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Bodó, Gergely; Riedle, Markus

DOI

[10.1112/jlms.70298](https://doi.org/10.1112/jlms.70298)

Publication date

2025

Document Version

Final published version

Published in

Journal of the London Mathematical Society

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Citation for published version (APA):

Bodó, G., & Riedle, M. (2025). Stochastic integration with respect to cylindrical Lévy processes in Hilbert spaces. *Journal of the London Mathematical Society*, 112(3), Article e70298. <https://doi.org/10.1112/jlms.70298>

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RESEARCH ARTICLE

Stochastic integration with respect to cylindrical Lévy processes in Hilbert spaces

Gergely Bodó¹ | Markus Riedle^{2,3} 

¹Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, The Netherlands

²Department of Mathematics, King's College London, London, UK

³Faculty of Mathematics, Institute of Mathematical Stochastics, TU Dresden, Dresden, Germany

Correspondence

Markus Riedle, Department of Mathematics, King's College, London WC2R 2LS, UK.

Email: markus.riedle@kcl.ac.uk

Abstract

In this work, we present a comprehensive theory of stochastic integration with respect to arbitrary cylindrical Lévy processes in Hilbert spaces. As cylindrical Lévy processes do not enjoy a semimartingale decomposition, our approach relies on an alternative approach to stochastic integration by decoupled tangent sequences. The space of deterministic integrands is identified as a modular space described in terms of the characteristics of the cylindrical Lévy process. The space of random integrands is described as the space of predictable processes whose trajectories are in the space of deterministic integrands almost surely. The derived space of random integrands is verified as the largest space of potential integrands, based on a classical definition of stochastic integrability. We apply the introduced theory of stochastic integration to establish a dominated convergence theorem.

MSC 2020

60H05, 60G20, 60G51, 28C20

1 | INTRODUCTION

Cylindrical Lévy processes serve as a natural generalisation of cylindrical Brownian motion, providing a unified framework for modelling a wide variety of different random perturbations

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of infinite-dimensional systems. Analogously to cylindrical Brownian motion, these processes generally do not exist as stochastic processes in the usual sense with values in the underlying infinite-dimensional space. Instead, they can only be interpreted in the generalised sense of Gel'fand and Vilenkin [9] or Segal [39]. Important and non-trivial examples of such generalised processes include standard symmetric α -stable cylindrical Lévy processes, see, for example, Kosmala and Riedle [17], and the class of orthogonal processes, frequently used in the SPDE community as models of random perturbations, as discussed in Peszat and Zabczyk [32], or Priola and Zabczyk [34].

The first systematic treatment of the concept of cylindrical Lévy processes in Hilbert and Banach spaces was undertaken by Applebaum and Riedle in their work [2]. Leveraging the theory of cylindrical measures, as outlined by Badrikian and Chevet [3] and Schwartz [38], the authors established a precise mathematical framework for understanding cylindrical Lévy processes. Within this framework, cylindrical Lévy processes has since found applications in modelling random perturbations of partial differential equations such as Bodó et al [5] and Kosmala and Riedle [17]. Specific instances of cylindrical Lévy processes as driving noise have also been explored in works by Priola and Zabczyk [34] and Peszat and Zabczyk [32]. However, all of these applications were made under restrictive assumptions, such as additive noise or specific types of noise, owing to the absence of a fully developed general theory of stochastic integration.

Our present work is motivated by the following problem. Let G and H be Hilbert spaces, and consider an abstract evolution equation of the form

$$\begin{aligned} dX(t) &= (AX(t) + F(X(t)))dt + B(X(t-))dL(t), & t \geq 0, \\ X(0) &= x_0, \end{aligned} \tag{1.1}$$

where A is the generator of a strongly continuous semigroup $(S(t))_{t \geq 0}$ on H , and $F: H \rightarrow H$, $B: H \rightarrow L_2(G, H)$ are given maps. The initial condition is specified by an H -valued, \mathcal{F}_0 -measurable random variable X_0 , and the noise is modelled by an arbitrary cylindrical Lévy process L on G . To formalise the meaning of Equation (1.1), we first need to develop a theory of stochastic integration with respect to a general cylindrical Lévy process L in G . However, due to the cylindrical nature of the noise and the absence of moment assumptions on L , standard fixed-point arguments, which typically relying on moment bounds for the stochastic integral, are not applicable. Therefore, our integration theory must be robust enough to yield a stochastic dominated convergence-type result, which we can then use to prove the existence and uniqueness of a solution to Equation (1.1). The remainder of this work is devoted to developing a comprehensive stochastic integration theory for general cylindrical Lévy processes that meets these requirements.

In finite-dimensional spaces, the integration theory for Lévy processes can be developed using the classical approach based on the semimartingale decomposition of the integrator; see Applebaum [1]. This method extends to genuine Lévy processes in Hilbert spaces, as shown by Peszat and Zabczyk [31].

However, the classical semimartingale-based approach cannot be applied to cylindrical Lévy processes, as they do not admit a semimartingale decomposition. For this reason, stochastic integration theories for cylindrical processes have been developed almost exclusively for cylindrical martingales. These include the Doléans measure approach by Métivier and Pellaumail in [23], the

construction of a family of reproducing kernel Hilbert spaces by Mikulevičius and Rozovskiĭ in [24], and the introduction of a novel type of quadratic variation for cylindrical continuous local martingales in UMD Banach spaces in Veraar and Yaroslavtsev [42].

In this work, we present a comprehensive theory of stochastic integration for random integrands with respect to arbitrary cylindrical Lévy processes. The robustness of our developed theory is showcased through establishing a typical dominated convergence theorem. Furthermore, we demonstrate that the class of random integrands, described by the characteristics of the integrator, constitutes the largest class of potential integrands, based on the classical definition of stochastic integrability introduced by Urbanic and Woyczynski in [40].

Our approach relies on a two-sided inequality for the metric of convergence in probability in Hilbert spaces applied to the sum of a decoupled tangent sequence representing the stochastic integral for simple integrands. This particular approach to stochastic integration in the real-valued case was pioneered in the late 1980s in a couple of publications by Kwapien and Woyczynski and cumulated in their monograph [20]. In the vector-valued setting, similar ideas were simultaneously pursued for stochastic integration in UMD-Banach spaces by McConnell in [22]. This work can be viewed as the precursor to the recently developed, comprehensive theory of stochastic integration with respect to cylindrical Brownian motion in UMD spaces by van Neerven, Veraar and Weis. This theory started with their work [27] and has led to various novel insights into stochastic partial differential equations, including sharp maximal inequalities.

The development of a stochastic integration theory for cylindrical Lévy processes was previously addressed only in the work Jakubowski and Riedle [12]. There, the underlying approach was, similarly as here, based on studying the sum of the decoupled tangent sequence but only applied to conclude relatively compactness in the Skorokhod space of the uncoupled sum. This approach led only to an integration theory for stochastic processes that are continuous from the left and have right limits, a restriction often insufficient for practical applications. In particular, this approach could not be extended to include the important case of predictable integrands. Our publication [4] serves as a precursor to the present work, wherein we tested our methods in the specific case of a canonical α -stable cylindrical process as the integrator. This restricted setting significantly simplifies the description of the space of admissible integrands. The current study builds upon this case study, offering a more comprehensive theory of stochastic integration for arbitrary cylindrical Lévy processes, overcoming limitations inherent in previous approaches.

The present work introduces the stochastic integral in two steps: first, for deterministic integrands and second, for random integrands. The largest space of deterministic integrands is derived in Theorem 4.4 as a modular space described in terms of the characteristics of the cylindrical Lévy process. The largest space of random integrands is derived in Theorem 7.1, and can be described as the space of predictable processes whose trajectories are in the space of deterministic integrands almost surely.

We briefly summarise our article: in Section 2, some fundamental results on cylindrical Lévy processes and infinitely divisible probability measures in Hilbert spaces are collected. Section 3 introduces the modular space that characterises the space of deterministic integrands, establishing its completeness, metrisability and linearity. The stochastic integral for deterministic integrands is presented in Section 4. The role of Itô's isometry for the classical stochastic integral with respect to a Brownian motion is taken by Proposition 4.5. Although it does not establish an isometry, it guarantees that a sequence of deterministic integrands is Cauchy if and only if the corresponding stochastic integrals are Cauchy in the semimartingale topology. Section 5 sets the foundation for integrating random integrands by introducing necessary definitions and elementary

results. Section 6 is devoted to the construction of the decoupled tangent sequence. In this section, we also briefly recall relevant definitions and results on decoupled tangent sequences from [20]. The space of random integrands is derived in Section 7. Analogous to the deterministic integrands, Corollary 7.3 establishes a relationship between the Cauchy-ness in the space of random integrands and the corresponding Cauchy-ness of stochastic integrals. We finish this section by applying the introduced theory of stochastic integration to establish a dominated convergence theorem.

2 | PRELIMINARIES

2.1 | Cylindrical Lévy processes

Let G and H be separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and corresponding norms $\|\cdot\|$. Let $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ be orthonormal bases of G and H , respectively. We identify the dual of a Hilbert space by the space itself. The Borel σ -algebra of H is denoted by $\mathfrak{B}(H)$ and the open unit ball by $B_H := \{h \in H : \|h\| < 1\}$ and the closed unit ball by $\bar{B}_H := \{h \in H : \|h\| \leq 1\}$.

The Banach space of bounded linear operators from G to H will be denoted by $L(G, H)$ with the operator norm $\|\cdot\|_{G \rightarrow H}$. Its subspace $L_2(G, H)$ of Hilbert–Schmidt operators is endowed with the norm $\|F\|_{\text{HS}}^2 := \sum_{k=1}^{\infty} \|F a_k\|^2$ for $F \in L_2(G, H)$.

Let (Ω, Σ, P) be a complete probability space. We will denote by $L_p^0(\Omega, H)$ the space of equivalence classes of measurable functions $X : \Omega \rightarrow H$, equipped with the topology of convergence in probability.

Let S be a subset of G . For each $n \in \mathbb{N}$, elements $g_1, \dots, g_n \in S$ and Borel sets $A \in \mathfrak{B}(\mathbb{R}^n)$, we define

$$C(g_1, \dots, g_n; A) := \{g \in G : (\langle g, g_1 \rangle, \dots, \langle g, g_n \rangle) \in A\}.$$

Such sets are called cylindrical sets with respect to A . The collection of all these cylindrical sets is denoted by $\mathcal{Z}(G, S)$, and it is a σ -algebra if S is finite and otherwise an algebra. We write $\mathcal{Z}(G)$ for $\mathcal{Z}(G, G)$.

A set function $\mu : \mathcal{Z}(G) \rightarrow [0, \infty]$ is called a cylindrical measure on $\mathcal{Z}(G)$ if for each finite-dimensional subset $S \subseteq G$, the restriction of μ to the σ -algebra $\mathcal{Z}(G, S)$ is a σ -additive measure. A cylindrical measure is said to be a cylindrical probability measure if $\mu(G) = 1$.

A cylindrical random variable X in G is a linear and continuous mapping $X : G \rightarrow L_p^0(\Omega, \mathbb{R})$. It defines a cylindrical probability measure μ_X by

$$\mu_X : \mathcal{Z}(G) \rightarrow [0, 1], \quad \mu_X(Z) = P((Xg_1, \dots, Xg_n) \in A)$$

for cylindrical sets $Z = C(g_1, \dots, g_n; A)$. The cylindrical probability measure μ_X is called the cylindrical distribution of X . We define the characteristic function of the cylindrical random variable X by

$$\varphi_X : G \rightarrow \mathbb{C}, \quad \varphi_X(g) = E[e^{iXg}].$$

Let $T : G \rightarrow H$ be a linear and continuous operator. By defining

$$TX : H \rightarrow L_p^0(\Omega, \mathbb{R}), \quad (TX)h = X(T^*h),$$

we obtain a cylindrical random variable on H . In the special case when T is a Hilbert–Schmidt operator and hence 0-Radonifying by [41, Theorem VI.5.2], it follows from [41, Proposition VI.5.3] that the cylindrical random variable TX is induced by a genuine random variable $Y : \Omega \rightarrow H$, that is $(TX)h = \langle Y, h \rangle$ for all $h \in H$. As shown in [4, Lemma 2.1], the inducing random variable Y depends continuously on the Hilbert–Schmidt operator.

A family $(L(t) : t \geq 0)$ of cylindrical random variables $L(t) : G \rightarrow L_p^0(\Omega, \mathbb{R})$ is called a cylindrical Lévy process if for each $n \in \mathbb{N}$ and $g_1, \dots, g_n \in G$, the stochastic process

$$((L(t)g_1, \dots, L(t)g_n) : t \geq 0)$$

is a Lévy process in \mathbb{R}^n . The filtration generated by $(L(t) : t \geq 0)$ is defined by

$$\mathcal{F}_t := \sigma(\{L(s)g : g \in G, s \in [0, t]\}) \quad \text{for all } t \geq 0.$$

Denote by $\mathcal{Z}_*(G)$ the collection

$$\{\{g \in G : (\langle g, g_1 \rangle, \dots, \langle g, g_n \rangle) \in B\} : n \in \mathbb{N}, g_1, \dots, g_n \in G, B \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})\}$$

of cylindrical sets, which forms an algebra of subsets of G . For fixed $g_1, \dots, g_n \in G$, let $\lambda_{g_1, \dots, g_n}$ be the Lévy measure of $((L(t)g_1, \dots, L(t)g_n) : t \geq 0)$. Define a function $\lambda : \mathcal{Z}_*(G) \rightarrow [0, \infty]$ by

$$\lambda(C) := \lambda_{g_1, \dots, g_n}(B) \quad \text{for } C = \{g \in G : (\langle g, g_1 \rangle, \dots, \langle g, g_n \rangle) \in B\} \text{ and } B \in \mathfrak{B}(\mathbb{R}^n).$$

It is shown in [2] that λ is well-defined. The set function λ is called the cylindrical Lévy measure of L .

