alpha f = C_\alpha C_{\cot \alpha} D_{\sin \alpha} \mathcal{F} C_{\cot \alpha}. \quad (5.7)$$

The fact that all operators in the right-hand side of (5.7) are unitary operators on $L^2(\mathbb{R})$ and that $|C_\alpha| = 1$ yields that $\mathcal{F}_\alpha$ is a unitary operator on $L^2(\mathbb{R})$, for all $\alpha \in \mathbb{R}$. Note, that $\mathcal{F}_0$ and $\mathcal{F}_\pi$ are also unitary, which follows immediately from Definition 5.1.1. As a consequence we also have Parseval's formula for the FRFT

$$\int_\mathbb{R} f(x) \overline{g(x)} \, dx = \int_\mathbb{R} \mathcal{F}_\alpha[f](x) \overline{\mathcal{F}_\alpha[g](x)} \, dx, \quad (5.8)$$

for all $\alpha \in \mathbb{R}$ and $f, g \in L^2(\mathbb{R})$. Furthermore, as a result we have Plancherel's formula for the FRFT

$$\int_\mathbb{R} |f(x)|^2 \, dx = \int_\mathbb{R} |\mathcal{F}_\alpha[f](x)|^2 \, dx, \quad (5.9)$$

for all $\alpha \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$. 

From the preceding derivations and the definition of $\mathcal{F}_0$ it follows that

$$G_{fr} = \{ \mathcal{F}_\alpha \mid \alpha \in \mathbb{R} \}$$

is a strongly continuous subgroup of unitary operators on $L^2(\mathbb{R})$. A cyclic subgroup of order 4 is given by the integer powers of the Fourier transform

$$\{ \mathcal{F}^n \mid n = 0, 1, 2, 3 \}.$$  

Consequently, the discrete cyclic group with generating element $\mathcal{F}$ is embedded in the continuous group $G_{fr}$.

A further relation with the classical Fourier transform on $L^2(\mathbb{R})$ can be observed by considering the formal derivation of the FRFT by Namias in [68]. His starting point was to consider the eigenvalues and eigenfunctions of the Fourier transform.

It is known, see e.g. [29], that the eigenfunctions of the Fourier transform are given by the Hermite functions

$$h_k(x) = (2^k k! \sqrt{\pi})^{-1/2} e^{-x^2/4} H_k(x),$$

where $H_k$ are the Hermite polynomials given by

$$H_k(x) = (-1)^k e^{x^2} \left( \frac{d}{dx} \right)^k e^{-x^2}.$$  

The Hermite functions form an orthonormal basis for $L^2(\mathbb{R})$ and they satisfy

$$\mathcal{F} h_k = e^{ik\pi/2} h_k.$$  

The first idea for the construction of the FRFT was to define an operator $\mathcal{F}_\alpha$, satisfying

$$\mathcal{F}_\alpha h_k = e^{ik\alpha} h_k,$$  

for $\alpha \in \mathbb{R}$. For $\alpha = m\pi/2$, with $m \in \mathbb{Z}$, we have $\mathcal{F}_{m\pi/2} = \mathcal{F}^m$. Particularly, if $m \mod 4 = 0$, then $\mathcal{F}^m = \mathcal{I}$. For a formal representation of $\mathcal{F}_\alpha$, with $0 < \alpha < \pi/2$, we follow Namias in [68].

We write $f \in L^2(\mathbb{R})$ as $f = \sum_{k=0}^{\infty} (f, h_k) h_k$. Consequently, we have

$$\mathcal{F}_\alpha[f](x) = \sum_{k=0}^{\infty} (f, h_k) \mathcal{F}_\alpha[h_k](x) = \sum_{k=0}^{\infty} (f, h_k) e^{ik\alpha} h_k(x)$$

$$= \int_{\mathbb{R}} f(u) \left( \sum_{k=0}^{\infty} e^{ik\alpha} h_k(u) h_k(x) \right) du$$

$$= \int_{\mathbb{R}} f(u) \left( \sum_{k=0}^{\infty} e^{ik\alpha} 2^k k! \sqrt{\pi} H_k(u) H_k(x) e^{-u^2/2-x^2/2} \right) du.$$
The latter expression can be rewritten using Mehler's formula, see [64],
\[
\sum_{k=0}^{\infty} \frac{z^k}{2^k k! \sqrt{\pi}} H_k(u) H_k(x) = \frac{1}{\sqrt{\pi(1-z^2)}} \exp \left( \frac{2xuz - z^2(x^2 + u^2)}{1 - z^2} \right). \tag{5.13}
\]
We observe that the series converges in \(L^2\) with respect to \(u\), for all \(x\) and \(z\), see [29]. Using Mehler's formula in the previous result yields
\[
F_\alpha[f](x) = \frac{1}{\sqrt{\pi e^{i\alpha} \cdot \sqrt{e^{-i\alpha}} - e^{i\alpha}}} \int_{\mathbb{R}} f(u) \exp \left( i \frac{2ixu - i(e^{i\alpha} + e^{-i\alpha})(x^2 + u^2)/2}{e^{i\alpha} - e^{-i\alpha}} \right) du
\]
\[
= \frac{e^{i\pi^2/4 - i\alpha/2}}{\sqrt{2\pi \sin \alpha}} \int_{\mathbb{R}} f(u) e^{i((u^2+x^2) \cdot (\cot \alpha)/2 - uz \csc \alpha)} du.
\]
For a rigorous framework in which this formal work of Namias can be studied we refer to [53, 61].

5.1.2 The FRFT and the Wigner Plane

For time-frequency analysis it is interesting to consider the relation of the FRFT with time-frequency operators like the Wigner distribution. Therefore, we compute the Wigner distribution of the FRFT. This will give us insight in how the FRFT acts in phase space.

For this computation we need the following lemma.

**Lemma 5.1.2** Let \(T_b\) and \(M_\omega\), \(b, \omega \in \mathbb{R}\), denote the shift operator and frequency modulation on \(L^2(\mathbb{R})\) as given in (2.12) and (2.13) respectively. Furthermore, let \(F_\alpha\), \(\alpha \in \mathbb{R}\), the fractional Fourier transform on \(L^2(\mathbb{R})\) as given in Definition 5.1.1. Then
\[
F_\alpha T_b = e^{ib^2(\sin 2\alpha)/4} M_{-\sin \alpha} T_b \cos \alpha F_\alpha,
\]
\[
F_\alpha M_\omega = e^{-i\omega^2(\sin 2\alpha)/4} M_{\omega \cos \alpha} T_{\cos(\alpha/2)} \sin \alpha F_\alpha. \tag{5.15}
\]

**Proof**

For \(\alpha = 0\) both results are trivial, since \(F_0 = I\). For \(\alpha = \pi\) both results follow directly from Definition 5.1.1. Furthermore, equation (5.15) follows from (5.14) by observing that \(FM_\omega = T_\omega F\), with \(F\) the Fourier transform. Indeed, if (5.14) holds, this observation yields
\[
F_\alpha M_\omega = F_\alpha F^* T_\omega F = F_{\alpha - \pi/2} T_\omega F_{\pi/2}
\]
\[
= e^{i\omega^2(\sin(2\alpha - \pi))/4} M_{-\omega \sin(\alpha - \pi/2)} T_{\omega \cos(\alpha - \pi/2)} F_{\alpha - \pi/2} F_{\pi/2}
\]
\[
= e^{-i\omega^2(\sin 2\alpha)/4} M_{\omega \cos \alpha} T_\omega \sin \alpha F_\alpha,
\]
using the index law for the FRFT. Consequently, the proof is established by showing that (5.14) holds for $0 < |\alpha| < \pi$. We derive for $f \in L^2(\mathbb{R})$, $b \in \mathbb{R}$ and $0 < |\alpha| < \pi$

$$
\mathcal{F}_\alpha T_b[f](x) = \frac{C_\alpha}{\sqrt{2\pi |\sin \alpha|}} \int_R f(u - b) e^{i((u^2 + x^2)\cdot(cot \alpha)/2 - uz \csc \alpha)} du
$$

$$
= \frac{C_\alpha}{\sqrt{2\pi |\sin \alpha|}} \int_R f(u) e^{i((u^2 + x^2 + b^2 + 2ub)\cdot(cot \alpha)/2 - (u + b)z \csc \alpha)} du
$$

$$
= \frac{C_\alpha}{\sqrt{2\pi |\sin \alpha|}} e^{i(b^2 \cdot(cot \alpha)/2 - bx)(1 - \cos \alpha) \csc \alpha} \times
$$

$$
\int_R f(u) e^{i((u^2 + (x - b \cos \alpha)\cdot(cot \alpha)/2 - (u(x - b \cos \alpha)) \csc \alpha)} du
$$

$$
= e^{i(b^2 \cdot(cot 2\alpha)/4 - bx \sin \alpha)} \mathcal{F}_\alpha [f](x - b \cos \alpha)
$$

$$
= e^{ib^2 \cdot(cot 2\alpha)/4} M_{-b \sin \alpha} \mathcal{T}_b \mathcal{F}_\alpha [f](x).
$$

Using this lemma, we can compute the action of the FRFT in phase space by means of the Wigner distribution. For this we write

$$
\mathcal{WV}[f](x, \omega) = \frac{1}{2\pi} \int_R f(x + t/2)f(x - t/2)e^{-it\omega} dt
$$

$$
= \frac{1}{\pi} \int_R f(t + x)f(x - t)e^{-2it\omega} dt
$$

$$
= (\mathcal{M}_{-\omega} \mathcal{T}_z f, \mathcal{M}_0 \mathcal{T}_z \mathcal{F}_\omega f)/\pi.
$$

