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A FUNCTIONAL MODEL APPROACH TO LINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

Sjoerd M. Verduyn Lunel and Dmitry V. Yakubovich *

We consider a linear neutral functional equation of the form

$$\frac{d}{dt} Mx_t = Lx_t, \quad x(0) = c, \quad x_0 = \varphi,$$

where $M, L : L^2([-h, 0], \mathbb{C}^n) \rightarrow \mathbb{C}^n$ are unbounded linear operators of certain type and $x_t(\theta) = x(t + \theta)$, $-h \leq \theta \leq 0$. Under certain conditions, the solution semigroup $\{T(t)\}_{t \geq 0}$ is well-defined on the space $Z = \mathbb{C}^n \times L^2([-h, 0], \mathbb{C}^n)$ of initial data. Let A be the infinitesimal generator of this semigroup. We shall establish the following model representation for $\{T(t)\}_{t \geq 0}$ and A : there exists a half-plane $\Pi_- = \{\operatorname{Re} z < a\}$, an $n \times n$ matrix function δ in $H^\infty(\Pi_-, \mathbb{C}^{n \times n})$ and a linear isomorphism U of Z onto the quotient space $H^2(\Pi_-, \mathbb{C}^{n \times n}) / \delta H^2(\Pi_-, \mathbb{C}^{n \times n})$ that transforms the operators $T(t)$ and A into multiplication operators by e^{tz} and z , respectively. We explain the relations between this result and special factorizations of A that have been studied in [7]. The proof is based on a certain duality scheme rather than the Nagy—Foiş model theory. As applications, we give a criterium for the eigenspaces of A to form a Riesz basis and recover the results on completeness and small solutions for the case of L^2 -spaces (see [15, 16, 17]). Another application concerns γ -periodic solutions.

1 The main result

In this paper we construct a “spectral” model for linear autonomous neutral functional differential equations

$$\begin{aligned} \frac{d}{dt} Mx_t &= Lx_t, & t &\geq 0, \\ x(0) &= c, \quad x(\theta) = \varphi(\theta), & -h &\leq \theta < 0. \end{aligned} \tag{1.1}$$

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Here $x \in L^2_{\text{loc}}([-h, \infty), \mathbb{C}^n)$ is an unknown vector-valued function, h is the delay, x_t is a function in $L^2([-h, 0], \mathbb{C}^n)$, defined by

$$x_t(\theta) = x(t + \theta), \quad -h \leq \theta \leq 0$$

and $M, L : L^2([-h, 0], \mathbb{C}^n) \rightarrow \mathbb{C}^n$ are (unbounded) linear operators of certain type. Before we specify the assumptions on M and L , we introduce some notation.

For a function class $\mathcal{B}$, we denote by $\mathcal{B}_n$ and $\mathcal{B}_{m \times n}$ the corresponding classes of n -dimensional vector-valued functions and $m \times n$ matrix-valued functions whose components are in $\mathcal{B}$. Whenever $\mathcal{B}$ is a Banach (Hilbert) space, we assume that $\mathcal{B}_n$ and $\mathcal{B}_{m \times n}$ are of the same type. In the sequel, we make two hypotheses on M and L .

(H1) $M : C[-h, 0]_n \rightarrow \mathbb{C}^n$ and $L : W^2_1[-h, 0]_n \rightarrow \mathbb{C}^n$ are bounded operators.

From this hypothesis it follows that M and L admit representations

$$\begin{aligned} M\varphi &= \int_0^h d\mu(\theta)\varphi(-\theta), \\ L\varphi &= \int_0^h \zeta(\theta)\varphi(-\theta) d\theta + \int_0^h \eta(\theta)\dot{\varphi}(-\theta) d\theta, \end{aligned} \quad (1.2)$$

where $\zeta, \eta \in L^2[0, h]_{n \times n}$ and μ is an $n \times n$ matrix function on $\mathbb{R}$, constant on $\mathbb{R} \setminus [0, h]$, whose entries are of bounded variation.

The operator M is said to be *atomic at* $s \in [0, h]$ if the jump at s , $\mu(s+) - \mu(s-)$, is a nonsingular matrix.

(H2) M is atomic at 0.

Without loss of generality, we will assume that $\mu(s+) - \mu(s-) = I$, which implies that

$$M\varphi = \varphi(0) + \int_0^h d\mu_1(\theta)\varphi(-\theta),$$

where μ_1 is an $n \times n$ matrix function on $\mathbb{R}$, constant on $\mathbb{R} \setminus [0, h]$, whose entries are of bounded variation and such that $\mu_1(0+) - \mu_1(0-) = 0$ (μ_1 is continuous at 0).

From a result by Burns, Herdman and Stech [2], it follows that the initial value problem (1.1) is well-posed in the following sense: for any initial condition $(c, \varphi)^T \in Z$, there is a unique generalized solution $x(t) = x(t; c, \varphi)$ of Eq. (1.1) which belongs to $L^2_{\text{loc}}[-h, \infty)_n$. Furthermore, the semigroup $T(t) : Z \rightarrow Z$, $t \geq 0$, defined by

$$T(t) \begin{pmatrix} c \\ \varphi \end{pmatrix} = \begin{pmatrix} Mx_t(\cdot; c, \varphi) \\ x_t(\cdot; c, \varphi) \end{pmatrix}, \quad \begin{pmatrix} c \\ \varphi \end{pmatrix} \in Z,$$

is a $\mathcal{C}_0$ -semigroup with infinitesimal generator given by

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} c \\ \varphi \end{pmatrix} \in Z : c = M\varphi \right\}, \quad A \begin{pmatrix} c \\ \varphi \end{pmatrix} = \begin{pmatrix} L\varphi \\ \dot{\varphi} \end{pmatrix}. \quad (1.3)$$

Let

$$\begin{aligned}\Delta(z) &= zMe^{z\cdot} - Le^{z\cdot} \\ &= z\left(\int_0^h e^{-z\theta} d\mu(\theta) - \int_0^h \eta(\theta)e^{-z\theta} d\theta\right) - \int_0^h e^{-z\theta} \zeta(\theta) d\theta.\end{aligned}\quad (1.4)$$

The matrix function Δ is an $n \times n$ entire matrix function and plays a key role in the study of Eq. (1.1). It is called the *characteristic matrix* of the operator A (compare [7]).

Consider, for example, the following functional differential equation

$$\begin{aligned}\dot{x}_1(t) - \dot{x}_1(t-1) &= x_2(t-1) \\ \dot{x}_2(t) &= x_1(t-1)\end{aligned}$$

with initial conditions

$$\begin{aligned}x_1(0) &= c_1, & x_1(\theta) &= \varphi_1(\theta), & -1 \leq \theta < 0, \\ x_2(0) &= c_2, & x_2(\theta) &= \varphi_2(\theta), & -1 \leq \theta < 0.\end{aligned}$$

If we define $\varphi = (\varphi_1, \varphi_2)^T$, $c = (c_1, c_2)^T$,

$$M\varphi = \begin{pmatrix} \varphi_1(0) - \varphi_1(-1) \\ \varphi_2(0) \end{pmatrix} \quad \text{and} \quad L\varphi = \begin{pmatrix} \varphi_2(-1) \\ \varphi_1(-1) \end{pmatrix},$$

then this equation has the form (1.1). Let us consider special solutions of (1.1) of the form $x(t) = e^{\lambda t}c$, which is equivalent to looking for eigenvectors of A that satisfy $A(c, \varphi)^T = \lambda(c, \varphi)^T$. We obtain the following equations

$$\begin{aligned}\lambda e^{\lambda t}c_1 - \lambda e^{\lambda(t-1)}c_1 &= e^{\lambda(t-1)}c_2 \\ \lambda e^{\lambda t}c_2 &= e^{\lambda(t-1)}c_1,\end{aligned}$$

or $\Delta(\lambda)c = 0$, where in this case

$$\Delta(\lambda) = \begin{pmatrix} \lambda - \lambda e^{-\lambda} & -e^{-\lambda} \\ -e^{-\lambda} & \lambda \end{pmatrix}.$$

So solutions of the form $x(t) = e^{\lambda t}c$ exist if and only if $\det \Delta(\lambda) = 0$.

It is known [4] that in general, the spectrum of A equals to its point spectrum and is given by

$$\sigma(A) = \{\lambda \in \mathbb{C} : \det \Delta(\lambda) = 0\}.$$

In this paper we shall give the precise connection between the matrix function Δ and the infinitesimal generator A , which shows, in particular, that the characteristic function Δ captures all the information about A .

Before we formulate the main result we introduce some more notation. Put

$$a_0 = \inf\{a : \det \Delta(z) \neq 0 \text{ and } \|\Delta(z)^{-1}\| \leq \frac{C}{1+|z|} \text{ for } \operatorname{Re} z > a\}. \quad (1.5)$$

It is easy to deduce from (H2) that a_0 is finite.

Take any $a \in \mathbb{R}$ and $b \in \mathbb{C}$ with $a_0 < a < \operatorname{Re} b$ and put

$$\begin{aligned} \Delta_-(z) &= e^{hz} \Delta(z), & \delta(z) &= (z-b)^{-1} \Delta_-(z) \\ \Pi_- &= \{z : \operatorname{Re} z < a\}, & \Pi_+ &= \{z : \operatorname{Re} z > a\}, & \ell &= \partial \Pi_+ = \partial \Pi_-, \\ H_{\pm,n}^2 &= H^2(\Pi_{\pm})_n. \end{aligned} \quad (1.6)$$

Then $\delta \in H_{n \times n}^\infty(\Pi_-)$ and the definition of a implies that $\|\delta^{-1}\|$ is bounded on the vertical line ℓ . Therefore $\delta H_{-,n}^2$ and $\delta^T H_{-,n}^2$ are closed subspaces of $H_{-,n}^2$. The quotient spaces $H_{-,n}^2/\delta H_{-,n}^2$ and $H_{-,n}^2/\delta^T H_{-,n}^2$ will be called *functional model spaces*. Define

$$\begin{aligned} \mathcal{D}_\delta &= \{x_- + \delta H_{-,n}^2 : zx_- \in H_{-,n}^2\} \\ \mathcal{D}_{\delta^T} &= \{x_- + \delta^T H_{-,n}^2 : zx_- \in H_{-,n}^2\}. \end{aligned}$$

The linear sets $\mathcal{D}_\delta, \mathcal{D}_{\delta^T}$ are the domains of definition of the unbounded operator defined by multiplication by the independent variable

$$M_z f := z f,$$

acting, respectively, on $H_{-,n}^2/\delta H_{-,n}^2$ and $H_{-,n}^2/\delta^T H_{-,n}^2$.

