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MULTIVARIABLE BIG AND LITTLE q -JACOBI POLYNOMIALS*

JASPER V. STOKMAN†

Abstract. A four-parameter family of multivariable big q -Jacobi polynomials and a three-parameter family of multivariable little q -Jacobi polynomials are introduced. For both families, full orthogonality is proved with the help of a second-order q -difference operator which is diagonalized by the multivariable polynomials. A link is made between the orthogonality measures and R. Askey's q -extensions of Selberg's multidimensional beta-integrals.

Key words. big q -Jacobi polynomials, little q -Jacobi polynomials, BC_n -type Askey–Wilson polynomials, multivariable orthogonal polynomials, q -extensions of Selberg's multidimensional beta-integrals

AMS subject classifications. 33D45, 33D80

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1. Introduction. In the one-variable case, big (resp. little) q -Jacobi polynomials depend apart from q on (essentially) three (resp. two) parameters. The big and little q -Jacobi polynomials are orthogonal with respect to inner products which are both given by a Jackson (q -) integral over a positive weight function. The associated orthogonality measures therefore have positive weights on infinitely many discrete mass points.

The one-variable big and little q -Jacobi polynomials are q -analogues of the classical Jacobi polynomials in the sense that when q tends to 1, the big and little q -Jacobi polynomials tend (up to a possible translation and dilation of the variable) to the classical Jacobi polynomials.

The families of one-variable big and little q -Jacobi polynomials are members of the Askey–Wilson hierarchy. The Askey–Wilson hierarchy consists of families of orthogonal polynomials which are joint eigenfunctions of a second-order q -difference operator. Some families can be obtained from others by limit transitions or by specializations of parameters. This induces the hierarchy structure between the families. In this point of view, the four-parameter family of Askey–Wilson polynomials is on top of the hierarchy and the families of big (resp. little) q -Jacobi polynomials are directly below the Askey–Wilson polynomials. Suitable limit transitions are known from the Askey–Wilson polynomials to the big (resp. little) q -Jacobi polynomials (cf. [12]). Furthermore, the little q -Jacobi polynomials can be obtained from the big q -Jacobi polynomials by a suitable limit transition.

Recently, Koornwinder introduced in [11] a multivariable (BC_n -type) generalization of the family of Askey–Wilson polynomials by extending the three-parameter family of Macdonald polynomials of type (BC_n, B_n) to a five-parameter family of orthogonal polynomials. Four of these parameters play the same role as in the one-variable case, while the fifth parameter is an extra deformation parameter. The BC_n -type Askey–Wilson polynomials are again joint eigenfunctions of a second q -difference operator. Koornwinder remarked in [11] that the whole Askey–Wilson hierarchy could probably be generalized to the BC_n case as well as the limit transitions between the families.

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In this paper, a four-parameter family of multivariable BC_n -type big q -Jacobi polynomials and a three-parameter family of multivariable BC_n -type little q -Jacobi polynomials are introduced which have the following properties:

- (1) compared with the one-variable case, there is an extra deformation parameter involved;
- (2) the multivariable big (resp. little) q -Jacobi polynomials are joint eigenfunctions of a second-order q -difference operator;
- (3) the multivariable big (resp. little) q -Jacobi polynomials are mutually orthogonal with respect to an inner product, which is essentially given by a multidimensional Jackson integral over a positive weight function;
- (4) in a paper by the author and Koornwinder (cf. [18]), limit transitions from multivariable Askey–Wilson polynomials to multivariable big (resp. little) q -Jacobi polynomials and from multivariable big q -Jacobi polynomials to multivariable little q -Jacobi polynomials are proved which generalize the limit transitions in the one-variable case, and it is proved that the multivariable big (resp. little) q -Jacobi polynomials are q -analogues of generalized Jacobi polynomials (see [19]) (which are related with BC_n -type Heckman–Opdam polynomials by a suitable change of variables).

This paper is organized as follows. In section 2, the definitions of big and little q -Jacobi polynomials in one variable are given. In section 3, we will consider two formal limits of the BC_n -type Askey–Wilson polynomials, which generalize the limits from Askey–Wilson polynomials to big (resp. little) q -Jacobi polynomials in the one-variable case. We will obtain two second-order q -difference operators D_B (resp. D_L), and in section 4, it will be proved that D_B and D_L are triangular with respect to the basis of monomial symmetric functions. In section 5, the multivariable big and little q -Jacobi polynomials will be introduced. We will use techniques introduced by Macdonald in [15] to prove full orthogonality of the polynomials. First, it will be proved that the big (resp. little) q -Jacobi polynomials are joint eigenfunctions of D_B (resp. D_L) by proving the self-adjointness of D_B (resp. D_L). Full orthogonality of the big (resp. little) q -Jacobi polynomials will then be a consequence of the fact that the eigenvalues are sufficiently different.

Furthermore, it will be shown in section 5 that for special values of the extra deformation parameter, the multidimensional Jackson integrals over the weight functions are essentially the q -extensions of Selberg’s multidimensional beta-integrals which were introduced by Askey [3]. Askey’s conjectured evaluations of these multidimensional Jackson integrals have recently been proved [6], [8], [10]. Section 6 contains some proofs which were omitted in section 5.

Notation and conventions. Throughout this paper, we work with a fixed $q \in (0, 1)$. $\mathbb{N} = \{1, 2, \dots\}$ denotes the natural numbers and $\mathbb{N}_0$ denotes the natural numbers together with 0. The convention will be used that $\prod_{i=l}^k a_i = 1$ if $k < l$ for $k, l \in \mathbb{N}_0$. If there is no confusion possible, the dependence on the parameters will be omitted in the formulas.

2. One variable big and little q -Jacobi polynomials. Let $a, b \in \mathbb{R}$, $a < b$, and f be a function defined on the points $\{aq^k, bq^k \mid k \in \mathbb{N}_0\}$. Define the Jackson (q -) integral of f over $[a, b]$ by

$$\begin{aligned} \int_a^b f(x) d_q x &:= \int_0^b f(x) d_q x - \int_0^a f(x) d_q x, \\ \int_0^b f(x) d_q x &:= (1 - q) \sum_{k=0}^{\infty} f(bq^k) bq^k, \end{aligned}$$

provided that the infinite sums in the definition of the q -integral from 0 to a and in the definition of the q -integral from 0 to b are absolutely convergent. In the special case that $a = bq^{k+1}$ for some $k \in \mathbb{N}_0$, we have

$$(2.1) \quad \int_{bq^{k+1}}^b f(x) d_q x = (1-q) \sum_{m=0}^k f(bq^m) bq^m,$$

so we can then use (2.1) as definition of the q -integral from bq^{k+1} to b without worrying about convergence.

Define the q -shifted factorial by

$$(a; q)_b := \frac{(a; q)_\infty}{(q^b a; q)_\infty}, \quad (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k),$$

for $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \mathbb{N}_0$ such that $q^b a \neq q^{-k}$ for all $k \in \mathbb{N}_0$. For $l \in \mathbb{N}_0$, we set $(a; q)_l := \prod_{k=0}^{l-1} (1 - aq^k)$. Denote

$$(a_1, \dots, a_r; q)_b := \prod_{j=1}^r (a_j; q)_b.$$

Let $c, d > 0$, $a \in (-c/dq, 1/q)$, and $b \in (-d/cq, 1/q)$ or $a = cz$ and $b = -d\bar{z}$ with $z \in \mathbb{C} \setminus \mathbb{R}$. Denote V_B^q for the set of parameters (a, b, c, d) which satisfy these conditions. Define

$$(2.2) \quad w_B(x; a, b, c, d; q) := \frac{(qx/c, -qx/d; q)_\infty}{(qax/c, -qbx/d; q)_\infty};$$

then $w_B(x; a, b, c, d; q)$ is positive for $x \in [-d, c]$, and

$$(2.3) \quad \langle f, g \rangle_{B,1,q}^{a,b,c,d} := \int_{-d}^c f(x)g(x)w_B(x; a, b, c, d; q) d_q x, \quad f, g \in \mathbb{R}[x],$$

is a well-defined inner product on $\mathbb{R}[x]$.

DEFINITION 2.1. *The big q -Jacobi polynomials $\{P_m(x; a, b, c, d; q) \mid m \in \mathbb{N}_0\}$ are defined by the following two conditions:*

- (1) $P_m(x)$ is a monic polynomial of degree m in x ;
- (2) $\langle P_m(x), x^l \rangle_{B,1} = 0$ if $l < m$.

Consequently, the big q -Jacobi polynomials are mutually orthogonal with respect to $\langle \cdot, \cdot \rangle_{B,1}$. Explicit expressions for the big q -Jacobi polynomials are given by

$$P_m(x; a, b, c, d; q) = \frac{(qa; q)_m (-qad/c; q)_m}{(q^{m+1}ab; q)_m (qa/c)^m} {}_3\phi_2 \left[\begin{matrix} q^{-m}, q^{m+1}ab, qxa/c \\ qa, -qad/c \end{matrix}; q, q \right],$$

with the q -hypergeometric series defined by

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z \right] := \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_{r+1}; q)_k}{(b_1, \dots, b_r, q; q)_k} z^k$$

(cf. [2]).

Note that $P_m(x; a, b, c/d, 1; q) = d^{-m} P_m(dx; a, b, c, d; q)$, so the big q -Jacobi polynomials depend (apart from q) essentially on a, b , and the ratio c/d . The second-order q -difference operator

$$(2.4) \quad \left(D_{1,q}^{a,b,c,d} f \right)(x) := q \left(a - \frac{c}{qx} \right) \left(b + \frac{d}{qx} \right) (f(qx) - f(x)) \\ + \left(1 - \frac{c}{x} \right) \left(1 + \frac{d}{x} \right) (f(q^{-1}x) - f(x)) \quad (f \in \mathbb{R}[x])$$

is diagonalized by the big q -Jacobi polynomials

$$(2.5) \quad \left(D_{1,q}^{a,b,c,d} P_m(\cdot; a, b, c, d; q) \right)(x) = a_m^{a,b,q} P_m(x; a, b, c, d; q) \quad \forall m \in \mathbb{N}_0$$

with eigenvalues

$$(2.6) \quad a_m^{a,b,q} := qab(q^m - 1) + (q^{-m} - 1).$$

Note that D_1 is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{B,1}$ because $\{P_m(x) \mid m \in \mathbb{N}_0\}$ is an orthogonal basis of $\mathbb{R}[x]$ with respect to $\langle \cdot, \cdot \rangle_{B,1}$ which consists of eigenfunctions of D_1 .

The little q -Jacobi polynomials can be introduced in a similar way. Let $0 < a < 1/q$ and $b < 1/q$, and denote by V_L^q the set of parameters (a, b) which satisfy these conditions. Define

$$(2.7) \quad v_L(x; a, b; q) := \frac{(qx; q)_\infty}{(qbx; q)_\infty} x^\alpha \quad (a = q^\alpha);$$

then $v_L(x; a, b; q)$ is positive for $x \in [0, 1]$ and

$$(2.8) \quad \langle f, g \rangle_{L,1,q}^{a,b} := \int_0^1 f(x)g(x)v_L(x; a, b; q) d_q x, \quad f, g \in \mathbb{R}[x],$$

is an inner product on $\mathbb{R}[x]$.

DEFINITION 2.2. *The little q -Jacobi polynomials $\{p_m(x; a, b; q) \mid m \in \mathbb{N}_0\}$ are defined by the following two conditions:*

- (1) $p_m(x) \in \mathbb{R}[x]$ is a monic polynomial of degree m in x ;
- (2) $\langle p_m(x), x^l \rangle_{L,1} = 0$ if $l < m$.

Consequently, the little q -Jacobi polynomials are mutually orthogonal with respect to $\langle \cdot, \cdot \rangle_{L,1}$. Explicit expressions for the little q -Jacobi polynomials are given by

$$p_m(x; a, b; q) := \frac{(-1)^m q^{\binom{m}{2}} (qa; q)_m}{(q^{m+1}ab; q)_m} {}_2\phi_1 \left[\begin{matrix} q^{-m}, q^{m+1}ab \\ qa \end{matrix}; q, qx \right]$$

(cf. [1]). The little q -Jacobi polynomials are eigenfunctions of the q -difference operator $D_{1,q}^{b,a,1,0}$ with the same eigenvalues as in the big q -Jacobi case ((2.5) and (2.6)):

$$(2.9) \quad \left(D_{1,q}^{b,a,1,0} p_m(\cdot; a, b; q) \right)(x) = a_m^{a,b,q} p_m(x; a, b; q) \quad \forall m \in \mathbb{N}_0,$$

so $D_{1,q}^{b,a,1,0}$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{L,1,q}^{a,b}$.

Remark 2.3. In this section, we defined the one-variable big and little q -Jacobi polynomials as monic polynomials because the multivariable generalizations will be monic. However, it is more common to normalize the big and little q -Jacobi polynomials differently. The big q -Jacobi polynomials are usually defined by

$$\tilde{P}_m(x; a, b, c, d; q) = {}_3\phi_2 \left[\begin{matrix} q^{-m}, q^{m+1}ab, qxa/c \\ qa, -qad/c \end{matrix}; q, q \right]$$

and the little q -Jacobi polynomials are usually defined by

$$\tilde{p}_m(x; a, b; q) = {}_2\phi_1 \left[\begin{matrix} q^{-m}, q^{m+1}ab \\ qa \end{matrix}; q, qx \right].$$

For more details about the one-variable big and little q -Jacobi polynomials, see [1], [2], [7], and [13].