Using Lemma 5.1.2 we compute

$$
\mathcal{F}_{-\alpha} \mathcal{M}_0 \mathcal{T}_z f = e^{i(\omega^2 - x^2)\cdot(cot 2\alpha)/4} \mathcal{M}_0 \mathcal{F}_\alpha \mathcal{T}_z \sin \alpha M_\alpha \sin \alpha \mathcal{T}_z \cos \alpha \mathcal{F}_{-\alpha}
$$

$$
= e^{i(\omega^2 - x^2)\cdot(cot 2\alpha)/4} e^{iz\omega \sin \alpha^2} M_\alpha \sin \alpha + \omega \cos \alpha \mathcal{T}_z \cos \alpha - \omega \sin \alpha \mathcal{F}_{-\alpha}.
$$

Combining these two results yields

$$
\mathcal{WV}[\mathcal{F}_\alpha f](x, \omega)
$$

$$
= (\mathcal{M}_{-\omega} \mathcal{T}_z \mathcal{F}_\alpha f, \mathcal{M}_0 \mathcal{T}_z \mathcal{F}_\omega \mathcal{F}_\alpha f)/\pi
$$

$$
= (\mathcal{F}_{-\alpha} \mathcal{M}_{-\omega} \mathcal{T}_z \mathcal{F}_\alpha f, \mathcal{F}_{-\alpha} \mathcal{M}_0 \mathcal{T}_z \mathcal{F}_\omega \mathcal{F}_\alpha f)/\pi
$$

$$
= (\mathcal{M}_{-\omega} \mathcal{T}_z \mathcal{F}_\alpha \mathcal{F}_{-\alpha} f, \mathcal{M}_0 \mathcal{T}_z \mathcal{F}_\omega \mathcal{F}_{-\alpha} f)/\pi
$$

$$
= \mathcal{WV}[f](x \cos \alpha - \omega \sin \alpha, x \sin \alpha + \omega \cos \alpha) = \mathcal{WV}[f](R_\alpha(x, \omega)),
$$

(5.16)
where $R_\alpha(x, \omega)$ represents the matrix vector product with matrix

$$R_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$ (5.17)

We conclude from this derivation that the FRFT of order $\alpha$ acts like a rotation by $\alpha$ in the Wigner plane. In particular, we have a rotation by $\pi/2$ in the Wigner plane for $F_{\pi/2}$. We observe, that this result coincides with the action of the Fourier transform in the Wigner plane as given in (2.32).

The action of the FRFT in the Wigner plane leads us in a natural way to the question which operators on $L^2(\mathbb{R})$ act like a linear transformation in the Wigner plane. The sequel of this chapter is devoted to this question. However, instead of operators on $L^2(\mathbb{R})$ we consider operators acting on $L^2(\mathbb{R}^n)$. It will turn out that finding a solution for the multi-dimensional problem does not follow straightforwardly from the solution for the one-dimensional case.

Since we want to give an answer to our problem for operators on $L^2(\mathbb{R}^n)$, we introduce the fractional Fourier transform on $L^2(\mathbb{R}^n)$ by

$$\mathcal{F}_{\alpha_1, \ldots, \alpha_n} = \mathcal{F}_{\alpha_1} \cdots \mathcal{F}_{\alpha_n},$$ (5.18)

for $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Computing the multi-dimensional Wigner distribution of this FRFT yields

$$\mathcal{WV}[\mathcal{F}_{\alpha_1, \ldots, \alpha_n} f](x, \omega) = \mathcal{WV}[f](R_{\alpha_1, \ldots, \alpha_n}(x, \omega)),$$ (5.19)

with

$$R_{\alpha_1, \ldots, \alpha_n} = \begin{pmatrix} \cos \alpha_1 & 0 & -\sin \alpha_1 & 0 \\ & \ddots & \ddots & \ddots \\ 0 & \cos \alpha_n & 0 & -\sin \alpha_n \\ \sin \alpha_1 & 0 & \cos \alpha_1 & 0 \\ & \ddots & \ddots & \ddots \\ 0 & \sin \alpha_n & 0 & \cos \alpha_n \end{pmatrix}.$$ (5.20)

This result follows in a straightforward way from (5.16).

### 5.2 Affine Transformations in the Wigner Plane

Inspired by the fractional Fourier transform and its action in the Wigner plane, we search for linear operators $\mathcal{V}$ on $L^2(\mathbb{R}^n)$ such that there exist a matrix $A \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$ for which

$$\mathcal{WV}[\mathcal{V} f](x, \omega) = \mathcal{WV}[f](A(x, \omega) + b),$$ (5.21)
holds for all \( f \in L^2(\mathbb{R}^n) \). We observe, that De Bruijn already considered this problem in [9] using a new class of generalized functions. Here we will follow an approach based on group theory, see [86, 87, 100]. These results will be placed within the concept of the FRFT in order to embed this transform in a larger class of unitary transformations. Also new results will be added.

We restrict ourselves to matrices \( A \) for which \( \det A = \pm 1 \). For these matrices we have

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{W} \mathcal{V}[f](A[x, \omega] + b) \, d\omega \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{W} \mathcal{V}[f](x, \omega) \, d\omega \, dx.
\]

We shall refer to such affine transformations in the Wigner plane as energy preserving affine transformations. For these transformations the corresponding operators \( \mathcal{V} \) on \( L^2(\mathbb{R}^n) \) satisfy

\[
(\mathcal{V}f, \mathcal{V}f) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{W} \mathcal{V}[\mathcal{V}f](x, \omega) \, d\omega \, dx
\]

\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{W} \mathcal{V}[f](A(x, \omega) + b) \, d\omega \, dx
\]

\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{W} \mathcal{V}[f](x, \omega) \, d\omega \, dx = (f, f),
\]

for \( f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) or \( \hat{f} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) which follows from (2.41). We observe that \( L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) is a dense subspace of \( L^2(\mathbb{R}^n) \). Concluding, an operator on \( L^2(\mathbb{R}^n) \) that yields an energy preserving affine transformation in the Wigner plane has to be an isometry on \( L^2(\mathbb{R}^n) \). On the other hand, Equation (5.21) follows directly from applying (2.41) on both sides of the equation \( (\mathcal{V}f, \mathcal{V}f) = (f, f) \), for \( f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) or \( \hat{f} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \).

Before dealing with a classification of all unitary operators that satisfy (5.21), we present some well-known operators for which (5.21) holds.

**Multiplication**

We start our set of unitary operators on \( L^2(\mathbb{R}^n) \) with a trivial one, namely multiplication by a constant \( C \) with \( |C| = 1 \). Result (2.51) already showed that \( \mathcal{W} \mathcal{V}[f] = \mathcal{W} \mathcal{V}[Cf] \), for all \( |C| = 1 \). Consequently, this multiplication operator satisfies (5.21) with \( A = I_{2n} \), the \((2n \times 2n)\) identity matrix, and \( b = 0 \).
Complex conjugation

Besides linear operators there also exists a non-linear operator for which (5.21) holds, namely the operator \( f \mapsto \bar{f} \). For the one-dimensional case we have already seen in (2.30) that
\[
\mathcal{W}V[f](x, \omega) = \mathcal{W}V[f](x, -\omega).
\]
For \( f \in L^2(\mathbb{R}^n) \) this result also holds. This follows from a straightforward generalization of (2.30). We conclude, that taking the complex conjugate also satisfies (5.21) with
\[
A = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \quad \text{and} \quad b = 0.
\]
We observe that we have \( \det A = (-1)^n \) for the complex conjugation. Later in this section it will turn out that a necessary condition on a linear operator \( V \), such that (5.21) holds, is given by \( \det A = 1 \).

Space and frequency shift

For \( x_0, \omega_0 \in \mathbb{R}^n \) we introduce the shift operator and the frequency shift operator on \( L^2(\mathbb{R}^n) \) by
\[
T_{x_0}[f](x) = f(x - x_0) \quad \text{and} \quad M_{\omega_0}[f](x) = e^{i(\omega_0, x)} f(x)
\]
respectively, with \( f \in L^2(\mathbb{R}^n) \). Remark, that these operators coincide with the shift and frequency shift operators (2.12) and (2.13) in the one-dimensional case.

We combine the introduced unitary operators \( T_{x_0} \) and \( M_{\omega_0} \) into one unitary operator on \( L^2(\mathbb{R}^n) \), given by
\[
\mathcal{N}(x_0, \omega_0)[f](x) = T_{x_0} M_{\omega_0}[f](x) = e^{i(\omega_0, x)} f(x - x_0).
\]
(5.22)

Computing the Wigner transform of this operator yields
\[
\mathcal{W}V[\mathcal{N}(x_0, \omega_0)f](x, \omega) = \mathcal{W}V[f](x - x_0, \omega - \omega_0),
\]
which is a result we have seen before in discussing the one-dimensional Wigner distribution. From this result we conclude, that (5.21) holds for \( \mathcal{N}(x_0, \omega_0) \), namely by taking \( A = 0 \) and \( b = (x_0, \omega_0) \).