We define bilinear pairings on $\mathbb{C}^n \times \mathbb{C}^n$, $Z \times Z$ and the space $H_{-,n}^2/\delta H_{-,n}^2 \times H_{-,n}^2/\delta^T H_{-,n}^2$ by, respectively,

$$\begin{aligned} \langle c, d \rangle &= \sum_{j=1}^n c_j d_j, \quad c = (c_1, \dots, c_n), \quad d = (d_1, \dots, d_n) \in \mathbb{C}^n \\ \left\langle \begin{pmatrix} c \\ \varphi \end{pmatrix}, \begin{pmatrix} d \\ \psi \end{pmatrix} \right\rangle &= \langle c, d \rangle + \int_{-h}^0 \langle \varphi(t), \psi(t) \rangle dt, \quad \begin{pmatrix} c \\ \varphi \end{pmatrix}, \begin{pmatrix} d \\ \psi \end{pmatrix} \in Z \end{aligned} \quad (1.7)$$

$$\begin{aligned} \langle f + \delta H_{-,n}^2, g + \delta^T H_{-,n}^2 \rangle &= \frac{1}{2\pi i} \int_\ell \langle \delta^{-1}(z) f(z), g(z) \rangle dz, \quad f \in H_{-,n}^2/\delta H_{-,n}^2, \\ &g \in H_{-,n}^2/\delta^T H_{-,n}^2. \end{aligned} \quad (1.8)$$

(We use the convention that we integrate along ℓ from $-\infty$ to ∞ .) For any $F, G \in H^2(\Pi_{\pm})$, we have $FG \in H^1(\Pi_{\pm})$, and also

$$\int_\ell r(z) dz = 0, \quad \text{for all } r \in H^1(\Pi_{\pm}). \quad (1.9)$$

So the pairing defined in (1.8) does not depend on the particular choice of f and g as representants of the equivalence classes. Therefore the pairing is well defined.

For a Banach space B , we denote by B^* its adjoint in the sense of the space of linear continuous functionals on B . Then it is easy to see that the pairings defined in (1.7) and (1.8) give rise to the following identifications

$$\begin{aligned} Z^* &= Z, \\ (H_{-,n}^2/\delta H_{-,n}^2)^* &= H_{-,n}^2/\delta^T H_{-,n}^2. \end{aligned}$$

Set

$$m(z) = M e^{z \cdot} = \int_0^h e^{-z\theta} d\mu(\theta). \quad (1.10)$$

Define the differential operator D_- on $L^2[-h, 0]_n$ by

$$\mathcal{D}(D_-) = \{\varphi \in W_1^2[-h, 0]_n : \varphi(-h) = 0\}, \quad D_- \varphi = \varphi', \quad (1.11)$$

then $(zI - D_-)^{-1}$ is bounded for all $z \in \mathbb{C}$ and

$$((zI - D_-)^{-1} \varphi)(x) = - \int_{-h}^x e^{z(x-t)} \varphi(t) dt, \quad \varphi \in L^2[-h, 0]_n. \quad (1.12)$$

We are now in a position to formulate the main result of this paper.

Theorem 1.1. *Let $Z = \mathbb{C}^n \times L^2([-h, 0], \mathbb{C}^n)$, δ and $H_{\pm,n}^2$ be defined by (1.6), M and L by (1.2), D_- by (1.11) and m by (1.10). The operators*

$$U : Z \rightarrow H_{-,n}^2/\delta H_{-,n}^2, \quad \text{and} \quad V : Z \rightarrow H_{-,n}^2/\delta^T H_{-,n}^2,$$

given by

$$U \begin{pmatrix} c \\ \varphi \end{pmatrix} (z) = \frac{1}{z-b} (c - (zM - L)(zI - D_-)^{-1} \varphi) \Big|_{\Pi_-}, \quad (1.13)$$

$$V \begin{pmatrix} d \\ \psi \end{pmatrix} (z) = ((e^{hz} m^T(z) - \delta^T(z))d + \int_{-h}^0 e^{z(h+s)} \psi(s) ds) \Big|_{\Pi_-}. \quad (1.14)$$

are isomorphisms. Furthermore

$$\begin{aligned} \mathcal{UD}(A) &= \mathcal{D}_\delta = \{x_- + \delta H_{-,n}^2 : zx_- \in H_{-,n}^2\} \\ \mathcal{VD}(A^*) &= \mathcal{D}_{\delta^T} = \{x_- + \delta^T H_{-,n}^2 : zx_- \in H_{-,n}^2\} \end{aligned}$$

and the following intertwining formulas hold:

$$\begin{aligned} UAv &= M_z Uv, \quad v \in \mathcal{D}(A), \\ VA^*w &= M_z Vw, \quad w \in \mathcal{D}(A^*), \end{aligned} \quad (1.15)$$

and

$$UT(t) = M_{e^{zt}}U, \quad VT^*(t) = M_{e^{zt}}V. \quad (1.16)$$

Thus the operator M_z on the functional model space $H_{-,n}^2/\delta H_{-,n}^2$ ($H_{-,n}^2/\delta^T H_{-,n}^2$) is a functional model for A (A^*). These models are dual in the sense that the following identity holds

$$\langle Ux, Vy \rangle = \langle x, y \rangle, \quad x, y \in Z. \quad (1.17)$$

One easily calculates the adjoint operators of U and V :

$$U^*(g + \delta^T H_{-,n}^2) = \begin{pmatrix} d \\ \psi \end{pmatrix},$$

where

$$\begin{aligned} d &= \frac{1}{2\pi i} \int_{\ell} \Delta_-^T(z) g(z) dz, \\ \psi(t) &= \int_0^{-t} d\mu(\theta) \frac{1}{2\pi i} \int_{\ell} e^{-z(\theta+t)} z \Delta_-^T(z)^{-1} g(z) dz - \eta(-t)d \\ &\quad - \int_0^{-t} d\theta \frac{1}{2\pi i} \int_{\ell} e^{-z(\theta+t)} (\zeta(\theta) + z\eta(\theta)) \Delta_-^T(z)^{-1} g(z) dz, \quad -h \leq t \leq 0; \end{aligned} \quad (1.18)$$

and

$$V^*(f + \delta H_{-,n}^2) = \begin{pmatrix} c \\ \varphi \end{pmatrix},$$

where

$$\begin{aligned} c &= \frac{1}{2\pi i} \int_{\ell} (e^{hz} m(z) \delta(z)^{-1} - I) f(z) dz, \\ \varphi(\theta) &= \frac{1}{2\pi i} \int_{\ell} e^{(\theta+h)z} \delta(z)^{-1} f(z) dz, \quad -h \leq \theta \leq 0. \end{aligned} \quad (1.19)$$

Here Δ_- , δ and m are defined in (1.6) and (1.10). Since, by (1.17), $U^{-1} = V^*$ and $V^{-1} = U^*$, these are in fact explicit formulas for the inverse transformations U^{-1} and V^{-1} .

Remark. We observe that the idea of constructing model representations for A and for A^* in dual functional spaces so that the duality formula (1.17) holds has already been exploited by the second author in [18] for Toeplitz operators with smooth symbols and related classes of operators. However, the model spaces constructed there were quite different.

As a consequence of our theorem we conclude that all information about the generator A is contained in the characteristic matrix $\Delta(z)$.

2 A connection with the factorization of A

In this section we derive some preliminary results necessary in the proof the main theorem and explain how the expressions for the transformations U and V are related with the factorization of A as given by Kaashoek-Verduyn Lunel [7].

For Banach spaces B and B_1 we denote by $\mathcal{L}(B, B_1)$ the Banach space of all bounded operators from B to B_1 . If $B_1 = B$ we set $\mathcal{L}(B, B) = \mathcal{L}(B)$.

In [7], the concept of a characteristic matrix for an unbounded closed operator A was introduced. An analytic $n \times n$ matrix-valued function $\tilde{\Delta}$ is called a *characteristic matrix* for A on $\Omega \subset \mathbb{C}$ if there exist holomorphic functions $E : \Omega \rightarrow \mathcal{L}(Z, Z_A)$ and $F : \Omega \rightarrow \mathcal{L}(Z)$ whose values are bijective mappings such that

$$(zI - A)^{-1} = E(z) \begin{pmatrix} \tilde{\Delta}(z)^{-1} & 0 \\ 0 & I \end{pmatrix} F(z), \quad z \in \Omega. \quad (2.1)$$

Here Z_A denotes the Banach space defined by the domain of A provided with the graph norm. In [7] it was shown that the operator A defined by (1.2) has a characteristic matrix on $\Omega = \mathbb{C}$ and explicit formulas for E and F and their inverses were given.

In the sequel we shall use (2.1) in the equivalent form

$$F(z)(zI - A) = \begin{pmatrix} \tilde{\Delta}(z) & 0 \\ 0 & I \end{pmatrix} H(z), \quad (2.2)$$

where $H(z) = E(z)^{-1}$ and we shall use the explicit formulas for F and E^{-1} derived in [7]. In order to give these formulas we first recall some notation from [7].

Let

$$\mathcal{D}(D) = W_1^2([-h, 0])_n, \quad D\varphi = \varphi'$$

and

$$j : \mathbb{C}^n \rightarrow W_1^2([-h, 0])_n, \quad (jc)(x) = c.$$

Let D_0 be any restriction of D such that

- (i) the regular set of D_0 is $\mathbb{C}$;
- (ii) $W_1^2([-h, 0])_n = j\mathbb{C}^n + \mathcal{D}(D_0)$.