3. Formal limits of multivariable Askey–Wilson polynomials. Let A be the algebra of Laurent polynomials in the independent indeterminates $x_1, \dots, x_n$. The Weyl group W corresponding to the root system of type BC_n acts in a natural way on A . Let A^W be the subalgebra of A consisting of W -invariant Laurent polynomials. Let P^+ be the partitions of length $\leq n$, so

$$(3.1) \quad P^+ := \{\lambda = (\lambda_1, \dots, \lambda_n) \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}.$$

W acts on $\mathbb{Z}^n$ by sign changes and permutations of the coordinates. The monomials $\{\tilde{m}_\lambda \mid \lambda \in P^+\}$, with $\tilde{m}_\lambda := \sum_{\mu \in W\lambda} x^\mu$, form a basis of A^W . Let $a, b, c, d, t \in \mathbb{C}$ and define the weight function $\delta(x; a, b, c, d; q, t)$ by

$$\delta(x_1, \dots, x_n) := \delta^+(x_1, \dots, x_n) \delta^+(x_1^{-1}, \dots, x_n^{-1}),$$

$$\delta^+(x) := \prod_{i=1}^n \frac{(x_i^2; q)_\infty}{(ax_i, bx_i, cx_i, dx_i; q)_\infty} \prod_{1 \leq k < l \leq n} \frac{(x_k x_l^{-1}, x_k x_l; q)_\infty}{(tx_k x_l^{-1}, tx_k x_l; q)_\infty}.$$

Assume that $|a|, |b|, |c|, |d| \leq 1$ and that if a, b, c , and d are complex, then they appear in conjugate pairs. Assume furthermore that the pairwise products of a, b, c , and d are not equal to 1. Denote $du := du_1 \cdots du_n$ and $e^{iu} := (e^{iu_1}, \dots, e^{iu_n})$. Suppose that $t \in (0, 1)$; then

$$\langle f, g \rangle_{AW, t} := \int \cdots \int_{[-\pi, \pi]^n} f(e^{iu}) \overline{g(e^{iu})} \delta(e^{iu}; t) du, \quad f, g \in A^W,$$

is an Hermitian inner product on A^W . Define a partial order on P^+ in the following way: $\mu, \lambda \in P^+$. Then

$$(3.2) \quad \mu \leq \lambda \Leftrightarrow \sum_{j=1}^i \mu_j \leq \sum_{j=1}^i \lambda_j, \quad i = 1, \dots, n.$$

Remark 3.1. For the root system $R = R^+ \cup (-R^+)$ of type BC_n , choose the positive roots R^+ by

$$(3.3) \quad R^+ = \{e_i\}_{i=1}^n \cup \{e_i \pm e_j\}_{1 \leq i < j \leq n} \cup \{2e_i\}_{i=1}^n,$$

where $\{e_i\}_{i=1}^n$ is the standard orthonormal basis for $\mathbb{R}^n$, then P^+ coincides with the set of dominant weights, and $\lambda > \mu$ for $\lambda, \mu \in P^+$ iff $\lambda - \mu$ is a sum of positive roots (cf. [11]).

DEFINITION 3.2. Let $t \in (0, 1)$. The Askey–Wilson polynomials

$$\{Q_\lambda(x; a, b, c, d; q, t) \mid \lambda \in P^+\}$$

are defined by the following two conditions:

- (1) $Q_\lambda(t) = \tilde{m}_\lambda + \sum_{\mu < \lambda; \mu \in P^+} c_{\lambda, \mu}(t) \tilde{m}_\mu$, for certain $c_{\lambda, \mu}(t) \in \mathbb{C}$;
- (2) if $\mu < \lambda$ and $\mu \in P^+$, then $\langle Q_\lambda(t), \tilde{m}_\mu \rangle_{AW, t} = 0$.

Define a second-order q -difference operator $D_{AW, q, t}^{a, b, c, d}$ by

$$(D_{AW} f)(x) := \sum_{i=1}^n (\psi_i(x)(T_{q, i} f - f)(x) + \phi_i(x)(T_{q^{-1}, i} f - f)(x))$$

for $f \in A^W$, with

$$(3.4) \quad (T_{q, i} f)(x) := f(x_1, \dots, x_{i-1}, qx_i, x_{i+1}, \dots, x_n),$$

the q -shift in the i th component, and with $\psi_i(x; a, b, c, d; q, t)$ and $\phi_i(x; a, b, c, d; q, t)$ given by

$$\psi_i(x) := \frac{(1 - ax_i)(1 - bx_i)(1 - cx_i)(1 - dx_i)}{(1 - x_i^2)(1 - qx_i^2)} \prod_{l \neq i} \frac{(1 - tx_i x_l)(1 - tx_i x_l^{-1})}{(1 - x_i x_l)(1 - x_i x_l^{-1})}$$

$$\phi_i(x) := \psi_i(x_1^{-1}, \dots, x_n^{-1}).$$

Koornwinder proved the following theorem in [11].

THEOREM 3.3. Let $t \in (0, 1)$. Define $b_\lambda(a, b, c, d; q, t)$ for $\lambda \in P^+$ by

$$b_\lambda := \sum_{j=1}^n (q^{-1} abcd t^{2n-j-1} (q^{\lambda_j} - 1) + t^{j-1} (q^{-\lambda_j} - 1));$$

then $D_{AW, t} Q_\lambda(t) = b_\lambda(t) Q_\lambda(t)$ for all $\lambda \in P^+$ and $\langle Q_\lambda(t), Q_\mu(t) \rangle_{AW, t} = 0$ if $\lambda \neq \mu$.

For the one-variable case ($n = 1$), explicit expressions of the Askey–Wilson polynomials $\{Q_m(x; a, b, c, d; q) \mid m \in \mathbb{N}_0\}$ are given by

$$Q_m(x; a, b, c, d; q) = \frac{(ab, ac, ad; q)_m}{a^m (q^{m-1} abcd; q)_m} {}_4\phi_3 \left[\begin{matrix} q^{-m}, q^{m-1} abcd, ax, ax^{-1} \\ ab, ac, ad \end{matrix}; q, q \right]$$

(cf. [4], [13]) and the following limit transitions hold:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \left(\frac{\epsilon(cd)^{\frac{1}{2}}}{q^{\frac{1}{2}}} \right)^m Q_m \left(\frac{q^{\frac{1}{2}} x}{\epsilon(cd)^{\frac{1}{2}}}; \epsilon a(qd/c)^{\frac{1}{2}}, \epsilon^{-1}(qc/d)^{\frac{1}{2}}, -\epsilon^{-1}(qd/c)^{\frac{1}{2}}, -\epsilon b(qc/d)^{\frac{1}{2}}; q \right) \\ = P_m(x; a, b, c, d; q), \end{aligned}$$

$$\lim_{\epsilon \rightarrow 0} \left(\frac{\epsilon}{q^{\frac{1}{2}}} \right)^m Q_m \left(\frac{q^{\frac{1}{2}} x}{\epsilon}; \epsilon q^{\frac{1}{2}} b, \epsilon^{-1} q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}} a; q \right) = p_m(x; a, b; q).$$

(See [12, Propositions 6.1 and 6.2] and take into account that the Askey–Wilson polynomials used in those limit transitions are written as function of $(x + x^{-1})/2$ and that the polynomials used in [12] are not monic.)

The most obvious generalizations of these two limits to the n -variable case give two new second-order q -difference operators and a new set of eigenvalues.

Let $x = (x_1, \dots, x_n)$ and denote $cx := (cx_1, \dots, cx_n)$ for $c \in \mathbb{C}$; then we have the following limits for the big q -Jacobi case:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \psi_i \left(\frac{q^{\frac{1}{2}}x}{\epsilon(cd)^{\frac{1}{2}}}; \epsilon a(qd/c)^{\frac{1}{2}}, \epsilon^{-1}(qc/d)^{\frac{1}{2}}, -\epsilon^{-1}(qd/c)^{\frac{1}{2}}, -\epsilon b(qc/d)^{\frac{1}{2}}; q, t \right) \\ = h_i(x; a, b, c, d; q, t), \end{aligned}$$

with $h_i(x; a, b, c, d; q, t)$ given by

$$(3.5) \quad h_i(x; a, b, c, d; q, t) := qt^{n-1} \left(a - \frac{c}{qx_i} \right) \left(b + \frac{d}{qx_i} \right) \prod_{l \neq i} \frac{x_l - tx_i}{x_l - x_i},$$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \phi_i \left(\frac{q^{\frac{1}{2}}x}{\epsilon(cd)^{\frac{1}{2}}}; \epsilon a(qd/c)^{\frac{1}{2}}, \epsilon^{-1}(qc/d)^{\frac{1}{2}}, -\epsilon^{-1}(qd/c)^{\frac{1}{2}}, -\epsilon b(qc/d)^{\frac{1}{2}}; q, t \right) \\ = g_i(x; c, d; q, t) \end{aligned}$$

with $g_i(x; c, d; q, t)$ given by

$$(3.6) \quad g_i(x; c, d; q, t) := \left(1 - \frac{c}{x_i} \right) \left(1 + \frac{d}{x_i} \right) \prod_{l \neq i} \frac{x_i - tx_l}{x_i - x_l},$$

and

$$\lim_{\epsilon \rightarrow 0} b_\lambda \left(\epsilon a(qd/c)^{\frac{1}{2}}, \epsilon^{-1}(qc/d)^{\frac{1}{2}}, -\epsilon^{-1}(qd/c)^{\frac{1}{2}}, -\epsilon b(qc/d)^{\frac{1}{2}}; q, t \right) = a_\lambda(a, b; q, t),$$

with

$$(3.7) \quad a_\lambda(a, b; q, t) := \sum_{j=1}^n (qabt^{2n-j-1}(q^{\lambda_j} - 1) + t^{j-1}(q^{-\lambda_j} - 1)).$$

For the little q -Jacobi case, we have the following limits:

$$\lim_{\epsilon \rightarrow 0} \psi_i \left(\frac{q^{\frac{1}{2}}x}{\epsilon}; \epsilon q^{\frac{1}{2}}b, \epsilon^{-1}q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}a; q, t \right) = h_i(x; b, a, 1, 0; q, t),$$

$$\lim_{\epsilon \rightarrow 0} \phi_i \left(\frac{q^{\frac{1}{2}}x}{\epsilon}; \epsilon q^{\frac{1}{2}}b, \epsilon^{-1}q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}a; q, t \right) = g_i(x; 1, 0; q, t),$$

and

$$\lim_{\epsilon \rightarrow 0} b_\lambda \left(\epsilon q^{\frac{1}{2}}b, \epsilon^{-1}q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}a; q, t \right) = a_\lambda(a, b; q, t).$$

Therefore, define the q -difference operator $D_{n,q,t}^{a,b,c,d}$ by

$$(3.8) \quad (D_n f)(x) := \sum_{j=1}^n (h_j(x)(T_{q,j}f - f)(x) + g_j(x)(T_{q^{-1},j}f - f)(x));$$

then D_{AW} tends to $D_{n,q,t}^{a,b,c,d}$ (resp. to $D_{n,q,t}^{b,a,1,0}$) in the two limits we have just considered, and the eigenvalues $\{b_\lambda(a, b, c, d; q, t) \mid \lambda \in P^+\}$ tend to $\{a_\lambda(a, b; q, t) \mid \lambda \in P^+\}$. For $n = 1$, $D_{1,q,t}^{a,b,c,d}$ and $D_{1,q,t}^{b,a,1,0}$ correspond with the second-order q -difference operators for which the one-variable big (resp. little) q -Jacobi polynomials are joint eigenfunctions ((2.4) and (2.5) (resp. (2.9))), and $\{a_m(a, b; q, t) \mid m \in \mathbb{N}_0\}$ is exactly the corresponding set of eigenvalues (formula (2.6)). Therefore, we denote

$$(3.9) \quad D_{B,q,t}^{a,b,c,d} := D_{n,q,t}^{a,b,c,d}$$

and

$$(3.10) \quad D_{L,q,t}^{a,b} := D_{n,q,t}^{b,a,1,0}.$$

In section 5, we will see that the multivariable big (resp. little) q -Jacobi polynomials are joint eigenfunctions of D_B (resp. D_L) with eigenvalues $\{a_\lambda \mid \lambda \in P^+\}$. In [18], it is shown that the formal limit transitions of the second-order q -difference operator D_{AW} that we discussed in this section can be used to prove limit transitions from multivariable Askey–Wilson polynomials to multivariable big and little q -Jacobi polynomials.

Remark 3.4. Van Diejen mentioned similar limit transitions in [5] but did not look for eigenfunctions of the newly obtained q -difference operators. In his terminology, the limit transitions correspond to sending the center of mass in an n -particle-difference Calogero–Moser system with trigonometric potentials (Hamiltonian given by D_{AW}) to infinity.

4. Triangularity of the second-order q -difference operator $D_{n,q,t}^{a,b,c,d}$. Note that

$$(4.1) \quad \Delta(x)^{-1}(T_{t,i}\Delta)(x) = \prod_{l \neq i} \frac{x_l - tx_i}{x_l - x_i},$$

where $\Delta(x)$ is the Vandermonde determinant $\Delta(x) := \prod_{1 \leq i < j \leq n} (x_i - x_j)$. Therefore, we can rewrite the second-order q -difference operator $\tilde{D}_n$ (given by (3.8)) in the following form:

$$(4.2) \quad (D_n f)(x) = \Delta(x)^{-1}(\tilde{D}_n f)(x),$$

with

$$(4.3) \quad (\tilde{D}_n f)(x) := \sum_{i=1}^n \left(\tilde{h}_i(x)(T_{q,i}f - f)(x) + \tilde{g}_i(x)(T_{q^{-1},i}f - f)(x) \right),$$

$$(4.4) \quad \tilde{h}_i(x) = q \left(a - \frac{c}{qx_i} \right) \left(b + \frac{d}{qx_i} \right) t^{n-1}(T_{t,i}\Delta)(x),$$

$$(4.5) \quad \tilde{g}_i(x) = \left(1 - \frac{c}{x_i} \right) \left(1 + \frac{d}{x_i} \right) t^{n-1}(T_{t^{-1},i}\Delta)(x).$$

Denote $\mathbb{C}[x_1, \dots, x_n]$ for the $\mathbb{C}$ -algebra of polynomials in the variables $x_1, \dots, x_n$. We have the following result.