We observe that all possible translations \( b \in \mathbb{R}^n \) in (5.21) can be obtained from \( \mathcal{N}_0 \). This means, that if we are looking for a unitary operator \( V \) on \( L^2(\mathbb{R}^n) \) such that (5.21) holds, then we only have to find a linear operator \( \mathcal{U} \) on \( L^2(\mathbb{R}^n) \) such that
\[
\mathcal{W}V[\mathcal{U}f](x, \omega) = \mathcal{W}V[f](A(x, \omega)),
\]
(5.23)
for all \( f \in \mathbb{R}^n \). The operator \( V \) we are looking for is then given by \( V = \mathcal{N}_0 \mathcal{U} \). Therefore, we will restrict ourselves from now on to operators \( \mathcal{U} \) that satisfy (5.23) with \( \det A = \pm 1 \).
The Fourier transform

In Section 2.3 we already derived for the Fourier transform $F$ on $L^2(\mathbb{R})$

$$\mathcal{WV}[Ff](x,\omega) = \mathcal{WV}[f](\omega, x).$$ \hfill (5.24)

For $f \in L^2(\mathbb{R}^n)$ and the $n$-dimensional Fourier transform $F$ this relation remains the same, which follows straightforwardly from a generalization of Relation (2.31) for the multi-dimensional Wigner distribution. Consequently, the Fourier transform on $L^2(\mathbb{R}^n)$ satisfies (5.23) with $A = J_n^T$. Here $J_n$ denotes the $(2n \times 2n)$ matrix given by

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$ \hfill (5.25)

In the sequel of this section this matrix will play an important role in classifying all unitary operators $U$ that satisfy (5.23).

The dilation operator

For $B \in \mathbb{R}^{n \times n}$, with $\det B \neq 0$, the dilation operator $D_B$ on $L^2(\mathbb{R}^n)$ is defined by

$$D_B[f](x) = \frac{1}{\sqrt{|\det B|}} f(B^{-1}x),$$ \hfill (5.26)

with inverse

$$D_B^{-1}[f](x) = \sqrt{|\det B|} f(Bx).$$

We use the definition of the Wigner distribution to derive the action of $D_B$ in the Wigner plane. We compute

$$\mathcal{WV}[D_Bf](x,\omega)$$

$$= \frac{1}{(2\pi)^n|\det B|} \int_{\mathbb{R}^n} f(B^{-1}(x + \tau/2)) \overline{f(B^{-1}(x - \tau/2))} e^{-i(\tau,\omega)} d\tau$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(B^{-1}x + \tau/2) \overline{f(B^{-1}x - \tau/2)} e^{-i(\tau, B^T\omega)} d\tau$$

$$= \mathcal{WV}[f](B^{-1}x, B^T\omega).$$ \hfill (5.27)

Concluding, also $D_B$ corresponds to a linear transformation in the Wigner plane. For $D_B$ Relation (5.23) holds with

$$A = \begin{pmatrix} B^{-1} & 0 \\ 0 & B^T \end{pmatrix}. $$
Multiplication with a chirp

The last example of a unitary operator that satisfies (5.23) is the operator that multiplies a function in $L^2(\mathbb{R}^n)$ with a quadratic chirp. This operator is given by

$$C_S[f](x) = e^{i(Sx,x)/2}f(x),$$  \hspace{1cm} (5.28)

with $S \in \mathbb{R}^{n \times n}$ symmetric. Remark, that we have seen this operator already for the one-dimensional case in (5.6), which coincides with (5.28) for $n = 1$. Obviously its inverse is given by

$$C_S^{-1}[f](x) = e^{-i(Sx,x)/2}f(x).$$

We use (2.52) to derive the action of $C_S$ in the Wigner plane

$$\mathcal{W}[C_S f](x,\omega) = (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ((C_S^* \mu(p,q,0)C_S)f,f) e^{-i(p,x)} e^{-i(q,\omega)} dp dq.$$

In a direct way we get

$$(C_S^* \mu(p,q,0)C_S)[f](x) = e^{-i(Sx,x)/2} e^{i(p,x)} e^{i(p,q)/2} e^{i(S(x+q),x+q)/2} f(x+q)$$

$$= e^{i(p+q,x)} e^{i(p+S(x,q)))/2} f(x+q)$$

$$= \mu(p+S q, q, 0)[f](x),$$

which yields

$$\mathcal{W}[C_S f](x,\omega) = (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mu(p+q,x,0)f,f) e^{-i(p,x)} e^{-i(q,\omega)} dp dq$$

$$= (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mu(p,q,0)f,f) e^{-i(p,q), A(x,\omega))} dp dq$$

$$= \mathcal{W}[f](A(x,\omega)),$$  \hspace{1cm} (5.29)

with $A = \left( \begin{array}{cc} I_n & 0 \\ -S & I_n \end{array} \right)$. Consequently, also $C_S$ satisfies (5.23) with $A$ as given before.

5.2.1 A Group Theoretical Approach

In the last example of the previous subsection we have already seen that the relation between a unitary operator on $L^2(\mathbb{R}^n)$ and its affine action in the Wigner plane can be given by using (2.52). This relation can also be used to translate our problem in terms of group theory. This can be done in the following way.
Given a unitary operator $\mathcal{V}$ on $L^2(\mathbb{R}^n)$, we define a unitary representation $\rho$ of the Heisenberg group $H_n$ by $\rho(g) = \mathcal{V}^{*}\mu(g)\mathcal{V}$, for all $g \in H_n$ and $\mu$ the Schrödinger representation. Then by (2.52) we have for such $\rho$ and $\mathcal{V}$

$$\mathcal{W}[\mathcal{V}f](x, \omega) = (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ((\mathcal{V}^{*}\mu(p, q, 0)\mathcal{V})f, f) e^{-i(p, x)} e^{-i(q, \omega)} \, dp \, dq.$$ 

$$= (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\rho(p, q, 0)f, f) e^{-i(p, x)} e^{-i(q, \omega)} \, dp \, dq.$$ 

Consequently, if there exists a linear transformation $A$ such that $\mu(g, 0) = \rho(A^Tg, 0)$ for all $g \in H'_n$, with

$$H'_n = \{ g \in \mathbb{R}^{2n} \mid \forall t \in \mathbb{R}, (g, t) \in H_n \},$$

then

$$\mathcal{W}[\mathcal{V}f](x, \omega) = (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\mu(A^{-T}(p, q), 0)f, f) e^{-i(p, x)} e^{-i(q, \omega)} \, dp \, dq.$$ 

$$= |\det A| \cdot \mathcal{W}[f](A(x, \omega)), \quad (5.30)$$

using the notation $A^{-T} = (A^{-1})^T$.

This derivation shows that the problem we are considering is equivalent to problem of finding operators $\mathcal{V} \in U(L^2(\mathbb{R}^n))$ for which there exist matrices $A \in \mathbb{R}^{n \times n}$ such that

$$\mathcal{V}^{*}\mu(g, t)\mathcal{V} = \mu(A^{-T}g, t), \quad (5.31)$$

for all $g \in H'_n$ and $t \in \mathbb{R}$.

Besides the Lie groups that have been discussed in Example 2.4.2 we will use another Lie group for solving this problem, namely the symplectic group $Sp(n)$. This group is defined by

$$Sp(n) = \{ M \in GL(2n) \mid J_n M^T J_n^T = M^{-1} \}, \quad (5.32)$$

with $J_n$ as given in (5.25). Note that by definition $M^T \in Sp(n)$ and $\det M = \pm 1$ for any $M \in Sp(n)$. Moreover, it can be shown that $Sp(n)$ is connected, see [29]. This yields that $\det M = 1$ if $M \in Sp(n)$. Furthermore, we observe, that $Sp(n) \subset SL(2n)$, but $Sp(1) = SL(2)$. It will turn out later in this section, that this property of the symplectic group causes the fact that solutions for the multi-dimensional problem do not follow straightforwardly from the solution for the one-dimensional case.

To solve our problem we start with the introduction of $G$, the subgroup of $U(L^2(\mathbb{R}^n))$ given by

$$G = \{ \mathcal{V} \in U(L^2(\mathbb{R}^n)) \mid \forall \ g \in \mathbb{R}^{2n} \forall t \in \mathbb{R} \exists g' \in \mathbb{R}^{2n} : \mathcal{V}^{*}\mu(g, t)\mathcal{V} = \mu(g', t) \}. \quad (5.33)$$
Obviously, $G$ is a semi-group. Later we will show that every $g \in G$ has an inverse element in $G$, which yields that $G$ is a group. This group can be equipped with the strong operator topology of $U(L^2(\mathbb{R}^n))$. Furthermore, it is clear from (2.45) that $g'$ in (5.33) is uniquely determined. So a mapping $\nu(V) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ can be defined, which depends on $V \in G$. This $\nu(V)$ is given by $\nu(V)(p, q) = (p', q')$, with $p, p', q$ and $q'$ as in (5.33). Also $\nu(V)$ is a homomorphism for all $V \in G$. This is shown in the following way.

For $\alpha, \beta \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in \mathbb{R}^n$ we have

$$V^*\mu(\alpha p_1, \alpha q_1, 0) \mu(\beta p_2, \beta q_2, 0)V = V^*\mu(\alpha p_1 + \beta p_2, \alpha q_1 + \beta q_2, (\alpha q_1, \beta p_2) / 2 - (\alpha p_1, \beta q_2) / 2)V = \mu(\nu(V)(\alpha p_1 + \beta p_2, \alpha q_1 + \beta q_2), (\alpha J_n(p_1, q_1), \beta(p_2, q_2)) / 2).$$

On the other hand we also have

$$V^*\mu(\alpha p_1, \alpha q_1, 0) \mu(\beta p_2, \beta q_2, 0)V = \mu(\alpha \nu(V)(p_1, q_1), 0) \mu(\beta \nu(V)(p_2, q_2), 0) = \mu(\alpha \nu(V)(p_1, q_1) + \beta \nu(V)(p_2, q_2), (y_1, x_2) / 2 - (x_1, y_2) / 2),$$

with $(x_1, y_1) = \alpha \nu(V)(p_1, q_1)$ and $(x_2, y_2) = \beta \nu(V)(p_2, q_2)$. Taking these results together yields

$$\mu(\nu(V)(\alpha p_1 + \beta p_2, \alpha q_1 + \beta q_2), (\alpha J_n(p_1, q_1), \beta(p_2, q_2)) / 2) = \mu(\alpha \nu(V)(p_1, q_1) + \beta \nu(V)(p_2, q_2), (y_1, x_2) / 2 - (x_1, y_2) / 2).$$

A necessary condition such that (5.34) holds for all $\alpha, \beta, p_1, p_2, q_1$ and $q_2$ is given by the linearity of $\nu(V)$ for all $V \in G$. Consequently, $\nu(V) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is a homomorphism, that satisfies

$$V^*\mu(p, q, t)V = \mu(\nu(V)(p, q), t).$$

Using this relation we can show, that $\nu(V)$ is also injective. To do this, we assume $\nu(V)g = 0$, or equivalently $\mu(g, t)V = \mu(0, t)$. Then

$$\mu(g, t) = V\mu(0, t)V^* = \mu(0, t),$$

which yields $g = 0$.