The characteristic function of a cylindrical Lévy process L in G takes for each $t \geq 0$ the form

$$\varphi_{L(t)} : G \rightarrow \mathbb{C}, \quad \varphi_{L(t)}(g) = \exp(tS(g)),$$

where the mapping $S : G \rightarrow \mathbb{C}$ is called the cylindrical symbol of L , and satisfies

$$S(g) = ia(g) - \frac{1}{2}\langle Qg, g \rangle + \int_G \left(e^{i\langle g, h \rangle} - 1 - i\langle g, h \rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle g, h \rangle) \right) \lambda(dh),$$

where $a : G \rightarrow \mathbb{R}$ is a continuous mapping with $a(0) = 0$, $Q : G \rightarrow G$ is a positive and symmetric operator, and λ is a cylindrical Lévy measure on G . We call the triplet (a, Q, λ) the cylindrical characteristics of L . For this and related results, see [35].

2.2 | Infinitely divisible measures and their characteristics

Infinitely divisible measures on a Hilbert space H can be defined as in the Euclidean space; see [30]. As in finite dimensions, the characteristic function of any infinitely divisible measure μ on $\mathfrak{B}(H)$ satisfies $\varphi_\mu(h) = \exp(S(h))$, where the symbol $S : H \rightarrow \mathbb{C}$ is of the form

$$S(h) = i\langle b^\kappa, h \rangle - \frac{1}{2}\langle Qh, h \rangle + \int_H \left(e^{i\langle h, g \rangle} - 1 - i\langle h, \kappa(g) \rangle \right) \lambda(dg),$$

where $b^\kappa \in H$, the mapping $Q : H \rightarrow H$ is nuclear, symmetric and non-negative, the Lévy measure λ is a σ -finite measure on $\mathfrak{B}(H)$ satisfying $\int_H (\|h\|^2 \wedge 1) \lambda(dh) < \infty$ and $\kappa : H \rightarrow H$ is a function which is bounded and satisfies $\kappa(h) = h$ in a neighbourhood of 0. Such a function κ is called a truncation function. The triplet (b^κ, Q, λ) is called characteristics of μ . For different truncation functions κ , one obtains the same representation of the symbol S but only the term b^κ depends on κ . When dealing with limit theorems, for technical reasons, it is often preferable to use a continuous function κ . A specific example of a continuous truncation function, which will play an important role in the rest of this work, is the truncation function

$$\theta : H \rightarrow H, \quad \theta(h) = \begin{cases} h & \text{if } \|h\| \leq 1; \\ \frac{h}{\|h\|} & \text{if } \|h\| > 1. \end{cases}$$

Let $\kappa : H \rightarrow H$ be a continuous truncation function. A sequence of infinitely divisible measures $\mu_n = (b_n^\kappa, Q_n, \lambda_n)$ with associated sequence $(T_n)_{n \in \mathbb{N}}$ of so-called S -operators $T_n : H \rightarrow H$, which are defined by

$$\langle T_n h_1, h_2 \rangle = \langle Q_n h_1, h_2 \rangle + \int_{\|u\| \leq 1} \langle h_1, u \rangle \langle h_2, u \rangle \lambda_n(du) \quad \text{for all } h_1, h_2 \in H,$$

converges weakly to an infinitely divisible measure $\mu = (b^\kappa, Q, \lambda)$ if and only if the following conditions hold:

$$(1) \quad b^\kappa = \lim_{n \rightarrow \infty} b_n^\kappa; \tag{2.1}$$

$$(2) \quad \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \int_{\|h\| \leq \delta} \langle h, u \rangle^2 \lambda_n(du) + \langle Q_n h, h \rangle = \langle Qh, h \rangle \text{ for all } h \in H; \tag{2.2}$$

$$(3) \quad \lambda_n \rightarrow \lambda \text{ weakly outside of every closed neighbourhood of the origin}; \tag{2.3}$$

$$(4) \quad (T_n)_{n \in \mathbb{N}} \text{ is compact in the space of nuclear operators.} \tag{2.4}$$

Remark 2.1. Let $(\mathcal{I}, \|\cdot\|_0)$ denote the collection of H -valued, infinitely divisible random variables endowed with a translation invariant metric $\|\cdot\|_0$ generating the topology of convergence in probability. Define the mapping

$$g : \mathcal{I} \rightarrow H, \quad g(X) = b_X^\theta,$$

where b_X^θ denotes the first characteristic of X with respect to the truncation function θ . Then the function g is continuous according to Equation (2.1), and hence, by the topological characterisation of continuity, for all $\epsilon > 0$ there exists $\delta > 0$, depending only on ϵ and the metric $\|\cdot\|_0$, such that $\|X\|_0 < \delta$ implies $\|b_X^\theta\| < \epsilon$ for all $X \in \mathcal{I}$.

Let $(\pi_n)_{n \in \mathbb{N}}$ be a sequence of partitions of the interval $[s, t]$ of the form

$$\pi_n = \{s = p_{0,n} < p_{1,n} < \dots < p_{N(n),n} = t\}.$$

We say that $(\pi_n)_{n \in \mathbb{N}}$ is a nested normal sequence of partitions if:

- (1) $\pi_n \subseteq \pi_m$ for all $n \leq m$;
- (2) $\lim_{n \rightarrow \infty} \max_{i \in \{1, \dots, N(n)\}} |p_{i,n} - p_{i-1,n}| = 0$.

The following result enables us to express Lévy characteristics as limits of certain series, which will play a key role in the sequel.

Theorem 2.2. *Let L be an H -valued Lévy process with characteristics (b^θ, Q, λ) , and let $(\pi_n)_{n \in \mathbb{N}}$ be a nested normal sequence of partitions of $[s, t]$, where for each fixed $n \in \mathbb{N}$ we have $\pi_n = \{s = p_{0,n} < p_{1,n} < \dots < p_{N(n),n} = t\}$. If we put $d_{i,n} = L(p_{i,n}) - L(p_{i-1,n})$, then we have*

- (1) $\lim_{n \rightarrow \infty} \sum_{\pi_n} E [\theta(d_{i,n})] = (t - s)b^\theta$;
- (2) $\lim_{n \rightarrow \infty} \sum_{\pi_n} E [\|d_{i,n}\|^2 \wedge 1] = (t - s) \left(\int_H (\|h\|^2 \wedge 1) \lambda(dh) + \text{Tr}(Q) \right)$.

Proof. For a proof, see [29, Lemma 3.4]. □

3 | THE MODULAR SPACE

Originally introduced by Nakano [25], modular spaces serve as natural generalisations of metric spaces. Prominent and non-trivial examples are Lebesgue–Bochner spaces, Orlicz spaces and generalised Musielak–Orlicz spaces. While numerous different definitions appear in the literature, in this work we will always use the following adaption of Nakano’s original definition of a generalised modular; see [26]. Our main objective in this section, apart from defining these spaces, is to establish them as complete, metrisable linear spaces and to demonstrate denseness of simple functions. Although various abstract conditions on the modular are known in the literature, guaranteeing one of these properties, we found it easier to establish these properties directly. The metrisability is achieved through recent results on K -quasi-metric spaces in [7].

Definition 3.1. Let V be a real vector space. A function $\Delta : V \rightarrow [0, \infty]$ is called a modular if

- (1) $\Delta(-v) = \Delta(v)$ for all $v \in V$;
- (2) $\inf_{\alpha > 0} \Delta(\alpha v) = 0$ for all $v \in V$;
- (3) $\Delta(\alpha v) \leq \Delta(\beta v)$ for all $0 \leq \alpha \leq \beta$ and $v \in V$;
- (4) there exists a constant $c > 0$ such that

$$\Delta(v + w) \leq c(\Delta(v) + \Delta(w)) \quad \text{for all } v, w \in V.$$

A function satisfying Condition (4) of Definition 3.1 is said to be of moderate growth.

As Hilbert–Schmidt operators between Hilbert spaces map cylindrical random variables to genuine random variables, they transform cylindrical Lévy processes to genuine Lévy processes.

Lemma 3.2. *Let $(L(t) : t \geq 0)$ be a cylindrical Lévy process in G with cylindrical characteristics (a, Q, λ) , and let $F \in L_2(G, H)$ be a Hilbert–Schmidt operator. Then there exists an H -valued Lévy process $(F(L)(t) : t \geq 0)$ satisfying $\langle F(L)(t), h \rangle = L(t)(F^*h)$ for all $t \in [0, T]$ and $h \in H$. Moreover, $F(L)$ has characteristics $(b_F, FQF^*, \lambda \circ F^{-1})$, where for all $u \in H$*

$$\langle b_F, u \rangle = a(F^*u) + \int_H \langle h, u \rangle \left(\mathbb{1}_{B_H}(h) - \mathbb{1}_{B_{\mathbb{R}}}(\langle h, u \rangle) \right) (\lambda \circ F^{-1})(dh).$$

Proof. Existence of the H -valued Lévy process $F(L)$ follows from [11, Theorem A]. To derive the characteristics, first apply [36, Lemma 5.4] to obtain the cylindrical characteristics of $F(L)$, and then use [36, Lemma 5.8] to convert the cylindrical characteristics into genuine characteristics. \square

Remark 3.3. In the Lemma above and throughout this article, we use $\lambda \circ F^{-1}$ to denote the classical Lévy measure of the genuine H -valued Lévy process $F(L)$. In particular, $\lambda \circ F^{-1}$ is a σ -finite measure on the Borel σ -algebra $\mathfrak{B}(H)$. Formally, one obtains this σ -finite measure by extending the image cylindrical measure of λ under the mapping F from the cylindrical algebra $\mathcal{Z}_*(H)$ to the Borel σ -algebra $\mathfrak{B}(H)$.

Remark 3.4. Note that in the special case, when the truncation function is θ , see Subsection 2.2 of the Preliminaries, the first characteristic b_F^θ satisfies for all $u \in H$ that

$$\langle b_F^\theta, u \rangle = a(F^*u) + \int_H \left(\langle \theta(h), u \rangle - \langle h, u \rangle \mathbb{1}_{B_{\mathbb{R}}}(\langle h, u \rangle) \right) (\lambda \circ F^{-1})(dh). \quad (3.1)$$

For the rest of this chapter, we fix a cylindrical Lévy process L with cylindrical characteristics (a, Q, λ) .

Definition 3.5. Let L be a cylindrical Lévy process with cylindrical characteristics (a, Q, λ) and define functions $k_L, l_L : L_2(G, H) \rightarrow \mathbb{R}$ by

$$k_L(F) = \int_H (\|h\|^2 \wedge 1) (\lambda \circ F^{-1})(dh) + \text{Tr}(FQF^*);$$

$$l_L(F) = \sup_{O \in \bar{B}_{L(H)}} \|b_{OF}^\theta\|,$$

where $\bar{B}_{L(H)}$ denotes the collection of bounded linear operators $O : H \rightarrow H$ satisfying $\|O\|_{H \rightarrow H} \leq 1$, and the expression b_{OF}^θ denotes the first characteristic of the Radonified Lévy process $OF(L)$ for each $O \in \bar{B}_{L(H)}$ as defined in Equation (3.1).

Remark 3.6. It follows from the very definitions of k_L and l_L that for each fixed $F \in L_2(G, H)$ and for all $0 \leq \alpha \leq \beta$ we have that $k_L(\alpha F) \leq k_L(\beta F)$ and $l_L(\alpha F) \leq l_L(\beta F)$. This observation will be repeatedly used in the sequel.

Definition 3.7. For a measurable function $\psi : [0, T] \rightarrow L_2(G, H)$ define

$$m'_L(\psi) := \int_0^T (k_L(\psi(t)) + l_L(\psi(t))) dt;$$

$$m''(\psi) := \int_0^T \left(\|\psi(t)\|_{L_2(G, H)}^2 \wedge 1 \right) dt;$$

$$m_L(\psi) := m'_L(\psi) + m''(\psi),$$

where the fact that the above integrals are well-defined will follow from Lemma 3.12. We denote by $\mathcal{M}_{\text{det}, L}^{\text{HS}} := \mathcal{M}_{\text{det}, L}^{\text{HS}}(G, H)$ the space of Lebesgue almost everywhere equivalence classes of measurable functions $\psi : [0, T] \rightarrow L_2(G, H)$ for which $m_L(\psi) < \infty$.

Example 3.8. A cylindrical Lévy process L on G is called standard symmetric α -stable if its characteristic function is of the form

$$\varphi_{L(t)} : G \rightarrow \mathbb{C}, \quad \varphi_{L(t)}(g) = \exp(-\|g\|^\alpha),$$

for some $\alpha \in (0, 2)$, in which case the theory simplifies significantly. In particular, as L has cylindrical characteristics $(0, 0, \lambda)$, we have that l_L and the second term of k_L are always zero. Moreover, the first term of k_L can be equivalently controlled by an L^α norm, that is, by [4, Lemma 3.5.], [17, Lemma 3.1.] and [5, Lemma 2.1.], there exist constants $c_\alpha, d_\alpha > 0$ such that for all measurable functions $\psi : [0, T] \rightarrow L_2(G, H)$ we have

$$\begin{aligned} \frac{1}{c_\alpha} \int_0^T \|\psi(t)\|_{L_2(G,H)}^\alpha dt &\leq \int_0^T \int_H (\|h\|^2 \wedge 1) (\lambda \circ \psi(t)^{-1})(dh) dt \\ &\leq d_\alpha \int_0^T \|\psi(t)\|_{L_2(G,H)}^\alpha dt. \end{aligned}$$

Consequently, we obtain that $\mathcal{M}_{\det,L}^{\text{HS}} = L_{\text{Leb}}^\alpha([0, T], L_2(G, H))$.