LEMMA 4.1.

$$\tilde{D}_n(\mathbb{C}[x_1, \dots, x_n]) \subseteq \mathbb{C}[x_1, \dots, x_n].$$

Proof. For $f \in \mathbb{C}[x_1, \dots, x_n]$, define the backward partial q -derivative in the i th coordinate by

$$(D_q^{i,-} f)(x) := \frac{(f - T_{q,i} f)(x)}{(1 - q)x_i}.$$

Note that $D_q^{i,-}$ maps $\mathbb{C}[x_1, \dots, x_n]$ into itself. Now it can easily be checked that

$$(\tilde{D}_n f)(x) = \sum_{j=1}^n \left(A_j(x) \left(T_{q^{-1},j} \left((D_q^{j,-})^2 f \right) \right)(x) + B_j(x) \left(T_{q^{-1},j} (D_q^{j,-} f) \right)(x) \right),$$

with

$$A_j(x) = (1 - q)^2 q^{-2} (qax_j - c)(qbx_j + d)t^{n-1} (T_{t,j} \Delta)(x),$$

$$B_j(x) = \frac{(1 - q)}{q} t^{n-1} \left(\left(x_j + (d - c) - \frac{cd}{x_j} \right) (T_{t^{-1},j} \Delta)(x) - \left(q^2 abx_j + (qad - qbc) - \frac{cd}{x_j} \right) (T_{t,j} \Delta)(x) \right).$$

The lemma follows because $((T_{t,j} - T_{t^{-1},j}) \Delta)(x) \in \mathbb{C}[x_1, \dots, x_n]$ is divisible by x_j in $\mathbb{C}[x_1, \dots, x_n]$. $\square$

Let S_n be the permutation group of $\{1, \dots, n\}$. S_n acts on $\mathbb{C}[x_1, \dots, x_n]$ by permutation of the variables $x_1, \dots, x_n$. Denote $\mathbb{C}[x_1, \dots, x_n]^{S_n}$ for the subalgebra (over $\mathbb{C}$) of symmetric polynomials.

For $\lambda \in P^+$, define the symmetric monomial function m_λ by $m_\lambda(x) := \sum_{\mu \in S_n \lambda} x^\mu$, with $x^\mu := x_1^{\mu_1} \dots x_n^{\mu_n}$ and $w\lambda := (\lambda_{w^{-1}(1)}, \dots, \lambda_{w^{-1}(n)})$. Then $\{m_\lambda \mid \lambda \in P^+\}$ is a $\mathbb{C}$ -basis for $\mathbb{C}[x_1, \dots, x_n]^{S_n}$.

A second basis is given by the Schur functions $\{s_\lambda \mid \lambda \in P^+\}$, where

$$s_\lambda(x) := \Delta(x)^{-1} \sum_{w \in S_n} \det(w) x^{w(\lambda + \delta)},$$

where $\det(w)$ is the determinant of the linear map $w : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $w(e_i) := e_{w(i)}$ ($i = 1, \dots, n$) for an arbitrary basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$, and where

$$(4.6) \quad \delta := (n - 1, n - 2, \dots, 1, 0) \in P^+.$$

Let $\lambda \in P^+$; then

$$(4.7) \quad s_\lambda = m_\lambda + \sum_{\mu < \lambda; \mu \in P^+} c_{\lambda, \mu} m_\mu$$

for certain $c_{\lambda, \mu} \in \mathbb{R}$. See [17] for more details about Schur functions.

PROPOSITION 4.2. *Let $a, b, c, d, t \in \mathbb{C}$ and $\lambda \in P^+$. Then*

$$D_n m_\lambda = a_\lambda m_\lambda + \sum_{\mu < \lambda; \mu \in P^+} d_{\lambda, \mu} m_\mu$$

for certain $d_{\lambda, \mu} \in \mathbb{C}$, with $a_\lambda = a_\lambda(a, b; q, t)$ given by (3.7). $d_{\lambda, \mu}$ and a_λ depend polynomially on a, b, c, d , and t .

Proof. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^n$. Let S_n act on $\mathbb{R}^n$ by permutation of the basis $\{e_1, \dots, e_n\}$. Define

$$\tilde{P} := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n\},$$

and give $\tilde{P}$ the same partial order as P^+ (see (3.2)). For $\mu \in \mathbb{Z}^n$, define

$$J_\mu := \sum_{w \in S_n} \det(w) x^{w\mu}.$$

Then $J_\mu = 0$ unless $\mu = w(\nu + \delta)$ for certain $w \in S_n$ and $\nu \in \tilde{P}$, and in that case, we have $J_\mu = \det(w) J_{\nu + \delta}$. Write $\tilde{D}_n = \psi_+ + \psi_-$ with

$$(\psi_+ f)(x) := \sum_{j=1}^n \tilde{h}_j(x) (T_{q,j} f - f)(x), \quad (\psi_- f)(x) := \sum_{j=1}^n \tilde{g}_j(x) (T_{q^{-1},j} f - f)(x).$$

Let $S_n^\lambda \subseteq S_n$ be the stabilizer of $\lambda \in P^+$, so $S_n^\lambda := \{w \in S_n \mid w\lambda = \lambda\}$. Denote $T_{q,e_i} := T_{q,i}$ and $\sigma := e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Using the fact that

$$\Delta(x) = \sum_{w \in S_n} \det(w) x^{w\delta}$$

and

$$m_\lambda(x) = |S_n^\lambda|^{-1} \sum_{v \in S_n} x^{v\lambda} \quad (\lambda \in P^+),$$

we obtain, for $\lambda \in P^+$,

$$\begin{aligned} (\psi_+ m_\lambda)(x) &= \frac{1}{(n-1)!} \sum_{u \in S_n} q t^{n-1} \left(a - \frac{c}{q} x^{-u\sigma} \right) \left(b + \frac{d}{q} x^{-u\sigma} \right) (T_{t,u\sigma} \Delta)(x) \\ &\quad \times (T_{q,u\sigma} m_\lambda - m_\lambda)(x) \\ &= \frac{1}{(n-1)! |S_n^\lambda|} \sum_{u, v, w \in S_n} q t^{n-1} \left(a - \frac{c}{q} x^{-u\sigma} \right) \left(b + \frac{d}{q} x^{-u\sigma} \right) \det(w) \\ &\quad \times t^{\langle u\sigma, w\delta \rangle} (q^{\langle u\sigma, v\lambda \rangle} - 1) x^{w\delta + v\lambda} \\ &= \frac{1}{(n-1)! |S_n^\lambda|} \sum_{u', v' \in S_n} t^{\langle \sigma, v'\delta + \delta \rangle} (q^{\langle \sigma, v'u'\lambda \rangle} - 1) \left(qab \sum_{w \in S_n} \det(w) x^{w(\delta + u'\lambda)} \right. \\ &\quad \left. + (ad - bc) \sum_{w \in S_n} \det(w) x^{w(\delta + u'\lambda - v'^{-1}\sigma)} - \frac{cd}{q} \sum_{w \in S_n} \det(w) x^{w(\delta + u'\lambda - 2v'^{-1}\sigma)} \right). \end{aligned}$$

The third equality is obtained by the substitution of $u' = w^{-1}v$ and $v' = u^{-1}w$. Similarly, we have

$$(\psi_- m_\lambda)(x) = \frac{1}{(n-1)!|S_n^\lambda|} \sum_{u', v' \in S_n} t^{\langle \sigma, \delta - v' \delta \rangle} (q^{-\langle \sigma, v' u' \lambda \rangle} - 1) \left(\sum_{w \in S_n} \det(w) x^{w(\delta + u' \lambda)} \right. \\ \left. + (d-c) \sum_{w \in S_n} \det(w) x^{w(\delta + u' \lambda - v'^{-1} \sigma)} - cd \sum_{w \in S_n} \det(w) x^{w(\delta + u' \lambda - 2v'^{-1} \sigma)} \right).$$

Let $w, u', v' \in S_n$ and $\lambda \in P^+$; then $w(\delta + u' \lambda) \leq \delta + wu' \lambda \leq \delta + \lambda$, and

$$w(\delta + u' \lambda) = \delta + \lambda \Leftrightarrow w = (1) \quad \text{and} \quad u' \in S_n^\lambda.$$

Furthermore, we have $w(\delta + u' \lambda - v'^{-1} \sigma) \leq \delta + wu' \lambda - wv'^{-1} \sigma < \delta + wu' \lambda \leq \delta + \lambda$ and $w(\delta + u' \lambda - 2v'^{-1} \sigma) < \delta + \lambda$. Thus

$$\psi_\epsilon m_\lambda = \alpha_\lambda^\epsilon J_{\lambda+\delta} + \sum_{\mu < \lambda; \mu \in \bar{P}} \beta_{\lambda, \mu}^\epsilon J_{\mu+\delta}, \quad \epsilon = \pm,$$

with $\beta_{\mu, \lambda}^\epsilon \in \mathbb{C}$, $\alpha_\lambda^+ = \sum_{i=1}^n qabt^{2n-i-1}(q^{\lambda_i} - 1)$, and $\alpha_\lambda^- = \sum_{i=1}^n t^{i-1}(q^{-\lambda_i} - 1)$. Lemma 4.1 implies

$$\tilde{D}_n m_\lambda = \psi_+ m_\lambda + \psi_- m_\lambda = a_\lambda J_{\lambda+\delta} + \sum_{\mu < \lambda; \mu \in P^+} c_{\lambda, \mu} J_{\mu+\delta} \quad (\lambda \in P^+)$$

for certain $c_{\lambda, \mu} \in \mathbb{C}$. Formula (4.7) gives now the triangularity property.

Finally, note that the coefficients of J_ν ($\nu \in P^+$) in the expressions for $\psi_+ m_\lambda$ and $\psi_- m_\lambda$ depend polynomially on a, b, c, d , and t . Therefore, the coefficients of $D_n m_\lambda$ with respect to the basis of monomial symmetric functions depend polynomially on a, b, c, d , and t . $\square$

5. Multivariable big and little q -Jacobi polynomials. The $\mathbb{R}$ -algebra of symmetric polynomials in $x_1, \dots, x_n$ will be denoted by $\mathcal{A}^S$, so $\mathcal{A}^S := \mathbb{R}[x_1, \dots, x_n]^{S_n}$. We first define inner products $\langle \cdot, \cdot \rangle_{B, n, q, t}^{a, b, c, d}$ and $\langle \cdot, \cdot \rangle_{L, n, q, t}^{a, b}$ on $\mathcal{A}^S$, which generalize the inner products $\langle \cdot, \cdot \rangle_{B, 1}$ and $\langle \cdot, \cdot \rangle_{L, 1}$ ((2.3) and (2.8)) to the multivariable case.

For the big q -Jacobi case, we fix some $(a, b, c, d) \in V_B^q$ unless otherwise stated (V_B^q is defined in section 2). Define a symmetric bilinear form $\langle \cdot, \cdot \rangle_{B, n, q, t}^{a, b, c, d}$ for $t \in (0, 1)$ on $\mathcal{A}^S$ by

$$(5.1) \quad \langle f, g \rangle_{B, t} := \sum_{j=0}^n \langle f, g \rangle_{j, B, t}, \quad f, g \in \mathcal{A}^S,$$

with $\langle f, g \rangle_{j, B, t}$ given by the following multidimensional Jackson integral:

$$(5.2) \quad \int_{x_1=0}^c \int_{x_2=0}^{tx_1} \cdots \int_{x_j=0}^{tx_{j-1}} \int_{x_{j+1}=-dt^{n-j-1}}^0 \int_{x_{j+2}=-dt^{n-j-2}}^{qt^{-1}x_{j+1}} \cdots \int_{x_n=-d}^{qt^{-1}x_{n-1}} f(x)g(x)w_j(x;t)d_q x,$$

with $d_q x := d_q x_n \cdots d_q x_1$ and the weight function $w_j(x; a, b, c, d; q, t)$ given by

$$(5.3) \quad w_j(x; t) := d_j^\tau \left(\prod_{i=1}^n \frac{(qx_i/c, -qx_i/d; q)_\infty}{(qax_i/c, -qbx_i/d; q)_\infty} \right) \Delta_\tau^j(x),$$

with $t = q^\tau$ and

$$(5.4) \quad \Delta_\tau^j(x) := \Delta(x) \left(\prod_{\substack{1 \leq k < m \leq n \\ k \leq j}} |x_k|^{2\tau-1} \left(q^{1-\tau} \frac{x_m}{x_k}; q \right)_{2\tau-1} \right) \\ \times \prod_{j < k < m \leq n} |x_m|^{2\tau-1} \left(q^{1-\tau} \frac{x_k}{x_m}; q \right)_{2\tau-1}$$

and with $d_j^\tau = d_j^\tau(c, d)$ a positive constant given by

$$(5.5) \quad d_j^\tau := \prod_{\substack{1 \leq k < m \leq n \\ k \leq j}} |y_{mk}|^{2\tau-1} \frac{(q^{1-\tau} y_{mk}^{-1}; q)_{2\tau-1}}{(q^{1-\tau} y_{mk}; q)_{2\tau-1}}, \quad y_{mk} := \frac{-d}{c} q^{(n-m-k+1)\tau}.$$

In view of (2.1), we have that the measure associated with $\langle \cdot, \cdot \rangle_{j,B}$ has infinitely many discrete mass points given by the set

$$(5.6) \quad W_B^j := \{(x_1, \dots, x_n) \mid x_i = cq^{(i-1)\tau+k_i} \text{ if } i \leq j \text{ and } 0 \leq k_1 \leq \dots \leq k_j, \\ x_i = -dq^{(n-i)\tau+k_i} \text{ if } i > j \text{ and } 0 \leq k_n \leq \dots \leq k_{j+1}\}.$$

We have that $w_j(x; t) > 0$ for all $x \in W_B^j(t)$ and all $t \in (0, 1)$. Indeed, we only need to check that Δ_τ^j is positive because $(a, b, c, d) \in V_B^q$. For Δ_τ^j , it is easily checked that the terms of the form

$$(q^{1-\tau} x_p/x_r; q)_{2\tau-1} = \frac{(q^{1-\tau} x_p/x_r; q)_\infty}{(q^\tau x_p/x_r; q)_\infty}$$

are positive on the mass points since both the numerator and the denominator are positive on the mass points. Furthermore, note that for $x \in W_B^j$ we have the inequalities $x_1 > \dots > x_n$, so the Vandermonde determinant $\Delta(x)$ is positive for $x \in W_B^j$. Finally, it can be shown that w_j is bounded on W_B^j (in fact, we will see in the proof of Proposition 5.5 (section 6) that $w_j(x; q^\tau)$ is uniformly bounded on the set $\{(x, \tau) \mid \tau \in K, x \in W_B^j(q^\tau)\}$, with K an arbitrary compact subset of $(0, \infty)$), so $\langle \cdot, \cdot \rangle_{B,t}$ is a well-defined positive definite inner product for all $t \in (0, 1)$.