Furthermore, $\nu$ satisfies

$$\mu(\nu(CV)(p, q), t) = (CV^*)\mu(p, q, t)(CV) = \mu(\nu(V)(p, q), t)$$

and

$$\mu(\nu(V_1 V_2)(p, q), t) = \nu_2^*(\nu_1^*\mu(p, q, t)V_1)V_2 = \nu_2^*\nu(\nu(V_1)(p, q), t)V_2 = \mu(\nu(V_2)\nu(V_1)(p, q), t),$$
for all $V_1, V_2 \in U(L^2(\mathbb{R}^n))$ and $|C| = 1$. In the following lemma we deal with some other properties of the mapping $\nu$.

**Lemma 5.2.1** Let $G$ be the subgroup of $U(L^2(\mathbb{R}^n))$ as defined in (5.33) and let $\nu$ be the mapping as defined in (5.35). Then $\nu$ is a continuous mapping from $G$ onto $Sp(n)$ in the subspace topology of $G \subset U(L^2(\mathbb{R}^n))$. The kernel of $\nu$ is given by $\text{Ker } \nu = \{ C I ||C| = 1 \}$.

**Proof**
Since $p'$ and $q'$ are uniquely determined in (5.33) it follows that $\nu(V)$ is a non-singular mapping on $\mathbb{R}^{2n}$, or equivalently $\nu(V) \in GL(2n)$ for all $V \in G$. To show that $\nu(V) \in Sp(n)$, we take $T = \nu(V)$ and $p_1, p_2, q_1, q_2 \in \mathbb{R}^n$. Then by (5.34) we get for $\alpha = 1$ and $\beta = 1$

$$
\mu(T(p_1 + p_2, q_1 + q_2), (J_n(p_1, q_1), (p_2, q_2))/2) = \mu(T(p_1 + p_2, q_1 + q_2), (J_n(x_1, y_1), (x_2, y_2))/2) = \mu(T(p_1 + p_2, q_1 + q_2), (T^TJ_nT(p_1, q_1), (p_2, q_2))/2).
$$

This result must hold for all $p_1, p_2, q_1, q_2 \in \mathbb{R}^n$. This implies that $T^TJ_nTJ_n^T = I$, which is equivalent with the condition in (5.32).

To compute the kernel of $\nu$ we take $V$ such that $\nu(V) = I$. This yields $V\mu V^* = \mu$. Since $\mu$ is irreducible, we get from this equation $V = C I$, with $|C| = 1$.

To complete this proof we show the continuity of the mapping. Let $V_1, V_2 \in G$ and $W = V_2 - V_1$. Then for all $p, q \in \mathbb{R}^n$

$$
\mu((\nu(V_2) - \nu(V_1))(p, q), t) = \mu(\nu(V_2)(p, q), 0) \mu(\nu(V_1)(-p, -q), 0) = V_2^*\mu(p, q, 0)(W + V_1)V_1^*\mu(-p, -q, 0)V_1 = I - V_2^*W + V_2^*\mu(p, q, 0)WV_1^*\mu(-p, -q, 0)V_1,
$$

with $t = -(\nu(V_1)^TJ_n\nu(V_2)(p, q), (p, q))$. Consequently,

$$
\forall \epsilon > 0 \exists \delta > 0 \forall p, q \in \mathbb{R}^n : \|V_2 - V_1\|_2 < \delta \implies \|\mu((\nu(V_2) - \nu(V_1))(p, q), t) - \mu(0, 0, 0)\|_2 < \epsilon.
$$

It can be shown, see e.g. [100], that $\|\mu(p, q, t) - \mu(0, 0, 0)\|_2 \to 0$ implies $(p, q, t) \to (0, 0, 0)$. Since the latter result must hold for all $p, q \in \mathbb{R}^n$, we get $\|\nu(V_2) - \nu(V_1)\|_2 \to 0$. This condition is not only necessary to obtain $\|\mu(x, y, t) - \mu(0, 0, 0)\|_2 \to 0$. It is also sufficient, since $t \to -(\nu(V_1)^TJ_n\nu(V_1)(p, q), (p, q)) = -(J_n(p, q), (p, q)) = 0$, if $\nu(V_2) \to \nu(V_1)$.

For solving our original problem, namely to find unitary operators on $L^2(\mathbb{R}^n)$ that act like affine transformations in the Wigner plane, we combine (5.30), (5.31) and Lemma 5.2.1. This results into the following theorem.
Theorem 5.2.2 Let $V$ be a unitary operator on $L^2(\mathbb{R}^n)$ and $A$ a linear transformation on $\mathbb{R}^{2n}$. Then

$$\mathcal{W}_V[Vf](x,\omega) = \mathcal{W}_V[Af](x,\omega).$$

if and only if

(i) $V \in G$, with $G$ as defined in (5.33),

(ii) $A \in Sp(n),$

(iii) $A = \nu(V)^{-T}$, with $\nu$ the continuous mapping from $G$ onto $Sp(n)$ as defined in (5.35).

Theorem 5.2.2 tells us under which conditions unitary operators on $L^2(\mathbb{R}^n)$ act like affine transformations in the Wigner plane, namely if they belong to $G$. However, Theorem 5.2.2 does not tell us explicitly which unitary operators satisfy (5.36), e.g. by means of a representation formula for such operators. In the following examples we revisit three operators, that have been considered in the beginning of this section. We show that these three operators are elements of $G$ and we compute $\nu(V)$. These three operators will give us some insight in the type of operators, that $G$ consists of. In Section 5.3 we will present a representation formula that gives us an explicit formula for all operators in $G$.

Example 5.2.3 The first unitary operator we consider is the Fourier transform on $L^2(\mathbb{R}^n)$. We derive

$$(\mathcal{F}^*\mu(p,q,t)\mathcal{F})[f](x) = \int_{\mathbb{R}^n} \hat{f}(\omega + q)e^{i((p,\omega)+(x,\omega)+(p,q)/2+t)} d\omega$$

$$= \int_{\mathbb{R}^n} \hat{f}(\omega)e^{i((p,\omega)+(x,\omega)-(p,q)/2-(q,x)+t)} d\omega$$

$$= e^{i((-q,x)+(q,p)/2+t)} \int_{\mathbb{R}^n} \hat{f}(\omega)e^{i(x+p,\omega)} d\omega$$

$$= \mu(-q,p,t)[f](x),$$

for all $f \in L^2(\mathbb{R}^n)$. Consequently, $\mathcal{F} \in G$ and

$$\nu(\mathcal{F}) = J_n^T.$$

According to Theorem 5.2.2 the symplectic transformation in the Wigner plane corresponding to the Fourier transform is given by

$$A = \nu(\mathcal{F})^{-T} = (J_n^T)^{-T} = J_n^T,$$

which corresponds with (5.24).
Example 5.2.4 The second unitary operator we consider is the dilation operator $\mathcal{D}_B$ on $L^2(\mathbb{R}^n)$, with $B \in \mathbb{R}^{n \times n}$ and $\det B \neq 0$. We derive
\[
(D_B^* \mu(p, q, t) D_B)[f](x) = e^{i(p, Bx)} e^{i(t + (p, q)/2)} f(x + B^{-1}q) \\
= e^{i(t(B^T p, x))} e^{i((t(B^T p, B^{-1}q)/2))} f(x + B^{-1}q) \\
= \mu(B^T p, B^{-1}q, t)[f](x).
\]
This shows that also $\mathcal{D}_B \in G$ for $B \in GL(n)$. Moreover, we have
\[
\nu(D_B) = \begin{pmatrix} B^T & 0 \\ 0 & B^{-1} \end{pmatrix}. \tag{5.38}
\]
Now, Theorem 5.2.2 states that the action of the dilation operator in the Wigner plane is given by
\[
A = \nu(D_B)^{-T} = \begin{pmatrix} B^T & 0 \\ 0 & B^{-1} \end{pmatrix}^{-T} = \begin{pmatrix} B^{-1} & 0 \\ 0 & B^T \end{pmatrix}.
\]
We observe that this result corresponds to the linear transformation that we derived in (5.27).

Example 5.2.5 The last unitary operator we consider here is the operator $\mathcal{C}_S$ with $S \in \mathbb{R}^{n \times n}$ symmetric, as defined in (5.28). We have already seen
\[
(C_S^* \mu(p, q, t) C_S)[f](x) = \mu(p + Sq, q, t)[f](x),
\]
for $t = 0$. A straightforward computation shows that this result also holds for $t \neq 0$. This result yields that $\mathcal{C}_S \in G$ for $S \in \mathbb{R}^{n \times n}$ symmetric. Furthermore, we have
\[
\nu(C_S) = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}. \tag{5.39}
\]
Theorem 5.2.2 can also be applied to this operator. This yields
\[
A = \nu(C_S)^{-T} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-T} = \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix},
\]
which is the same result we derived in (5.29).