Then Theorem 3.1 of [7] yields that the factorization (2.1) can be obtained by defining

$$\begin{aligned} \tilde{\Delta}(z) &= -(zM - L)D_0(zI - D_0)^{-1}j, \quad z \in \mathbb{C} \\ F(z) \begin{pmatrix} c \\ \varphi \end{pmatrix} &= \begin{pmatrix} c - (zM - L)(zI - D_0)^{-1}\varphi \\ \varphi \end{pmatrix} \\ E(z) \begin{pmatrix} c \\ \varphi \end{pmatrix} &= \begin{pmatrix} -MD_0(zI - D_0)^{-1}jc + M(z - D_0)^{-1}\varphi \\ -D_0(z - D_0)^{-1}jc + (zI - D_0)^{-1}\varphi \end{pmatrix} \\ H(z) \begin{pmatrix} M\psi \\ \psi \end{pmatrix} &= \begin{pmatrix} j^{-1}(\psi - D_0^{-1}D\psi) \\ (zI - D)\psi \end{pmatrix}, \quad \begin{pmatrix} M\psi \\ \psi \end{pmatrix} \in \mathcal{D}(A). \end{aligned} \quad (2.3)$$

First we set $D_0 = D_-$ (where D_- is defined by (1.11)) and set F^- and H^- to be the corresponding operator functions defined by (2.3). For this choice of D_0 the characteristic matrix becomes

$$\tilde{\Delta}(z) = \Delta_-(z),$$

where $\Delta_-(z)$ is given by (1.6). Define

$$(U_1x)(z) = (I \ 0)F^-(z)x, \quad x \in Z. \quad (2.4)$$

By comparing formulas (2.3) and (1.13) we conclude that

$$U \begin{pmatrix} c \\ \varphi \end{pmatrix} (z) = \frac{1}{z-b} U_1 \begin{pmatrix} c \\ \varphi \end{pmatrix} (z)|_{\Pi_-} + \delta(z)H_{-,n}^2, \quad z \in \mathbb{C}. \quad (2.5)$$

The factorization (2.2) implies the following formula for the action of A :

$$U_1Av(z) = M_zU_1v(z) - \Delta_-(z)H_1(z)v, \quad v \in \mathcal{D}(A) \quad (2.6)$$

with

$$H_1(z) \begin{pmatrix} M\psi \\ \psi \end{pmatrix} = (I \ 0)H(z) \begin{pmatrix} M\psi \\ \psi \end{pmatrix} \equiv \psi(-h), \quad \begin{pmatrix} M\psi \\ \psi \end{pmatrix} \in \mathcal{D}(A).$$

The Fourier-Laplace transform

$$\mathcal{L}f(z) = \int_{\mathbb{R}} e^{tz} f(t) dt$$

has the property that $(\sqrt{2\pi})^{-1}\mathcal{L}$ is a unitary isomorphism of the weighted space

$$L^2(\mathbb{R}; e^{-\cdot a}) = \{f : e^{-\cdot a}f \in L^2(\mathbb{R})\} \text{ onto } L^2(\ell).$$

By the Paley-Wiener theorem, the spaces $L^2(\mathbb{R}_+; e^{-\cdot a})$ and $L^2(\mathbb{R}_-; e^{-\cdot a})$ viewed as subspaces of $L^2(\mathbb{R}; e^{-\cdot a})$ map onto $H^2(\Pi_+)$ and $H^2(\Pi_-)$, respectively.

For any bounded interval I of the real line, we define the Paley-Wiener space of entire functions

$$PW(I) = \{\mathcal{L}f : f \in L^2(I)\}.$$

In fact, if $I \subset \mathbb{R}_{\pm}$,

$$\{\varphi|_{\Pi_{\pm}} : \varphi \in PW(I)\} \subset H_{\pm}^2.$$

It is clear that

$$U_1 : Z \rightarrow {}^zPW[-h, 0]_n + \mathbb{C}^n \quad (2.7)$$

is a bounded operator. This implies that U is bounded. It is easy to see that we can rewrite (2.6) as an intertwining formula for the action of A

$$U_1Av(z) = M_zU_1v(z) - \Delta_-(z)\rho(U_1v(z)|_{\ell}), \quad v \in \mathcal{D}(A), \quad (2.8)$$

where the unbounded operator $\rho : (z - b)L^2(\ell)_n \rightarrow \mathbb{C}^n$ is defined by

$$\rho f = c \quad \text{if } f = g + e^{-zh}mc, \quad g \in L^2(\ell).$$

Next we define an unbounded operator D_+ by

$$\mathcal{D}(D_+) = \{\varphi \in W_1^2[-h, 0]_n : \varphi(0) = 0\}, \quad D_+\varphi = \varphi'.$$

If we set $D_0 = D_+$ we can define E^+ and F^+ to be the corresponding operator functions defined by (2.3). Passing to the adjoints in (2.1) we obtain the following analogue of (2.2)

$$E^+(z)^*(zI - A^*) = \begin{pmatrix} \Delta^T(z) & 0 \\ 0 & I \end{pmatrix} G^+(z)^*, \quad (2.9)$$

where $G^+(z) = F^+(z)^{-1}$. So, by applying the same scheme, we put

$$(V_1 y)(z) = (I \ 0)E^+(z)^*y, \quad y \in Z.$$

Next we substitute the identity

$$(z - D_0)^{-1}f(x) = \int_x^0 e^{z(x-t)}f(t) dt$$

into (2.3) and, after some calculations, we obtain

$$V_1 \begin{pmatrix} d \\ \psi \end{pmatrix} (z) = m^T(z)d + \int_{-h}^0 e^{zs}\psi(s) ds.$$

Hypothesis (H2) implies that $m^T(z)d \notin PW[0, h]_n$ for all $d \neq 0$ in $\mathbb{C}^n$. Hence

$$V_1 : Z \rightarrow PW[0, h]_n + m^T\mathbb{C}^n \quad (2.10)$$

is an isomorphism. One also sees that

$$V = V_2 V_1, \quad (2.11)$$

where $V_2 : PW[0, h]_n + m^T\mathbb{C}^n \rightarrow H_{-,n}^2/\delta^T H_{-,n}^2$ is defined by

$$V_2(m^T d + f) = (e^{hz}m^T - \delta^T)d + e^{hz}f|_{\Pi_-}, \quad f \in PW[0, h]_n, \quad d \in \mathbb{C}^n.$$

It is easy to see that V_2 and V are bounded.

We remark that

$$(I \ 0)G^+(z)^* \begin{pmatrix} d \\ \psi \end{pmatrix} = d$$

and, hence, does not depend on z . Therefore the intertwining formula for the action of A^* , which is obtained from (2.9), has a form similar to (2.8):

$$(V_1 A^* w)(z) = z(V_1 w)(z) - \Delta^T(z) \rho_*(V_1 w|_\ell), \quad w \in \mathcal{D}(A^*), \quad (2.12)$$

where $\rho_* : L^2(\ell)_n + m^T \mathbb{C}^n \rightarrow \mathbb{C}^n$ is given by

$$\rho_*(f + m^T c) = c, \quad f \in L^2(\ell), \quad c \in \mathbb{C}^n.$$

For sake of completeness we conclude this section with an explicit formula for A^* :

$$A^* \begin{pmatrix} d \\ \psi \end{pmatrix} = \begin{pmatrix} e \\ \chi \end{pmatrix},$$

where

$$\begin{aligned} e &= \eta^T(0)d + \psi(0) \\ \chi(\theta) &= \zeta^T(-\theta)d - \frac{d}{d\theta} [H_h(\theta)(\eta^T(h)d + \psi(-h)) \\ &\quad + \mu^1(-\theta)(\eta^T(0)d + \psi(0)) + \eta^T(-\theta)d + \psi(\theta)], \quad -h \leq \theta \leq 0 \end{aligned}$$

and

$$\mathcal{D}(A^*) = \left\{ \begin{pmatrix} d \\ \psi \end{pmatrix} \in Z : \chi \in L^2[-h, 0] \right\}.$$

(Here H_h denotes the Heaviside function with jump in h .)

3 The proof of the theorem

Before we prove Theorem 1.1 we state the following lemmas

Lemma 3.1. *The operator $V : Z \rightarrow H_{-,n}^2 / \delta^T H_{-,n}^2$ defined by (1.14) is an isomorphism.*

Lemma 3.2. *The identity (1.16) holds.*

Before we prove these lemmas we shall first prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.2 it follows that $U = (V^*)^{-1}$. Therefore Lemma 3.1 implies that U and V are isomorphisms.

The first identity in (1.15) follows if we divide (2.8) by $(z - b)$.

For any $\lambda \in \Pi_+$ we set $v = (A - \lambda I)^{-1}u$ in the first identity of (1.15) to obtain

$$M_{(z-\lambda)^{-1}} U u = U (A - \lambda I)^{-1} u \quad \text{for all } u \in Z. \quad (3.1)$$

Therefore

$$U \mathcal{D}(A) = U \text{Range}(A - \lambda I)^{-1} = (z - \lambda)^{-1} U Z = \mathcal{D}_\delta.$$

Passing to the adjoints in (3.1), we get

$$U^* M_{(z-\lambda)^{-1}} = (A^* - \lambda I)^{-1} U^*,$$

or

$$M_{(z-\lambda)^{-1}} V = V(A^* - \lambda I)^{-1};$$

this implies the second identity in (1.15) and the relation

$$V\mathcal{D}(A^*) = \mathcal{D}_{\delta T}.$$

From semigroup theory it follows that the $(zI - A)^{-1}$ is the Laplace transform of the semigroup $T(t)$ in the right half plane $\operatorname{Re} z > \omega(A)$, where $\omega(A)$ denotes the growth bound of $T(t)$. Together with (3.1) this implies

$$\begin{aligned} \int_0^\infty e^{-\lambda\xi} U T(\xi) U^{-1} d\xi &= U(\lambda I - A)^{-1} U^{-1} \\ &= M_{(\lambda - z)^{-1}} \\ &= \int_0^\infty e^{-\lambda\xi} M_{e^{z\xi}} d\xi, \quad \operatorname{Re} \lambda > \max(\omega(A), a). \end{aligned}$$

(Here the operators $M_{(\lambda - z)^{-1}}$ and $M_{e^{z\xi}}$ act on $H_{-,n}^2/\delta H_{-,n}^2$.) The uniqueness property of the Laplace transform now yields (1.16). $\square$

We need some more preparations in order to prove Lemma 3.1 and 3.2. We first observe that an entire function F belongs to $PW[-h, 0]$ if and only if $F|_{\Pi_-} \in H^2(\Pi_-)$ and $e^{-h \cdot} F|_{\Pi_+} \in H^2(\Pi_+)$. Also, the Paley-Wiener theorem implies a direct sum decomposition

$$L^2(\ell)_n = H_{-,n}^2 + H_{+,n}^2;$$

the corresponding projections will be denoted by P_- and P_+ .

Lemma 3.3. *The mapping $J_\delta : PW[-h, 0]_n + (z - b)^{-1}\mathbb{C}^n \rightarrow H_{-,n}^2/\delta H_{-,n}^2$ defined by*

$$J_\delta x := x|_{\Pi_-} + \delta H_{-,n}^2$$

is an isomorphism.

Proof. It is clear that J_δ is bounded.

1) To prove that J_δ is one-to-one, suppose that $J_\delta x = 0$ for a certain $x \in PW[-h, 0]_n + (z - b)^{-1}\mathbb{C}^n$. Then $x|_{\Pi_-} = \delta y_-$ for some $y_- \in H_{-,n}^2$. Put

$$y_+ = \delta^{-1} x|_{\Pi_+} = \Delta^{-1}[(z - b)e^{-hz} x]|_{\Pi_+}$$

and take an arbitrary point $b_- \in \Pi_-$. Since

$$(z - b)e^{-hz}x \in (z - b_-)H_{+,n}^2,$$

it follows that $y_+ \in H_{+,n}^2$. But $y_+|_\ell = y_-|_\ell$ and therefore $y_+ = y_- = 0$, because $H_{+,n}^2 \cap H_{-,n}^2 = 0$.