For the little q -Jacobi case, we fix $(a, b) \in V_L^q$ unless otherwise stated (V_L^q is defined in section 2). Define a symmetric bilinear form $\langle \cdot, \cdot \rangle_{L,n,q,t}^{a,b}$ on $\mathcal{A}^S$ by

$$(5.7) \quad \langle f, g \rangle_{L,t} := \int_{x_1=0}^1 \int_{x_2=0}^{tx_1} \cdots \int_{x_n=0}^{tx_{n-1}} f(x)g(x)v(x;t)d_q x, \quad f, g \in \mathcal{A}^S,$$

with the weight function $v(x; a, b; q, t)$ given by

$$(5.8) \quad v(x; t) := \left(\prod_{i=1}^n \frac{(qx_i; q)_\infty}{(qbx_i; q)_\infty} x_i^\alpha \right) \Delta_\tau(x) \quad (a = q^\alpha, t = q^\tau),$$

$$(5.9) \quad \Delta_\tau(x) := \Delta(x) \prod_{1 \leq i < j \leq n} |x_i|^{2\tau-1} \left(q^{1-\tau} \frac{x_j}{x_i}; q \right)_{2\tau-1}.$$

The measure associated with the inner product $\langle \cdot, \cdot \rangle_L$ has infinitely many discrete mass points given by the set

$$(5.10) \quad W_L := \{(x_1, \dots, x_n) \mid x_i = q^{(i-1)\tau + k_i} \text{ with } 0 \leq k_1 \leq \dots \leq k_n\}.$$

By a similar argument as in the big q -Jacobi case, we have that $v(x; t) > 0$ for all $x \in W_L(t)$ and all $t \in (0, 1)$ because $(a, b) \in V_L^q$. Furthermore, $v(x)(\prod_{i=1}^n x_i^{-\alpha})$ is bounded on W_L (we will see in the proof of Proposition 5.5 (section 6) that $v(x; q^\tau)(\prod_{i=1}^n x_i^{-\alpha})$ is uniformly bounded on $\{(x, \tau) \mid \tau \in K, x \in W_L(q^\tau)\}$, with K an arbitrary compact subset of $(0, \infty)$), so $\langle \cdot, \cdot \rangle_{L,t}$ is well defined because $\alpha > -1$ and positive definite for all $t \in (0, 1)$.

DEFINITION 5.1. *Let $t \in (0, 1)$. The big q -Jacobi polynomials*

$$\{P_\lambda^B(\cdot; a, b, c, d; q, t) \mid \lambda \in P^+\}$$

are defined by the following two conditions: let $\lambda \in P^+$; then

- (1) $P_\lambda^B(t) = m_\lambda + \sum_{\mu < \lambda; \mu \in P^+} c_{\lambda, \mu}(t) m_\mu$ for some $c_{\lambda, \mu}(t) \in \mathbb{R}$;
- (2) $\langle P_\lambda^B(t), m_\mu \rangle_{B,t} = 0$ if $\mu < \lambda$, $\mu \in P^+$.

DEFINITION 5.2. *Let $t \in (0, 1)$. The little q -Jacobi polynomials*

$$\{P_\lambda^L(\cdot; a, b; q, t) \mid \lambda \in P^+\}$$

are defined by the following two conditions: let $\lambda \in P^+$; then

- (1) $P_\lambda^L(t) = m_\lambda + \sum_{\mu < \lambda; \mu \in P^+} d_{\lambda, \mu}(t) m_\mu$ for some $d_{\lambda, \mu}(t) \in \mathbb{R}$;
- (2) $\langle P_\lambda^L(t), m_\mu \rangle_{L,t} = 0$ if $\mu < \lambda$, $\mu \in P^+$.

For $n = 1$, the inner products $\langle \cdot, \cdot \rangle_B$ and $\langle \cdot, \cdot \rangle_L$ are the same as the inner products given by (2.3) and (2.8), respectively. Thus for $n = 1$, the big (resp. little) q -Jacobi polynomials given by Definition 5.1 (resp. Definition 5.2) are exactly the one-variable big (resp. little) q -Jacobi polynomials as defined in section 2 (Definition 2.1 (resp. Definition 2.2)).

Observe that the multivariable big q -Jacobi polynomials depend (apart from q) only on a, b, t , and the ratio c/d . Indeed, let $f \in \mathcal{A}^S$ and define $f_d \in \mathcal{A}^S$ by $f_d(x) := f(dx)$; then

$$\langle f, g \rangle_{B,n,q,t}^{a,b,c,d} = d^{2\tau \binom{n}{2} + n} \langle f_d, g_d \rangle_{B,n,q,t}^{a,b,\frac{c}{d},1}, \quad f, g \in \mathcal{A}^S,$$

because

$$(5.11) \quad \int_0^\alpha h(u) d_q u = \alpha \int_0^1 h(\alpha u) d_q u \quad (\alpha \neq 0),$$

and $d_j^\tau(c, d) = d_j^\tau(c/d, 1)$, so $w_j(dx; a, b, c, d; q, t) = d^{2\tau \binom{n}{2}} w_j(x; a, b, c/d, 1; q, t)$. Therefore, we have that

$$d^{-|\lambda|} P_\lambda^B(dx; a, b, c, d; q, t) = P_\lambda^B\left(x; a, b, \frac{c}{d}, 1; q, t\right), \quad \lambda \in P^+,$$

where $|\lambda| := \sum_{i=1}^n \lambda_i$.

Remark 5.3. If we compare the weight functions w_j with the function w given by

$$(5.12) \quad w(x) := \left(\prod_{i=1}^n \frac{(qx_i/c, -qx_i/d; q)_\infty}{(qax_i/c, -qbx_i/d; q)_\infty} \right) \tilde{\Delta}_\tau(x),$$

with

$$(5.13) \quad \tilde{\Delta}_\tau(x) := \Delta(x) \prod_{1 \leq i < j \leq n} \operatorname{sgn}(x_i) |x_i|^{2\tau-1} \left(q^{1-\tau} \frac{x_j}{x_i}; q \right)_{2\tau-1},$$

and $\operatorname{sgn}(x_i) = 1$ if $x_i \geq 0$ and $= -1$ if $x_i < 0$, then we have that

$$(5.14) \quad \Delta_\tau^j(x) = \left(\prod_{k=1}^j \operatorname{sgn}(x_k)^{n-k} \right) \left(\prod_{j < k < m \leq n} \operatorname{sgn}(x_k) \psi_\tau \left(\frac{x_m}{x_k} \right) \right) \tilde{\Delta}_\tau(x)$$

with the function ψ_τ given by

$$(5.15) \quad \psi_\tau(z) := |z|^{2\tau-1} \frac{(q^{1-\tau} z^{-1}; q)_{2\tau-1}}{(q^{1-\tau} z; q)_{2\tau-1}}.$$

ψ_τ is a quasi-constant function, i.e., $\psi_\tau(qz) = \psi_\tau(z)$, so $w_j(x) = \phi_j(x)w(x)$ for some quasi-constant function ϕ_j ($T_{q,i}\phi_j = \phi_j$ for all i). The essential difference between $w(x)$ and $w_j(x)$ on W_B^j is that $w(x)$ can have poles on W_B^j , while $w_j(x)$ has no poles on W_B^j . Therefore, one can think of $\langle \cdot, \cdot \rangle_B$ as q -integration over the set of mass points $\cup_{j=0}^n W_B^j$ with respect to the weight function w , whereby one should resolve the poles of w on W_B^j when q -integrating over W_B^j by slightly modifying the weight function w (i.e., multiplying w with the quasi-constant function $d_j \phi_j$). The constants d_j in the definition of $w_j(x)$ will turn out to be crucial for the self-adjointness of D_B with respect to $\langle \cdot, \cdot \rangle_B$. Note that

$$(5.16) \quad d_j^\tau = \prod_{\substack{1 \leq k < m \leq n \\ k \leq j}} \psi_\tau(y_{mk}), \quad \text{with } y_{mk} := \frac{-d}{c} q^{(n-m-k+1)\tau},$$

so d_j^τ can also be expressed in terms of the quasi-constant function ψ_τ .

The inner products simplify when $\tau = k \in \mathbb{N}$. In that case, we have that $w_j(x) = w(x)$ on W_B^j for $j = 0, \dots, n$ (with $w(x)$ given by (5.12)). This follows from (5.14) and (5.16) since $\psi_k(z) = -\operatorname{sgn}(z)$. Furthermore, it holds that

$$(5.17) \quad \tilde{\Delta}_k(x) = (-1)^k \binom{n}{2}_q \binom{k}{2} \prod_{l=0}^{k-1} \prod_{i \neq j} (x_i - q^l x_j),$$

so $\tilde{\Delta}_k(x)$ is symmetric, and $\tilde{\Delta}_k(x) = 0$ if $x_i = q^l x_j$ for certain $i \neq j$ and certain $l \in \{0, \dots, k-1\}$. Thus when $\tau = k \in \mathbb{N}$, we have, for $f, g \in \mathcal{A}^S$,

$$(5.18) \quad \begin{aligned} \langle f, g \rangle_{B, q^k} &= \int_{x_1=-d}^c \int_{x_2=-d}^{x_1} \cdots \int_{x_n=-d}^{x_{n-1}} f(x) g(x) w(x; q^k) d_q x, \\ &= \frac{1}{n!} \int_{x_1=-d}^c \cdots \int_{x_n=-d}^c f(x) g(x) w(x; q^k) d_q x, \end{aligned}$$

$$(5.19) \quad \begin{aligned} \langle f, g \rangle_{L, q^k} &= \int_{x_1=0}^1 \int_{x_2=0}^{x_1} \cdots \int_{x_n=0}^{x_{n-1}} f(x) g(x) v(x; q^k) d_q x \\ &= \frac{1}{n!} \int_{x_1=0}^1 \cdots \int_{x_n=0}^1 f(x) g(x) v(x; q^k) d_q x. \end{aligned}$$

Here we have used that the weight functions $w(x; q^k)$ and $v(x; q^k)$ are zero for $x = (x_1, \dots, x_n)$ with $x_i = x_j$ for some $1 \leq i \neq j \leq n$, so if $x \in \mathbb{R}^n$ contributes to the support of the orthogonality measure, then the S_n orbit of $x \in \mathbb{R}^n$ has cardinality $n!$.

Finally, we may replace $w(x; a, b, c, d; q, q^k)$ in (5.18) by $\tilde{w}(x; a, b, c, d; q, q^k)$,

$$\tilde{w}(x) = \frac{n!}{\Gamma_{q^k}(n+1)} \left(\prod_{i=1}^n w_B(x_i) \right) \prod_{1 \leq i < j \leq n} x_i^{2k} \left(q^{1-k} \frac{x_j}{x_i}; q \right)_{2k},$$

and $v(x; a, b; q, q^k)$ in (5.19) by $\tilde{v}(x; a, b; q, q^k)$,

$$\tilde{v}(x) = \frac{n!}{\Gamma_{q^k}(n+1)} \left(\prod_{i=1}^n v_L(x_i) \right) \prod_{1 \leq i < j \leq n} x_i^{2k} \left(q^{1-k} \frac{x_j}{x_i}; q \right)_{2k},$$

with the q -gamma function $\Gamma_q(a)$ ($a \notin -\mathbb{N}_0$) defined by

$$\Gamma_q(a) := \frac{(q; q)_{a-1}}{(1-q)^{a-1}},$$

because w and v are symmetric functions such that

$$\tilde{w}(x) = \frac{n!}{\Gamma_{q^k}(n+1)} w(x) \prod_{i < j} \frac{x_i - q^k x_j}{x_i - x_j}, \quad \tilde{v}(x) = \frac{n!}{\Gamma_{q^k}(n+1)} v(x) \prod_{i < j} \frac{x_i - q^k x_j}{x_i - x_j},$$

and

$$\sum_{w \in S_n} \prod_{i < j} \frac{x_{w(i)} - q^k x_{w(j)}}{x_{w(i)} - x_{w(j)}} = \Gamma_{q^k}(n+1)$$

(cf. [8, p. 1479]).

Remark 5.4. For $t = q^k$, $k \in \mathbb{N}$, $\langle 1, 1 \rangle_L$ and $\langle 1, 1 \rangle_B$ are (up to the constant $1/\Gamma_{q^k}(n+1)$) the q -extensions of Selberg's multidimensional beta-integrals, introduced by Askey [3]. Askey's conjectured evaluations of these multidimensional q -integrals have recently been proved.

Let $t = q^k$, $k \in \mathbb{N}$, $a = q^\alpha$, and $b = q^\beta$; then Habsieger [8] and Kadell [10] have independently proved that

$$\langle 1, 1 \rangle_L = q^{k(\alpha+1)\binom{n}{2} + 2k^2\binom{n}{3}} \prod_{j=1}^n \frac{\Gamma_q(\alpha+1+(j-1)k) \Gamma_q(\beta+1+(j-1)k) \Gamma_q(jk)}{\Gamma_q(\alpha+\beta+2+(n+j-2)k) \Gamma_q(k)},$$

and Evans [6] has proved that

$$\begin{aligned} \langle 1, 1 \rangle_B &= q^{k^2\binom{n}{3} - \binom{k}{2}\binom{n}{2}} \prod_{j=1}^n \left(\frac{\Gamma_q(\alpha+1+(j-1)k) \Gamma_q(\beta+1+(j-1)k) \Gamma_q(jk)}{\Gamma_q(\alpha+\beta+2+(n+j-2)k) \Gamma_q(k)} \right. \\ &\quad \times \left. \frac{(-d/c; q)_\infty (-c/d; q)_\infty (cd)^{1+(j-1)k}}{((-d/c)q^{\alpha+1+(j-1)k}; q)_\infty ((-c/d)q^{\beta+1+(j-1)k}; q)_\infty (c+d)} \right). \end{aligned}$$

The inner products $\langle \cdot, \cdot \rangle_{B,t}$ and $\langle \cdot, \cdot \rangle_{L,t}$ depend continuously on $t \in (0, 1)$, in the following sense.