We observe that the fractional Fourier transform on $L^2(\mathbb{R}^n)$ is a combination of the three unitary operators discussed in the previous examples. We have for $0 < |\alpha_i| < \pi, i = 1, \ldots, n$,
\[
\mathcal{F}_{\alpha_1, \ldots, \alpha_n} = C_{\alpha_1} \cdots C_{\alpha_n} C_{S(\alpha)} D_{B(\alpha)} \mathcal{F} C_{S(\alpha)} \tag{5.40},
\]
with
\[
S(\alpha) = \text{diag}(\cot \alpha_1, \ldots, \cot \alpha_n) \quad \text{and} \quad B(\alpha) = \text{diag}(\sin \alpha_1, \ldots, \sin \alpha_n).
\]
Starting from (5.40) a limit process determines the FRFT with \( \alpha_i = 0 \) or \( \alpha_i = \pi \), for some, for some \( i = 1, \ldots, N \).

The following theorem classifies all possible elements of \( Sp(n) \). A proof of this result can be found in [29, 99].

**Theorem 5.2.6 (Bruhat Decomposition)** Let \( G \) be the group as defined in (5.33) and let \( \nu \) be the anti-homomorphism from \( G \) onto \( Sp(n) \) as defined in (5.35). Then \( \nu \) is surjective. Moreover, let \( J_n, \nu(D_B) \) and \( \nu(C_S) \) be the real valued \( (n \times n) \) matrices as given in (5.25), (5.38) and (5.39) and let

\[
G_1 = \{\nu(C_S) \mid S \in \mathbb{R}^{n \times n}, S^T = S\}
\]

and

\[
G_2 = \{\nu(D_B) \mid B \in \mathbb{R}^{n \times n}, \det B \neq 0\},
\]

then \( Sp(n) \) is generated by \( G_1 \cup G_2 \cup \{J_n\} \).

This result is a corollary of the generalized Bruhat decomposition with respect to a suitable maximal parabolic subgroup [102].

The next corollary combines Theorem 5.2.2 and Theorem 5.2.6. It characterizes all unitary operators on \( L^2(\mathbb{R}^n) \) that correspond to linear transformations in the Wigner plane.

**Corollary 5.2.7** Let \( f, g \in L^2(\mathbb{R}^n) \). Then

\[
\mathcal{WV}[g](x, \omega) = \mathcal{WV}[f](T(x, \omega)),
\]

for some \( T \in Sp(n) \) if and only if

\[
g = C \mathcal{U}_1 \cdots \mathcal{U}_N f,
\]

with \( |C| = 1 \) and \( \mathcal{U}_i = C_S, \mathcal{U}_i = D_B \) or \( \mathcal{U}_i = F \), with \( S \in \mathbb{R}^{n \times n} \) symmetric and \( B \in \mathbb{R}^{n \times n} \) non-singular, for \( i = 1, \ldots, N \), and \( N \in \mathbb{N} \).

We omit the proof of this corollary since it follows immediately from Theorem 5.2.2 and Theorem 5.2.6 by observing that \( \nu(F)^{-T} = \nu(F), \nu(D_B)^{-T} = \nu(D_B^{-T}) \) and \( \nu(C_S)^{-T} = J_n^T \nu(C_S) J_n = \nu(FC_SF^*) \).

The classification presented in Corollary 5.2.7 also holds for the mixed Wigner distribution. For a unitary operator \( V \) on \( L^2(\mathbb{R}^n) \) that corresponds to a linear transformation \( A \) in the Wigner plane we also have

\[
\mathcal{WV}[Vf, Vg](x, \omega) = \mathcal{WV}[f, g](A(x, \omega)),
\]

(5.41)
with $A \in Sp(n)$. This relation holds by polarization, i.e.,
\[
\mathcal{WV}[Vf, Vg](x, \omega) = (\mathcal{WV}[Vf](x, \omega) + \mathcal{WV}[Vg](x, \omega) - \mathcal{WV}[V(f + g)](x, \omega)) / 2
\]
\[
= (\mathcal{WV}[f](A(x, \omega)) + \mathcal{WV}[g](A(x, \omega)) - \mathcal{WV}[f + g](A(x, \omega))) / 2
\]
\[
= \mathcal{WV}[f, g](A(x, \omega)),
\]
for real-valued $f, g \in L^2(\mathbb{R}^n)$. For complex-valued functions we have to deal with the real and complex part separately.

In Section 5.3 this relation is used to come to a representation formula for the unitary operators as discussed in Corollary 5.2.7.

### 5.2.2 The FRFT Generalized

As we have seen in (5.40) the fractional Fourier transform on $L^2(\mathbb{R}^n)$ can be decomposed into four unitary operators, namely a chirp multiplication, the Fourier transform, a dilation and again a chirp multiplication. Both the chirp multiplications and the dilation depend on a set of parameters $\alpha_1, \ldots, \alpha_n$, that determine the FRFT. Therefore, a natural generalization of the FRFT is given by

\[
\mathcal{F}_{\Gamma, \Delta} = C_{\Gamma} D_{\Delta} FC_{\Gamma},
\]

for some $|C| = 1, \Gamma, \Delta \in \mathbb{R}^{n \times n}$, both symmetric and with $\Delta$ non-singular. We observe, that $\Delta$ is not required to be symmetric in (5.26). Here we require the symmetry of $\Delta$ to obtain a symmetrical representation formula for the generalized FRFT.

We observe, that (5.42) generalizes the multi-dimensional FRFT, which was introduced in Section 5.1.2. Indeed, by taking

\[
\Gamma = \text{diag}(\cot \alpha_1, \ldots, \cot \alpha_n) \quad \text{and} \quad \Delta = \text{diag}(\sin \alpha_1, \ldots, \sin \alpha_n)
\]

the generalized FRFT with the definition of the multi-dimensional FRFT.

As a consequence of Corollary 5.2.7, we have for all operators $\mathcal{F}_{\Gamma, \Delta}$

\[
\mathcal{WV}[\mathcal{F}_{\Gamma, \Delta} f](x, \omega) = \mathcal{WV}[f](A(x, \omega)),
\]

for some $A \in Sp(n)$. Using (5.37), (5.38) and (5.39) we compute straightforwardly

\[
A = \nu(C_{\Gamma} D_{\Delta} FC_{\Gamma})^{-T} = \begin{pmatrix}
\Delta \\
\Delta \\
-\Delta \\
\end{pmatrix}
\]

Taking $\Gamma$ and $\Delta$ as in (5.43) we arrive at the matrix $A$ as given in (5.20).
A special property of the FRFT is that for its corresponding transformation in the Wigner plane we have \( A \in Sp(n) \cap SO(2n) \), the orthonormal symplectic group. One may ask whether the generalized FRFT is also related to an orthogonal transformation in the Wigner plane. The answer to this question is given in the following lemma.

**Lemma 5.2.8** Let \( \mathcal{F}_{\Gamma,\Delta} \) be the generalized FRFT as defined in (5.42), for certain symmetric real valued \((n \times n)\) matrices \( \Gamma \) and \( \Delta \). Then \( A \) as given by (5.44) is orthogonal if and only if

(i) \( \Delta^{-2} - \Gamma^2 = I \),

(ii) \( \Gamma \Delta^{-1} \) is symmetric.

**Proof**

We compute

\[
A^T A = \begin{pmatrix} X & Y \\ YT & Z \end{pmatrix},
\]

with

\[
X = \Gamma \Delta \Gamma - \Gamma \Delta \Gamma^2 \Delta \Gamma + \Delta^{-2} - \Delta^{-1} \Gamma \Delta \Gamma - \Gamma \Delta \Gamma \Delta^{-1},
\]

\[
Y = \Delta^{-1} \Gamma \Delta - \Gamma \Delta^2 - \Gamma \Delta \Gamma^2 \Delta,
\]

\[
Z = \Delta + \Delta \Gamma^2 \Delta.
\]

For orthonormal \( A \) we should have \( X = Z = I \) and \( Y = 0 \). The condition \( Z = I \) yields \( \Delta^{-1} Z \Delta^{-1} = \Delta^{-2} \), which equals (i). Obviously, Condition (i) is also sufficient to guarantee \( Z = I \). Substituting (i) into the matrix \( Y \) yields

\[
Y = 0 \iff \Gamma \Delta^{-1} = \Delta^{-1} \Gamma \iff \Gamma \Delta^{-1} = (\Gamma \Delta^{-1})^T.
\]

After substituting Condition (i) and (ii) in the matrix \( X \) we get \( X = I \). So for the equation \( X = I \) no further conditions are required. \( \square \)

We observe that Conditions (i) and (ii) in Lemma 5.2.8 are equivalent with

\[
(\Delta^{-1} + \Gamma)(\Delta^{-1} - \Gamma) = I.
\]

It follows from this relation, that we have \( n^2/2 + n \) degrees of freedom for choosing symmetric matrices \( \Gamma \) and \( \Delta \), such that the matrix \( A \) corresponding to \( \mathcal{F}_{\Gamma,\Delta} \) is orthogonal. Therefore, for higher dimensional function spaces we may expect more variety in the class of operators \( \mathcal{F}_{\Gamma,\Delta} \) that yield orthogonal symplectic transformations in the Wigner plane. For the one-dimensional case the one-parameter family of the FRFT turns out to be the only transformation up to a constant, that is in the class of generalized FRFT and that acts like an orthogonal transform in the Wigner plane.
Lemma 5.2.9 Let $\mathcal{F}_{\Gamma,\Delta}$ be the unitary operator on $L^2(\mathbb{R})$ as given in (5.42), with $\Gamma, \Delta \in \mathbb{R}$. Then $A = \nu(\mathcal{F}_{\Gamma,\Delta})^{-T}$ is orthonormal if and only if $\mathcal{F}_{\Gamma,\Delta} = C \mathcal{F}_\alpha$, for some $\alpha \in \mathbb{R}$ and $C$ with $|C| = 1$.