Our main results in this section are Theorem 3.18 and Proposition 3.23. These results establish that the space $\mathcal{M}_{\det,L}^{\text{HS}}$ is a metrisable, separable and complete linear topological vector space. Convergence in the associated metric $\rho_L(\psi_1, \psi_2) \rightarrow 0$ is equivalent to $m_L(\psi_2 - \psi_1) \rightarrow 0$.

As a first step in this direction, we provide an alternative representation for l_L , which will play a central role in the subsequent analysis of various properties of the modular.

Lemma 3.9. *Let L be a cylindrical Lévy process in G with characteristics (a, Q, λ) . For all $F \in L_2(G, H)$ and $O \in L(H)$ it holds*

$$b_{OF}^\theta = Ob_F^\theta + \int_H (\theta(Oh) - O\theta(h)) (\lambda \circ F^{-1})(dh).$$

Proof. The term b_{OF}^θ must coincide with the corresponding term in the characteristics of the Lévy process which we obtain as the image of the Lévy process $F(L)$ under the map O . Using this observation, the formula follows immediately from [37, Proposition 11.10.]. \square

Remark 3.10. It follows from Lemma 3.9 that $l_L(F)$ is finite for each $F \in L_2(G, H)$. To see this, we first note that for all $h \in \bar{B}_H$ and $O \in \bar{B}_{L(H)}$ we have $\|\theta(Oh) - O\theta(h)\| = 0$, and for all $h \in H$ and $O \in \bar{B}_{L(H)}$ it holds that $\|\theta(Oh) - O\theta(h)\| \leq 2$. By combining these observations with Lemma 3.9, we obtain

$$\sup_{O \in \bar{B}_{L(H)}} \|b_{OF}^\theta\| \leq \|b_F^\theta\| + 2(\lambda \circ F^{-1})(\bar{B}_H^c) < \infty.$$

Before we could prove that our modular m_L is well-defined, we need to establish a relationship between weak convergence of infinitely divisible measures and convergence of the corresponding characteristics in the following sense:

Lemma 3.11. *Let $\mu_n \stackrel{D}{=} (b_n^\theta, Q_n, \lambda_n)$ be a sequence of infinitely divisible measures on $\mathfrak{B}(H)$ converging weakly to $\mu \stackrel{D}{=} (b^\theta, Q, \lambda)$. Then the following conditions hold:*

- (1) $\lim_{n \rightarrow \infty} \left(\int_H (\|h\|^2 \wedge 1) \lambda_n(dh) + \text{Tr}(Q_n) \right) = \int_H (\|h\|^2 \wedge 1) \lambda(dh) + \text{Tr}(Q).$
- (2) $\lim_{n \rightarrow \infty} \left\| b_n^\theta - b^\theta \right\| = 0.$

Proof. The fact that (2) holds follows directly from Equation (2.1). To prove (1), fix $\delta \in (0, 1]$ such that $\lambda(\|h\| = \delta) = 0$. By Equation (2.3), we have

$$\lim_{n \rightarrow \infty} \int_{\|h\| > \delta} (\|h\|^2 \wedge 1) \lambda_n(dh) = \int_{\|h\| > \delta} (\|h\|^2 \wedge 1) \lambda(dh).$$

Therefore, it remains only to deal with the limit of the integrals over $\bar{B}_H(\delta)$. Let $\epsilon > 0$ be fixed. It follows from properties of the Lebesgue integral that there exists a $\delta_1 \in (0, \delta]$ such that

$$\int_{\|h\| \leq \delta_1} \|h\|^2 \lambda(dh) < \frac{\epsilon}{12}. \tag{3.2}$$

Let $\{e_k\}_{k \in \mathbb{N}}$ be an orthonormal basis of H . As Q is a trace class operator, there exists $K_1 \in \mathbb{N}$ such that

$$\sum_{k=K_1+1}^\infty \langle Qe_k, e_k \rangle < \frac{\epsilon}{12}.$$

By compactness of the associated S -operators, see Condition (2.4), there exists $K_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$\sum_{k=K_2+1}^\infty \left(\int_{\|h\| \leq \delta} \langle e_k, h \rangle^2 \lambda_n(dh) + \langle Q_n e_k, e_k \rangle \right) < \frac{\epsilon}{4}. \tag{3.3}$$

Moreover, by an application of Condition (2.2), there exists a $\delta_2 < \delta_1$ and $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ and for all $k \leq K_2$ we have that

$$\left| \int_{\|h\| \leq \delta_2} \langle e_k, h \rangle^2 \lambda_n(dh) + \langle Q_n e_k, e_k \rangle - \langle Qe_k, e_k \rangle \right| < \frac{\epsilon}{12K}, \tag{3.4}$$

where $K := \max\{K_1, K_2\}$. Condition (2.3) guarantees that there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have

$$\left| \int_{\delta_2 < \|h\| \leq \delta} \|h\|^2 \lambda_n(dh) - \int_{\delta_2 < \|h\| \leq \delta} \|h\|^2 \lambda(dh) \right| < \frac{\epsilon}{2}. \tag{3.5}$$

By splitting the integration domain, we obtain

$$\begin{aligned} & \left| \int_{\|h\| \leq \delta} \|h\|^2 \lambda(dh) + \text{Tr}(Q) - \int_{\|h\| \leq \delta} \|h\|^2 \lambda_n(dh) - \text{Tr}(Q_n) \right| \\ & \leq \left| \int_{\delta_2 < \|h\| \leq \delta} \|h\|^2 \lambda(dh) - \int_{\delta_2 < \|h\| \leq \delta} \|h\|^2 \lambda_n(dh) \right| \\ & \quad + \left| \int_{\|h\| \leq \delta_2} \|h\|^2 \lambda(dh) + \text{Tr}(Q) - \int_{\|h\| \leq \delta_2} \|h\|^2 \lambda_n(dh) - \text{Tr}(Q_n) \right|. \end{aligned} \tag{3.6}$$

By Parseval’s identity, Equations (3.2)–(3.4) and a repeated application of the triangle inequality, we obtain for all $n \geq N := \max\{N_1, N_2\}$ that

$$\begin{aligned} & \left| \int_{\|h\| \leq \delta_2} \|h\|^2 \lambda(dh) + \text{Tr}(Q) - \int_{\|h\| \leq \delta_2} \|h\|^2 \lambda_n(dh) - \text{Tr}(Q_n) \right| \\ & \leq \left| \text{Tr}(Q) - \int_{\|h\| \leq \delta_2} \|h\|^2 \lambda_n(dh) - \text{Tr}(Q_n) \right| + \left| \int_{\|h\| \leq \delta_2} \|h\|^2 \lambda(dh) \right| \\ & \leq \left| \sum_{k=1}^K \left(\langle Qe_k, e_k \rangle - \int_{\|h\| \leq \delta_2} \langle h, e_k \rangle^2 \lambda_n(dh) - \langle Q_n e_k, e_k \rangle \right) \right| \\ & \quad + \left| \sum_{k=K+1}^\infty \left(\langle Qe_k, e_k \rangle - \int_{\|h\| \leq \delta_2} \langle h, e_k \rangle^2 \lambda_n(dh) - \langle Q_n e_k, e_k \rangle \right) \right| + \left| \int_{\|h\| \leq \delta_2} \|h\|^2 \lambda(dh) \right| \\ & \leq \frac{\epsilon}{2}. \end{aligned} \tag{3.7}$$

Hence, if $n \geq N$ then Equations (3.5)–(3.7) together imply

$$\left| \int_{\|h\| \leq \delta} \|h\|^2 \lambda(dh) + \text{Tr}(Q) - \int_{\|h\| \leq \delta} \|h\|^2 \lambda_n(dh) - \text{Tr}(Q_n) \right| < \epsilon.$$

As $\epsilon > 0$ was arbitrary, the result follows. □

Lemma 3.12. *Let $k_L, l_L : L_2(G, H) \rightarrow \mathbb{R}$ be as in Definition 3.5. Then we have:*

- (1) k_L is continuous;
- (2) l_L is lower-semicontinuous and continuous at 0.

Proof. Continuity of k_L follows immediately from [4, Lemma 2.1] and Lemma 3.11. To prove that l_L is lower-semicontinuous, we fix $F \in L_2(G, H)$ and a sequence $(F_n)_{n \in \mathbb{N}} \subseteq L_2(G, H)$ satisfying $\lim_{n \rightarrow \infty} \|F_n - F\|_{L_2(G, H)} = 0$. Let $\epsilon > 0$ be fixed. It follows from Remark 3.10 and the very definition of the supremum, that there exists $O_\epsilon \in \bar{B}_{L(H)}$ such that $\sup_{O \in \bar{B}_{L(H)}} \|b_{OF}^\theta\| \leq \|b_{O_\epsilon F}^\theta\| + \epsilon$. As $\lim_{n \rightarrow \infty} \|b_{O_\epsilon F_n}^\theta\| = \|b_{O_\epsilon F}^\theta\|$ by [4, Lemma 2.1] and Equation (2.1), we obtain

$$\sup_{O \in \bar{B}_{L(H)}} \|b_{OF}^\theta\| \leq \|b_{O_\epsilon F}^\theta\| + \epsilon = \lim_{n \rightarrow \infty} \|b_{O_\epsilon F_n}^\theta\| + \epsilon \leq \liminf_{n \rightarrow \infty} \sup_{O \in \bar{B}_{L(H)}} \|b_{OF_n}^\theta\| + \epsilon.$$

As $\epsilon > 0$ is arbitrary, the above shows $l_L(F) \leq \liminf_{n \rightarrow \infty} l_L(F_n)$ which proves lower-semicontinuity of l_L .

To show continuity of l_L at 0, note that by [4, Lemma 2.1] and Remark 2.1, for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|F\|_{L_2(G, H)} \leq \delta$ implies $\|b_F^\theta\| \leq \epsilon$. As $\|OF\|_{L_2(G, H)} \leq \|F\|_{L_2(G, H)}$ for all $O \in \bar{B}_{L(H)}$, we conclude that $\|F\|_{L_2(G, H)} \leq \delta$ implies $\sup_{O \in \bar{B}_{L(H)}} \|b_{OF}^\theta\| \leq \epsilon$, which concludes the proof. □

In preparation for showing that m_L is of moderate growth, see Definition 3.1/(4), we prove the following technical lemmas.

Lemma 3.13. *Let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis of G and let $P_n : G \rightarrow G$ be the projection onto $\text{Span}\{e_1, \dots, e_n\}$. Then we have for all $F \in L_2(G, H)$ that*

$$\lim_{n \rightarrow \infty} \|FP_n - F\|_{L_2(G,H)} = 0.$$

Proof. As $P_n e_i = e_i$ for $i \leq n$, and $P_n e_i = 0$ for $i > n$, we have

$$\|FP_n - F\|_{L_2(G,H)}^2 = \sum_{i=1}^{\infty} \|(FP_n - F)e_i\|_H^2 = \sum_{i=n+1}^{\infty} \|F e_i\|_H^2 \rightarrow 0 \text{ as } n \rightarrow \infty,$$

by the Hilbert–Schmidt property of F . □

Lemma 3.14. *For all $F, F_1, F_2 \in L_2(G, H)$ we have*

- (1) $k_L(F_1 + F_2) \leq 2(k_L(F_1) + k_L(F_2))$;
- (2) $\sup_{O \in \bar{B}_{L(H)}} k_L(O F) \leq k_L(F)$.

Proof. Let $P_n : G \rightarrow G$ denote the projections from Lemma 3.13. Using the inequality

$$(a + b)^2 \wedge 1 \leq 2[(a^2 \wedge 1) + (b^2 \wedge 1)] \text{ for all } a, b \in \mathbb{R}, \tag{3.8}$$

we observe for each $n \in \mathbb{N}$ that

$$\begin{aligned} & \int_H (\|h\|^2 \wedge 1) \left(\lambda \circ ((F_1 + F_2)P_n)^{-1} \right) (dh) \\ &= \int_G (\|(F_1 + F_2)g\|^2 \wedge 1) (\lambda \circ P_n^{-1}) (dg) \\ &\leq 2 \left(\int_G (\|F_1 g\|^2 \wedge 1) (\lambda \circ P_n^{-1}) (dg) + \int_G (\|F_2 g\|^2 \wedge 1) (\lambda \circ P_n^{-1}) (dg) \right) \\ &= 2 \left(\int_H (\|h\|^2 \wedge 1) (\lambda \circ (F_1 P_n)^{-1}) (dh) + \int_H (\|h\|^2 \wedge 1) (\lambda \circ (F_2 P_n)^{-1}) (dh) \right). \end{aligned} \tag{3.9}$$

Moreover, by symmetry and positivity of Q , and the very definition of the Hilbert–Schmidt inner product $\langle A, B \rangle_{L_2(G,H)} = \text{Tr}(AB^*)$ we obtain

$$\begin{aligned} & \text{Tr}(((F_1 + F_2)P_n)Q((F_1 + F_2)P_n)^*) \\ &= \left\| (F_1 + F_2)P_n Q^{1/2} \right\|_{L_2(G,H)}^2 \\ &\leq 2 \left(\left\| F_1 P_n Q^{1/2} \right\|_{L_2(G,H)}^2 + \left\| F_2 P_n Q^{1/2} \right\|_{L_2(G,H)}^2 \right) \\ &= 2(\text{Tr}((F_1 P_n)Q(F_1 P_n)^*) + \text{Tr}((F_2 P_n)Q(F_2 P_n)^*)). \end{aligned} \tag{3.10}$$

By adding the Inequalities in (3.9) and (3.10), we get

$$k_L((F_1 + F_2)P_n) \leq 2(k_L(F_1 P_n) + k_L(F_2 P_n)).$$

By taking limits on both sides, and using continuity of k_L , see Lemma 3.12/(1), the first part of this lemma is proved.