PROPOSITION 5.5. *Let $f, g \in \mathcal{A}^S$.*

- (1) $\langle f, g \rangle_{B,t}$ *is continuous in t for $t \in (0, 1)$.*
- (2) $\langle f, g \rangle_{L,t}$ *is continuous in t for $t \in (0, 1)$.*

We will omit the proof of the proposition in this section because it is rather long and technical. The proof will be given in section 6 (Proposition 6.1).

Let $\lambda \in P^+$. It is clear from the proof of Proposition 4.2, that the coefficients in the expansion of the symmetric polynomial $D_{n,q,t}^{a,b,c,d} m_\lambda$ with respect to the basis of monomials $\{m_\mu \mid \mu \in P^+\}$ are real for $(a, b, c, d) \in V_B^q$. Therefore, $D_B m_\lambda \in \mathcal{A}^S$. Similarly, we have that $D_L m_\lambda \in \mathcal{A}^S$.

THEOREM 5.6. *Let $t \in (0, 1)$.*

- (1) $D_{B,t}$ *is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{B,t}$.*
- (2) $D_{L,t}$ *is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{L,t}$.*

The proof will be given in section 6 (Theorem 6.5). The two essential ingredients for the proof are a special version of the q -partial integration rule (Lemma 6.4) and certain functional relations for the weight functions (Proposition 6.3). Self-adjointness is then a consequence of the fact that stock terms (which come from the q -partial integration rule) are zero or cancel. In the big q -Jacobi case, the specific positive constants d_j in the weight functions w_j are crucial for the cancellation of certain stock terms.

With the aid of Propositions 4.2 and 5.5 and Theorem 5.6, it is now straightforward to proof the main theorem. The proof is similar to proofs given by Macdonald in [15] and [16] (see also the second edition of [17]), and Koornwinder in [11].

THEOREM 5.7. *Let $t \in (0, 1)$.*

- (1) *Let $\lambda \in P^+$; then*

$$D_{B,t} P_\lambda^B(t) = a_\lambda(t) P_\lambda^B(t),$$

with a_λ given by (3.7). For $\lambda, \mu \in P^+$, we have

$$\langle P_\lambda^B(t), P_\mu^B(t) \rangle_{B,t} = 0 \quad \text{if } \lambda \neq \mu.$$

- (2) *Let $\lambda \in P^+$; then*

$$D_{L,t} P_\lambda^L(t) = a_\lambda(t) P_\lambda^L(t).$$

For $\lambda, \mu \in P^+$, we have

$$\langle P_\lambda^L(t), P_\mu^L(t) \rangle_{L,t} = 0 \quad \text{if } \lambda \neq \mu.$$

Proof. (1) Proposition 4.2 and Theorem 5.6 imply that $P_\lambda^B(t)$ is an eigenfunction of $D_{B,t}$ with eigenvalue $a_\lambda(t)$. Fix $(a, b, c, d) \in V_B^q$, fix $\mu, \lambda \in P^+$, $\mu \neq \lambda$, and fix $t \in (0, 1)$ such that $a_\lambda(a, b, q, t) \neq a_\mu(a, b, q, t)$. The self-adjointness of $D_{B,t}$ then gives that $\langle P_\lambda^B(t), P_\mu^B(t) \rangle_{B,t} = 0$. Note that $a_\lambda(a, b, q, t) \in \mathbb{R}[t]$ and $a_\lambda(a, b, q, t) = a_\mu(a, b, q, t)$ as polynomials in t iff $\lambda = \mu$ because $ab \notin \{q^{-2}, q^{-3}, \dots\}$. Therefore, (1) will be proved if we prove that $\langle P_\lambda^B(t), P_\mu^B(t) \rangle_{B,t}$ is continuous in t for $t \in (0, 1)$. This follows from Proposition 5.5. The proof of (2) is similar. $\square$

For $t = 1$, define

$$(5.20) \quad \langle f, g \rangle_{B,n,1} := \lim_{t \uparrow 1} \langle f, g \rangle_{B,n,t},$$

$$(5.21) \quad \langle f, g \rangle_{L,n,1} := \lim_{t \uparrow 1} \langle f, g \rangle_{L,n,t}$$

for $f, g \in \mathcal{A}^S$, provided that the limits exist. We then have the following proposition.

PROPOSITION 5.8. $\langle \cdot, \cdot \rangle_{B,n,1}$ and $\langle \cdot, \cdot \rangle_{L,n,1}$ (defined by (5.20) and (5.21)) are well-defined inner products on $\mathcal{A}^S$ and are explicitly given by

$$(5.22) \quad \langle f, g \rangle_{B,n,1} = \frac{1}{n!} \int_{x_1=-d}^c \cdots \int_{x_n=-d}^c f(x)g(x) \left(\prod_{i=1}^n w_B(x_i) \right) d_q x,$$

$$(5.23) \quad \langle f, g \rangle_{L,n,1} = \frac{1}{n!} \int_{x_1=0}^1 \cdots \int_{x_n=0}^1 f(x)g(x) \left(\prod_{i=1}^n v_L(x_i) \right) d_q x$$

for $f, g \in \mathcal{A}^S$. The corresponding multivariable big and little q -Jacobi polynomials (using Definitions 5.1 and 5.2 for $t = 1$) can be given explicitly in terms of the one-variable big and little q -Jacobi polynomials by the formulas ($\lambda \in P^+$)

$$(5.24) \quad P_\lambda^B(x; a, b, c, d; q, 1) = |S_n^\lambda|^{-1} \sum_{w \in S_n} \left(\prod_{i=1}^n P_{\lambda_{w^{-1}(i)}}(x_i; a, b, c, d; q) \right),$$

where P_n ($n \in \mathbb{N}_0$) are the one-variable big q -Jacobi polynomials (Definition 2.1) and $|S_n^\lambda| := \#\{w \in S_n \mid w\lambda = \lambda\}$, and

$$(5.25) \quad P_\lambda^L(x; a, b; q, 1) = |S_n^\lambda|^{-1} \sum_{w \in S_n} \left(\prod_{i=1}^n p_{\lambda_{w^{-1}(i)}}(x_i; a, b; q) \right),$$

where p_n ($n \in \mathbb{N}_0$) are the one-variable little q -Jacobi polynomials (Definition 2.2).

Proof. Fix $j \in \{0, \dots, n\}$. The set of mass points $W_B^j(q^\tau)$ (given by (5.6)) is in one-to-one correspondence with

$$(5.26) \quad V_j := \{p = (p_1, \dots, p_n) \mid 0 \leq p_1 \leq \dots \leq p_j, 0 \leq p_n \leq \dots \leq p_{j+1}\}$$

by the formula

$$(5.27) \quad x^{(j)}(p; \tau) = \left(cq^{p_1}, \dots, cq^{(j-1)\tau+p_j}, -dq^{(n-j-1)\tau+p_{j+1}}, \dots, -dq^{p_n} \right) \in W_B^j(q^\tau).$$

We first calculate $\lim_{\tau \downarrow 0} \Delta_\tau^j(x^{(j)}(p; \tau))$ for fixed $p \in V_j$, where $\Delta_\tau^j(x)$ is given by (5.4).

Rewrite Δ_τ^j as $\Delta_\tau^j = \rho_\tau^j D_\tau^j$ with

$$(5.28) \quad \rho_\tau^j(x) := \prod_{\substack{1 \leq i \leq k \leq n \\ i \leq j}} \frac{x_i - x_k}{x_i - q^\tau x_k} \prod_{j < l < m \leq n} \frac{x_m - x_l}{x_m - q^\tau x_l}$$

and

$$(5.29) \quad D_\tau^j(x) := \prod_{\substack{1 \leq i \leq k \leq n \\ i \leq j}} x_i^{2\tau} \left(q^{1-\tau} \frac{x_k}{x_i}; q \right)_{2\tau} \prod_{j < l < m \leq n} |x_m|^{2\tau} \left(q^{1-\tau} \frac{x_l}{x_m}; q \right)_{2\tau}.$$

For $p \in V_j$ and $\tau \in (0, \infty)$, define

$$(5.30) \quad g_{ik}^{(j)}(p, \tau) := \frac{(x_i^{(j)}(p; \tau) - x_k^{(j)}(p; \tau))}{(x_i^{(j)}(p; \tau) - q^\tau x_k^{(j)}(p; \tau))};$$

then we can write ρ_τ^j , evaluated at the mass point $x^{(j)}(p; \tau)$, as

$$(5.31) \quad \rho_\tau^j(x^{(j)}(p; \tau)) = \prod_{\substack{1 \leq i < k \leq n \\ i \leq j}} g_{ik}^{(j)}(p, \tau) \prod_{j < l < m \leq n} g_{ml}^{(j)}(p, \tau).$$

Let $p \in V_j$; then for $1 \leq i < k \leq n$ with $i \leq j$, we have

$$(5.32) \quad g_{ik}^{(j)}(p, 0) := \lim_{\tau \downarrow 0} g_{ik}^{(j)}(p, \tau) = \begin{cases} 1 & \text{if } x_i^{(j)}(p; 0) \neq x_k^{(j)}(p; 0), \\ \frac{k-i}{k-i+1} & \text{if } x_i^{(j)}(p; 0) = x_k^{(j)}(p; 0), \end{cases}$$

and for $j < l < m \leq n$, we have

$$(5.33) \quad g_{ml}^{(j)}(p, 0) := \lim_{\tau \downarrow 0} g_{ml}^{(j)}(p, \tau) = \begin{cases} 1 & \text{if } x_l^{(j)}(p; 0) \neq x_m^{(j)}(p; 0), \\ \frac{m-l}{m-l+1} & \text{if } x_l^{(j)}(p; 0) = x_m^{(j)}(p; 0). \end{cases}$$

Thus we have $\lim_{\tau \downarrow 0} \rho_\tau^j(x^{(j)}(p; \tau)) = n(x^{(j)}(p; 0))$ with $n(x)$ defined by

$$n(x) := \prod_{\{1 \leq l < m \leq n \mid x_l = x_m\}} \frac{m-l}{m-l+1}.$$

Fix $x \in \mathbb{R}^n$ with $x_1 \geq \cdots \geq x_n$, and denote $m(x) := \#\{w \in S_n \mid wx = x\}$. Let $(\lambda_1, \dots, \lambda_p)$ be the sequence of natural numbers such that $\lambda_1 + \cdots + \lambda_p = n$ and such that

$$x_1 = \cdots = x_{\lambda_1} > x_{\lambda_1+1} = \cdots = x_{\lambda_1+\lambda_2} > \cdots > x_{\lambda_1+\cdots+\lambda_{p-1}+1} = \cdots = x_n.$$

Then we have

$$\begin{aligned} n(x) &= \prod_{i=1}^p \left(\prod_{1 \leq l < m \leq \lambda_i} \frac{m-l}{m-l+1} \right) = \prod_{i=1}^p \left(\prod_{r=1}^{\lambda_i-1} \left(\frac{r}{r+1} \right)^{\lambda_i-r} \right) \\ &= \prod_{i=1}^p \frac{1}{\lambda_i!} = \frac{1}{m(x)}. \end{aligned}$$

It follows that $\lim_{\tau \downarrow 0} \rho_\tau^j(x^{(j)}(p; \tau)) = 1/m(x^{(j)}(p; 0))$ for all $p \in V_j$. Furthermore, it is easily checked that $\lim_{\tau \downarrow 0} D_\tau^j(x^{(j)}(p; \tau)) = 1$ for all $p \in V_j$. Hence we have

$$\lim_{\tau \downarrow 0} \Delta_\tau^j(x^{(j)}(p; \tau)) = 1/m(x^{(j)}(p; 0)), \quad p \in V_j.$$

Furthermore, we have $\lim_{\tau \downarrow 0} d_j^\tau = 1$ (with d_j^τ given by (5.5)), so for every $p \in V_j$,

$$\lim_{\tau \downarrow 0} w_j(x^{(j)}(p; \tau); q^\tau) = \frac{1}{m(x^{(j)}(p; 0))} \prod_{i=1}^n w_B(x_i^{(j)}(p; 0))$$

with $w_j(x)$ given by (5.3). Write $\langle f, g \rangle_{j, B, q^\tau}$ as a sum over $p \in V_j$ using formula (5.27); then we will see in Proposition 6.1 that we are allowed to pull the limit $\tau \downarrow 0$ through the infinite sum. Thus we obtain

$$(5.34) \quad \begin{aligned} \lim_{\tau \downarrow 0} \langle f, g \rangle_{j, B, q^\tau} &= \sum_{j=0}^n \int_{x_1=0}^c \int_{x_2=0}^{x_2} \cdots \int_{x_j=0}^{x_{j-1}} \int_{x_{j+1}=-d}^0 \int_{x_{j+2}=-d}^{qx_{j+1}} \\ &\quad \cdots \int_{x_n=-d}^{qx_{n-1}} f(x)g(x) \frac{1}{m(x)} \left(\prod_{i=1}^n w_B(x_i) \right) d_q x. \end{aligned}$$

(5.22) now follows by symmetrizing the right-hand side of this formula. Formula (5.24) then follows directly from the orthogonality of the one-variable big q -Jacobi polynomials. The proof of the proposition for the little q -Jacobi case is similar. $\square$

Note that Theorems 5.6 and 5.7 are still valid for $t = 1$. The two theorems are easy consequences of the corresponding results in the one-variable case. For $t = 1$, both theorems also follow by a continuity argument in t from the corresponding theorems for $t \in (0, 1)$.