2) Let us prove that J_δ is onto. Take any $u_- \in H_{-,n}^2$. We have to solve the equation

$$J_\delta x = u_- + \delta H_-^2.$$

Put $x_- = u_- - \delta P_-(\delta^{-1}u_-)$ and $x_+ = \delta P_+(\delta^{-1}u_-)$; then $x_+|_\ell = x_-|_\ell$. This implies that there exists a meromorphic function x that is analytic outside $z = b$ and has a simple pole at $z = b$, such that

$$x_- = x|_{\Pi_-}, \quad x_+ = x|_{\Pi_+}.$$

Set

$$y = x - (z - b)^{-1}c, \quad c = \operatorname{Res}_{z=b} x(z).$$

Then $y|_{\Pi_-} \in H_{-,n}^2$ and

$$e^{-hz}y|_{\Pi_+} = (z - b)^{-1}[\Delta v_+ - e^{-hz}c] \in H_{+,n}^2$$

(here $v_+ = P_+(\delta^{-1}u_-) \in H_{+,n}^2$). Therefore

$$y \in PW[-h, 0]_n, \quad x \in PW[-h, 0]_n + (z - b)^{-1}\mathbb{C}^n$$

and $J_\delta x = u_- + \delta H_-^2$. □

Remark. Similarly one can prove that $J_\delta = S_\delta W_\delta^{-1}$, where the isomorphisms $W_\delta : \mathcal{H}(\delta) \rightarrow H_{-,n}^2/\delta H_{-,n}^2$ and $S_\delta : \mathcal{H}(\delta) \rightarrow PW[-h, 0]_n + (z - b)^{-1}\mathbb{C}^n$ are defined as follows:

$$\mathcal{H}(\delta) = \{(x_-, x_+) \in H_{-,n}^2 \oplus H_{+,n}^2 : \delta x_+ = x_- \text{ on } \ell\},$$

$$W_\delta(x_-, x_+) = x_- + \delta H_{-,n}^2$$

and

$$S_\delta(x_-, x_+)(z) = \begin{cases} x_-(z), & z \in \bar{\Pi}_- \\ \delta(z)x_+(z), & z \in \bar{\Pi}_+. \end{cases}$$

Clearly, one can replace δ by δ^T in the above lemma and remark.

Lemma 3.4. *Let $[c, d]$ be an interval of $\mathbb{R}$,*

$$\varphi(z) = \mathcal{L}\nu := \int_{[c,d]} e^{-zt} d\nu(t),$$

where ν is a finite Borel measure supported on $[c, d]$. If

$$\psi(z) = \frac{\varphi(z) - \varphi(b)}{z - b}$$

then $\psi \in PW[c, d]$.

Proof. Integration by parts shows that $\psi = \mathcal{L}\xi$, where

$$\xi(t) = \int_t^d e^{b(t-s)} d\nu(s), \quad t \in [c, d]$$

and $\xi(t) = 0$ for $t \in \mathbb{R} \setminus [c, d]$. Since $\xi \in L^2[c, d]$, the lemma follows from the Paley-Wiener theorem. $\square$

Define an $n \times n$ matrix entire function β by

$$\beta(z) = \frac{\Delta_-(z) - \Delta_-(b)}{z - b} - e^{hz} m(z).$$

Since $\Delta_-(z) = e^{hz}(zm(z) - z\mathcal{L}\eta(z) - \mathcal{L}\zeta(z))$, it follows that

$$\beta(z) = e^{hz} \left[b \frac{m(z) - m(b)}{z - b} - b \frac{\mathcal{L}\eta - (\mathcal{L}\eta)(b)}{z - b} - \mathcal{L}\eta - \frac{\mathcal{L}\zeta - (\mathcal{L}\zeta)(b)}{z - b} \right].$$

So Lemma 3.4 implies that β belongs to $PW[-h, 0]_{n \times n}$.

Proof of Lemma 3.1. For any $g \in PW[-h, 0]_n$ and any $d \in \mathbb{C}^n$, one has

$$g + (z - b)^{-1}d = g_1 - (e^{hz}m^T - \delta^T)\Delta_-^T(b)^{-1}d,$$

where

$$g_1 := g - \beta^T \Delta_-^T(b)^{-1}d \in PW[-h, 0]_n.$$

In particular, this identity implies that

$$PW[-h, 0]_n + \frac{1}{z - b} \mathbb{C}^n = PW[-h, 0]_n + (e^{hz}m^T - \delta^T) \mathbb{C}^n.$$

Since, clearly, V is an isomorphism from Z onto $PW[-h, 0]_n + (e^{hz}m^T - \delta^T) \mathbb{C}^n$, the proof follows from Lemma 3.3. $\square$

Remark. Lemma 3.1 implies that V_2 is an isomorphism.

Before we can prove Lemma 3.2 we need one more observation. Let $f \in PW[-h, 0]$ and define

$$k(z) = \frac{f(z) - f(b)}{z - b}, \quad z \in \mathbb{C}.$$

From Lemma 3.4 it follows that $k \in PW[-h, 0]$. Furthermore, the following relation holds:

$$zf(z) = (z - b)g(z) + bf(b), \quad g(z) := f(z) + bk(z)$$

which gives

$$zPW[-h, 0]_n + \mathbb{C}^n = (z - b)PW[-h, 0]_n + \mathbb{C}^n. \quad (3.2)$$

Proof of Lemma 3.2. Let Z_0 be the subset of those vectors $(c, \varphi)^T$ in Z for which

$$\varphi \in C^\infty[-h, 0]_n \quad \text{and} \quad \varphi(0) = \varphi(-h) = 0.$$

Since U and V are bounded and Z_0 is dense in Z , it suffices to show that

$$V^*U \begin{pmatrix} c \\ \varphi \end{pmatrix} = \begin{pmatrix} c \\ \varphi \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} c \\ \varphi \end{pmatrix} \in Z_0.$$

For any $(c, \varphi)^T \in Z_0$ define

$$u = \frac{1}{z - b} U_1 \begin{pmatrix} c \\ \varphi \end{pmatrix}$$

Then, see (3.2),

$$u \in PW[-h, 0]_n + (z - b)^{-1} \mathbb{C}^n$$

and, by (2.5),

$$U \begin{pmatrix} c \\ \varphi \end{pmatrix} = J_\delta u.$$

Using the representation for V^* given in (1.19) we have

$$V^* J_\delta u = \begin{pmatrix} d \\ \psi \end{pmatrix},$$

where

$$\begin{aligned} d &= \frac{1}{2\pi i} \int_{\ell} (e^{hz} m(z) \delta(z)^{-1} - I) u(z) dz, \\ \psi(\theta) &= \frac{1}{2\pi i} \int_{\ell} e^{(\theta+h)z} \delta(z)^{-1} u(z) dz, \quad -h \leq \theta \leq 0. \end{aligned} \quad (3.3)$$

So we have to prove that $c = d$ and $\varphi = \psi$. Set $v_- = \mathcal{L}\varphi$. Using representation (1.4) for Δ and integration by parts, we can write

$$\begin{aligned} (\Delta v_-)(z) &= z \int_0^h d\mu(\theta) \int_{-h}^0 e^{z(-\theta-t)} \varphi(t) dt - \int_0^h \eta(\theta) d\theta \int_{-h}^0 e^{z(-\theta-t)} \varphi'(t) dt \\ &\quad - \int_0^h \zeta(\theta) d\theta \int_{-h}^0 e^{z(-\theta-t)} \varphi(t) dt. \end{aligned}$$

Define

$$\begin{aligned} M_- &= \int_0^h d\mu(\theta) \int_{-h}^{-\theta} e^{z(-\theta-t)} \varphi(t) dt, \\ M_+ &= \int_0^h d\mu(\theta) \int_{-\theta}^0 e^{z(-\theta-t)} \varphi(t) dt, \\ w_-(z) &= - \int_0^h \eta(\theta) d\theta \int_{-h}^{-\theta} e^{z(-\theta-t)} \varphi'(t) dt \\ &\quad - \int_0^h \zeta(\theta) d\theta \int_{-h}^{-\theta} e^{z(-\theta-t)} \varphi(t) dt \end{aligned}$$

and

$$\begin{aligned} w_+(z) &= - \int_0^h \eta(\theta) d\theta \int_{-\theta}^0 e^{z(-\theta-t)} \varphi'(t) dt \\ &\quad - \int_0^h \zeta(\theta) d\theta \int_{-\theta}^0 e^{z(-\theta-t)} \varphi(t) dt. \end{aligned}$$

It is easy to see that $M_{\pm} \in H_{\pm,n}^2$, $w_{\pm} \in H_{\pm,n}^2$ and

$$\begin{aligned} mv_- &= M_- + M_+, \\ \Delta v_- &= zmv_- + w_- + w_+. \end{aligned}$$

From the representation for U given in (1.13) it follows, using (1.12) and integration by parts, that

$$u = (z - b)^{-1}(c + w_- + zM_-).$$

So

$$\begin{aligned} \delta^{-1}(z)u(z) &= e^{-hz} \Delta^{-1}(z)(c + w_-(z) + zM_-(z)) \\ &= e^{-hz} \Delta^{-1}(z)(c - w_+(z) - zM_+(z)) + e^{-hz} v_-(z) \\ &= r_+(z) + e^{-hz} q_+(z), \end{aligned} \tag{3.4}$$

where

$$r_+(z) = e^{-hz} v_-(z) = \int_{-h}^0 \varphi(t) e^{-z(t+h)} dt \quad \text{and} \quad q_+(z) = \Delta^{-1}(z)(c - w_+(z) - zM_+(z)).$$

It is clear that $r_+ \in PW[0, h]$, $e^{-hz}q_+ \in e^{-hz}H_{+,n}^2$ and the inversion formula for the Laplace transform and the second equation in (3.3) implies that $\psi = \varphi$. From (3.4)

$$\begin{aligned} (e^{hz}m\delta^{-1} - I)u &= m\Delta(z)^{-1}(c - w_+ - zM_+) + M_- + M_+ \\ &\quad - \frac{1}{z-b}(c + w_- + zM_-) \\ &= \left[\frac{-b}{z-b}M_- - \frac{1}{z-b}w_- \right] + [M_+ - m\Delta(z)^{-1}(w_+ + zM_+)] \\ &\quad + \left[m\Delta(z)^{-1}c - \frac{1}{z-b}c \right]. \end{aligned} \quad (3.5)$$

Since the left hand side and the first and third expressions in the brackets on the right hand side of (3.5) belong to $L^1(\ell)_n$, it follows that the second expression in the brackets on the right hand side of (3.5) belong to $L^1(\ell)_n$ as well. So

$$\frac{-b}{z-b}M_- - \frac{1}{z-b}w_- \in H_{-,n}^1, \quad M_+ - m\Delta(z)^{-1}(w_+ + zM_+) \in H_{-,n}^1.$$

Integrating identity (3.5) along ℓ and bearing in mind (1.9) we get

$$d = \left[\frac{1}{2\pi i} \int_{\ell} \left(m(z)\Delta(z)^{-1} - \frac{1}{z-b}I \right) dz \right] c.$$

Let b_- be any point in Π_- . Since $(z-b_-)m-\Delta \in (z-b_-)L^2(\ell)_{n \times n}$ and $\Delta(z)^{-1} \in L^2(\ell)_{n \times n}$, it follows that

$$m(z)\Delta(z)^{-1} - \frac{1}{z-b_-}I = \frac{1}{z-b_-}[(z-b_-)m(z) - \Delta(z)]\Delta(z)^{-1} \in H_{+,n \times n}^1.$$

So

$$\begin{aligned} \frac{1}{2\pi i} \int_{\ell} \left(m(z)\Delta(z)^{-1} - \frac{1}{z-b}I \right) dz &= \frac{1}{2\pi i} \int_{\ell} \left(m(z)\Delta(z)^{-1} - \frac{1}{z-b_-}I \right) dz \\ &\quad + \frac{1}{2\pi i} \int_{\ell} \left(\frac{1}{z-b_-} - \frac{1}{z-b} \right) I dz \\ &= 0 + I \end{aligned}$$

and this shows $d = c$. □

Remark. In fact, U_1 maps Z onto $zPW[-h, 0]_n + \mathbb{C}^n$ (see (2.7)). Indeed, it follows from (3.2) that this assertion is equivalent to the assertion that the operator

$$\alpha \mapsto (z-b)^{-1}U_1\alpha$$

maps Z onto $PW[-h, 0]_n + (z-b)^{-1}\mathbb{C}^n$. But this follows from Lemma 3.3 and the fact that U is an isomorphism.