To prove the second part, we fix $F \in L_2(G, H)$ and obtain for all $O \in \bar{B}_{L(H)}$ and $n \in \mathbb{N}$ that

$$\begin{aligned} \int_H (\|h\|^2 \wedge 1) (\lambda \circ (OFP_n)^{-1})(dh) &= \int_G (\|(OF)g\|^2 \wedge 1) (\lambda \circ P_n^{-1})(dg) \\ &\leq \int_G (\|Fg\|^2 \wedge 1) (\lambda \circ P_n^{-1})(dg) \\ &= \int_H (\|h\|^2 \wedge 1) (\lambda \circ (FP_n)^{-1})(dh). \end{aligned} \tag{3.11}$$

Moreover, using the relationship between the Hilbert–Schmidt norm and the trace operator, we obtain for all $O \in \bar{B}_{L(H)}$ and $n \in \mathbb{N}$ that

$$\begin{aligned} \text{Tr}((OFP_n)Q(OFP_n)^*) &= \left\| OFP_n Q^{1/2} \right\|_{L_2(G,H)}^2 \\ &\leq \left\| FP_n Q^{1/2} \right\|_{L_2(G,H)}^2 = \text{Tr}((FP_n)Q(FP_n)^*). \end{aligned} \tag{3.12}$$

By adding Inequalities (3.11) and (3.12), and taking limits on both sides, the result follows from Lemmas 3.11 and 3.13. □

Lemma 3.15. For all $\psi_1, \psi_2 \in \mathcal{M}_{\det,L}^{\text{HS}}$, we have

$$m_L(\psi_1 + \psi_2) \leq 4(m_L(\psi_1) + m_L(\psi_2)).$$

Proof. Let $F_1, F_2 \in L_2(G, H)$ and $(\pi_n)_{n \in \mathbb{N}}$ be a nested normal sequence of partitions $\pi_n = (t_{i,n})_{i=1, \dots, N(n)}$ of the interval $[0,1]$. To simplify the notation, we define for each $n \in \mathbb{N}$ and $i \in \{1, \dots, N(n)\}$

$$A_{i,n} := F_1(L)(t_{i,n}) - F_1(L)(t_{i-1,n}) \text{ and } B_{i,n} := F_2(L)(t_{i,n}) - F_2(L)(t_{i-1,n}).$$

As $(F_1 + F_2)(L) = F_1(L) + F_2(L)$, Theorem 2.2 implies

$$\begin{aligned} &\left\| b_{F_1+F_2}^\theta \right\| \tag{3.13} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{N(n)} E[\theta((F_1 + F_2)(L)(t_{i,n}) - (F_1 + F_2)(L)(t_{i-1,n}))] \\ &= \lim_{n \rightarrow \infty} \left\| \sum_{i=1}^{N(n)} E[\theta(A_{i,n} + B_{i,n})] \right\| \\ &\leq \lim_{n \rightarrow \infty} \left(\left\| \sum_{i=1}^{N(n)} E[\theta(A_{i,n} + B_{i,n}) - \theta(A_{i,n}) - \theta(B_{i,n})] \right\| + \left\| \sum_{i=1}^{N(n)} E[\theta(A_{i,n}) + \theta(B_{i,n})] \right\| \right). \end{aligned}$$

Applying the inequality

$$\|\theta(h_1 + h_2) - \theta(h_1) - \theta(h_2)\| \leq 2(\theta(\|h_1\|)^2 + \theta(\|h_2\|)^2) \quad \text{for all } h_1, h_2 \in H,$$

let us conclude that

$$\begin{aligned} & \left\| \sum_{i=1}^{N(n)} E[\theta(A_{i,n} + B_{i,n}) - \theta(A_{i,n}) - \theta(B_{i,n})] \right\| \\ & \leq 2 \sum_{i=1}^{N(n)} E[\theta(\|A_{i,n}\|)^2] + 2 \sum_{i=1}^{N(n)} E[\theta(\|B_{i,n}\|)^2]. \end{aligned} \tag{3.14}$$

Applying this estimate to Inequality (3.13), we obtain

$$\begin{aligned} \|b_{F_1+F_2}^\theta\| & \leq \lim_{n \rightarrow \infty} \left(2 \sum_{i=1}^{N(n)} E[\theta(\|A_{i,n}\|)^2] + 2 \sum_{i=1}^{N(n)} E[\theta(\|B_{i,n}\|)^2] \right. \\ & \quad \left. + \left\| \sum_{i=1}^{N(n)} E[\theta(A_{i,n})] \right\| + \left\| \sum_{i=1}^{N(n)} E[\theta(B_{i,n})] \right\| \right). \end{aligned}$$

By taking the limit as $n \rightarrow \infty$ and using the limit characterisation of Lévy characteristics from Theorem 2.2, we obtain

$$\begin{aligned} \|b_{F_1+F_2}^\theta\| & \leq 2 \left(\int_H (\|h\|^2 \wedge 1) (\lambda \circ F_1^{-1})(dh) + \text{Tr}(F_1 Q F_1^*) \right) \\ & \quad + 2 \left(\int_H (\|h\|^2 \wedge 1) (\lambda \circ F_2^{-1})(dh) + \text{Tr}(F_2 Q F_2^*) \right) \\ & \quad + \|b_{F_1}^\theta\| + \|b_{F_2}^\theta\| \\ & = 2(k_L(F_1) + k_L(F_2)) + \|b_{F_1}^\theta\| + \|b_{F_2}^\theta\|. \end{aligned}$$

Taking supremum and using Lemma 3.14/(2) imply

$$l_L(F_1 + F_2) = \sup_{O \in \mathcal{B}_L(H)} \|b_{O(F_1+F_2)}^\theta\| \leq 2(k_L(F_1) + k_L(F_2)) + l_L(F_1) + l_L(F_2),$$

which let us conclude from Lemma 3.14/(1) that

$$k_L(F_1 + F_2) + l_L(F_1 + F_2) \leq 4(k_L(F_1) + k_L(F_2)) + l_L(F_1) + l_L(F_2). \tag{3.15}$$

Inequality (3.8) implies for all measurable functions $\psi_1, \psi_2 \in \mathcal{M}_{\text{det},L}^{\text{HS}}$ that

$$\begin{aligned} m_L(\psi_1 + \psi_2) & = \int_0^T k_L(\psi_1(t) + \psi_2(t)) + l_L(\psi_1(t) + \psi_2(t)) dt + \int_0^T (\|\psi_1(t) + \psi_2(t)\|_{L_2(G,H)}^2 \wedge 1) dt \\ & \leq 4 \left(\int_0^T k_L(\psi_1(t)) + l_L(\psi_1(t)) dt + \int_0^T k_L(\psi_2(t)) + l_L(\psi_2(t)) dt \right) \\ & \quad + 2 \left(\int_0^T (\|\psi_1(t)\|_{L_2(G,H)}^2 \wedge 1) dt + \int_0^T (\|\psi_2(t)\|_{L_2(G,H)}^2 \wedge 1) dt \right) \\ & \leq 4(m_L(\psi_1) + m_L(\psi_2)), \end{aligned}$$

which completes the proof. □

Lemma 3.16. *For all $r > 0$ there exists $c_r > 0$ such that*

$$\sup_{\|F\|_{L_2(G,H)} \leq r} (k_L(F) + l_L(F)) \leq c_r.$$

Proof. By Lemma 3.12, $k_L + l_L$ is continuous at 0, from which it follows that there exists a $\delta > 0$ such that $\|F\|_{L_2(G,H)} \leq \delta$ implies $(k_L + l_L)(F) \leq 1$. Let $r > 0$ be fixed. If we choose $N_r \in \mathbb{N}$ to be large enough so that $\frac{r}{N_r} \leq \delta$, then by a repeated use of Equation (3.15), we obtain for some $c_r > 0$ that

$$\begin{aligned} \sup_{\|F\|_{L_2(G,H)} \leq r} (k_L + l_L)(F) &= \sup_{\|F\|_{L_2(G,H)} \leq r} (k_L + l_L)\left(N_r \frac{F}{N_r}\right) \\ &\leq c_r \sup_{\|F\|_{L_2(G,H)} \leq r} (k_L + l_L)\left(\frac{F}{N_r}\right) \leq c_r, \end{aligned}$$

which completes the proof. □

Remark 3.17. Lemma 3.16 guarantees that each bounded function $\psi : [0, T] \rightarrow L_2(G, H)$ is in $\mathcal{M}_{\det,L}^{\text{HS}}$. Indeed, if $\sup_{t \in [0, T]} \|\psi(t)\|_{L_2(G,H)} \leq r$ for some $r > 0$, then Lemma 3.16 implies that there exists $c_r > 0$ such that $\sup_{\|F\|_{L_2(G,H)} \leq r} (k_L(F) + l_L(F)) \leq c_r$. Hence, we obtain

$$m'_L(\psi) := \int_0^T (k_L(\psi(t)) + l_L(\psi(t))) dt \leq T c_r < \infty.$$

As obviously $m''(\psi) < \infty$, it follows $m_L(\psi) < \infty$.

Having developed all the technical tools, we now present the main result of this section:

Theorem 3.18. $\mathcal{M}_{\det,L}^{\text{HS}}$ is a linear space and m_L is a modular on $\mathcal{M}_{\det,L}^{\text{HS}}$.

Proof. Lemma 3.15 shows for $\psi_1, \psi_2 \in \mathcal{M}_{\det,L}^{\text{HS}}$ that

$$m_L(\psi_1 + \psi_2) \leq 4(m_L(\psi_1) + m_L(\psi_2)) < \infty,$$

which implies that $\mathcal{M}_{\det,L}^{\text{HS}}$ is closed under addition. A similar argument as in Lemma 3.16 shows that $\mathcal{M}_{\det,L}^{\text{HS}}$ is closed under multiplication by scalars, which completes the proof that $\mathcal{M}_{\det,L}^{\text{HS}}$ is a vector space. Hence, it remains only to show that m_L satisfies the conditions of Definition 3.1. It follows directly from the definition of m_L that $m_L(-\psi) = m_L(\psi)$ for all $\psi \in \mathcal{M}_{\det,L}^{\text{HS}}$. Condition (2) of Definition 3.1 is a consequence of Lemma 3.12, Remark 3.6 and Lebesgue’s dominated convergence theorem. Condition (3) of Definition 3.1 follows from an argument similar to Lemma 3.14/(2) and the very definition of l_L . Finally, Condition (4) of Definition 3.1 is a direct consequence of Lemma 3.15. □

Remark 3.19. A sequence $(\psi_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}_{\det,L}^{\text{HS}}$ is said to converge to some $\psi \in \mathcal{M}_{\det,L}^{\text{HS}}$ in the modular m_L if we have $\lim_{n \rightarrow \infty} m_L(\psi_n - \psi) = 0$. As $m_L(\psi) = 0$ if and only if $\psi(t) = 0$ for Lebesgue almost all $t \in [0, T]$, we have that limits of sequences in the modular are Lebesgue almost everywhere uniquely determined. For this and further properties of modular convergence, see Section 2 of Nakano [26].

Later on, we will be interested in the space $L^0_P(\Omega, \mathcal{M}^{\text{HS}}_{\text{det},L})$ of $\mathcal{M}^{\text{HS}}_{\text{det},L}$ -valued random elements, which we verify as a Polish space in the sequel.

Lemma 3.20. *The modular space $(\mathcal{M}^{\text{HS}}_{\text{det},L}, m_L)$ is complete, that is, each modular Cauchy sequence on $\mathcal{M}^{\text{HS}}_{\text{det},L}$ is modular convergent.*

Proof. Let $(\psi_i)_{i \in \mathbb{N}} \subseteq \mathcal{M}^{\text{HS}}_{\text{det},L}$ be such that $\lim_{i,j \rightarrow \infty} m_L(\psi_i - \psi_j) = 0$. Then, for all $\epsilon \in (0, 1)$ we have by Markov's inequality that

$$\begin{aligned} & \lim_{i,j \rightarrow \infty} \text{Leb} \left(t \in [0, T] : \|\psi_i(t) - \psi_j(t)\|_{L_2(G,H)} > \epsilon \right) \\ & \leq \lim_{i,j \rightarrow \infty} \frac{1}{\epsilon^2} \int_0^T \left(\|\psi_i(t) - \psi_j(t)\|_{L_2(G,H)}^2 \wedge 1 \right) dt \leq \lim_{i,j \rightarrow \infty} \frac{1}{\epsilon^2} m_L(\psi_i - \psi_j) = 0, \end{aligned}$$

which implies that the sequence $(\psi_i)_{i \in \mathbb{N}}$ is Cauchy in Lebesgue measure. Hence, there exists a subsequence $(\psi_{i_n})_{n \in \mathbb{N}}$ converging Lebesgue almost everywhere to a measurable function $\psi : [0, T] \rightarrow L_2(G, H)$.