Similarly, we can express the multivariable big (resp. little) q -Jacobi polynomials for $t = q$ in terms of one-variable big (resp. little) q -Jacobi polynomials.

PROPOSITION 5.9. *For $\lambda \in P^+$, define $\tilde{\lambda} \in P^+$ by $\tilde{\lambda} := \lambda + \delta$ (δ given by (4.6)).*

(1) *Let $(a, b, c, d) \in V_B^q$ and $\lambda \in P^+$; then*

$$P_\lambda^B(x; a, b, c, d; q, q) = \Delta(x)^{-1} \sum_{w \in S_n} \det(w) \left(\prod_{i=1}^n P_{\tilde{\lambda}_{w^{-1}(i)}}(x_i; a, b, c, d; q) \right).$$

(2) *Let $(a, b) \in V_L^q$ and $\lambda \in P^+$; then*

$$P_\lambda^L(x; a, b; q, q) = \Delta(x)^{-1} \sum_{w \in S_n} \det(w) \left(\prod_{i=1}^n p_{\tilde{\lambda}_{w^{-1}(i)}}(x_i; a, b; q) \right).$$

Proof. The proposition follows from (4.7), (5.17), (5.18), (5.19), and the orthogonality in the one variable case. $\square$

The families of multivariable big (resp. little) q -Jacobi polynomials for $t = 1$ and $t = q$ are more or less the trivial families since full orthogonality of the multivariable big (resp. little) q -Jacobi polynomials for $t = 1$ and $t = q$ can easily be deduced from the orthogonality in the one-variable case. From this point of view, we can think of t as an extra (continuous) deformation parameter linking these two trivial families of multivariable big (resp. little) q -Jacobi polynomials.

6. Some proofs. In this section, we give the proofs that we omitted in section 5. We start with the proof of Proposition 5.5.

PROPOSITION 6.1. *Let $f, g \in \mathcal{A}^S$.*

(1) *$\langle f, g \rangle_{B,t}$ is continuous in t for $t \in (0, 1]$, where for $t = 1$, the inner product is given by (5.22).*

(2) *$\langle f, g \rangle_{L,t}$ is continuous in t for $t \in (0, 1]$, where for $t = 1$, the inner product is given by (5.23).*

Proof. It is sufficient to prove continuity in τ for $\tau \in [0, \infty)$ ($t = q^\tau$). If $h(z, \tau)$ is a function such that $h(uq^k, \tau)$ is continuous in $\tau \in [0, \infty)$ for all $k \in \mathbb{N}_0$ ($u \neq 0$), then

$$\int_0^u h(z, \tau) d_q z = (1 - q) \sum_{k=0}^{\infty} h(uq^k, \tau) uq^k$$

will be continuous in τ if for every compact subset K of $[0, \infty)$, there exists a $\epsilon_K < 1$ such that

$$\sup_{(k, \tau) \in \mathbb{N}_0 \times K} |q^{k\epsilon_K} h(uq^k, \tau)| < \infty.$$

For $\tau \in [0, \infty)$, define

$$(6.1) \quad w_j(x; q^\tau) := d_j^\tau \left(\prod_{i=1}^n w_B(x_i) \right) \Delta_\tau^j(x),$$

where d_j^τ is given by (5.5) and $\Delta_\tau^j(x) := \rho_\tau^j(x) D_\tau^j(x)$ for $\tau \in [0, \infty)$, with $D_j(x)$ defined by (5.29) for $\tau \in [0, \infty)$ and with $\rho_\tau^j(x)$ defined by (5.28) for $\tau \in (0, \infty)$ and $\rho_0^j(x) := m(x)^{-1}$ for $\tau = 0$ (where $m(x) := \{w \in S_n \mid wx = x\}$). The inner product $\langle \cdot, \cdot \rangle_{B, n, q^\tau}$ for $\tau \in [0, \infty)$ can now be given by formulas (5.1) and (5.2) if we use the weight function $w_j(x; q^\tau)$ given by (6.1). (Indeed, w_j is exactly the weight function (5.3) for $\tau > 0$, and for $\tau = 0$ we have $D_0^j(x) = 1$ and $d_j^0 = 1$, so it then follows from formula (5.34).)

Therefore, in the big q -Jacobi case, it will be sufficient to prove that for every $K \subset [0, \infty)$ compact and all $j \in \{0, \dots, n\}$,

$$\sup_{(p, \tau) \in V_j \times K} |\Delta_\tau^j(x^{(j)}(p; \tau))| < \infty, \quad j = 0, \dots, n,$$

with V_j and $x^{(j)}(p; \tau)$ defined by (5.26) and (5.27), respectively. (Clearly, $d_j^\tau(c, d, q)$ is continuous in τ for $\tau \in [0, \infty)$.)

Similarly, in the little q -Jacobi case, we can take (5.7) as the definition of $\langle \cdot, \cdot \rangle_{L, n, q^\tau}$ for all $\tau \in [0, \infty)$ if we take the function $v(x; 1) := (\prod_{i=1}^n v_L(x_i)) \Delta_0(x)$ with $\Delta_0(x) := \rho_0^n(x) D_0^n(x) = m(x)^{-1}$ as the weight function for $\tau = 0$ in (5.7).

Thus in the little q -Jacobi case, it will be sufficient to prove that for arbitrary $K \subset [0, \infty)$ compact,

$$\sup_{(p, \tau) \in V_n \times K} |\Delta_\tau(\tilde{x}(p; \tau))| < \infty$$

(where $\tilde{x}(p; \tau) := (q^{p_1}, q^{\tau+p_2}, \dots, q^{(n-1)\tau+p_n})$) because if $-1 < \alpha < 0$, then the factor x_i^α in the weight function can be compensated by taking $\epsilon_{K, i} = -\alpha$ ($i = 1, \dots, n$).

We will prove that for all $j \in \{0, \dots, n\}$ and all $K \subset [0, \infty)$ compact,

$$(6.2) \quad \sup_{(p, \tau) \in V_j \times K} |\rho_\tau^j(x^{(j)}(p; \tau))| < \infty$$

and

$$(6.3) \quad \sup_{(p, \tau) \in V_j \times K} |D_\tau^j(x^{(j)}(p; \tau))| < \infty.$$

Then we are ready because the little q -Jacobi case follows from (6.2) and (6.3) with $j = n$ and $c = 1$. We use the expression for $\rho_\tau^{(j)}$ evaluated at a specific mass point $x^{(j)}(p; \tau) \in W_B^j(q^\tau)$ as was given in the proof of Proposition 5.8 (formula (5.31)),

$$(6.4) \quad \rho_\tau^j(x^{(j)}(p; \tau)) = \prod_{\substack{1 \leq i < k \leq n \\ i \leq j}} g_{ik}^{(j)}(p, \tau) \prod_{j < l < m \leq n} g_{ml}^{(j)}(p, \tau)$$

with $g_{rs}^{(j)}(p, \tau)$ defined by (5.30) for $\tau > 0$ and by (5.32) and (5.33) for $\tau = 0$.

Proof of (6.2). We look at the factors of the form $g_{rs}^{(j)}(l, \tau)$ in the expression for $\rho_\tau^j(x^{(j)}(l, \tau))$ (see (6.4)).

Case (1): $k \leq j < m$. We have

$$g_{km}^{(j)}(l, \tau) = \frac{1 + (d/c)q^{(n-m-k+1)\tau} q^{l_m - l_k}}{1 + (d/c)q^{(n-m-k+2)\tau} q^{l_m - l_k}}$$

for $l \in V_j$ and $\tau \in K$. Thus $\sup_{(l,\tau) \in V_j \times K} |g_{km}^{(j)}(l, \tau)| < \infty$ because the map $\psi : [0, \infty) \times K \rightarrow \mathbb{R}$ given by

$$\psi(x, \tau) = \frac{1 + (d/c)xq^{(n-m-k+1)\tau}}{1 + (d/c)xq^{(n-m-k+2)\tau}}$$

is continuous, and $\lim_{x \rightarrow \infty} \psi(x, \tau) = q^{-\tau}$ uniformly for $\tau \in K$.

Case (2): $k < m \leq j$. Let $l \in V_j$ and $\tau \in K$. We have

$$(6.5) \quad g_{km}^{(j)}(l, \tau) = \frac{1 - q^{(m-k)\tau} q^{l_m - l_k}}{1 - q^{(m-k+1)\tau} q^{l_m - l_k}}$$

if $\tau > 0$ and $l_m \geq l_k$ or if $\tau = 0$ and $l_m > l_k$. Furthermore, we have $g_{km}^{(j)}(l, 0) = (m-k)/(m-k+1)$ if $l_k = l_m$. We have to prove that $\sup_{(l,\tau) \in V_j \times K} |g_{km}^{(j)}(l, \tau)| < \infty$. First, consider the supremum over $V_j^0 \times K$, where V_j^0 is the subset of V_j defined by $V_j^0 := \{l \in V_j \mid l_k = l_m\}$. Since $g_{km}^{(j)}(l, \tau) = g_{km}(0, \tau)$ independently of $l \in V_j^0$, and since $g_{km}^{(j)}(0, \tau)$ is continuous in $\tau \in [0, \infty)$, we have $\sup_{(l,\tau) \in V_j^0 \times K} |g_{km}^{(j)}(l, \tau)| < \infty$. Furthermore, $\sup_{(l,\tau) \in V_j^1 \times K} |g_{km}^{(j)}(l, \tau)| < \infty$ with $V_j^1 := V_j \setminus V_j^0$ follows from the fact that the map $\psi_{km} : [0, q] \times K \rightarrow \mathbb{R}$ given by

$$\psi_{km}(x, \tau) := \frac{1 - q^{(m-k)\tau} x}{1 - q^{(m-k+1)\tau} x}$$

is continuous, and $[0, q] \times K$ is compact.

Case (3): $j < k < m$. Similar arguments as in case (2) gives uniform boundness of $g_{mk}^{(j)}(l, \tau)$ for $l \in V_j$ and $\tau \in K$.

Proof of (6.3). We examine the factors of the form $|x_r|^{2\tau} (q^{1-\tau} x_s / x_r; q)_{2\tau}$ in the expression for $D_B^\tau(x)$ for $x \in W_B^j(q^\tau)$ (see (5.29)).

Case (1): $k \leq j < m$. For $x \in W_B^j(q^\tau)$, we have $x_k = cq^{(k-1)\tau + l_k}$ and $x_m = -dq^{(n-m)\tau + l_m}$ for some $l_m, l_k \in \mathbb{N}_0$. Using the formula

$$q^{2k\tau} (q^{1-\tau-k} z; q)_{2\tau} = \frac{(q^\tau z^{-1}; q)_k}{(q^{-\tau} z^{-1}; q)_k} (q^{1-\tau} z; q)_{2\tau},$$

we get

$$x_k^{2\tau} \left(q^{1-\tau} \frac{x_m}{x_k}; q \right)_{2\tau} = \left(cq^{(k-1)\tau} \right)^{2\tau} \frac{(q^{\tau-l_m} w_{km}^{-1}; q)_{l_k}}{(q^{-\tau-l_m} w_{km}^{-1}; q)_{l_k}} (q^{1-\tau+l_m} w_{km}; q)_{2\tau}$$

with $w_{km} := (-d/c)q^{(n-m-k+1)\tau}$. Then

$$\left| \frac{(q^{\tau-l_m} w_{km}^{-1}; q)_{l_k}}{(q^{-\tau-l_m} w_{km}^{-1}; q)_{l_k}} \right| \leq 1$$

for all $l_k, l_m \in \mathbb{N}_0$ and $\tau \in K$ because

$$1 - q^{-\tau-l_m+i} w_{km}^{-1} \geq 1 - q^{\tau-l_m+i} w_{km}^{-1} > 0 \quad \forall \tau \in K, l_m \in \mathbb{N}_0, i \in \{0, \dots, l_k - 1\}.$$

Choose $N_K \in \mathbb{N}_0$ such that $2\tau \leq N_K$ for all $\tau \in K$. Then

$$\left| (q^{1-\tau+l_m} w_{km}; q)_{2\tau} \right| \leq \left| (q^{1-\tau+l_m} w_{km}; q)_{N_K} \right|,$$

and arguments similar to those in Case (2) of the proof of (6.2) show that

$$\sup_{(l_m, \tau) \in \mathbb{N}_0 \times K} \left| (q^{1-\tau+l_m} w_{km}; q)_{N_K} \right| < \infty.$$

Case (2): $k < m \leq j$. For $x \in W_B^j(q^\tau)$, we have that $x_k = cq^{(k-1)\tau+l_k}$ and $x_m = cq^{(m-1)\tau+l_m}$ for some $l_k, l_m \in \mathbb{N}_0$ with $l_k \leq l_m$, so

$$\left| x_k^{2\tau} \left(q^{1-\tau} \frac{x_m}{x_k}; q \right)_{2\tau} \right| = \left(cq^{(k-1)\tau} \right)^{2\tau} q^{2l_k\tau} \left| \frac{(q^{1-\tau} (q^{(m-k)\tau+l_m-l_k}); q)_\infty}{(q^{1+\tau} (q^{(m-k)\tau+l_m-l_k}); q)_\infty} \right|.$$

We have

$$\left| \frac{(q^{1-\tau} (q^{(m-k)\tau+l_m-l_k}); q)_\infty}{(q^{1+\tau} (q^{(m-k)\tau+l_m-l_k}); q)_\infty} \right| \leq 1$$

for $\tau \in K$ and $l_m, l_k \in \mathbb{N}_0$ with $l_k \leq l_m$ because

$$1 - q^{1+\tau} (q^{(m-k)\tau+l_m-l_k}) q^i \geq 1 - q^{1-\tau} (q^{(m-k)\tau+l_m-l_k}) q^i > 0$$

for all $\tau \in K$, all $l_m, l_k \in \mathbb{N}_0$ with $l_k \leq l_m$, and all $i \in \mathbb{N}_0$.