4 Applications

In this section we interpret some known facts in the theory of delay equation in our model and also derive from it certain new results.

4.1 Inner-outer factorization

We start to recall some basic facts about factorizations, both in the scalar and the matrix setting. The basic references are [5] and [8].

The Beurling-Lax-Halmos theorem implies that nonzero closed invariant subspaces of M_z (or, more rigorously, of $M_{(z-b)^{-1}}$) in $H^2_{-,n}$ are precisely the spaces $\theta H^2(\Pi_-)_m$, where $1 < m \leq n$ and $\theta \in H^\infty(\Pi_-)_{n \times m}$ is *inner* that is $\theta(z)$ is an isometry for almost every $z \in \ell$. The product of two $m \times m$ matrix inner functions is again an inner function. A matrix function $g \in H^\infty(\Pi_-)_{m \times n}$ is called *outer* if the closure of $gH^2(\Pi_-)_n$ coincides with $H^2(\Pi_-)_m$; this is only possible if $m \leq n$. Each nonzero matrix function $\varphi \in H^\infty(\Pi_-)_{n \times n}$ admits an inner-outer factorization $\varphi = \varphi_i \varphi_e$, where $\varphi_i \in H^\infty(\Pi_-)_{n \times m}$ is inner, $\varphi_e \in H^\infty(\Pi_-)_{m \times n}$ is outer and $1 \leq m \leq n$. Moreover, the inner-outer factorization is unique up to multiplication by a constant unitary $m \times m$ matrix.

Let us remind the Nevanlinna theory of factorization of scalar inner functions. For any $\lambda \in \Pi_-$ define $\lambda_* := 2a - \bar{\lambda}$ to be the symmetric point to λ with respect to ℓ . The function

$$b_\lambda(z) = \frac{z - \lambda}{z - \lambda_*} \frac{|1 - (\lambda - a)^2|}{1 - (\lambda - a)^2}$$

maps Π_- onto the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and is called a Blaschke factor (the second factor is omitted if $\lambda = a - 1$). A countable subset τ of Π_- and a multiplicity function $k : \tau \rightarrow \mathbb{N}$ are said to satisfy the *Blaschke condition* if

$$\sum_{\lambda \in \tau} k(\lambda) (\operatorname{Im} \lambda - a) (1 + |\lambda|^2)^{-1} < \infty. \quad (4.1)$$

If condition (4.1) holds the infinite product

$$b(z) = \xi \prod_{\lambda \in \tau} b_\lambda^{k(\lambda)}(z), \quad \xi \text{ constant and } |\xi| = 1 \quad (4.2)$$

converges absolutely on compact subsets of Π_- and represents an inner function. Functions of the type (4.2) are called Blaschke products. An inner function s in Π_- is called *singular* if s has no zeros in Π_- . A singular inner function has the following integral representation

$$s(z) = \xi e^{\alpha(z-a)} \exp \left[\frac{1}{\pi i} \int_0^\infty \left(\frac{1}{t + i(z-a)} - \frac{t}{1+t^2} \right) d\mu(t) \right], \quad (4.3)$$

where ξ is constant, $|\xi| = 1$, $\alpha > 0$ and μ is a nonnegative singular Borel measure on $\mathbb{R}$. Each (scalar) inner function φ_i has a representation

$$\varphi_i(z) = b(z)s(z),$$

where b is a Blaschke product and s is a singular inner function. The representation is unique up to multiplication by a constant with modulus one.

If θ and θ_1 are scalar inner functions, we say that θ divides θ_1 if $\theta = \theta_1\theta_2$ for an inner function θ_2 . Every family $\{\theta_\alpha\}_{\alpha \in A}$ of inner functions has a unique (up to multiplication by a constant with modulus one) greatest common divisor $\theta = \gcd(\{\theta_\alpha\})$ which is the inner function defined by the properties

- (i) $\theta_\alpha | \theta$ for all $\alpha \in A$;
- (ii) if $\theta_\alpha | \theta_*$ for all $\alpha \in A$ then $\theta | \theta_*$.

One can find θ using the relation $\theta H^2 = \text{span} \{\theta_\alpha H^2 : \alpha \in A\}$ or using the Nevanlinna parameterization of inner functions described above.

Analogous results hold if Π_- is replaced by any other half plane.

4.2 A relation with the Sz. Nagy—Foiş model theory

An (unbounded) operator B on a Hilbert space H with dense domain is called *dissipative* if $\text{Im}(Bf, f) \geq 0$ for all $f \in \mathcal{D}(B)$. If in addition, $(B + iI)\mathcal{D}(B) = H$, B is called *maximal dissipative*. B is called (maximal) *accretive* if iB is (maximal) dissipative.

If $\delta_1(z) = \delta(a + iz)$ where δ is defined as in (1.6), then the operator $-i(M_z - a)$ on $H^2_{-,n}/\delta H^2_{-,n}$ can be equivalently represented as

$$M_z \quad \text{on } H_n^2(\Pi_u)/\delta_1 H_n^2(\Pi_u),$$

where

$$\Pi_u = \{z \in \mathbb{C} : \text{Im } z > 0\}$$

is the upper half-plane. In the sequel we shall denote this operator by B .

Consider the inner-outer factorization $\delta_1 = \delta_{1i}\delta_{1e}$ of δ_1 . Since $\delta_1(z)$ is invertible for some $z \in \Pi_u$, it follows that $m = n$ in this factorization. Furthermore, in our case, the matrices $\delta_1(z)^{-1}$ exist everywhere on $\mathbb{R}$ and their norms are uniformly bounded. So δ_{1e} is invertible in the algebra $H_{n \times n}^\infty(\Pi_u)$ and

$$\delta_1 H_n^2(\Pi_u) = \delta_{1i} H_n^2(\Pi_u).$$

If we consider the usual Hilbert space structure on $H_n^2(\Pi_u)$, the quotient space

$$H_n^2(\Pi_u)/\delta_{1i} H_n^2(\Pi_u)$$

becomes isomorphic to the orthogonal complement

$$K = H_n^2(\Pi_u) \ominus \delta_{1i} H_n^2(\Pi_u). \quad (4.4)$$

On K , we have the following representation for the resolvent of B :

$$(B - \lambda I)^{-1}f = P_K(z - \lambda)^{-1}f, \quad f \in K, \quad \text{Re } \lambda < 0; \quad (4.5)$$

here P_K denotes the orthogonal projection onto K . From (4.4) and (4.5) it follows that B considered as an operator on K is maximal dissipative, given already by its Sz-Nagy—Foiş model. The characteristic function is unitarily equivalent to δ_{1i} .

In combination with Theorem 1.1 this yields the following result.

Proposition 4.1. *The operator $-i(A - aI)$ is similar to a maximal dissipative operator.*

The model theory gives an explicit expression for the characteristic function of a dissipative operator. However, unlike the situation with δ and δ_1 , we have no explicit expression for δ_{1i} which is the characteristic function for B when considered as an operator on K . Therefore it seems hard to obtain our model representation for A directly from the Sz.-Nagy—Foiş model theory.

An equally strong relation exists with the theory of contractions. To each maximal dissipative operator B there corresponds a unique contraction T with $1 \notin \sigma_p(T)$ given by the Cayley transform:

$$B = i(I + T)(I - T)^{-1}$$

(see Section 4.4 of [11].) For the operator B defined above, the corresponding contraction is modeled by

$$Tg = P_{K'}(zg), \quad g \in H^2(\mathbb{D})_n,$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $K' = H^2(\mathbb{D})_n \ominus \delta_2 H^2(\mathbb{D})_n$ and

$$\delta_2(\xi) = \delta_1\left(i\frac{1+\xi}{1-\xi}\right).$$

T is called the operator of restricted shift. So we can also say that

$$(A - (a - 1)I)(A - (a + 1)I)^{-1}$$

is similar to a restricted shift. Using these conformal transformations, we can apply the well-developed spectral theory for accretive, dissipative operators and for contractions to study the operator A defined by (1.2). We shall use the theorems originally stated for contractions, but using the Cayley transform, the symmetry $z \mapsto a - z$ and the similarity-invariance of the results, we can apply them directly to the operator A defined by (1.2). In particular, we shall extensively use the results from the book by Nikolski [8] which is especially devoted to the study of the model operator of restricted shift.

4.3 The H^∞ -calculus

By applying the Sz.-Nagy—Foiş model theory one can introduce the H^∞ -calculus for A as follows. For $\psi \in H^\infty(\Pi_-)$ the operator M_ψ given by

$$M_\psi(f + \delta H_{-,n}^2) := \psi f + \delta H_{-,n}^2$$

is well-defined and bounded on the model space $H_{-,n}^2/\delta H_{-,n}^2$. Therefore, if we define

$$\psi(A) = U^{-1}M_\psi U, \quad \psi \in H^\infty(\Pi_-), \quad (4.6)$$

it follows that

$$\|\psi(A)\| \leq C\|\psi\|_{H^\infty(\Pi_-)}, \quad (4.7)$$

where $C = \|U\| \|U^{-1}\|$.

Let δ^{adj} be the transpose associate matrix to δ . Since $\det \delta I = \delta \delta^{\text{adj}}$, the operator $M_{\det \delta}$ maps $H_n^2(\Pi_u)$ into $\delta H_n^2(\Pi_u)$. Therefore

$$(\det \delta)(A) = 0. \quad (4.8)$$

In the terminology of Sz.-Nagy—Foiş, one can say that $aI - A$ is similar to a C_0 -accretive operator. Consider all scalar inner functions θ such that $\theta(A) = 0$ and $\theta_0 = \gcd\{\theta : \theta(A) = 0\}$. Using the results from Section 3.4 of [11] one has $\theta_0(A) = 0$. The function θ_0 is called the minimal annihilator of A .