**Proof**

In the case that $\Gamma$ and $\Delta$ are scalars, the conditions in Lemma 5.2.8 reduce to

$$\Delta^{-2} = 1 + \Gamma^2.$$ 

This equation can be parameterized by taking $\Gamma = \cot \alpha$ and $\Delta = \sin \alpha$, for some $\alpha \in \mathbb{R}$. Substituting this parameterization into (5.42) leaves the FRFT $\mathcal{F}_\alpha$ up to a constant of absolute value 1, which does not affect $A$. \hfill \Box

As we expected from the considerations before Lemma 5.2.9, this lemma cannot be extended in a canonical way to higher dimensions. This is shown by the following example for $n = 2$. Moreover, by extending the example to higher dimensions in a natural way it follows that the preceding lemma can only hold for $\mathcal{F}_{\Gamma,\Delta} \in U(L^2(\mathbb{R}))$.

**Example 5.2.10** We consider $\mathcal{F}_{\Gamma,\Delta}$ on $L^2(\mathbb{R}^2)$, with

$$\Gamma = \begin{pmatrix} r_1^2 \cos^2 \alpha + r_2^2 \sin^2 \alpha & (r_1 - r_2) \cos \alpha \sin \alpha \\ (r_1 - r_2) \cos \alpha \sin \alpha & r_1^2 \sin^2 \alpha + r_2^2 \cos^2 \alpha \end{pmatrix} \quad \text{and}$$

$$\Delta = \begin{pmatrix} \rho_1^2 \cos^2 \alpha + \rho_2^2 \sin^2 \alpha & (\rho_1 - \rho_2) \cos \alpha \sin \alpha \\ (\rho_1 - \rho_2) \cos \alpha \sin \alpha & \rho_1^2 \sin^2 \alpha + \rho_2^2 \cos^2 \alpha \end{pmatrix}^{-1},$$

with $\alpha \in \mathbb{R}$ and $\rho_i^2 = 1 + r_i^2$, $i = 1, 2$. Then

$$\Delta^{-2} - \Gamma^2 = \begin{pmatrix} \rho_1^2 - r_1^2 & 0 \\ 0 & \rho_2^2 - r_2^2 \end{pmatrix} = I,$$

and

$$\Gamma \Delta^{-1} = \begin{pmatrix} r_1^2 \rho_1^2 \cos^2 \alpha + r_2^2 \rho_2^2 \sin^2 \alpha & (r_1 \rho_1 - r_2 \rho_2) \cos \alpha \sin \alpha \\ (r_1 \rho_1 - r_2 \rho_2) \cos \alpha \sin \alpha & r_1^2 \rho_1^2 \sin^2 \alpha + r_2^2 \rho_2^2 \cos^2 \alpha \end{pmatrix} = (\Gamma \Delta^{-1})^T.$$ 

Consequently, the matrices $\Gamma$ and $\Delta$ satisfy the conditions in Lemma 5.2.8. The orthogonal symplectic transformation in the Wigner plane, that corresponds to $\mathcal{F}_{\Gamma,\Delta}$ is now given by $A = U(\alpha)^T M U(\alpha)$, with

$$M = \begin{pmatrix} -r_1/\rho_1 & 0 & -1/\rho_1 & 0 \\ 0 & -r_2/\rho_2 & 0 & -1/\rho_2 \\ 1/\rho_1 & 0 & -r_1/\rho_1 & 0 \\ 0 & 1/\rho_2 & 0 & -r_2/\rho_2 \end{pmatrix}.$$
and

\[ U(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha & \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha & -\sin \alpha & \cos \alpha \\ \cos \alpha & \sin \alpha & \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha & -\sin \alpha & \cos \alpha \end{pmatrix}. \]

Resuming, we have extended the FRFT to a unitary transformation on \( L^2(\mathbb{R}^n) \) given by \( \mathcal{F}_{\Gamma,\Delta} \), where \( \Gamma, \Delta \in \mathbb{R}^{n \times n} \), both symmetric and \( \Delta \) non-singular. So the set of all generalizations of the FRFT on \( L^2(\mathbb{R}^n) \) of this kind are given by the set

\[ V_n = \{ \mathcal{F}_{\Gamma,\Delta} \mid \Gamma, \Delta \in \mathbb{R}^{n \times n} \text{ symmetric}, \det \Delta \neq 0 \}. \]

Furthermore, a subset of \( V_n \) is defined consisting of all \( \mathcal{F}_{\Gamma,\Delta} \in V_n \) that act like orthogonal transformations in the Wigner plane. This subset is given by

\[ W_n = \{ \mathcal{F}_{\Gamma,\Delta} \in V_n \mid \Delta^{-2} - \Gamma^2 = I, \Gamma \Delta = (\Gamma \Delta)^T \}. \]

For the FRFT we have \( \mathcal{F}_{\alpha_1,\ldots,\alpha_n} \in W_n \subset V_n \). Moreover, for the one-dimensional case we have

\[ W_1 = \{ C \mathcal{F}_{\alpha} \mid \alpha \in \mathbb{R}, |C| = 1 \} \]

and

\[ W_n \setminus \{ \mu \mathcal{F}_{\alpha_1,\ldots,\alpha_n} \mid \alpha_1, \ldots, \alpha_n \in \mathbb{R}, |\mu| = 1 \} \neq \emptyset, \]

for \( n \geq 2 \).

### 5.3 A Representation Formula

In this section we present a representation formula for all unitary operators \( \mathcal{V} \) on \( L^2(\mathbb{R}^n) \) for which there exists a transformation \( A \) on \( \mathbb{R}^{2n} \) such that

\[ \mathcal{W} \mathcal{V} \mathcal{V} f, \mathcal{V} g \mathcal{V} \mathcal{V} (x, \omega) = \mathcal{W} \mathcal{V} f, \mathcal{V} g \mathcal{V} (A(x, \omega)). \]  

We observe, that for the particular choice \( f = g \), (5.45) coincides with (5.23). We have already shown that (5.45) can only be realized for symplectic transformations \( A \). Therefore, we start with some properties of symplectic matrices.

Given a matrix \( A \in Sp(n) \), then we can represent \( A \) by its \( 2 \times 2 \) block decomposition

\[ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \]  

(5.46)
Since $A$ is symplectic, it has to satisfy (5.32). This yields for the block decomposition

$$A^{-1} = \begin{pmatrix} A_{22}^T & -A_{12}^T \\ -A_{21}^T & A_{11}^T \end{pmatrix},$$

or equivalently

$$A_{22}^T A_{11} - A_{12}^T A_{21} = I, \quad (5.48)$$

$$A_{11}^T A_{21} - A_{21}^T A_{11} = 0, \quad (5.49)$$

$$A_{22}^T A_{12} - A_{12}^T A_{22} = 0. \quad (5.50)$$

Using these relations we prove the following less known properties of symplectic matrices.

**Lemma 5.3.1** Let $A \in \text{Sp}(n)$ be given by its $2 \times 2$ block decomposition (5.46). Then the following relations hold

(i) $(A_{22}^T)^{-1}(\text{Ran}(A_{12}^T)) = \text{Ran}(A_{12})$,

(ii) $\dim A_{22}(\text{Ker}(A_{12})) = \dim \text{Ker}(A_{12})$,

(iii) $A_{22}(\text{Ker}(A_{12})) = (\text{Ran}(A_{12}))^\perp$,

with $\text{Ker}(B)$ and $\text{Ran}(B)$ denoting respectively the null space and range of a linear transformation $B$ and with $B^\perp(W)$ denoting the inverse image of a subspace $W$ under the linear transformation $B$.

**Proof**

Let $v \in (A_{22}^T)^{-1}(\text{Ran}(A_{12}^T))$. Then there exists an $u \in \mathbb{R}^n$ such that $A_{22}^T v + A_{12}^T u = 0$. Hence,

$$A^T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_{11}^T u + A_{21}^T v \\ 0 \end{pmatrix}.$$  

Since $A$ is symplectic, we can apply (5.47). This yields

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} \begin{pmatrix} A_{21}^T v + A_{11}^T u \\ 0 \end{pmatrix}.$$  

Consequently, $v = -A_{12} (A_{21}^T v + A_{11}^T u) \in \text{Ran}(A_{12})$. On the other hand, if $v \in \text{Ran}(A_{12})$, then there exists a $w \in \mathbb{R}^n$ such that $v = A_{12} w$. Using (5.50) we derive

$$A_{22}^T v = A_{22}^T A_{12} w = A_{12}^T A_{22} w \in \text{Ran}(A_{12}^T),$$

which proves Property (i).

In order to prove (ii), it is sufficient to show that, if $A_{22} u = 0$, for $u \in \text{Ker}(A_{12})$, then $u = 0$. Using (5.48), this follows from

$$u = I_n u = A_{11}^T A_{22} u - A_{21}^T A_{12} u = 0.$$
In [63] we have shown that, given a linear transformation $B$ in $\mathbb{R}^n$ and a linear subspace $V$ in $\mathbb{R}^n$, we have

$$\dim(B^T(V^\perp)) = \dim V^\perp \implies B^T(V^\perp) = (B^+(V))^\perp.$$ 

Now, replacing $B$ by $A^T_{22}$ and $V$ by $\text{Ran}(A^T_{12})$ yields

$$(A_{22}(\text{Ker}(A_{12})))^\perp = (A^T_{22})^{-1}(\text{Ran}(A^T_{12})) = \text{Ran}(A_{12}),$$

which proves Property (iii).