Case (3): $j < k < m$. Similar arguments as in case (2) give a uniform boundedness of

$$\left| |x_m|^{2\tau} \left(q^{1-\tau} \frac{x_k}{x_m}; q \right)_{2\tau} \right|$$

for $\tau \in K, x_k \in \{-dq^{(n-k)\tau+l_k}\}_{l_k \in \mathbb{N}_0}, x_m \in \{-dq^{(n-m)\tau+l_m}\}_{l_m \in \mathbb{N}_0}, l_m \leq l_k$. $\square$

We have the following corollary.

COROLLARY 6.2. *Let $f, g \in \mathcal{A}^S$.*

(1) $\langle D_{B,t} f, g \rangle_{B,t}$ is continuous in t for $t \in (0, 1]$.

(2) $\langle D_{L,t} f, g \rangle_{L,t}$ is continuous in t for $t \in (0, 1]$.

Proof. Let $\lambda \in P^+$. The coefficients in the expansion of the symmetric polynomial $D_{n,q,t}^{a,b,c,d} m_\lambda$ with respect to the basis of monomials $\{m_\mu \mid \mu \in P^+\}$ are continuous in $t \in (0, 1]$ for arbitrary fixed $a, b, c, d \in \mathbb{C}$ because they depend polynomially on t (Proposition 4.2). Now apply Proposition 6.1. $\square$

Define the forward and backward partial q -derivatives in the i th coordinate by

$$(6.6) \quad (D_q^{i,+} f)(x) := \frac{(T_{q^{-1},i} f - f)(x)}{(1-q)x_i} \quad \text{and} \quad (D_q^{i,-} f)(x) := \frac{(f - T_{q,i} f)(x)}{(1-q)x_i},$$

respectively. In the one-variable case, we will use the notation D_q^+ and D_q^- , respectively. $D_{n,q,t}^{a,b,c,d}$ can now be written in the following form:

$$(D_n f)(x) = \sum_{i=1}^n (p_i(x) (D_q^{i,-} f)(x) + q_i(x) (D_q^{i,+} f)(x)),$$

with

$$p_i(x; a, b, c, d; q, t) := \hat{h}(x_i; a, b, c, d; q) t^{n-1} \Delta(x)^{-1} (T_{t,i} \Delta)(x),$$

$$\hat{h}(y; a, b, c, d; q) := -q(1-q)y \left(a - \frac{c}{qy} \right) \left(b + \frac{d}{qy} \right)$$

and

$$q_i(x; c, d; q, t) := \hat{g}(x_i; c, d; q) t^{n-1} \Delta(x)^{-1} (T_{t^{-1}, i} \Delta)(x),$$

$$\hat{g}(y; c, d; q) := (1-q)y \left(1 - \frac{c}{y} \right) \left(1 + \frac{d}{y} \right).$$

We have the following result.

PROPOSITION 6.3. (1) *Let $i \in \{1, \dots, n\}$ and $j \in \{0, \dots, n\}$; then*

$$(6.7) \quad \begin{aligned} & (T_{q^{-1}, i}(p_i(\cdot; a, b, c, d; q, t)w_j(\cdot; a, b, c, d; q, t)))(x) \\ & = -q_i(x; c, d; q, t)w_j(x; a, b, c, d; q, t). \end{aligned}$$

(2) *Let $i \in \{1, \dots, n\}$; then*

$$(6.8) \quad \begin{aligned} & (T_{q^{-1}, i}(p_i(\cdot; b, a, 1, 0; q, t)v(\cdot; a, b; q, t)))(x) \\ & = -q_i(x; 1, 0; q, t)v(x; a, b; q, t). \end{aligned}$$

Proof. In Remark 5.3, we saw that $w_j(x) = \phi_j(x)w(x)$ with w given by (5.12),

$$w(x; a, b, c, d; q, t) = \left(\prod_{j=1}^n w_B(x_j; a, b, c, d; q) \right) \tilde{\Delta}_\tau(x),$$

and with ϕ_j a quasi-constant function (w_B given by (2.2)). Thus for (1), it is sufficient to prove (6.7) with w_j replaced by w . For (2), it will be sufficient to prove (6.8) with v replaced by

$$\left(\prod_{j=1}^n v_L(x_j; a, b; q) \right) \tilde{\Delta}_\tau(x)$$

(with v_L given by (2.7)).

For every $i \in \{1, \dots, n\}$, we have

$$\left(T_{q^{-1}, i} \left(\frac{(T_{t, i} \Delta)}{\Delta} \tilde{\Delta}_\tau \right) \right) (x) = \frac{(T_{t^{-1}, i} \Delta)(x)}{\Delta(x)} \tilde{\Delta}_\tau(x),$$

which follows from a straightforward calculation using (4.1). Thus the proposition follows from

$$\hat{h}(q^{-1}y; a, b, c, d; q)w_B(q^{-1}y; a, b, c, d; q) = -\hat{g}(y; c, d; q)w_B(y; a, b, c, d; q)$$

and

$$\hat{h}(q^{-1}y; b, a, 1, 0; q)v_L(q^{-1}y; a, b; q) = -\hat{g}(y; 1, 0; q)v_L(y; a, b; q). \quad \square$$

The self-adjointness of D_B with respect to $\langle \cdot, \cdot \rangle_B$ and the self-adjointness of D_L with respect to $\langle \cdot, \cdot \rangle_L$ can now be proved with the help of the following special

version of the q -partial integration rule. The proof is similar to the proof of the usual q -partial integration rule (cf. [13]).

LEMMA 6.4. *Let $\alpha \neq 0$ and let f_i be functions in one variable ($i = 1, \dots, 4$). Suppose that f_1 and f_4 are defined on $\{\alpha q^k \mid k \in \mathbb{N}_0\}$ and that f_2 and f_3 are defined on $\{\alpha q^k \mid k \in \mathbb{N}_0 \cup \{-1\}\}$. Suppose that*

$$\lim_{k \rightarrow \infty} (f_1(\alpha q^{k+1})f_2(\alpha q^k) + f_3(\alpha q^k)f_4(\alpha q^{k+1}))$$

exists and is equal to L . Then

$$\begin{aligned} \int_0^\alpha ((D_q^- f_1)(x)f_2(x) + (D_q^+ f_3)(x)f_4(x)) d_q x &= f_1(\alpha)f_2(q^{-1}\alpha) \\ &+ f_3(q^{-1}\alpha)f_4(\alpha) - L - \int_0^\alpha (f_1(x)(D_q^+ f_2)(x) + f_3(x)(D_q^- f_4)(x)) d_q x. \end{aligned}$$

THEOREM 6.5. *Let $t \in (0, 1]$.*

(i) $D_{B,t}$ *is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{B,t}$.*

(ii) $D_{L,t}$ *is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{L,t}$.*

Proof. (i) Fix $j \in \{0, \dots, n\}$. Define

$$W_j(\tau) := \{x \mid x_k = cq^{(k-1)\tau+l_k} (k \leq j), x_k = -dq^{(n-k)\tau+l_k} (k > j) \text{ and } l_k \in \mathbb{N}_0\}.$$

First, let us check that if $\tau \in (0, \infty) \setminus \cup_{p=1}^n (1/p)\mathbb{N}$, then $w_j(x; q^\tau) \neq 0$ for $x \in W_j(\tau)$ iff $x \in W_B^j(q^\tau)$.

Therefore, let $x \in W_j(\tau)$. Then $(q^{1-\tau}x_{k+1}/x_k; q)_\infty = 0$ if $1 \leq k < j$ and $l_k > l_{k+1}$, and $(q^{1-\tau}x_k/x_{k+1}; q)_\infty = 0$ if $j < k < n$ and $l_k < l_{k+1}$. Furthermore, if we assume that $\tau \in (0, \infty) \setminus \cup_{p=1}^n (1/p)\mathbb{N}$, then $(q^\tau x_m/x_k; q)_\infty \neq 0$ for $1 \leq k < m \leq j$ and $(q^\tau x_k/x_m; q)_\infty \neq 0$ for $j < k < m \leq n$. Therefore, if $\tau \in (0, \infty) \setminus \cup_{p=1}^n (1/p)\mathbb{N}$, then $\Delta_\tau^j(x) \neq 0$ for $x \in W_j(\tau)$ iff $x \in W_B^j(q^\tau)$, and so this also holds for $w_j(x; q^\tau)$.

As a consequence, we have that

$$(6.9) \quad \langle f, g \rangle_{j,B,t} = \int_{x_1=0}^c \cdots \int_{x_j=0}^{ct^{j-1}} \int_{x_{j+1}=-dt^{n-j-1}}^0 \cdots \int_{x_n=-d}^0 f(x)g(x)w_j(x;t)d_q x$$

for all $f, g \in \mathcal{A}^S$ if $\tau \in (0, \infty) \setminus \cup_{p=1}^n (1/p)\mathbb{N}$. We will prove self-adjointness for $\tau \notin \cup_{p=1}^n (1/p)\mathbb{N}$; then Corollary 6.2 asserts self-adjointness for $\tau \in [0, \infty)$. Therefore, fix $\tau \in (0, \infty) \setminus \cup_{p=1}^n (1/p)\mathbb{N}$. We will apply Lemma 6.4 repeatedly on the right-hand side of the formula

$$(6.10) \quad \begin{aligned} \langle D_{B,t} f, g \rangle_{B,t} &= \sum_{j=0}^n \sum_{l=1}^n \int_{x_1=0}^c \cdots \int_{x_j=0}^{ct^{j-1}} \int_{x_{j+1}=-dt^{n-j-1}}^0 \\ &\quad \cdots \int_{x_n=-d}^0 (p_l(x)(D_q^{l,+} f)(x) + q_l(x)(D_q^{l,-} f)(x))g(x)w_j(x)d_q x. \end{aligned}$$

Formula (6.10) is valid because the proof of Proposition 6.1, together with the fact that $\tau \notin \cup_{p=1}^n (1/p)\mathbb{N}$, shows that the multisums in the right-hand side of (6.10) converge absolutely for each $j \in \{0, \dots, n\}$ and each $l \in \{1, \dots, n\}$. We are therefore also allowed to interchange the order of q -integration in the right-hand side of (6.10).

For $x = (x_1, \dots, x_n)$, we denote $\hat{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, and we denote $\hat{x}_i(u) := (x_1, \dots, x_{i-1}, u, x_{i+1}, \dots, x_n)$. Define W_j^i by

$$W_j^i(\tau) := \{\hat{x}_i \mid x \in W_j(\tau)\}.$$

Note that $W_{i-1}^i = W_i^i$ for all $i \in \{1, \dots, n\}$. Let $f, g \in \mathcal{A}^S$. For $i \leq j$, we apply Lemma 6.4 on

$$(6.11) \quad \int_{x_i=0}^{ct^{i-1}} (p_i(x)(D_q^{i,-}f)(x) + q_i(x)(D_q^{i,+}f)(x))g(x)w_j(x)d_qx_i,$$

and for $i > j$, we apply Lemma 6.4 to

$$(6.12) \quad \int_{x_i=0}^{-dt^{n-i}} (p_i(x)(D_q^{i,-}f)(x) + q_i(x)(D_q^{i,+}f)(x))g(x)w_j(x)d_qx_i$$

with fixed $\hat{x}_i \in W_j^i$ for the variable x in the integrand of (6.11) and (6.12). Therefore, define

$$\begin{aligned} f_1^{i,j,\hat{x}_i}(y) &:= f(\hat{x}_i(y)), & f_2^{i,j,\hat{x}_i}(y) &:= p_i(\hat{x}_i(y))w_j(\hat{x}_i(y))g(\hat{x}_i(y)), \\ f_3^{i,j,\hat{x}_i}(y) &:= f(\hat{x}_i(y)), & f_4^{i,j,\hat{x}_i}(y) &:= q_i(\hat{x}_i(y))w_j(\hat{x}_i(y))g(\hat{x}_i(y)). \end{aligned}$$

Then formula (6.7) gives

$$(6.13) \quad \begin{aligned} & \int_{x_i=0}^{ct^{i-1}} \left(f_1^{i,j,\hat{x}_i}(x_i) \left(D_q^+ f_2^{i,j,\hat{x}_i} \right)(x_i) + f_3^{i,j,\hat{x}_i}(x_i) \left(D_q^- f_4^{i,j,\hat{x}_i} \right)(x_i) \right) d_qx_i \\ &= - \int_{x_i=0}^{ct^{i-1}} f(x) (p_i(x)(D_q^{i,-}g)(x) + q_i(x)(D_q^{i,+}g)(x))w_j(x)d_qx_i \end{aligned}$$

if $i \leq j$ and

$$(6.14) \quad \begin{aligned} & \int_{x_i=0}^{-dt^{n-i}} \left(f_1^{i,j,\hat{x}_i}(x_i) \left(D_q^+ f_2^{i,j,\hat{x}_i} \right)(x_i) + f_3^{i,j,\hat{x}_i}(x_i) \left(D_q^- f_4^{i,j,\hat{x}_i} \right)(x_i) \right) d_qx_i \\ &= - \int_{x_i=0}^{-dt^{n-i}} f(x) (p_i(x)(D_q^{i,-}g)(x) + q_i(x)(D_q^{i,+}g)(x))w_j(x)d_qx_i \end{aligned}$$

if $i > j$, with $\hat{x}_i \in W_j^i$ fixed for the variable x in the integrands. Define

$$h^{i,j,\hat{x}_i}(\gamma) := f_1^{i,j,\hat{x}_i}(\gamma)f_2^{i,j,\hat{x}_i}(q^{-1}\gamma) + f_3^{i,j,\hat{x}_i}(q^{-1}\gamma)f_4^{i,j,\hat{x}_i}(\gamma).$$

Then we will prove the following:

- (1a) $h^{i,j,\hat{x}_i}(cq^{(i-1)\tau}) = 0$ if $i \leq j$ and $\hat{x}_i \in W_j^i$;
- (1b) $h^{i,j,\hat{x}_i}(-dq^{(n-i)\tau}) = 0$ if $i > j$ and $\hat{x}_i \in W_j^i$;
- (2a) $\lim_{l_i \rightarrow \infty} h^{i,j,\hat{x}_i}(cq^{(i-1)\tau+l_i+1}) = 0$ for $i \in \{1, \dots, n\}$, $j \geq i+1$, and $\hat{x}_i \in W_j^i$;
- (2b) $\lim_{l_i \rightarrow \infty} h^{i,j,\hat{x}_i}(-dq^{(n-i)\tau+l_i+1}) = 0$ for $i \in \{1, \dots, n\}$, $j \leq i-2$, and $\hat{x}_i \in W_j^i$;
- (2c) $\lim_{l_i \rightarrow \infty} h^{i,i,\hat{x}_i}(cq^{(i-1)\tau+l_i+1})$ and $\lim_{l_i \rightarrow \infty} h^{i,i-1,\hat{x}_i}(-dq^{(n-i)\tau+l_i+1})$ exist and have the same limit for all $i \in \{1, \dots, n\}$ and all $\hat{x}_i \in W_i^i = W_{i-1}^i$.