4.4 Completeness of generalized eigenspaces of A

In this section we deduce a completeness result by the first author [16, 17] from our model representation for A (in the Hilbert space setting). Let $\mathcal{A}(\Pi_-)$ be the class of functions F that are holomorphic in Π_- , of exponential type and such that

$$\frac{\log |F|}{1 + |z|^2} \in L^1(\partial\Pi_-).$$

For each $F \in \mathcal{A}(\Pi_-)$, there exists a finite number $\alpha(F)$ such that for almost all $\vartheta \in (\pi/2, 3\pi/2)$,

$$\lim_{r \rightarrow \infty} \frac{\log |F(re^{i\vartheta})|}{r} = \alpha(F) |\cos \vartheta|.$$

We will refer to $\alpha(F)$ as the *mean type* of F . For $F, G \in \mathcal{A}(\Pi_-)$, $FG \in \mathcal{A}(\Pi_-)$ and it follows that $\alpha(FG) = \alpha(F)\alpha(G)$ (see [1] for details). As explained above, each function in $H^\infty(\Pi_-)$ admits a factorization

$$F(z) = g(z)b(z)s(z),$$

where g is outer, b is a Blaschke product and s is singular. Furthermore, in this case, $\alpha(g) = \alpha(b) = 0$ and $\alpha(F) = \alpha(s) = -\alpha$, where $\alpha > 0$ is the number in (4.3).

For any eigenvalue λ of A , put k_λ to be the length of the maximal chain of the generalized eigenvectors corresponding to λ . The generalized eigenspace of A is defined as

$$\mathcal{M}_\lambda = \text{Ker}(A - \lambda I)^{k_\lambda}.$$

We also have $\mathcal{M}_\lambda = \text{Ker}(A - \lambda I)^k$ for $k > k_\lambda$. The linear span

$$\mathcal{M} = \text{span}\{\mathcal{M}_\lambda : \lambda \in \sigma_p(A)\}$$

is called the *generalized eigenspace* of A . The generalized eigenvectors of A are said to be complete if $\overline{\mathcal{M}} = Z$.

Theorem 4.2. (Verduyn Lunel [16, 17]). *The generalized eigenvectors of A are complete if and only if $\alpha(\det \Delta) = nh$.*

Before we shall give an alternative proof of this theorem, we give a description of the generalized eigenspace of A in our model. First we introduce some definitions. An $n \times n$ inner function Ψ is called *singular* if $\det \Psi$ is singular, a *Blaschke-Potapov product* if $\det \Psi$ is a Blaschke product. Let $\delta = \delta_i \delta_e$ be the inner-outer factorization of δ where δ is defined as in (1.6). A theorem by Potapov [10] yields factorizations

$$\delta_i = \tilde{B} \tilde{S} = SB, \quad (4.9)$$

where B and $\tilde{B}$ are Blaschke-Potapov products, S and $\tilde{S}$ are singular. The factorizations are unique up to multiplication by constant unitary factors. Each Blaschke-Potapov product can be decomposed as a product of Blaschke-Potapov factors:

$$B(\lambda) = \xi \prod_n (b_{\lambda_n} P_n + (I - P_n)), \quad \xi \text{ constant, } |\xi| = 1,$$

where the λ_n satisfy the Blaschke condition (4.1) and the P_n are orthogonal projections in $\mathbb{C}^n$. (For a multiplicative representation of singular functions we refer to [10].)

Let

$$\det \delta_i = sb$$

denote the decomposition of $\det \delta_i$ into its singular inner part s and a scalar Blaschke product b . Then the above mentioned unitary factors can be chosen so that

$$b = \det B = \det \tilde{B}, \quad s = \det S = \det \tilde{S}$$

We need the following lemma.

Lemma 4.3. *In the model representation*

$$UM = SH_{-,n}^2 / \delta_i H_{-,n}^2. \quad (4.10)$$

This lemma essentially follows from [8, Lecture 4], but we include a proof for completeness.

Proof. Define the scalar Blaschke product

$$b_0(z) = \prod_{\lambda \in \sigma(A)} b_\lambda(z)^{k(\lambda)}.$$

The order of the pole of δ_i^{-1} at $\lambda \in \sigma_p(A)$ is the least number $l(\lambda)$ such that

$$\text{Ker}(M_z - \lambda I)^m = \text{Ker}(M_z - \lambda I)^{l(\lambda)} \quad \text{for } m \geq l(\lambda).$$

Therefore this order equals $k(\lambda)$ and it follows that

$$b(z) = \prod_{\lambda \in \sigma(A)} b_\lambda(z)^{k'(\lambda)} \quad \text{for some } k'(\lambda) \geq k(\lambda).$$

Applying the results from [8, Lecture 4, Sect. 5], one derives that

$$U\mathcal{M}_\lambda = \text{Ker } b_\lambda^{k(\lambda)}(M_z), \quad U\mathcal{M} = \text{Ker } b_0(M_z) = \text{Ker } b(M_z).$$

Next we verify the two inclusions in (4.10).

1) If $f = Sf_1 \in SH_{-,n}^2$ then

$$bf = SBB^{\text{adj}} f_1 \in \delta_i H_{-,n}^2$$

and so

$$f + \delta H_{-,n}^2 \in \text{Ker } b(M_z) = U\mathcal{M}.$$

2) Conversely, if $bf = SBg$ for some $g \in H_{-,n}^2$, then

$$S^{\text{adj}} bf = sBg.$$

Since s and b are relatively prime, we conclude that $S^{\text{adj}} f = sr$ for some $r \in H_{-,n}^2$ which implies $f = Sr$. $\square$

From the lemma it follows, in particular, that the generalized eigenvectors of A are complete if and only if S is a constant. But S is a constant if and only if $\det \delta_i$ is a Blaschke product. Since $m_A | \det \delta_i | m_A^n$ another equivalent condition is that m_A is a Blaschke product. This latter criterium (in a more general situation) is proved in Lecture 4 of [8].

Proof of Theorem 4.2. Let $\delta = \delta_i \delta_e$ be the inner-outer factorization of δ in Π_- . Since the scalar inner function $\det \delta_i(z)$ is analytic on $\partial\Pi_-$, the singular part of $\det \delta_i(z)$ necessarily has the form

$$s(z) = e^{\beta(z-a)},$$

where $\beta = -\alpha(\det \delta_i) \geq 0$. Therefore the equality $\overline{\mathcal{M}} = Z$ holds if and only if $\alpha(\det \delta_i) = 0$. Since $|\det \delta_e| \geq \epsilon > 0$ in Π_- , $\alpha(\det \delta_e) = 0$. From (1.3)

$$\det \delta(z) = e^{nhz} (z-b)^{-n} \det \Delta(z).$$

Hence

$$\alpha(\det \Delta) = nh + \alpha(\det \delta) = nh + \alpha(\det \delta_i) \quad (4.11)$$

and this proves the theorem. $\square$

We remark that (4.11) implies

$$\beta = -\alpha(\det \delta_i) = nh - \alpha(\det \Delta).$$

Since $\alpha(\det \Delta) \geq 0$, one has $\beta \leq nh$.

4.5 Small solutions

In this section we recover some of the results by the first author [15] on the existence of small solutions and the relation with completeness of the generalized eigenvectors of A (in the case of Hilbert spaces) by applying our model.

A solution x of system (1.1) is called *small* if

$$\lim_{t \rightarrow \infty} e^{\alpha t} \|x_t\| = 0 \quad \text{for all } \alpha \in \mathbb{R}.$$

A small solution that is not identically zero is called a *nontrivial* small solution. Let $\mathcal{S}$ denote the linear subspace of those data $(c, \varphi)^T \in Z$ that give rise to small solutions. We shall prove the following theorem

Theorem 4.4. *Define $\tilde{B}$, $\tilde{S}$ as given in the factorization (4.9).*

(i) *In the model representation one has the following formula for $\mathcal{S}$*

$$U\mathcal{S} = \tilde{B}H_{-,n}^2/\delta H_{-,n}^2.$$

(ii) *Nontrivial small solutions exist if and only if the generalized eigenspaces of A are not complete.*

(iii) *Each small solution x of (1.1) satisfies $x(t) = 0$ for $t \geq nh$.*

(iv) *If $\mathcal{M}$ denotes the generalized eigenspace of A , then*

$$\mathcal{S} \cap \overline{\mathcal{M}} = \{0\} \quad \text{and} \quad \overline{\mathcal{S} + \mathcal{M}} = Z.$$

The proof will be based on the following lemmas.

Lemma 4.5. *If $u \in H_{-,n}^2$ then $\delta^{-1}u$ is analytic in Π_- if and only if $u \in \tilde{B}H_{-,n}^2$.*

Lemma 4.6.

$$U\mathcal{S} = \{u \in H_{-,n}^2/\delta H_{-,n}^2 : \delta^{-1}u \text{ is analytic in } \Pi_-\}.$$

Proof of Lemma 4.5. The linear manifold

$$X = \{u \in H_{-,n}^2 : \delta^{-1}u \text{ is analytic in } \Pi_-\}$$

is invariant under $M_{(z-b)^{-1}}$ and closed. Indeed, the condition on u amounts to a finite number of linear equations in each of the points of $\sigma(A)$. Therefore $X = B_1 H_{-,m}^2$ for some $n \times m$ inner function B_1 . Since $X \supset \tilde{B}H_{-,n}^2$, it follows that $n = m$ and $\tilde{B} = B_1 B_2$ for an inner function B_2 . Since $\det B_2 | \det \tilde{B}$, $\det B_2$ is a Blaschke product.

For each $c \in \mathbb{C}^n$, $B_1 c \in X$ which yields that $\tilde{B}^{-1}B_1 = B_2^{-1}$ is analytic in Π_- . Therefore $\theta = \det B_2$ is a Blaschke product with the property that θ^{-1} is analytic in Π_- . But this implies that B_2 is a constant. $\square$

Proof of Lemma 4.6. 1) “ $\subset$ ”. If $u \in H_{-,n}^2$ and $\delta^{-1}u$ is not analytic in Π_- , then $U^{-1}(u + \delta H_{-,n}^2)$ cannot give rise to a small solution. To prove this claim, suppose that $\delta^{-1}u$ has exactly a k^{th} -order pole at a point $\lambda_0 \in \sigma(A)$. Define a linear operator $\psi : H_{-,n}^2/\delta H_{-,n}^2 \rightarrow \mathbb{C}^n$ by

$$\psi(v + \delta H_{-,n}^2) = c_{-k}, \quad \text{where } \delta^{-1}(z)v(z) = \sum_{n=-\infty}^{\infty} c_n(z - \lambda_0)^n.$$

Since

$$\psi(M_{e^{tz}}u) = e^{t\lambda_0}\psi(u),$$

it follows that there is an $\epsilon > 0$ such that

$$\|M_{e^{tz}}u\|_{H_{-,n}^2/\delta H_{-,n}^2} \geq \epsilon e^{t\operatorname{Re} \lambda_0}.$$

If $(c, \varphi)^T = U^{-1}u$ and $x = x(\cdot; c, \varphi)$ denotes the solution of (1.1) corresponding to $(c, \varphi)^T$, then

$$\left\| \begin{pmatrix} Mx_t \\ x_t \end{pmatrix} \right\|_Z = \|U^{-1}M_{e^{tz}}u\|_Z \geq \epsilon \|U\|^{-1} e^{t\operatorname{Re} \lambda_0}$$

which proves our claim.