For deriving a representation formula we also need the following result.

**Lemma 5.3.2** Let $W$ be a subspace of $\mathbb{R}^n$ and let $B$ be a linear transformation on $\mathbb{R}^n$, such that $\dim(B(W)) = \dim(W) = d$. Then

$$\int_{W} f(Bx) \, dx = \frac{1}{q_w(B)} \int_{B(W)} f(x) \, dx, \quad \forall f \in S(\mathbb{R}^n),$$

with $q_w(B)$ the $d$-dimensional volume of the simplex generated by $Be_1, \ldots, Be_d$, with $e_1, \ldots, e_d$ an orthonormal basis in $W$.

The proof of this lemma is omitted, since it is straightforward. We observe, that $q_w(B)$ is positive. Furthermore, if $W$ is the null space and $B$ is non-singular, then by setting $q_w(B) = 1$ the definition of $q_w(B)$ is extended in a consistent way.

The last lemma we need to derive our representation formula is as follows.

**Lemma 5.3.3** Let $f \in S(\mathbb{R}^n)$ and $A \in \text{Sp}(n)$ with block decomposition (5.46). Furthermore, let $\dim \text{Ran}(A_{12}) = d > 0$. Then,

$$\int_{\text{Ker}(A_{12})} \int_{\mathbb{R}^n} f(u) e^{i(v,A^T_{22}u)} \, du \, dv = \frac{(2\pi)^{n-d}}{q_{\text{Ker}(A_{12})}(A_{22})} \int_{\text{Ran}(A_{12})} f(v) \, dv.$$

**Proof**

Since $\dim A_{22}(\text{Ker}(A_{12})) = \dim \text{Ker}(A_{12}) = n - d$, cf. Property (ii) of Lemma 5.3.1, we may apply Lemma 5.3.2. This yields

$$\int_{\text{Ker}(A_{12})} \left( \int_{\mathbb{R}^n} f(u) e^{i(v,A^T_{22}u)} \, du \right) \, dv = (2\pi)^{n/2} \int_{\text{Ker}(A_{12})} \hat{f}(A_{22}v) \, dv =$$

$$\frac{(2\pi)^{n/2}}{q_{\text{Ker}(A_{12})}(A_{22})} \int_{A_{22}(\text{Ker}(A_{12}))} \hat{f}(v) \, dv.$$
for all \( f \in S(\mathbb{R}^n) \) and linear subspaces \( W \) of \( \mathbb{R}^n \). By taking \( W = A_{22}(\text{Ker}(A_{12})) \) this result becomes

\[
\int_{A_{22}(\text{Ker}(A_{12}))} \hat{f}(v) \, dv = (2\pi)^{n/2-d} \int_{A_{22}(\text{Ker}(A_{12}))} f(v) \, dv.
\]

Since \( A_{22}(\text{Ker}(A_{12}))^\perp = \text{Ran}(A_{12}) \), we have, cf. Property (iii) of Lemma 5.3.1,

\[
\int_{A_{22}(\text{Ker}(A_{12}))} \hat{f}(v) \, dv = (2\pi)^{n/2-d} \int_{\text{Ran}(A_{12})} f(v) \, dv.
\]

In combination with (5.53) the latter result establishes the proof. \( \square \)

The starting point for the derivation of our representation formula is the characteristic function of the Wigner distribution (2.35). For the \( n \)-dimensional mixed Wigner distribution, we can also define a characteristic function by

\[
M[f,g](\theta,t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(u + t/2) g(u-t/2) e^{i(u,\theta)} \, du,
\]

or equivalently

\[
M[f,g](\theta,t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(u + t) g(u) e^{i(u+t/2,\theta)} \, du,
\]

with \( f, g \in L^2(\mathbb{R}^n) \). By the inverse Fourier transform we have

\[
f(x)g(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} M[f,g](\theta,x-y) e^{-i(\theta,x+y)/2} \, d\theta.
\]

For the \( n \)-dimensional mixed Wigner distribution we have

\[
\mathcal{W}[f](x,\omega) = (2\pi)^{-3n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} M[f](\theta,t) e^{-i(\theta,x)} e^{-i(t,\omega)} \, d\theta \, dt.
\]

Now, let \( \mathcal{V} \) be a unitary operator satisfying (5.45). It follows from (5.56) together with (5.45) that

\[
M[\mathcal{V}f, \mathcal{V}g] = M[f, g] \circ (A^{-1})^T.
\]

Combining (5.57) with (5.47) and (5.55) we arrive at

\[
\mathcal{V}[f](x) \, \mathcal{V}[g](y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} M[f,g]((A^{-1})^T(\theta, x-y)) e^{-i(\theta,x+y)/2} \, d\theta.
\]
Combining (5.57) with (5.47) and (5.55) we arrive at

\[ \mathcal{V}[f](x) \mathcal{V}[g](y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} M[f, g]((A^{-1})^T(\theta, x-y)) e^{-i(\theta, x+y)/2} d\theta \]

\[ = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u - A_{12} \theta/2 + A_{11}(x-y)/2) \times \]

\[ g(u + A_{12} \theta/2 - A_{11}(x-y)/2) E_0(u, \theta, x, y) du d\theta, \text{a.e...} \]

for all \( f, g \) in \( L^2(\mathbb{R}^n) \), with

\[ E_0(u, \theta, x, y) = \exp(i(A_{22} \theta - A_{21}(x-y), u) - i(\theta, x+y)/2). \]

This last relation only holds formally for general \( f, g \in L^2(\mathbb{R}^n) \), but it holds rigorously for \( f, g \in S(\mathbb{R}^n) \). Therefore, we assume \( f, g \in S(\mathbb{R}^n) \) from now on. After this derivation, we will show that the representation formula also hold for \( f \in L^2(\mathbb{R}^n) \).

By taking \( v = u - A_{11}(x+y)/2 \) in the previous result, we have

\[ \mathcal{V}[f](x) \mathcal{V}[g](y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u - A_{12} \theta/2 + A_{11}(x)) g(v + A_{12} \theta/2 + A_{11}(y)) \times \]

\[ \exp(i E_1(v, \theta, x, y)) dv d\theta, \]

with \( E_1(v, \theta, x, y) = (A_{22} \theta - A_{21}(x-y), v + A_{11}(x+y)/2 - (\theta, x+y)/2 \). Using Relations (5.48) - (5.50), we can write \( E_1 \) as

\[ E_1(v, \theta, x, y) = (A_{22} \theta - A_{21}(x-y), v) + (A_{12} \theta, A_{21}(x+y))/2 - (A_{21} x, A_{11} x)/2 + (A_{21} y, A_{11} y)/2. \]

Hence, \( \mathcal{V}[f](x) \mathcal{V}[g](y) \) can be rewritten as

\[ \mathcal{V}[f](x) \mathcal{V}[g](y) = e^{-i(A_{21} x, A_{11} x)/2} e^{i(A_{21} y, A_{11} y)/2} \mathcal{H}[f, g](x, y), \quad \text{(5.58)} \]

with

\[ \mathcal{H}[f, g](x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(v - A_{12} \theta/2 + A_{11} x) g(v + A_{12} \theta/2 + A_{11} y) \times \]

\[ e^{i(A_{12} \theta, A_{21} (x+y))/2} e^{i(A_{22} \theta - A_{21}(x-y), v)} dv d\theta. \]

Our aim is now to write \( \mathcal{H} \) in a possible degenerate form. If this is established, then the representation formula for \( \mathcal{V}f \) can be read off from this form. To come to such a form we
We are now in the position to apply Lemma 5.3.3 with respect to the function
\[ f(u - A_{12} \theta_1/2 + A_{11} x)g(v + A_{12} \theta_1/2 + A_{11} y) e^{i(A_{22} \theta_1 - A_{21} (x-y), v)}. \]

By applying this lemma, we arrive at
\[
\mathcal{H}[f, g](x, y) = \frac{1}{(2\pi)^{-d}} \int_{\text{Ran}(A_{12}^T)} \int_{\text{Ran}(A_{12})} f(u - A_{12} \theta_1/2 + A_{11} x) \times
\]
\[ g(v + A_{12} \theta_1/2 + A_{11} y) \times e^{i((A_{12} \theta_1, A_{21} (x+y))/2 + (A_{22} \theta_1 - A_{21} (x-y), v))} dv \, d\theta_1, \]

with \( d = \dim \text{Ran}(A_{12}). \) Since \( u \in \text{Ran}(A_{12}), \) we may substitute \( u = A_{12} w \) with \( w \in \text{Ran}(A_{12}^T), \) since \( A_{12} \) restricted to \( \text{Ran}(A_{12}^T) \) is a linear bijection onto \( \text{Ran}(A_{12}). \) We obtain
\[
\mathcal{H}[f, g](x, y) = C_A^2 \int_{\text{Ran}(A_{12}^T)} \int_{\text{Ran}(A_{12}^T)} f(A_{12} w - A_{12} \theta_1/2 + A_{11} x) \times
\]
\[ g(A_{12} w + A_{12} \theta_1/2 + A_{11} y) \times e^{i((A_{12} \theta_1, A_{21} (x+y))/2 + (A_{22} \theta_1 - A_{21} (x-y), A_{12} w))} dw \, d\theta_1, \]

with
\[ C_A = \sqrt{\frac{s(A_{12})}{(2\pi)^{d} q_{\text{Ker}(A_{12})}(A_{22})}}. \tag{5.59} \]

Here \( s(A_{12}) \) denotes the product of the nonzero singular values of \( A_{12}, \) or equivalently
\[ s(A_{12}) = q_{\text{Ran}(A_{12}^T)}(A_{12}). \]