Let $\epsilon > 0$ be fixed. By assumption, there exists $N \in \mathbb{N}$ such that for all $i, j \geq N$ we have $m_L(\psi_i - \psi_j) < \epsilon/2$. As by Lemma 3.12, k_L is continuous and l_L is lower-semicontinuous, Fatou's lemma implies for all $i \geq N$ that

$$\begin{aligned} m'_L(\psi_i - \psi) &= \int_0^T (k_L + l_L)(\psi_i(t) - \psi(t)) dt \\ &\leq \int_0^T \liminf_{n \rightarrow \infty} (k_L + l_L)(\psi_i(t) - \psi_{i_n}(t)) dt \\ &\leq \liminf_{n \rightarrow \infty} \int_0^T (k_L + l_L)(\psi_i(t) - \psi_{i_n}(t)) dt \leq \liminf_{n \rightarrow \infty} m_L(\psi_i - \psi_{i_n}) < \frac{\epsilon}{2}. \end{aligned} \tag{3.16}$$

As $(\psi_{i_n})_{n \in \mathbb{N}}$ converges Lebesgue almost everywhere to ψ , using the dominated convergence theorem we obtain

$$\begin{aligned} m''(\psi_i - \psi) &= \int_0^T \left(\|\psi_i(t) - \psi(t)\|_{L_2(G,H)}^2 \wedge 1 \right) dt \\ &= \int_0^T \lim_{n \rightarrow \infty} \left(\|\psi_i(t) - \psi_{i_n}(t)\|_{L_2(G,H)}^2 \wedge 1 \right) dt \\ &= \lim_{n \rightarrow \infty} \int_0^T \left(\|\psi_i(t) - \psi_{i_n}(t)\|_{L_2(G,H)}^2 \wedge 1 \right) dt \leq \lim_{n \rightarrow \infty} m_L(\psi_i - \psi_{i_n}) < \frac{\epsilon}{2}. \end{aligned} \tag{3.17}$$

Equations (3.16) and (3.17) establish that ψ_i converge to ψ in the modular topology. Finally, to see that $\psi \in \mathcal{M}^{\text{HS}}_{\text{det},L}$, fix $i_0 \in \mathbb{N}$ such that $m_L(\psi_{i_0} - \psi) \leq 1$. It follows that

$$m_L(\psi) \leq 4 \left(m_L(\psi - \psi_{i_0}) + m_L(\psi_{i_0}) \right) \leq 4 \left(1 + m_L(\psi_{i_0}) \right) < \infty,$$

which concludes the proof. □

Remark 3.21. Note that Lemma 3.20 explains the role of m'' in the modular m_L . In particular, m'' is needed to establish completeness of the modular space $(\mathcal{M}_{\det,L}^{\text{HS}}, m_L)$ by allowing the identification of a potential m_L -limit of an m_L -Cauchy sequence.

Our next goal is to establish that step functions are dense in the modular space $(\mathcal{M}_{\det,L}^{\text{HS}}, m_L)$. In particular, this will immediately yield that the modular space is separable.

Lemma 3.22. *The collection of Hilbert–Schmidt operator-valued step functions of the form*

$$\psi : [0, T] \rightarrow L_2(G, H), \quad \psi(t) = F_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} F_i \mathbb{1}_{(t_i, t_{i+1}]}(t),$$

where $0 = t_1 < \dots < t_n = T$, $F_i \in L_2(G, H)$ for each $i \in \{0, \dots, n - 1\}$, is dense in $(\mathcal{M}_{\det,L}^{\text{HS}}, m_L)$. Moreover, the modular space $(\mathcal{M}_{\det,L}^{\text{HS}}, m_L)$ is separable.

Proof. First, it follows from Remark 3.17 that each step function of the above form is an element of the modular space $\mathcal{M}_{\det,L}^{\text{HS}}$. To prove the claimed result, we first assume that $\psi \in \mathcal{M}_{\det,L}^{\text{HS}}$ is bounded, that is, there exists a constant $r > 0$ such that $\sup_{t \in [0, T]} \|\psi(t)\|_{L_2(G, H)} \leq r$. By [10, Lemma 1.2.19], there exists a sequence $(\psi_n)_{n \in \mathbb{N}}$ of step functions satisfying:

- (1) $\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \|\psi_n(t)\|_{L_2(G, H)} \leq r$;
- (2) $(\psi_n)_{n \in \mathbb{N}}$ converges to ψ Lebesgue almost everywhere

As $\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \|\psi_n(t) - \psi(t)\|_{L_2(G, H)} \leq 2r$, Lemma 3.16 guarantees that there exists a constant $c > 0$ such that

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} (k_L + l_L)(\psi_n(t) - \psi(t)) \leq c. \tag{3.18}$$

Lebesgue’s dominated convergence theorem and Lemma 3.12 imply that

$$\lim_{n \rightarrow \infty} \int_0^T (k_L + l_L)(\psi_n(t) - \psi(t)) dt = \int_0^T \lim_{n \rightarrow \infty} (k_L + l_L)(\psi_n(t) - \psi(t)) dt = 0.$$

Applying Lebesgue’s dominated convergence theorem to $m''(\psi_n - \psi)$ shows that the step functions ψ_n converge to ψ in the modular m_L .

In the general case of an arbitrary ψ in $\mathcal{M}_{\det,L}^{\text{HS}}$, we define a sequence of functions

$$\psi_n : [0, T] \rightarrow L_2(G, H), \quad \psi_n(t) = \begin{cases} \psi(t) & \text{if } \|\psi(t)\|_{L_2(G, H)} \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the very definition of ψ_n that for every $n \in \mathbb{N}$ and $t \in [0, T]$ we have

$$(k_L + l_L)(\psi_{n+1}(t) - \psi(t)) \leq (k_L + l_L)(\psi_n(t) - \psi(t)) \leq (k_L + l_L)(\psi(t)).$$

As $m_L(\psi) < \infty$, we get

$$\int_0^T (k_L + l_L)(\psi_1(t) - \psi(t)) dt \leq \int_0^T (k_L + l_L)(\psi(t)) dt \leq m_L(\psi) < \infty.$$

The monotone convergence theorem implies

$$\lim_{n \rightarrow \infty} \int_0^T (k_L + l_L)(\psi_n(t) - \psi(t)) dt = 0. \tag{3.19}$$

As Lebesgue’s dominated convergence theorem shows $m''(\psi_n - \psi) \rightarrow 0$, we obtain $m_L(\psi_n - \psi) \rightarrow 0$. By the first part of this lemma, for each $n \in \mathbb{N}$, there exists a sequence $(\psi_{n,i})_{i \in \mathbb{N}}$ of step functions converging to ψ_n in the modular m_L as $i \rightarrow \infty$. For each $n \in \mathbb{N}$, we can choose $i_n \in \mathbb{N}$ such that $m_L(\psi_n - \psi_{n,i_n}) < \frac{1}{n}$. It follows from Lemma 3.15 that

$$\lim_{n \rightarrow \infty} m_L(\psi - \psi_{n,i_n}) \leq \lim_{n \rightarrow \infty} 4(m_L(\psi - \psi_n) + m_L(\psi_n - \psi_{n,i_n})) = 0.$$

As one might require that the approximating sequence of step functions are defined on rational partitions of the time domain and, by separability of $L_2(G, H)$, only take values in a countable dense subset of $L_2(G, H)$, separability of $(\mathcal{M}_{\text{det},L}^{\text{HS}}, m_L)$ follows. \square

Proposition 3.23. *There exists a translation invariant metric ρ_L on $\mathcal{M}_{\text{det},L}^{\text{HS}}$ satisfying:*

- (1) $(\mathcal{M}_{\text{det},L}^{\text{HS}}, \rho_L)$ is a Polish space;
- (2) for any sequence $(\psi_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}_{\text{det},L}^{\text{HS}}$ and $\psi \in \mathcal{M}_{\text{det},L}^{\text{HS}}$ we have

$$\lim_{n \rightarrow \infty} m_L(\psi_n - \psi) = 0 \iff \lim_{n \rightarrow \infty} \rho_L(\psi_n, \psi) = 0.$$

Proof. It follows from Lemma 3.15 and basic properties of the modular m_L that the mapping $(\psi_1, \psi_2) \mapsto m_L(\psi_1 - \psi_2)$ defines a K -quasi-metric on $\mathcal{M}_{\text{det},L}^{\text{HS}}$ in the sense of [7, Definition 2.2.]. Hence, by [7, Theorem 3.10.], there exists a translation-invariant metric ρ_L on $\mathcal{M}_{\text{det},L}^{\text{HS}}$ and $p \in (0, 1)$ such that

$$\rho_L(\psi_1, \psi_2) \leq m_L(\psi_1 - \psi_2)^p \leq 2\rho_L(\psi_1, \psi_2) \quad \text{for all } \psi_1, \psi_2 \in \mathcal{M}_{\text{det},L}^{\text{HS}}. \tag{3.20}$$

Combining Equation (3.20) with Lemma 3.20, Theorem 3.18 and Lemma 3.22, we obtain that $(\mathcal{M}_{\text{det},L}^{\text{HS}}, \rho_L)$ is a complete and separable metric linear space. \square

4 | STOCHASTIC INTEGRALS WITH DETERMINISTIC INTEGRANDS

The definition of the stochastic integral for deterministic integrands with respect to a cylindrical Lévy process L depends heavily on two classes of step functions. We give a precise definition of what is meant by a step function in the following.

Definition 4.1.

- (1) An $L_2(G, H)$ -valued step function is of the form

$$\psi : [0, T] \rightarrow L_2(G, H), \quad \psi(t) = F_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} F_i \mathbb{1}_{(t_i, t_{i+1}]}(t), \tag{4.1}$$

where $0 = t_1 < \dots < t_n = T$, $F_i \in L_2(G, H)$ for each $i \in \{0, \dots, n - 1\}$. The space of $L_2(G, H)$ -valued step functions is denoted by $S_{\det}^{\text{HS}} := S_{\det}^{\text{HS}}(G, H)$.

(2) An $L(H)$ -valued step function is of the form

$$\gamma : [0, T] \rightarrow L(H), \quad \gamma(t) = F_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} F_i \mathbb{1}_{(t_i, t_{i+1}]}(t), \tag{4.2}$$

where $0 = t_1 < \dots < t_n = T$ and $F_i \in L(H)$ for each $i \in \{0, \dots, n - 1\}$. The space of $L(H)$ -valued step functions with $\sup_{t \in [0, T]} \|\gamma(t)\|_{H \rightarrow H} \leq 1$ is denoted by $S_{\det}^{1, \text{op}} := S_{\det}^{1, \text{op}}(H, H)$.

Let $L(t_{i+1}) - L(t_i)$ be an increment of the cylindrical Lévy process L and assume that $F_i \in L_2(G, H)$ for each $i \in \{1, \dots, n - 1\}$. As Hilbert–Schmidt operators are 0-Radonifying by [41, Theorem VI.5.2], it follows from [41, Proposition VI.5.3] that there exist genuine random variables $F_i(L(t_{i+1}) - L(t_i)) : \Omega \rightarrow H$ for each $i \in \{1, \dots, n - 1\}$ satisfying

$$(L(t_{i+1}) - L(t_i))(F_i^* h) = \langle F_i(L(t_{i+1}) - L(t_i)), h \rangle \quad P\text{-almost surely for all } h \in H.$$

We call the random variables $F_i(L(t_{i+1}) - L(t_i))$ Radonified increments for each $i \in \{1, \dots, n - 1\}$. The stochastic integral is defined for any $\psi \in S_{\det}^{\text{HS}}$ with representation (4.1) as the sum of the Radonified increments

$$I(\psi) := \int_0^T \psi \, dL = \sum_{i=1}^{n-1} F_i(L(t_{i+1}) - L(t_i)).$$

Thus, the integral $I(\psi) : \Omega \rightarrow H$ is a genuine H -valued random variable.

The following definition of the stochastic integral can be traced back to the theory of vector measures, and was adapted to the probabilistic setting in [40] by Urbanik and Woyczyński.

Definition 4.2. A function $\psi : [0, T] \rightarrow L_2(G, H)$ is L -integrable for a given cylindrical Lévy process L on G if there exists a sequence $(\psi_n)_{n \in \mathbb{N}}$ of elements of S_{\det}^{HS} satisfying

- (1) $(\psi_n)_{n \in \mathbb{N}}$ converges to ψ Lebesgue almost everywhere;
- (2) $\lim_{m, n \rightarrow \infty} \sup_{\gamma \in S_{\det}^{1, \text{op}}} E \left[\left\| \int_0^T \gamma(\psi_m - \psi_n) \, dL \right\| \wedge 1 \right] = 0.$

In this case, the stochastic integral of the deterministic function ψ is defined by

$$I(\psi) := \int_0^T \psi \, dL := \lim_{n \rightarrow \infty} \int_0^T \psi_n \, dL \quad \text{in } L_P^0(\Omega, H).$$

The class of all deterministic L -integrable Hilbert–Schmidt operator-valued functions is denoted by $\mathcal{I}_{\det, L}^{\text{HS}} := \mathcal{I}_{\det, L}^{\text{HS}}(G, H)$.

Remark 4.3. If Conditions (1) and (2) in Definition 4.2 are satisfied, then the completeness of $L_P^0(\Omega, H)$ implies the existence of the limit. Furthermore, the integral process $(\int_0^t \psi \, dL)_{t \geq 0}$, defined by $\int_0^t \psi \, dL := \int_0^T \mathbb{1}_{[0, t]} \psi \, dL$, has càdlàg paths. To see this, note that for each $m, n \in \mathbb{N}$ the process $(\int_0^t (\psi_m - \psi_n) \, dL)_{t \geq 0}$ has càdlàg paths for a sequence $(\psi_n)_{n \in \mathbb{N}}$ of step functions $\psi_n \in S_{\det}^{\text{HS}}$. By the Lévy–Ottaviani’s inequality, it follows that

$$\lim_{m,n \rightarrow \infty} P\left(\sup_{0 \leq t \leq T} \left\| \int_0^t (\psi_m - \psi_n) dL \right\| > \epsilon\right) \leq 3 \lim_{m,n \rightarrow \infty} \sup_{0 \leq t \leq T} P\left(\left\| \left(\int_0^t \psi_m - \psi_n \right) dL \right\| > \frac{\epsilon}{3}\right) = 0,$$

see [20, Proposition 8.2.1] for details. By passing to a suitable subsequence if necessary, we obtain a subsequence $(\int_0^\cdot \psi_{n_k} dL)_{k \in \mathbb{N}}$ that converges uniformly almost surely, which guarantees that the limiting process has càdlàg paths.

The following is the main result of this section identifying the largest space of L -integrable Hilbert–Schmidt operator-valued functions with the modular space $\mathcal{M}_{\det,L}^{\text{HS}}$.

Theorem 4.4. *The space $\mathcal{I}_{\det,L}^{\text{HS}}$ of deterministic functions integrable with respect to the cylindrical Lévy process L in G coincides with the modular space $\mathcal{M}_{\det,L}^{\text{HS}}$.*

The remainder of this section is devoted to proving the above theorem. As a first step, we prove a key lemma, which shows that convergence of step functions in the modular topology is equivalent to convergence of the corresponding stochastic integrals in the following sense.