2) Conversely, if $\delta^{-1}u$ is analytic in Π_- then by Lemma 4.5, $u = \tilde{B}v$ for a certain $v \in H_{-,n}^2$. Since $\det \tilde{S} = \xi e^{\beta(z-a)}$ and $\beta \leq nh$, we have

$$e^{nhz}u = e^{(nh-\beta)z+\beta a}\xi^{-1}\tilde{B}\tilde{S}\tilde{S}^{\operatorname{adj}}v \in \delta H_{-,n}^2$$

(here $\xi = \text{const}$, $|\xi| = 1$). Therefore

$$M_{e^{nhz}}(u + \delta H_{-,n}^2)$$

is the zero element of the model space. Therefore

$$T(t)U^{-1}u = 0 \quad \text{for } t \geq nh$$

and, in particular, $U^{-1}u$ is a small solution. $\square$

Proof of Theorem 4.4. The assertions (i) and (iii) follow immediately from Lemmas 4.5 and 4.6 and their proofs. Using (i), we have $\mathcal{S} \neq \{0\}$ if and only if $\det \delta_i$ is a Blaschke product if and only if the generalized eigenspaces of A are complete. This shows (ii).

To prove (iv) first note that

$$US = \tilde{B}H_{-,n}^2/\delta H_{-,n}^2 \quad \text{and} \quad U\mathcal{M} = SH_{-,n}^2/\delta H_{-,n}^2.$$

Define $n \times n$ inner functions Ψ and Φ by the formulas

$$\tilde{B}H_{-,n}^2 \cap SH_{-,n}^2 = \Psi H_{-,n}^2$$

and

$$\overline{\tilde{B}H_{-,n}^2 + SH_{-,n}^2} = \Phi H_{-,n}^2.$$

(A rank argument easily shows that Ψ and Φ are indeed square matrix functions.) The inclusion

$$\Phi H_{-,n}^2 \supset \tilde{B}H_{-,n}^2$$

implies that $\tilde{B} = \Phi \tilde{B}_1$ for an $n \times n$ matrix inner function $\tilde{B}_1$. Similarly, there exist $n \times n$ inner functions S_1, X and Y such that

$$S = \Phi S_1, \quad \Psi = \tilde{B}X \quad \text{and} \quad \Psi = SY.$$

By taking determinants, we obtain

$$b = (\det \Phi)(\det \tilde{B}_1) \quad \text{and} \quad s = (\det \Phi)(\det S_1).$$

This implies $\det \Phi$ is constant and $\Phi H_{-,n}^2 = H_{-,n}^2$. Similarly, $\det \Psi|b$ and $\det \Psi|s$ imply that $\det \Psi|\det \delta_i$ and together with the inclusion $\Psi H_{-,n}^2 \supset \delta_i H_{-,n}^2$ this yields $\Psi H_{-,n}^2 = \delta_i H_{-,n}^2$. $\square$

4.6 Riesz bases of eigenspaces and of generalized eigenspaces of A

In this section we assume that the generalized eigenspaces $\mathcal{M}_\lambda$ of A are complete and we shall obtain criteria for $\mathcal{M}_\lambda$ to form a Riesz basis. We start with some definitions.

Let $\{L_n\}_{n=1}^\infty$ be closed subspaces of a Hilbert space H . The family $\{L_n\}$ forms a *Riesz basis in their linear span* if there exists a linear isomorphism G of $\overline{\text{span}}\{L_n\}$ onto a Hilbert space H_1 such that $\{GL_n\}$ are orthogonal subspaces of H_1 . If $\{L_n\}$ forms a Riesz basis, each $f \in \overline{\text{span}}\{L_n\}$ admits a unique decomposition $f = \sum_{n=1}^\infty l_n$, where $l_n \in L_n$ and $\sum \|l_n\|^2 < \infty$. Conversely, each series $\sum_{n=1}^\infty l_n$ with $l_n \in L_n$ and $\sum \|l_n\|^2 < \infty$ gives rise to an element f in $\overline{\text{span}}\{L_n\}$.

The system $\{L_n\}$ is called *minimal* if for each n , the equations

$$Q_n|_{L_n} = I \quad \text{and} \quad Q_n|_{L_k} = 0, \quad k \neq n,$$

define a bounded projection Q_n on $\overline{\text{span}}\{L_n\}$. The system $\{L_n\}$ is called *uniformly minimal* if

$$\sup_n \|Q_n\| < \infty.$$

From the definitions it follows that if $\{L_n\}$ forms a Riesz basis then $\{L_n\}$ is uniformly minimal.

For any $\lambda \in \sigma(A)$, the results by Potapov allow to put all the Blaschke-Potapov factors corresponding to λ in the factorization of δ either on the left or on the right. So we arrive at the factorizations

$$\delta = R_\lambda B_\lambda = \tilde{B}_\lambda \tilde{R}_\lambda \quad (4.12)$$

with the properties

- (i) $\det B_\lambda = \det \tilde{B}_\lambda$ is a natural power of the Blaschke factor b_λ ;
- (ii) $R(\lambda)$ and $\tilde{R}(\lambda)$ are invertible.

We remark that

$$\text{Ker}(A - \lambda I) = \{f_{\lambda,c} : \delta(\lambda)c = 0\}, \quad (4.13)$$

where

$$f_{\lambda,c} = \begin{pmatrix} m(\lambda)c \\ e^{\lambda x} c \end{pmatrix}. \quad (4.14)$$

It follows from (1.8) that the images of these eigenvectors are given by

$$U f_{\lambda,c} = e^{-\lambda h} \frac{\delta(z)c}{z - \lambda}. \quad (4.15)$$

(as expected, $(M_z - \lambda I)U f_{\lambda,c} = 0$). If $k_\lambda = 1$ then $\mathcal{M}_\lambda = \text{Ker}(A - \lambda I)$.

Lemma 4.7. *The generalized eigenspace $\mathcal{M}_\lambda$ in the model representation has the form*

$$U\mathcal{M}_\lambda = R_\lambda H_{-,n}^2 / \delta H_{-,n}^2.$$

Proof. Using the fact that

$$\mathcal{M}_\lambda = \text{Ker } b_\lambda^{k_\lambda}(A)$$

(see [8, Lect. 4, Sect. 5]), one can proceed in the same way as in the proof of Lemma 4.3. $\square$

Let $\delta = \delta_i \delta_e$ be the inner-outer factorization of δ and

$$B_\lambda \delta_e^{-1} = \delta_{\lambda,e}^{(1)} B_\lambda^{(1)}$$

the outer-inner factorization of $B_\lambda \delta_e^{-1}$, then $\det B_\lambda^{(1)} = \det B_\lambda$ and $\delta_{\lambda,e}^{(1)}$ is outer and invertible in $H_{n \times n}^\infty(\Pi_-)$. From (4.12) we derive the following representations

$$\delta_i = R_{i,\lambda} B_\lambda^{(1)} = \tilde{B}_\lambda \tilde{R}_{i,\lambda}, \quad (4.16)$$

where

$$R_{i,\lambda} = R_\lambda \delta_{\lambda,e}^{(1)}, \quad \tilde{R}_{i,\lambda} = \tilde{R}_\lambda \delta_e^{-1}$$

and all factors in (4.16) are Blaschke-Potapov products.

The next theorem is a direct consequence of the result that a system of spaces $R_j H_{-,n}^2 \ominus \theta H_{-,n}^2$, where R_j and θ are $n \times n$ inner functions is a Riesz basis if and only if it is uniformly minimal. The origin of this result dates back to Newmann, Carleson, Shapiro, Shields, Nikol'skiĭ and Pavlov in the case when these subspaces are eigenspaces of a scalar restricted shift; Vasyunin [14] has obtained this result for the general scalar case and Treil [12] in the vector valued case which we shall use here. We refer to [8, Lect. 6] and to [12] for much more information on Riesz basis of invariant subspaces of model operators and for a bibliography.

Theorem 4.8. *Suppose that the generalized eigenspaces $\mathcal{M}_\lambda$ of A are complete.*

(i) *The system $\{\mathcal{M}_\lambda\}_{\lambda \in \sigma(A)}$ is a Riesz basis if and only if*

$$\inf_{\lambda \in \sigma(A)} \inf_{\substack{z \in \Pi_- \\ g \in \mathbb{C}^n, \|g\|=1}} (\|B_\lambda(z)g\| + \|\tilde{R}_\lambda(z)g\|) > 0. \quad (4.17)$$

(ii) *If $k_\lambda = 1$ for all $\lambda \in \sigma(A)$, then θ^{-1} has a first order pole in each point of $\sigma(A)$ and (4.17) can be replaced by a simpler condition*

$$\sup_{\lambda \in \sigma(A)} (a - \operatorname{Re} \lambda) \|\operatorname{Res}_{z=\lambda} \delta^{-1}\| < \infty. \quad (4.18)$$

(iii) *If $k(\lambda) = 1$ for all $\lambda \in \sigma(A)$ and the following two conditions hold*

$$\sum_{\lambda \in \sigma(A) \cap B(\mu, r)} (a - \operatorname{Re} \lambda) \leq Cr, \quad \mu \in \partial \Pi_- \quad (4.19)$$

and

$$\inf_{\lambda, \lambda' \in \sigma(A)} |b_\lambda(\lambda')| > 0, \quad (4.20)$$

where $B(\mu, r)$ denotes a ball with radius r and center μ and C is independent of μ and r , then the system $\{\mathcal{M}_\lambda\}$ is a Riesz basis.

In particular, it follows from (iii) that, if $\sigma(A)$ is contained in a strip $a' < \operatorname{Re} z < a$, then a sufficient condition for $\{\mathcal{M}_\lambda\}$ to be a Riesz basis is

$$\inf_{\lambda, \lambda' \in \sigma(A), \lambda \neq \lambda'} |\lambda - \lambda'| > 0.$$

Notice that $\sigma(A)$ is contained in a strip $a' < \operatorname{Re} z < a$ if, in addition to the hypothesis (H2), the map M is also atomic at h .