Observe that if these five statements are valid, then the multisum over $\hat{x}_i \in W_i^i$ of the function

$$\hat{h}_i(\hat{x}_i) := \left(\prod_{j \neq i} |x_j| \right) \left(\lim_{l_i \rightarrow \infty} h^{i,i,\hat{x}_i}(cq^{(i-1)\tau+l_i+1}) \right)$$

is absolutely convergent for $i = 1, \dots, n$. This follows from the formula

$$\begin{aligned} \hat{h}_i(\hat{x}_i) := & \left(\prod_{j \neq i} |x_j| \right) \left(\int_{x_i=0}^{ct^{i-1}} f(x) (p_i(x)(D_q^{i,-}g)(x) + q_i(x)(D_q^{i,+}g)(x)) w_i(x) d_q x_i \right. \\ & \left. - \int_{x_i=0}^{ct^{i-1}} (p_i(x)(D_q^{i,-}f)(x) + q_i(x)(D_q^{i,+}f)(x)) g(x) w_i(x) d_q x_i \right), \end{aligned}$$

which is a consequence of Lemma 6.4, (6.13), (1a), and (2c). The self-adjointness of $D_{B,t}$ with respect to $\langle \cdot, \cdot \rangle_{B,t}$ then follows directly from this observation and these five statements, in view of Lemma 6.4, (6.10), (6.13), and (6.14).

For the proof of the five statements, we use the fact that

$$h^{i,j,\hat{x}_i}(\gamma) = (f(\hat{x}_i(\gamma))g(\hat{x}_i(q^{-1}\gamma)) - f(\hat{x}_i(q^{-1}\gamma))g(\hat{x}_i(\gamma))) p_i(\hat{x}_i(q^{-1}\gamma)) w_j(\hat{x}_i(q^{-1}\gamma))$$

for $\gamma \in \{cq^{(i-1)\tau+l_i}\}_{l_i \in \mathbb{N}_0}$ if $i \leq j$ and for $\gamma \in \{-dq^{(n-i)\tau+l_i}\}_{l_i \in \mathbb{N}_0}$ if $i > j$ with fixed $\hat{x}_i \in W_i^i$. This formula is a consequence of (6.7).

(1a) If $i = 1$, then $w_j(\hat{x}_1(cq^{-1})) = 0$ for $j \geq i$ because $(qx_1/c; q)_\infty$ is zero when $x_1 = cq^{-1}$. If $1 < i \leq n$, then $w_j(\hat{x}_i(cq^{(i-1)\tau-1})) = 0$ if $j \geq i$ because $x_{i-1} = cq^{(i-2)\tau+l_{i-1}}$ for certain $l_{i-1} \in \mathbb{N}_0$, so $(q^{1-\tau}cq^{(i-1)\tau-1}/x_{i-1}; q)_\infty = 0$.

(1b) This is similar to the proof of (1a).

(2a) If $i \in \{1, \dots, n\}$ and $j \geq i+1$, then $x_{i+1} = cq^{i\tau+l_{i+1}}$ for certain $l_{i+1} \in \mathbb{N}_0$, so $(q^{1-\tau}x_{i+1}/cq^{(i-1)\tau+l_i}; q)_\infty = 0$ if $l_i > l_{i+1}$, and therefore $w_j(\hat{x}_i(cq^{(i-1)\tau+l_i})) = 0$ if $l_i > l_{i+1}$.

(2b) This is similar to the proof of (2a).

(2c) f, g are polynomials, so

$$\frac{f(x)(T_{q^{-1},i}g)(x) - (T_{q^{-1},i}f)(x)g(x)}{x_i} \in \mathbb{R}[x_1, \dots, x_n]$$

is continuous as a function of x_i in $x_i = 0$ for arbitrary fixed $\hat{x}_i$. It is therefore sufficient to prove that

$$\lim_{l_i \rightarrow \infty} \left(-dq^{(n-i)\tau+l_i} p_i(\hat{x}_i(-dq^{(n-i)\tau+l_i})) w_{i-1}(\hat{x}_i(-dq^{(n-i)\tau+l_i})) \right)$$

and

$$\lim_{l_i \rightarrow \infty} \left(cq^{(i-1)\tau+l_i} p_i(\hat{x}_i(cq^{(i-1)\tau+l_i})) w_i(\hat{x}_i(cq^{(i-1)\tau+l_i})) \right)$$

exist and that they have the same limit for all $\hat{x}_i \in W_i^i$ and all $i \in \{1, \dots, n\}$. Fix $i \in \{1, \dots, n\}$ and $\hat{x}_i \in W_i^i$. Since $x_i p_i(x)$ is continuous as a function of x_i in $x_i = 0$, it is sufficient to prove that

$$\lim_{l_i \rightarrow \infty} d_i^\tau \Delta_\tau^i(\hat{x}_i(cq^{(i-1)\tau+l_i})) \quad \text{and} \quad \lim_{l_i \rightarrow \infty} d_{i-1}^\tau \Delta_\tau^{i-1}(\hat{x}_i(-dq^{(n-i)\tau+l_i}))$$

exist and that they have the same limit. We have

$$\begin{aligned} \Delta_\tau^i(x) &= \phi_i(x) \prod_{1 \leq k < i} (x_k - x_i) |x_k|^{2\tau-1} \left(q^{1-\tau} \frac{x_i}{x_k}; q \right)_{2\tau-1} \\ &\quad \times \prod_{i < m \leq n} (x_i - x_m) |x_i|^{2\tau-1} \left(q^{1-\tau} \frac{x_m}{x_i}; q \right)_{2\tau-1} \end{aligned}$$

with

$$\begin{aligned} \phi_i(x) &:= \left(\prod_{\substack{1 \leq k < m \leq n \\ k < i, m \neq i}} (x_k - x_m) |x_k|^{2\tau-1} \left(q^{1-\tau} \frac{x_m}{x_k}; q \right)_{2\tau-1} \right) \\ &\quad \times \prod_{i < k < m \leq n} (x_k - x_m) |x_m|^{2\tau-1} \left(q^{1-\tau} \frac{x_k}{x_m}; q \right)_{2\tau-1} \end{aligned}$$

independent of x_i and

$$\begin{aligned} \Delta_\tau^{i-1}(x) &= \phi_i(x) \prod_{1 \leq k < i} (x_k - x_i) |x_k|^{2\tau-1} \left(q^{1-\tau} \frac{x_i}{x_k}; q \right)_{2\tau-1} \\ &\quad \times \prod_{i < m \leq n} (x_i - x_m) |x_m|^{2\tau-1} \left(q^{1-\tau} \frac{x_i}{x_m}; q \right)_{2\tau-1}. \end{aligned}$$

Therefore, it is sufficient to prove that

$$\lim_{l_i \rightarrow \infty} d_i^\tau \sigma_i(\hat{x}_i(cq^{(i-1)\tau+l_i})) \quad \text{and} \quad \lim_{l_i \rightarrow \infty} d_{i-1}^\tau \rho_i(\hat{x}_i(-dq^{(n-i)\tau+l_i}))$$

exist and that they have the same limit, with

$$\begin{aligned} \sigma_i(x) &:= \prod_{i < m \leq n} (x_i - x_m) |x_i|^{2\tau-1} \left(q^{1-\tau} \frac{x_m}{x_i}; q \right)_{2\tau-1}, \\ \rho_i(x) &:= \prod_{i < m \leq n} (x_i - x_m) |x_m|^{2\tau-1} \left(q^{1-\tau} \frac{x_i}{x_m}; q \right)_{2\tau-1}. \end{aligned}$$

Clearly,

$$\lim_{l_i \rightarrow \infty} \rho_i(\hat{x}_i(-dq^{(n-i)\tau+l_i})) = \prod_{i < m \leq n} |x_m|^{2\tau},$$

and the formula

$$q^{k(2\tau-1)} (q^{1-\tau-k} z; q)_{2\tau-1} = \frac{(q^\tau z^{-1}; q)_k}{(q^{1-\tau} z^{-1}; q)_k} (q^{1-\tau} z; q)_{2\tau-1}$$

gives that

$$\lim_{l_i \rightarrow \infty} \sigma_i(\hat{x}_i(cq^{(i-1)\tau+l_i})) = \prod_{i < m \leq n} |x_m|^{2\tau} \frac{(cq^{(i-1)\tau})^{2\tau-1} (q^{1-\tau} (x_m/cq^{(i-1)\tau}); q)_{2\tau-1}}{|x_m|^{2\tau-1} (q^{1-\tau} (cq^{(i-1)\tau}/x_m); q)_{2\tau-1}}.$$

Thus

$$\lim_{l_i \rightarrow \infty} \sigma_i(\hat{x}_i(cq^{(i-1)\tau+l_i})) = \frac{d_{i-1}^\tau}{d_i^\tau} \prod_{i < m \leq n} |x_m|^{2\tau}$$

for $\hat{x}_i \in W_i^i$ because the function ψ_τ given by formula (5.15) is a quasi-constant function.

(ii) Fix $\tau \in (0, \infty) \setminus \cup_{p=1}^n (1/p)\mathbb{N}$ and $f, g \in \mathcal{A}^S$; then a similar argument as in the proof of (i) gives that

$$\langle f, g \rangle_L = \int_{x_1=0}^1 \cdots \int_{x_n=0}^{q^{(n-1)\tau}} f(x)g(x)v(x)d_q x,$$

and the order of q -integration may be changed because of absolute convergence. Fix $i \in \{1, \dots, n\}$ and fix $x_k \in \{q^{(k-1)\tau+l_k}\}_{l_k \in \mathbb{N}_0} (k \neq i)$. Define

$$\begin{aligned} f_1^i(x_i) &:= f(x), & f_2^i(x_i) &:= p_i(x; b, a, 1, 0; q, t)v(x; a, b; q, t)g(x), \\ f_3^i(x_i) &:= f(x), & f_4^i(x_i) &:= q_i(x; 1, 0; q, t)v(x; a, b; q, t)g(x) \end{aligned}$$

($x = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$). It follows from formula (6.8) and Lemma 6.4 that the proof of (ii) is complete if the following two formulas are valid:

$$(6.15) \quad f_1^i(q^{(i-1)\tau})f_2^i(q^{(i-1)\tau-1}) + f_3^i(q^{(i-1)\tau-1})f_4^i(q^{(i-1)\tau}) = 0,$$

$$(6.16) \quad \lim_{k \rightarrow \infty} \left(f_1^i(q^{(i-1)\tau+k+1})f_2^i(q^{(i-1)\tau+k}) + f_3^i(q^{(i-1)\tau+k})f_4^i(q^{(i-1)\tau+k+1}) \right) = 0.$$

For the proof, we use the formula

$$\begin{aligned} f_1^i(\gamma)f_2^i(q^{-1}\gamma) + f_3^i(q^{-1}\gamma)f_4^i(\gamma) &= (f(\hat{x}_i(\gamma))g(\hat{x}_i(q^{-1}\gamma)) - f(\hat{x}_i(q^{-1}\gamma))g(\hat{x}_i(\gamma))) \\ &\quad \times p_i(\hat{x}_i(q^{-1}\gamma); b, a, 1, 0; q, t)v(\hat{x}_i(q^{-1}\gamma); a, b; q, t) \end{aligned}$$

with $\gamma \in \{q^{(i-1)\tau+l_i}\}_{l_i \in \mathbb{N}_0}$. This formula is a consequence of (6.8). The proof of (6.15) is similar to the proof of (1a), and (6.16) holds because

$$\lim_{l_i \rightarrow \infty} q^{(i-1)\tau+l_i} p_i(\hat{x}_i(q^{(i-1)\tau+l_i}); b, a, 1, 0; q, t)v(\hat{x}_i(q^{(i-1)\tau+l_i}); a, b; q, t)$$

is zero for $i = 1, \dots, n-1$ because $v(\hat{x}_i(q^{(i-1)\tau+l_i}); a, b; q, t) = 0$ for l_i sufficiently big and is also zero if $i = n$ since

$$\lim_{l_n \rightarrow \infty} v(\hat{x}_n(q^{(n-1)\tau+l_n}); a, b; q, t) \left(q^{(n-1)\tau+l_n} \right)^{-\alpha}$$

exists, and

$$\lim_{l_n \rightarrow \infty} \left(q^{(n-1)\tau+l_n} \right)^{\alpha+1} p_i(\hat{x}_n(q^{(n-1)\tau+l_n}); b, a, 1, 0; q, t) = 0$$

since $\alpha > -1$. $\square$

Note added in proof. The evaluation formula for $\langle 1, 1 \rangle_{L, q^k}$ which we presented in Remark 5.4 for the special parameter values $k \in \mathbb{N}$ is in fact valid for all $k \in (0, \infty)$. This follows from a modified form of Askey, Habsieger, and Kadell's formula which has recently been proved by K. Aomoto in his preprint "On elliptic product formulas for Jackson integrals associated with reduced root systems."

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