Proposition 4.5. *Let L be a cylindrical Lévy process in G , and $(\psi_n)_{n \in \mathbb{N}}$ a sequence in S_{\det}^{HS} . Then the following are equivalent.*

- (a) $\lim_{n \rightarrow \infty} m_L(\psi_n) = 0$.
- (b) $\lim_{n \rightarrow \infty} \sup_{\gamma \in S_{\det}^{1, \text{op}}} E \left[\left\| \int_0^T \gamma \psi_n dL \right\| \wedge 1 \right] = 0$ and $\lim_{n \rightarrow \infty} m''(\psi_n) = 0$.

The proof of the implication (a) \Rightarrow (b) relies on two technical lemmas. The first of these gives a limit representation of the modular. Recall the notation $F(L)$ of the Radonified Lévy process for an operator $F \in L_2(G, H)$ and a cylindrical Lévy process L in G from Lemma 3.2.

Lemma 4.6. *Let $(L(t) : t \geq 0)$ be a cylindrical Lévy process in G with cylindrical characteristics (a, Q, λ) and assume that $\psi \in S_{\det}^{\text{HS}}$ has the representation as in (4.1). If $(\pi_k)_{k \in \mathbb{N}}$ is a nested normal sequence of partitions of $[0, T]$ containing the time points over which ψ is defined, then we have:*

- (a) $\lim_{k \rightarrow \infty} \sum_{i=1}^{n-1} \sum_{\substack{p_{j,k} \in \pi_k \\ t_i < p_{j,k} \leq t_{i+1}}} E \left[\theta(F_i(L)(p_{j,k}) - F_i(L)(p_{j-1,k})) \right] = \int_0^T b_{\psi(t)}^\theta dt$;
- (b) $\lim_{k \rightarrow \infty} \sum_{i=1}^{n-1} \sum_{\substack{p_{j,k} \in \pi_k \\ t_i < p_{j,k} \leq t_{i+1}}} E \left[\left\| \theta(F_i(L)(p_{j,k}) - F_i(L)(p_{j-1,k})) \right\|^2 \right] \\ = \int_0^T \int_H (\|h\|^2 \wedge 1) (\lambda \circ \psi(t)^{-1})(dh) dt + \int_0^T \text{Tr}(\psi(t)Q\psi(t)^*) dt$.

Proof. The proof is a direct application of Lemma 3.2 and the limit characterisation of Lévy characteristics in Theorem 2.2. □

Another ingredient of the proof of Proposition 4.5 is the following general result from [20].

Lemma 4.7. *For all $\epsilon > 0$ there exists $\delta > 0$ such that for any sequence $(X_n)_{n \in \{1, \dots, N\}}$ of independent H -valued random variables X_n satisfying*

$$\left\| \sum_{n=1}^N E[\theta(X_n)] \right\| < \delta \quad \text{and} \quad \sum_{n=1}^N E \left[\|\theta(X_n)\|^2 \right] < \delta,$$

it follows that

$$E \left[\left\| \sum_{n=1}^N X_n \right\| \wedge 1 \right] < \epsilon.$$

Proof. See [20, Proposition 8.1.1/(ii)]. □

Proof of (a) ⇒ (b) in Proposition 4.5. Let $\epsilon > 0$ be fixed and choose $\delta > 0$ so that the implication in Lemma 4.7 holds. The hypothesis guarantees that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $m(\psi_n) < \delta$. Let $n_0 \geq N$ and $\gamma_0 \in S_{\det}^{1,op}$ be fixed. We assume the representation

$$\gamma_0 \psi_{n_0}(t) = O_{0,n_0} F_{0,n_0} \mathbb{1}_{\{0\}}(t) + \sum_{i=0}^{N(n_0)-1} O_{i,n_0} F_{i,n_0} \mathbb{1}_{(t_{i,n_0}, t_{i+1,n_0}]}(t),$$

where $0 = t_{0,n_0} < t_{1,n_0} < \dots < t_{N(n_0),n_0} = T$, $O_{i,n_0} \in \bar{B}_{L(H)}$ and $F_{i,n_0} \in L_2(G, H)$. Let $(\pi_k)_{k \in \mathbb{N}}$ be a nested normal sequence of partitions containing the points over which $\gamma_0 \psi_{n_0}$ is defined. As by Lemma 3.14 and the very definition of l_L we have $m_L(\gamma_0 \psi_{n_0}) \leq m_L(\psi_{n_0}) < \delta$, Lemma 4.6 guarantees that there exists a $K \in \mathbb{N}$ such that the partition π_K satisfies

$$\left\| \sum_{i=0}^{N(n_0)-1} \sum_{\substack{p_{j,K} \in \pi_K \\ t_{i,n_0} < p_{j,K} \leq t_{i+1,n_0}}} E \left[\theta \left(O_{i,n_0} F_{i,n_0}(L)(p_{j,K}) - O_{i,n_0} F_{i,n_0}(L)(p_{j-1,K}) \right) \right] \right\| < \delta$$

and

$$\sum_{i=0}^{N(n_0)-1} \sum_{\substack{p_{j,K} \in \pi_K \\ t_{i,n_0} < p_{j,K} \leq t_{i+1,n_0}}} E \left[\left\| \theta \left(O_{i,n_0} F_{i,n_0}(L)(p_{j,K}) - O_{i,n_0} F_{i,n_0}(L)(p_{j-1,K}) \right) \right\|^2 \right] < \delta.$$

As π_K contains the time points over which $\gamma_0 \psi_{n_0}$ is defined, Lemma 4.7 implies

$$E \left[\left\| \int_0^T \gamma_0 \psi_{n_0} dL \right\| \wedge 1 \right] = E \left[\left\| \sum_{i=0}^{N(n_0)-1} O_{i,n_0} F_{i,n_0} \left(L(t_{i+1,n_0}) - L(t_{i,n_0}) \right) \right\| \wedge 1 \right] < \epsilon.$$

This concludes the proof of the implication. □

The proof of the reverse implication (b) ⇒ (a) in Proposition 4.5 relies on three technical lemmas, which we present below.

Lemma 4.8. Fix some $e \in H$ with $\|e\| = 1$ and define $A_h := \text{Span}\{e, h\}$ for each $h \in H$. Define the mapping

$$f : H \rightarrow \bar{B}_{L(H)}, \quad f(h)h' = \begin{cases} \sigma_{A_h}(h'_{A_h}) + h'_{A_h^\perp}, & \text{if } h \in H \setminus \text{Span}\{e\} \\ \text{sgn}(\beta)h', & \text{if } h = \beta e \text{ for some } \beta \in \mathbb{R}, \end{cases}$$

where h'_{A_h} and $h'_{A_h^\perp}$ denote the projections of h' onto the subspace A_h and its orthogonal complement A_h^\perp , respectively, and σ_{A_h} denotes the rotation on the plane A_h around the origin by the angle which

rotates the vector h into e . It follows that

$$\langle e, b_{f(b_F^\theta)}^\theta \rangle = \|b_F^\theta\| \quad \text{for all } F \in L_2(G, H).$$

Proof. We first show that $f(h) : H \rightarrow H$ is an isometry for each $h \in H$. If $h \in H \setminus \text{Span}\{e\}$, we obtain for every $h' \in H$, by using the fact that rotations are isometries and the orthogonality of both $\sigma_{A_h}(h'_{A_h})$ with $h'_{A_h^\perp}$ and h'_{A_h} with $h_{A_h^\perp}$, that

$$\|f(h)(h')\|^2 = \|\sigma_{A_h}(h'_{A_h})\|^2 + \|h'_{A_h^\perp}\|^2 = \|h'_{A_h}\|^2 + \|h'_{A_h^\perp}\|^2 = \|h'\|^2.$$

As the case $h \in \text{Span}\{e\}$ is obvious, we conclude that $f(h)$ is indeed an isometry for every $h \in H$.

As the isometry property guarantees $\theta(f(h)h') = f(h)\theta(h')$ for all $h, h' \in H$, Lemma 3.9 implies

$$b_{f(h)F}^\theta = f(h)b_F^\theta \quad \text{for all } F \in L_2(G, H), h \in H. \tag{4.3}$$

Next, as $\sigma_{A_h}(h) = \|h\|e$ and $h_{A_h^\perp} = 0$ if $h \in H \setminus \text{Span}\{e\}$, and $\langle e, f(h)h \rangle = \|\beta\|$ if $h = \beta e$ for some $\beta \in \mathbb{R}$, we obtain $\langle e, f(h)h \rangle = \|h\|$ for each $h \in H$. Applying this equality together with (4.3) for $h = b_F^\theta$ completes the proof. \square

Lemma 4.9. *If $\psi \in S_{\det}^{\text{HS}}$, then*

$$\int_0^T \sup_{O \in \bar{B}_{L(H)}} \|b_{O\psi(t)}^\theta\| dt = \sup_{\gamma \in S_{\det}^{1, \text{op}}} \left\| \int_0^T b_{\gamma\psi(t)}^\theta dt \right\|.$$

Proof. In this proof, we use the notation $L_2 := L_2(G, H)$ and equip the space $\bar{B}_{L(H)}$ with the strong operator topology that is metrisable, say, with a metric d . Thus, $(\bar{B}_{L(H)}, d)$ is a Polish space; see [14, p. 14].

For fixed $\epsilon > 0$, define the multi-valued function $g : L_2 \rightarrow \bar{B}_{L(H)}$ by

$$g(F) = \left\{ O \in \bar{B}_{L(H)} : \sup_{Q \in \bar{B}_{L(H)}} \|b_{QF}^\theta\| - \|b_{OF}^\theta\| < \frac{\epsilon}{T} \right\}.$$

Let μ be a probability measure on $\mathfrak{B}(L_2)$ and denote by $(L_2, \mathfrak{B}_\mu(L_2), \mu_0)$ the completion of the probability space $(L_2, \mathfrak{B}(L_2), \mu)$. As the function $R \mapsto RF$ is continuous as a function from $(\bar{B}_{L(H)}, d)$ to L_2 for each $F \in L_2$, Lemma 3.11 implies that the mapping $R \mapsto b_{RF}^\theta$ is also continuous as a function from $(\bar{B}_{L(H)}, d)$ to H . Consequently, the set $g(F)$ is open in $\bar{B}_{L(H)}$ for each $F \in L_2$, which shows that

$$g(L_2) = \bigcup_{F \in L_2} g(F)$$

is open in $(\bar{B}_{L(H)}, d)$. It follows that the graph of g , which is defined as the set $\{(F, g(F)) : F \in L_2\}$, belongs to $\mathfrak{B}(L_2) \otimes \mathfrak{B}(\bar{B}_{L(H)})$. By the Aumann measurable selection theorem, see, for example, [6, Theorem 6.9.13], there exists a measurable mapping $i : (L_2, \mathfrak{B}_\mu(L_2)) \rightarrow (\bar{B}_{L(H)}, \mathfrak{B}(\bar{B}_{L(H)}))$ with $i(F) \in g(F)$ for all $F \in L_2$. As ψ attains only finitely many values, we obtain

$\psi : ([0, T], \mathfrak{B}([0, T])) \rightarrow (L_2, \mathfrak{B}_\mu(L_2))$ is measurable. This enables us to define the measurable function

$$\rho : ([0, T], \mathfrak{B}([0, T])) \rightarrow (L_2, \mathfrak{B}(L_2)), \quad \rho(t) := i(\psi(t))\psi(t).$$

By using the function $f : H \rightarrow \bar{B}_{L(H)}$ defined in Lemma 4.8, we introduce

$$\eta : [0, T] \rightarrow \bar{B}_{L(H)}, \quad \eta(t) = f\left(b_{\rho(t)}^\theta\right).$$

The map η is measurable as it is the composition of measurable functions. It follows from Lemma 4.8 that

$$\left\| \int_0^T b_{\eta\rho(t)}^\theta dt \right\| \geq \int_0^T \langle e, b_{\eta\rho(t)}^\theta \rangle dt = \int_0^T \|b_{\rho(t)}^\theta\| dt \geq \int_0^T \left(\sup_{R \in \bar{B}_{L(H)}} \|b_{R\psi(t)}^\theta\| - \frac{\epsilon}{T} \right) dt.$$

As $\epsilon > 0$ is arbitrary, we obtain

$$\left\| \int_0^T b_{\eta\rho(t)}^\theta dt \right\| \geq \int_0^T \sup_{R \in \bar{B}_{L(H)}} \|b_{R\psi(t)}^\theta\| dt. \tag{4.4}$$

Furthermore, define $c := \sup_{t \in [0, T]} \|b_{\eta\rho(t)}^\theta\|$, which is finite as ψ attains only finitely many values. It follows from [10, Lemma 1.2.19] and [10, Remark 1.2.20] that for each $\epsilon > 0$ there exist $A \in \mathfrak{B}([0, T])$ with $\text{Leb}(A^c) \leq \epsilon$ and a sequence $(\gamma_k)_{k \in \mathbb{N}}$ in $S_{\text{det}}^{1, \text{op}}$ converging uniformly to η on A . As Lebesgue’s theorem on dominated convergence implies

$$\lim_{k \rightarrow \infty} \left\| \int_0^T (b_{\gamma_k \rho(t)}^\theta - b_{\eta\rho(t)}^\theta) dt \right\| \leq \lim_{k \rightarrow \infty} \left\| \int_A (b_{\gamma_k \rho(t)}^\theta - b_{\eta\rho(t)}^\theta) dt \right\| + c\epsilon = c\epsilon,$$

it follows from (4.4) that

$$\sup_{\gamma \in S_{\text{det}}^{1, \text{op}}} \left\| \int_0^T b_{\gamma\psi(t)}^\theta dt \right\| \geq \int_0^T \sup_{R \in \bar{B}_{L(H)}} \|b_{R\psi(t)}^\theta\| dt.$$

As the reverse inequality follows from basic properties of the Bochner integral, the proof is complete. □

Lemma 4.10. *If a sequence $(\psi_n)_{n \in \mathbb{N}} \subseteq S_{\text{det}}^{\text{HS}}$ satisfies*

$$\lim_{n \rightarrow \infty} \sup_{\gamma \in S_{\text{det}}^{1, \text{op}}} E \left[\left\| \int_0^T \gamma \psi_n dL \right\| \wedge 1 \right] = 0,$$

then

$$\lim_{n \rightarrow \infty} \sup_{\gamma \in S_{\text{det}}^{1, \text{op}}} \left\| \int_0^T b_{\gamma\psi_n(t)}^\theta dt \right\| = 0.$$

Proof. Assume, aiming for a contradiction, that this is not the case. Then, by passing on to a suitable subsequence if necessary, there exists an $\epsilon > 0$ and a sequence $(\gamma_n)_{n \in \mathbb{N}} \subseteq S_{\text{det}}^{1, \text{op}}$ satisfying

for all $n \in \mathbb{N}$ that

$$\left\| \int_0^T b_{\gamma_n \psi_n(t)}^\theta dt \right\| > \epsilon. \tag{4.5}$$

On the other hand, the hypothesis implies that the sequence $(I(\gamma_n \psi_n))_{n \in \mathbb{N}}$ of infinitely divisible random variables $I(\gamma_n \psi_n)$, whose first characteristics equals $\int_0^T b_{\gamma_n \psi_n(t)}^\theta dt$, converges to 0 in probability. It follows from Lemma 3.11 that

$$\lim_{n \rightarrow \infty} \int_0^T b_{\gamma_n \psi_n(t)}^\theta dt = 0,$$

which contradicts Equation (4.5). Hence, the result follows. □

Having provided all the necessary preliminary results, we now proceed to proving the reverse implication in Proposition 4.5.