The factors in (4.12), which figure in the Theorem, can be computed explicitly from the power series of δ at points $\lambda \in \sigma(A)$. In particular, if $k_\lambda = 1$ both B_λ and $\tilde{B}_\lambda$ are Blaschke-Potapov factors (in general, different):

$$B_\lambda = b_\lambda \Delta_\lambda + (I - \Delta_\lambda), \quad \tilde{B}_\lambda = b_\lambda \tilde{\Delta}_\lambda + (I - \tilde{\Delta}_\lambda), \quad (4.21)$$

where Δ_λ and $\tilde{\Delta}_\lambda$ are orthogonal projections in $\mathbb{C}^n$ onto $\text{Ker } \delta(\lambda)$ and $\text{Ker } \delta(\lambda)^*$, respectively (see [8, p. 143]).

Proof of Theorem 4.8. 1) To apply the result by Treil [12], we introduce the sesquilinear product

$$(f, g) := \int_{\ell} f^T(\lambda) \overline{g(\lambda)} |d\lambda| \quad \text{in } L^2(\ell)_n$$

and we consider orthogonal complements instead of quotient spaces.

The isometry $f \mapsto \delta_i^* f$ maps $H_{-,n}^2 \ominus \delta_i H_{-,n}^2$ onto $H_{+,n}^2 \ominus \delta_i^* H_{+,n}^2$ (we remark that $\delta_i^*|_{\partial\Pi_+}$ is the boundary limit value function of the $n \times n$ matrix inner function $z \mapsto \delta_i^*(2a - \bar{z})$ in Π_+). From Lemma 4.7 we conclude

$$U\mathcal{M}_\lambda = R_{i,\lambda} H_{-,n}^2 \ominus \delta_i H_{-,n}^2$$

and the spaces $\mathcal{M}_\lambda$ form a Riesz basis if and only if the spaces

$$\delta_i^*(R_{i,\lambda} H_{-,n}^2 \ominus \delta_i H_{-,n}^2) = H_{+,n}^2 \ominus B_\lambda^{(1)*} H_{+,n}^2 \quad (4.22)$$

form a Riesz basis. The span of the latter spaces is $H_{+,n}^2 \ominus \delta_i^* H_{+,n}^2$. If we take the adjoint of the factorization in (4.16) and apply the condition from Treil [12, Thm. 9.1], then we can conclude that the system $\mathcal{M}_\lambda$ forms a Riesz basis if and only if

$$\inf_{\lambda \in \sigma(A)} \inf_{\substack{z \in \Pi_- \\ f \in \mathbb{C}^n, \|f\|=1}} (\|B_\lambda^{(1)}(z)f\| + \|\tilde{R}_{i,\lambda}(z)f\|) > 0.$$

Put

$$f = \frac{\delta_e(\lambda)g}{\|\delta_e(\lambda)g\|}.$$

Since $\delta_e, \delta_e^{-1} \in H^\infty(\Pi_-)_{n \times n}$, and

$$\sup_{\lambda \in \sigma(A)} \|\delta_{\lambda,e}^{(1)}\|_{H^\infty(\Pi_-)_{n \times n}} < \infty, \quad \sup_{\lambda \in \sigma(A)} \|(\delta_{\lambda,e}^{(1)})^{-1}\|_{H^\infty(\Pi_-)_{n \times n}} < \infty,$$

we arrive at the necessary and sufficient condition (4.17).

2) The above reduction to the spaces in (4.22) and the result by Treil imply that the system $\{\mathcal{M}_\lambda\}$ is a Riesz basis if and only if it is uniformly minimal.

Since $k(\lambda) = 1$, $B_\lambda^{(1)}$ is a Blaschke-Potapov factor:

$$B_\lambda^{(1)} = b_\lambda \Delta_\lambda^{(1)} + (I - \Delta_\lambda^{(1)}),$$

where $\Delta_\lambda^{(1)}$ is the orthogonal projection onto $\text{Ker } \delta_i(\lambda)$. Let $\{Q_{\lambda^*}\}_{\lambda \in \sigma(A)}$ be the system of spectral projections, corresponding to the system of spaces (4.22). From the results in

[8, pp. 145–147], one derives that the system of spaces (4.22) is uniformly minimal if and only if

$$\begin{aligned} \sup_{\lambda \in \sigma(A)} \|Q_{\lambda^*}\| &= \sup_{\lambda \in \sigma(A)} \|(R_{i,\lambda}(\lambda)^*)^{-1} \Delta_\lambda^{(1)}\| \\ &= \sup_{\lambda \in \sigma(A)} \|\Delta_\lambda^{(1)} R_{i,\lambda}(\lambda)^{-1}\| < \infty. \end{aligned} \quad (4.23)$$

But

$$(B_\lambda^{(1)})^{-1} = b_\lambda^{-1} \Delta_\lambda^{(1)} + (I - \Delta_\lambda^{(1)})$$

and this implies that for $\lambda \in \sigma(A)$

$$\begin{aligned} \|\operatorname{Res}_{z=\lambda} \delta_i^{-1}\| &= \|\operatorname{Res}_{z=\lambda} b_\lambda^{-1} \Delta_\lambda^{(1)} R_{i,\lambda}^{-1}(\lambda)\| \\ &= \frac{1}{2(a - \operatorname{Re} \lambda)} \|\Delta_\lambda^{(1)} R_{i,\lambda}^{-1}(\lambda)\|. \end{aligned}$$

So (4.23) can be rewritten as

$$\sup_{\lambda \in \sigma(A)} (a - \operatorname{Re} \lambda) \|\operatorname{Res}_{z=\lambda} \delta_i^{-1}\| < \infty.$$

Since $\operatorname{Res}_{z=\lambda} \delta_i^{-1} = \delta_e^{-1}(\lambda) \operatorname{Res}_{z=\lambda} \delta^{-1}$ and the norms $\|\delta_e(\lambda)\|$ and $\|\delta_e(\lambda)^{-1}\|$ are uniformly bounded, this last condition is equivalent to (4.18).

3) It follows from formulas (4.13)–(4.15) that if the spaces $(z - \lambda)^{-1} \mathbb{C}^n$ form a Riesz basis in their linear span in $L^2(\ell)_n$ then $\{\mathcal{M}_\lambda\}$ are also a Riesz basis in their linear span. A necessary and sufficient condition for $\{(z - \lambda)^{-1} \mathbb{C}^n\}_{\lambda \in \sigma(A)}$ (or the corresponding scalar rational functions $\{(z - \lambda)^{-1}\}$ in $L^2(\ell)$) to form a Riesz basis is the Carleson condition

$$\inf_{\lambda \in \sigma(A)} \prod_{\lambda' \in \sigma(A) \setminus \{\lambda\}} |b_{\lambda'}(\lambda)| > 0. \quad (4.24)$$

Furthermore, (4.24) is equivalent to (4.19) and (4.20) (see [8, Lect. VII and XI]). In the case that $\sigma(A)$ is contained in a vertical strip (4.20) reduces to the condition

$$\inf_{\lambda, \lambda' \in \sigma(A), \lambda \neq \lambda'} |\lambda - \lambda'| > 0$$

and (4.20) implies (4.19). $\square$

Remark. If $k(\lambda) = 1$ for $\lambda \in \sigma(A)$, the system $\{\mathcal{M}_\lambda\}$ forms a Riesz basis if and only if (4.19) together with the following condition hold:

- (i) There exist $\epsilon > 0$ and $\kappa > 0$ such that for each $\mu \in \Pi_-$, the hyperbolic disc

$$D(\mu, \epsilon) = \{z : |b_\mu(z)| < \epsilon\}$$

contains no more than n points $\lambda \in \sigma(A)$ and the constant of uniform minimality of the spaces $\operatorname{Ker} \delta(\lambda)$ corresponding to these points is less than κ .

This can be deduced from the results by Vasyunin [13] and Treil [12]. One can also apply Lemma 15.3 from [12] to obtain a similar result for arbitrary multiplicities $k(\lambda)$.

4.7 (τ, γ) -periodic solutions

Let $\tau > 0$ and $\gamma \in \mathbb{C}$, $\gamma \neq 0$. A solution x of equation (1.1) is called (τ, γ) -periodic if

$$x(t + \tau) = \gamma x(t), \quad -h \leq t < \infty.$$

It is immediate that the initial data corresponding to the (τ, γ) -periodic solution are precisely the eigenvectors of $T(\tau)$ corresponding to the eigenvalue γ .

Define, next, the set

$$\Lambda_{\tau, \gamma} = \{\lambda \in \sigma(A) : e^{\tau\lambda} = \gamma\}.$$

For any eigenvector f of A corresponding to an eigenvalue $\lambda \in \Lambda_{\tau, \gamma}$ one has

$$T(\tau)f = e^{\tau\lambda}f = \gamma f,$$

which shows that the solution of equation (1.1) corresponding to the initial data f is (τ, γ) -periodic. From the spectral mapping theorem for the point spectrum (see [9]) it follows that there exist (τ, γ) -periodic solutions if and only if $\Lambda_{\tau, \gamma} \neq \emptyset$. Using our model we have the following more precise result.

Proposition 4.9. *Let $\tau > 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$.*

- (i) *The eigenspaces $\text{Ker}(A - \lambda I)$ that correspond to points $\lambda \in \Lambda_{\tau, \gamma}$ form a Riesz basis in their linear span.*
- (ii) *A solution $x = x(\cdot; c, \varphi)$ corresponding to the initial data $f = (c, \varphi)^T \in Z$ is (τ, γ) -periodic if and only if*

$$f = \sum_n f_{\lambda_n}, \quad (4.25)$$

where $\alpha_n \in \Lambda_{\tau, \gamma}$, $f_{\lambda_n} \in \text{Ker}(A - \lambda_n I)$ and $\sum_n \|f_{\lambda_n}\|^2 < \infty$.

Proof. We can assume $\Lambda_{\tau, \gamma} \neq \emptyset$. Since $\Lambda_{\tau, \gamma}$ satisfies both (4.19) and (4.20), (i) follows. So each f of the form (4.25) gives rise to a (τ, γ) -periodic solution.

Conversely, let $f = (c, \varphi)^T$, $x = x(\cdot; c, \varphi)$ and $g(z) = e^{\tau z} - \gamma$, then in the canonical factorization $g = g_i g_e$ the function g_i is the Blaschke product with simple zeros in the points belonging to $\Lambda_{\tau, \gamma}$. Since

$$g(A)f = 0 \quad \text{if and only if} \quad g_i(A)f = 0,$$

we conclude that x is (τ, γ) -periodic if and only if $g_i(A)f = 0$. But the results from [8, Lect. 4, Sect. 5] yield

$$\text{Ker } g_i(A) = \overline{\text{span}}_{\lambda \in \Lambda_{\tau, \gamma}} \text{Ker}(A - \lambda I).$$

□

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