Our next step is to substitute \( t_1 = w - \theta_1/2 \) and \( t_2 = w + \theta_1/2. \) Then, by using (5.48) - (5.50) one has
\[
(A_{12} (t_2 - t_1), A_{21} (x+y))/2 + (A_{22} \theta_1 - A_{21} (x-y), A_{12} w) =
\]
\[ (A_{21} (x-y), A_{12} (t_1 + t_2))/2 =
\]
\[ -(A_{22} t_1, A_{12} t_2)/2 + (A_{22} t_2, A_{12} t_2)/2 - (A_{12} t_1, A_{21} x) + (A_{21} y, A_{12} t_2). \]

With this result we can rewrite \( \mathcal{H}[f, g](x, y) \) in the degenerate form
\[
\mathcal{H}[f, g](x, y) = C_A^2 \mathcal{H}_0[f](x) \mathcal{H}_0[g](y). \tag{5.60} \]
\[(A_{12} (t_2 - t_1), A_{21} (x + y))/2 + (A_{22} (t_2 - t_1), A_{12} (t_1 + t_2))/2 - \]
\[-(A_{22} t_1, A_{12} t_1)/2 + (A_{22} t_2, A_{12} t_2)/2 - (A_{12} t_1, A_{21} x) + (A_{21} y, A_{12} t_2).\]

With this result we can rewrite \(\mathcal{H}[f, g](x, y)\) in the degenerate form
\[
\mathcal{H}[f, g](x, y) = C_A^2 \mathcal{H}_0[f](x) \mathcal{H}_0[g](y),
\]
with
\[
\mathcal{H}_0[f](x) = \int_{\text{Ran}(A_{T_2}^\dagger)} f(A_{12} t + A_{11} x) e^{-i \left((A_{22} t, A_{12} t)/2 + (A_{12} t, A_{21} x)\right)} \, dt.
\]

Finally, combining (5.58) and (5.60) yields the degenerate form for \(\mathcal{V}[f](x) \mathcal{V}[g](y)\)
\[
\mathcal{V}[f](x) \mathcal{V}[g](y) = C_A^2 \mathcal{H}_0[f](x) \mathcal{H}_0[g](y).
\]

In a natural way this derivation results into the definition of an operator \(\mathcal{F}_A\) that satisfies (5.45). We will define this operator on \(L^2(\mathbb{R}^n)\) and show that it indeed corresponds to the unitary operator we have been searching for.

**Definition 5.3.4** Let \(A \in \text{Sp}(n)\) with block decomposition (5.46). Then the linear operator \(\mathcal{F}_A\) on \(L^2(\mathbb{R}^n)\) is defined as follows. If \(\dim(\text{Ran}(A_{12})) > 0\), then
\[
\mathcal{F}_A[f](x) = C_A e^{-i \left(A_{T_1}^\dagger A_{21} x, z\right)/2} \times \\
\int_{\text{Ran}(A_{T_2}^\dagger)} f(A_{12} t + A_{11} x) e^{-i \left(A_{T_2}^\dagger A_{22} t, t\right)/2 - i \left(t, A_{T_2}^\dagger A_{21} x\right)} \, dt,
\]
for all \(f \in L^2(\mathbb{R}^n)\) and with \(C_A\) as given in (5.59). Furthermore, if \(\dim(\text{Ran}(A_{12})) = 0\) then
\[
\mathcal{F}_A[f](x) = \sqrt{\det A_{11}} \left| e^{-i \left(A_{T_1}^\dagger A_{21} x, z\right)/2} f(A_{11} x),
\]
for all \(f \in L^2(\mathbb{R}^n)\).

The main theorem of this section can be stated as follows.

**Theorem 5.3.5** Let \(A \in \text{Sp}(n)\) and \(\mathcal{F}_A\) be given as in Definition 5.3.4. Then
\[
\mathcal{W}[\mathcal{F}_A f, \mathcal{F}_A g](x, \omega) = W[f, g](A(x, \omega)),
\]
for all \(f, g \in L^2(\mathbb{R}^n)\).
for all \( f \in L^2(\mathbb{R}^n) \). The proof for \( \dim(\text{Ran}(A_{12})) > 0 \) is completed by assuming, that \( \mathcal{V} \) satisfies (5.45).

If \( \dim(\text{Ran}(A_{12})) = 0 \), we have \( A_{12} = 0 \). Then (5.48) and (5.49) yield, that \( A_{11} \) is non-singular and that \( A_{11}^{-1} = A_{22}^T \). Moreover, \( A_{11}^T A_{21} \) is symmetric. Using these observations, we compute the mixed Wigner distribution of \( \mathcal{F}_A f \) and \( \mathcal{F}_A g \) as follows.

\[
W[\mathcal{F}_A f, \mathcal{F}_A g](x, \omega) = \frac{|\det A_{11}|}{(2\pi)^n} \int_{\mathbb{R}^n} f(A_{11} x + A_{11} t/2) \times \\
g(A_{11} x - A_{11} t/2) e^{-i(A_{11}^T A_{21} x, t)} e^{-i(t, \omega)} dt = \\
(2\pi)^{-n} \int_{\mathbb{R}^n} f(A_{11} x + t/2) g(A_{11} x - t/2) e^{-i((A_{11}^T A_{21} x, A_{11}^{-1} t) + (A_{11}^{-1} t, \omega))} dt.
\]

Hence,

\[
W[\mathcal{F}_A f, \mathcal{F}_A g](x, \omega) = W[f, g](A_{11} x, A_{21} x + A_{22} \omega).
\]

This establishes the proof for \( \dim(\text{Ran}(A_{12})) = 0 \). \( \square \)

At the end of this section, we present two well-known examples of unitary operators, that satisfy (5.45).

**Example 5.3.6** We recall, that for a set of parameters \( \alpha_1, \ldots, \alpha_n \in (0, \pi) \) the \( n \)-dimensional fractional Fourier transform is given by

\[
\mathcal{F}_{\alpha_1, \ldots, \alpha_n}[f](x) = \frac{C_{\alpha} e^{i(B x, x)/2}}{\sqrt{(2\pi)^n |\sin \alpha_1 \cdots \sin \alpha_n|}} \int_{\mathbb{R}^n} f(u) e^{i((Bu, u)/2 - (Cx, u))} du, \tag{5.64}
\]

with \( B = \text{diag}(\cot \alpha_1, \ldots, \cot \alpha_n) \), \( C = \text{diag}(\csc \alpha_1, \ldots, \csc \alpha_n) \) and \( C_{\alpha} = C_{\alpha_1} \cdots C_{\alpha_n} \), where \( C_{\alpha} \) is given by (5.2). The symplectic matrix, that corresponds to this transform in the Wigner plane is given by the rotation matrix \( R_{\alpha_1, \ldots, \alpha_n} \) as given in (5.20). We observe, that in this particular case \( A_{12} \) is non-singular. This yields \( q_{\text{Ker}(A_{12})}(A_{22}) = 1 \) and \( s(A_{12}) = \det(A_{12}) \). Using these simplifications and the substitution \( u = A_{12} t + A_{11} x \), Formula (5.62) simplifies to

\[
\mathcal{F}_A[f](x) = \frac{e^{-i(A_{12}^{-1} A_{11} x, x)/2}}{(2\pi)^{n/2} \sqrt{\det A_{12}}} \int_{\mathbb{R}^n} f(u) e^{-i((A_{22} A_{12}^{-1} u, u)/2 - (x, A_{12}^{-1} u))} du.
\]

Taking \( A_{11} = A_{22} = \text{diag}(\cos \alpha_1, \ldots, \cos \alpha_n) \) and \( A_{12} = \text{diag}(- \sin \alpha_1, \ldots, - \sin \alpha_n) \), the latter representation formula turns into the \( n \)-dimensional FRFT as given in (5.64).
det($A_{12}$). Using these simplifications and the substitution $u = A_{12}t + A_{11}x$, Formula (5.62) simplifies to

$$\mathcal{F}_A[f](x) = \frac{e^{-i(A_{12}^{-1}A_{11}z, z)/2}}{(2\pi)^{n/2}\sqrt{\det A_{12}}} \int_{\mathbb{R}^n} f(u) e^{-i((A_{22}A_{12}^{-1}u, u)/2 - (x, A_{12}^{-1}u))} du.$$ 

Taking $A_{11} = A_{22} = \text{diag}(\cos \alpha_1, \ldots, \cos \alpha_n)$ and $A_{12} = \text{diag}(-\sin \alpha_1, \ldots, -\sin \alpha_n)$, the latter representation formula turns into the $n$-dimensional FRFT as given in (5.64).

**Example 5.3.7** The second example is the unitary operator on $L^2(\mathbb{R}^2)$, which corresponds in the Wigner plane to the symplectic matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$ 

Remark, that all matrices in the block decomposition of $A$ are singular.

It can be verified in a straightforward way, that $q_{\text{Ker}(A_{12})}(A_{22}) = 1$ and $s(A_{12}) = 1$. By substituting the block matrices of $A$ into (5.62), the unitary operator, we are dealing with, reads

$$\mathcal{F}_A[f](x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x_1, \xi) e^{-i\xi x_2} d\xi,$$ 

which is the one-dimensional Fourier-transform of $f(x_1, \cdot)$. We observe, that this operator can also be derived from (5.64) by taking $\alpha_1 \to 0$ and $\alpha_2 \to \pi/2$.

We observe that in [29] and [33] also a representation formula is presented for unitary operators that correspond to symplectic transformations in the Wigner plane. However, both references do not give a formula that can also handle symplectic transformations with a block decomposition, that consists of four singular block matrices, which is the case in the second example.
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