Proof of (b) \Rightarrow (a) in Proposition 4.5. As for each $n \in \mathbb{N}$, ψ_n has a representation of the form

$$\psi_n(t) = F_0^n \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{N(n)-1} F_i^n \mathbb{1}_{(t_i^n, t_{i+1}^n]}(t),$$

where $0 = t_1^n < \dots < t_{N(n)}^n = T$, and $F_i^n \in L_2(G, H)$ for each $i \in \{0, \dots, N(n) - 1\}$, the integral $I(\psi_n)$ takes the form

$$I(\psi_n) = \sum_{i=1}^{N(n)-1} F_i^n (L(t_{i+1}^n) - L(t_i^n)).$$

As the integral $I(\psi_n)$ is the sum of independent infinitely divisible random variables, $I(\psi_n)$ is also infinitely divisible and has characteristics

$$\left(\sum_{i=1}^{N(n)-1} (t_{i+1}^n - t_i^n) b_{F_i^n}^\theta, \sum_{i=1}^{N(n)-1} (t_{i+1}^n - t_i^n) F_i^n Q(F_i^n)^*, \sum_{i=1}^{N(n)-1} (t_{i+1}^n - t_i^n) (\lambda \circ (F_i^n)^{-1}) \right).$$

As the hypothesis implies that $\lim_{n \rightarrow \infty} I(\psi_n) = 0$ in $L_p^0(\Omega, H)$, we conclude from Lemma 3.11 and [4, Lemma 3.6] that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_0^T k_L(\psi(t)) dt \\ &= \lim_{n \rightarrow \infty} \int_H (\|h\|^2 \wedge 1) \left((\lambda \otimes \text{Leb}) \circ \kappa_{\psi_n}^{-1} \right) (dh) + \text{Tr} \left(\int_0^T (\psi_n(t) Q \psi_n(t)^*) dt \right) = 0. \end{aligned} \tag{4.6}$$

Furthermore, the hypothesis implies by Lemmas 4.9 and 4.10 that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^T l_L(\psi_n(t)) dt &= \lim_{n \rightarrow \infty} \int_0^T \sup_{O \in \mathcal{B}_L(H)} \|b_{O \psi_n(t)}^\theta\| dt \\ &= \lim_{n \rightarrow \infty} \sup_{\gamma \in S_{\det}^{1, \text{op}}} \left\| \int_0^T b_{\gamma \psi_n(t)}^\theta dt \right\| = 0. \end{aligned} \tag{4.7}$$

Equations (4.6) and (4.7) together imply that $\lim_{n \rightarrow \infty} m'_L(\psi_n) = 0$. As by assumption, we have $\lim_{n \rightarrow \infty} m''(\psi_n) = 0$, we obtain that $\lim_{n \rightarrow \infty} m_L(\psi_n) = 0$. \square

We are now ready to present the proof of the main result of this section characterising the largest space of deterministic Hilbert–Schmidt operator-valued functions which are integrable with respect to a cylindrical Lévy process L in Hilbert space.

Proof of Theorem 4.4. If $\psi \in \mathcal{I}_{\det,L}^{\text{HS}}$ then by the very definition of integrability, see Definition 4.2, there exists a sequence $(\psi_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}_{\det}^{\text{HS}}$ such that $\psi_n \rightarrow \psi$ Lebesgue almost everywhere and $\sup_{\gamma \in \mathcal{S}_{\det}^{1,\text{op}}} E[\|I(\gamma(\psi_n - \psi_m))\| \wedge 1] \rightarrow 0$. By Proposition 4.5, this implies that $m_L(\psi_n - \psi_m) \rightarrow 0$. Completeness of the modular space $\mathcal{M}_{\det,L}^{\text{HS}}$, see Lemma 3.20, and the fact that $\psi_n \rightarrow \psi$ Lebesgue almost everywhere allows us to conclude that $\psi \in \mathcal{M}_{\det,L}^{\text{HS}}$.

Conversely, if $\psi \in \mathcal{M}_{\det,L}^{\text{HS}}$, then Lemma 3.22 implies that there exists a sequence $(\psi_n)_{n \in \mathbb{N}}$ of elements in $\mathcal{S}_{\det}^{\text{HS}}$ such that $\psi_n \rightarrow \psi$ Lebesgue almost everywhere and $m_L(\psi_n - \psi) \rightarrow 0$. It follows that $m_L(\psi_n - \psi_m) \rightarrow 0$, which implies $\sup_{\gamma \in \mathcal{S}_{\det}^{1,\text{op}}} E[\|I(\gamma(\psi_n - \psi_m))\| \wedge 1] \rightarrow 0$ by Proposition 4.5 and establishes that $\psi \in \mathcal{I}_{\det,L}^{\text{HS}}$. \square

5 | STOCHASTIC INTEGRALS WITH PREDICTABLE INTEGRANDS

For the remainder of this section, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. As in the case of deterministic integrands, we begin by introducing two classes of functions on which our definition of the stochastic integral depends on.

Definition 5.1.

(1) An $L_2(G, H)$ -valued predictable step process $\Psi : \Omega \times [0, T] \rightarrow L_2(G, H)$ is of the form

$$\Psi(\omega, t) = \left(\sum_{k=1}^{N(0)} F_{0,k} \mathbb{1}_{A_{0,k}}(\omega) \right) \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} \left(\sum_{k=1}^{N(i)} F_{i,k} \mathbb{1}_{A_{i,k}}(\omega) \right) \mathbb{1}_{(t_i, t_{i+1}]}(t), \tag{5.1}$$

where $0 = t_1 < \dots < t_n = T$, $A_{0,k} \in \mathcal{F}_0$ and $F_{0,k} \in L_2(G, H)$ for all $k = 1, \dots, N(0)$, $A_{i,k} \in \mathcal{F}_{t_i}$ and $F_{i,k} \in L_2(G, H)$ for all $i = 1, \dots, n - 1$ and $k = 1, \dots, N(i)$. The space of all $L_2(G, H)$ -valued predictable step processes is denoted by $\mathcal{S}_{\text{prd}}^{\text{HS}} := \mathcal{S}_{\text{prd}}^{\text{HS}}(G, H)$.

(2) An $L(H)$ -valued predictable step process $\Gamma : \Omega \times [0, T] \rightarrow L(H, H)$ is of the form

$$\Gamma(\omega, t) = \left(\sum_{k=1}^{N(0)} O_{0,k} \mathbb{1}_{A_{0,k}}(\omega) \right) \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} \left(\sum_{k=1}^{N(i)} O_{i,k} \mathbb{1}_{A_{i,k}}(\omega) \right) \mathbb{1}_{(t_i, t_{i+1}]}(t), \tag{5.2}$$

where $0 = t_1 < \dots < t_n = T$, $A_{0,k} \in \mathcal{F}_0$ and $O_{0,k} \in L(H)$ for all $k = 1, \dots, N(0)$, $A_{i,k} \in \mathcal{F}_{t_i}$ and $O_{i,k} \in L(H)$ for all $i = 1, \dots, n - 1$ and $k = 1, \dots, N(i)$. The space of all $L(H)$ -valued predictable step processes with

$$\sup_{(\omega,t) \in \Omega \times [0,T]} \|\Gamma(\omega, t)\|_{H \rightarrow H} \leq 1$$

is denoted by $\mathcal{S}_{\text{prd}}^{1,\text{op}} := \mathcal{S}_{\text{prd}}^{1,\text{op}}(H, H)$.

Let $\Psi \in \mathcal{S}_{\text{prd}}^{\text{HS}}$ be of the form (5.1) and L a cylindrical Lévy process in G . As explained after Definition 4.1, there exists an H -valued random variable $F_{i,k}(L(t_{i+1}) - L(t_i)) : \Omega \rightarrow H$ for each $i = 1, \dots, n - 1$ and $k = 1, \dots, N(i)$, satisfying

$$(L(t_{i+1}) - L(t_i))(F_{i,k}^* h) = \langle F_{i,k}(L(t_{i+1}) - L(t_i)), h \rangle \quad P\text{-almost surely for all } h \in H.$$

In this case, the stochastic integral of Ψ is defined by

$$I(\Psi) := \int_0^T \Psi(t) dL(t) := \sum_{i=1}^{n-1} \sum_{k=1}^{N(i)} \mathbb{1}_{A_{i,k}} F_{i,k}(L(t_{i+1}) - L(t_i)).$$

Thus, the integral $I(\Psi) : \Omega \rightarrow H$ is a genuine H -valued random variable. In contrast to the deterministic case in Section 4, the integral $I(\Psi)$ is not necessarily infinitely divisible.

For the purposes of this section, it is convenient to introduce the measure space $(\Omega \times [0, T], \mathcal{P}, P_T)$, where \mathcal{P} denotes the predictable σ -algebra and the measure P_T is defined by $P_T := P \otimes \text{Leb}|_{[0,T]}$.

Definition 5.2. We say that a predictable process Ψ is L -integrable if there exists a sequence $(\Psi_n)_{n \in \mathbb{N}}$ of processes in $\mathcal{S}_{\text{prd}}^{\text{HS}}$ such that

- (1) $(\Psi_n)_{n \in \mathbb{N}}$ converges P_T -almost everywhere to Ψ ;
- (2) $\lim_{m,n \rightarrow \infty} \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E \left[\left\| \int_0^T \Gamma(\Psi_m - \Psi_n) dL \right\| \wedge 1 \right] = 0.$

In this case, the stochastic integral of Ψ is defined by

$$I(\Psi) := \int_0^T \Psi dL = \lim_{n \rightarrow \infty} \int_0^T \Psi_n dL \quad \text{in } L_P^0(\Omega, H).$$

The class of all L -integrable $L_2(G, H)$ -valued predictable processes will be denoted by $\mathcal{I}_{\text{prd},L}^{\text{HS}} := \mathcal{I}_{\text{prd},L}^{\text{HS}}(G, H)$. As usual, for $t \in [0, T]$, we define $\int_0^t \Psi dL := \int_0^t \mathbb{1}_{\Omega \times (0,t]} \Psi dL$.

Remark 5.3. An extension of [19, Lemma 2.3] to H -valued random variables shows that Condition (2) of Definition 5.2 implies

$$\lim_{m,n \rightarrow \infty} \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E \left[\sup_{t \in [0,T]} \left\| \int_0^t \Gamma(\Psi_m - \Psi_n) dL \right\| \wedge 1 \right] = 0.$$

Hence, the notion of convergence introduced in Definition 5.2 is stronger than convergence in the ucp topology. It follows that for each $\Psi \in \mathcal{I}_{\text{prd},L}^{\text{HS}}$, the process $(\int_0^t \Psi dL)_{t \in [0,T]}$ has càdlàg paths. For details, see the end of Remark 4.3.

Let L be a cylindrical Lévy process in G and denote by ρ_L the metric on the modular space $\mathcal{M}_{\text{det},L}^{\text{HS}}$ making this space Polish; see Proposition 3.23. Let $L_P^0(\Omega, \mathcal{M}_{\text{det},L}^{\text{HS}})$ denote the collection of all Borel-measurable mappings $\Psi : \Omega \rightarrow \mathcal{M}_{\text{det},L}^{\text{HS}}$ which we endow with the translation invariant metric

$$\bar{\rho}_L(\Psi_1, \Psi_2) := E[\rho_L(\Psi_1, \Psi_2) \wedge 1] \quad \text{for } \Psi_1, \Psi_2 \in L_P^0(\Omega, \mathcal{M}_{\text{det},L}^{\text{HS}}),$$

which guarantees its completeness.

Lemma 5.4. *Let Ψ be a predictable stochastic process in $L^0_P(\Omega, \mathcal{M}^{\text{HS}}_{\text{det},L})$. Then there exists a sequence $(\Psi_k)_{k \in \mathbb{N}}$ of elements of $S^{\text{HS}}_{\text{prd}}$ converging to Ψ both in the metric $\bar{\rho}_L$ and P_T -almost everywhere*

Proof. If Ψ is bounded, then $\Psi \in L^\infty_{P_T}(\Omega \times [0, T], L_2(G, H))$. As the algebra of sets

$$\mathcal{A}' = \{(s, t] \times B : s < t, B \in \mathcal{F}_s\} \cup \{\{0\} \times B : B \in \mathcal{F}_0\}$$

generates \mathcal{P} , we conclude from [10, Lemma 1.2.19] and [10, Remark 1.2.20] that there exists a sequence $(\Psi_k)_{k \in \mathbb{N}}$ of uniformly bounded processes in $S^{\text{HS}}_{\text{prd}}$ such that $\Psi_k \rightarrow \Psi$ P_T -almost everywhere. Thus, there exists a set $N \in \mathcal{P}$ such that $P_T(N) = 0$ and $(\Psi_k(\omega, t) - \Psi(\omega, t)) \rightarrow 0$ for all $(\omega, t) \in N^c$. Fubini's theorem implies that

$$0 = P_T(N) = P \otimes \text{Leb}|_{[0,T]}(N) = \int_{\Omega} \text{Leb}|_{[0,T]}(N_\omega) P(d\omega),$$

where for each fixed $\omega \in \Omega$ we define

$$N_\omega := \left\{ t \in [0, T] : (\Psi_k(\omega, t) - \Psi(\omega, t))_{m \in \mathbb{N}} \text{ does not converge to } 0 \right\}.$$

The above implies that $\text{Leb}|_{[0,T]}(N_\omega) = 0$ for almost all $\omega \in \Omega$, that is, there exists an $\Omega_0 \subseteq \Omega$ with $P(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ we have

$$\text{Leb}|_{[0,T]}(t \in [0, T] : (\Psi_k(\omega, t) - \Psi(\omega, t))_{m \in \mathbb{N}} \text{ does not converge to } 0) = 0.$$

Because $(\Psi_k)_{k \in \mathbb{N}}$ is uniformly bounded and Ψ is bounded, we can conclude from Lebesgue's dominated convergence theorem that $\lim_{n \rightarrow \infty} m_L(\Psi_k(\omega, \cdot) - \Psi(\omega, \cdot)) = 0$ for each $\omega \in \Omega_0$. As m_L and ρ_L generate the same topology on $\mathcal{M}^{\text{HS}}_{\text{det},L}$, we also have $\rho_L(\Psi_k(\omega, \cdot), \Psi(\omega, \cdot)) \rightarrow 0$ as $k \rightarrow \infty$ for each $\omega \in \Omega_0$. Another application of Lebesgue's dominated convergence theorem yields

$$\lim_{k \rightarrow \infty} \bar{\rho}_L(\Psi_k, \Psi) = \lim_{k \rightarrow \infty} \int_{\Omega} (\rho_L(\Psi_k(\omega, \cdot), \Psi(\omega, \cdot)) \wedge 1) dP = 0,$$

which shows the claim if Ψ is bounded. In the case of a general Ψ , we define

$$\Psi_n : \Omega \times [0, T] \rightarrow L_2(G, H), \quad \Psi_n(\omega, t) = \begin{cases} \Psi(\omega, t) & \text{if } \|\Psi(\omega, t)\|_{L_2(G,H)} \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\lim_{n \rightarrow \infty} \bar{\rho}_L(\Psi, \Psi_n) = 0$. The first part of the proof shows that for each $n \in \mathbb{N}$ there exists a sequence $(\Psi_{n,k})_{k \in \mathbb{N}} \subseteq S^{\text{HS}}_{\text{prd}}$ converging to Ψ_n as $k \rightarrow \infty$ in $\bar{\rho}_L$ and P_T -almost everywhere. For each $n \in \mathbb{N}$ choose $k_n \in \mathbb{N}$ such that $\bar{\rho}_L(\Psi_n, \Psi_{n,k_n}) < \frac{1}{n}$. It follows that

$$\lim_{n \rightarrow \infty} \bar{\rho}_L(\Psi, \Psi_{n,k_n}) \leq \lim_{n \rightarrow \infty} (\bar{\rho}_L(\Psi, \Psi_n) + \bar{\rho}_L(\Psi_n, \Psi_{n,k_n})) = 0,$$

which completes the proof, as by passing on to a suitable subsequence, we also have convergence P_T -almost everywhere □

6 | CONSTRUCTION OF THE DECOUPLED TANGENT SEQUENCE

The technique of constructing decoupled tangent sequences is a powerful tool to obtain strong results on a sequence of possibly dependent random variables. In this section, we briefly recall the fundamental definition, see, for example, Kwapien and Woyczyński [20] or de la Peña and Giné [33], and construct the decoupled tangent sequence in our setting which will enable us to identify the largest space of predictable integrands in the next section.

Remark 6.1. We repeatedly use the fact in the following that given a random variable X on (Ω, \mathcal{F}, P) and another probability space $(\Omega', \mathcal{F}', P')$, the random variable X can always be considered as a random variable on the product space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \otimes P')$ by defining

$$X(\omega, \omega') = X(\omega) \quad \text{for all } (\omega, \omega') \in \Omega \times \Omega'.$$

In this case, if X is real-valued and P -integrable we have $E_P[X] = E_{P \otimes P'}[X]$.

In the next definition, we follow closely [20, chapter 4.3].

Definition 6.2. Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n \in \mathbb{N}})$ be a filtered probability space and $(X_n)_{n \in \mathbb{N}}$ an (\mathcal{F}_n) -adapted sequence of H -valued random variables. If $(\Omega', \mathcal{F}', P', (\mathcal{F}'_n)_{n \in \mathbb{N}})$ is another filtered probability space, then a sequence $(Y_n)_{n \in \mathbb{N}}$ of H -valued random variables defined on $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \otimes P', (\mathcal{F}_n \otimes \mathcal{F}'_n)_{n \in \mathbb{N}})$ is said to be a decoupled tangent sequence to $(X_n)_{n \in \mathbb{N}}$ if

- (1) for each $\omega \in \Omega$, we have that $(Y_n(\omega, \cdot))_{n \in \mathbb{N}}$ is a sequence of independent random variables on $(\Omega', \mathcal{F}', P')$;
- (2) the sequences $(X_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ satisfy for each $n \in \mathbb{N}$ that

$$\mathcal{L}(X_n | \mathcal{F}_{n-1} \otimes \mathcal{F}'_{n-1}) = \mathcal{L}(Y_n | \mathcal{F}_{n-1} \otimes \mathcal{F}'_{n-1}) \quad P \otimes P' - \text{almost surely.}$$

The importance of decoupled tangent sequences lies in the existence of a collection of inequalities, frequently called decoupling inequalities, which relate convergence of an adapted sequence of random variables to convergence of their decoupled tangent sequence. We provide two such inequalities in the following result, which will be repeatedly used in the sequel.

Proposition 6.3. *There exist constants $c_1, c_2 > 0$ such that for all finite adapted H -valued sequences $(X_n)_{n=1, \dots, N}$ of random variables with corresponding decoupled tangent sequence $(Y_n)_{n=1, \dots, N}$, the following inequalities hold:*

$$E_P \left[\left\| \sum_{n=1}^N X_n \right\| \wedge 1 \right] \leq c_1 E_{P \otimes P'} \left[\left\| \sum_{n=1}^N Y_n \right\| \wedge 1 \right]; \tag{6.1}$$

$$E_{P \otimes P'} \left[\left\| \sum_{n=1}^N Y_n \right\| \wedge 1 \right] \leq c_2 \sup_{\epsilon_n \in \{\pm 1\}} E_P \left[\left\| \sum_{n=1}^N \epsilon_n X_n \right\| \wedge 1 \right]. \tag{6.2}$$

Proof. See [20, Proposition 5.7.1.(ii)] and [20, Proposition 5.7.2]. □

The main tool for establishing the stochastic integral in the next section is a cylindrical Lévy process \tilde{L} on an enlarged probability space, whose Radonified increments are decoupled to the

Radonified increments of the original cylindrical Lévy process. This cylindrical Lévy process \tilde{L} is explicitly constructed in the following result.

Proposition 6.4. *Let L be a cylindrical Lévy process in G , $0 = t_0 \leq \dots \leq t_N = T$ be a partition of $[0, T]$ and for each $n = 1, \dots, N$ we define $\Theta_n := \sum_{k=1}^{M(n)} F_{n,k} \mathbb{1}_{A_{n,k}}$ where $F_{n,k} \in L_2(G, H)$, $A_{n,k} \in \mathcal{F}_{t_{n-1}}$ for all $k = 1, \dots, M(n)$. By defining cylindrical random variables*

$$\tilde{L}(t) : G \rightarrow L_{P \otimes P}^0(\Omega \times \Omega; \mathbb{R}), \quad (\tilde{L}(t)g)(\omega, \omega') = (L(t)g)(\omega'),$$

it follows that $(\tilde{L}(t) : t \geq 0)$ is a cylindrical Lévy process on G and the sequence of its Radonified increments

$$(\Theta_n(\tilde{L}(t_n) - \tilde{L}(t_{n-1})))_{n \in \{1, \dots, N\}}$$

defined on $(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P, (\mathcal{F}_{t_n} \otimes \mathcal{F}_{t_n})_{n \in \{0, \dots, N\}})$ is a decoupled tangent sequence to the sequence of Radonified increments

$$(\Theta_n(L(t_n) - L(t_{n-1})))_{n \in \{1, \dots, N\}}$$

defined on $(\Omega, \mathcal{F}, P, (\mathcal{F}_{t_n})_{n \in \{0, \dots, N\}})$.

Proof. To make it easier to follow this proof, we define $\Omega' = \Omega$, $\mathcal{F}' = \mathcal{F}$, $P' = P$ and $\mathcal{F}'_{t_n} = \mathcal{F}_{t_n}$ for all $n \in \{0, \dots, N\}$ and instead of denoting the filtered product space by

$$(\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, P \otimes P, (\mathcal{F}_{t_n} \otimes \mathcal{F}_{t_n})_{n \in \{0, \dots, N\}}),$$

we write

$$(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \otimes P', (\mathcal{F}_{t_n} \otimes \mathcal{F}'_{t_n})_{n \in \{0, \dots, N\}}).$$

The fact that for each $t \geq 0$ the mapping $\tilde{L}(t) : G \rightarrow L_{P \otimes P'}^0(\Omega \times \Omega', \mathbb{R})$ is continuous follows directly from the definition of \tilde{L} and Remark 6.1. Thus, \tilde{L} is a cylindrical stochastic process. To prove that it is in fact a cylindrical Lévy process, let us fix $n \in \mathbb{N}$ and $g_1, \dots, g_n \in G$ and consider the n -dimensional processes Y and Z defined by $Y(t) = (\tilde{L}(t)g_1, \dots, \tilde{L}(t)g_n)$ and $Z(t) = (L(t)g_1, \dots, L(t)g_n)$. It is enough to show that for any $m \in \mathbb{N}$ and times $0 \leq t_0 < \dots < t_m \leq T$ the random variables $Y(t_m) - Y(t_{m-1}), \dots, Y(t_1) - Y(t_0)$ and $Z(t_m) - Z(t_{m-1}), \dots, Z(t_1) - Z(t_0)$ have the same distribution. Here we only prove that for any $0 \leq s < t \leq T$ the random variables $Y(t) - Y(s)$ and $Z(t) - Z(s)$ have the same distribution. The general case follows analogously. To see this, let $A \in \mathfrak{B}(\mathbb{R}^n)$ be arbitrary. The very definition of \tilde{L} shows

$$\begin{aligned} & (P \otimes P')(Y(t) - Y(s) \in A) \\ &= (P \otimes P')((\tilde{L}(t)g_1 - \tilde{L}(s)g_1, \dots, \tilde{L}(t)g_n - \tilde{L}(s)g_n) \in A) \\ &= (P \otimes P')(\Omega \times \{(L(t)g_1 - L(s)g_1, \dots, L(t)g_n - L(s)g_n) \in A\}) \\ &= P'((L(t)g_1 - L(s)g_1, \dots, L(t)g_n - L(s)g_n) \in A) \\ &= P(Z(t) - Z(s) \in A). \end{aligned}$$

To show that the Radonified increments of \tilde{L} satisfy Condition (1) of Definition 6.2, fix some $\omega \in \Omega$. Then $\Theta_n(\omega)$ is a (deterministic) Hilbert–Schmidt operator and $(\tilde{L}(t)(\omega, \cdot) : t \geq 0)$ is a cylindrical Lévy process in G . Thus, for a fixed $\omega \in \Omega$ and $n \in \{1, \dots, N\}$, the Radonified increment $\Theta_n(\omega)(\tilde{L}(t_n)(\omega, \cdot) - \tilde{L}(t_{n-1})(\omega, \cdot))$ is an \mathcal{F}'_{t_n} -measurable H -valued random variable on $(\Omega', \mathcal{F}', P')$ independent of $\mathcal{F}'_{t_{n-1}}$. It follows for each $\omega \in \Omega$ that

$$(\Theta_n(\omega)(\tilde{L}(t_n)(\omega, \cdot) - \tilde{L}(t_{n-1})(\omega, \cdot)))_{n \in \{1, \dots, N\}}$$

is a sequence of independent random variables.

For establishing Condition (2) of Definition 6.2, we define for each $n \in \{1, \dots, N\}$ the H -valued random variables

$$X_n := \Theta_n(L(t_n) - L(t_{n-1})) = \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k}} F_{n,k}(L(t_n) - L(t_{n-1})),$$

$$Y_n := \Theta_n(\tilde{L}(t_n) - \tilde{L}(t_{n-1})) = \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k}} F_{n,k}(\tilde{L}(t_n) - \tilde{L}(t_{n-1})),$$

where $F_{n,k}(L(t_n) - L(t_{n-1}))$ and $F_{n,k}(\tilde{L}(t_n) - \tilde{L}(t_{n-1}))$ refer to the Radonified increments, and by taking another representation of Θ_n if necessary, we may assume that for each $n \in \mathbb{N}$ the representation of Θ_n satisfies $A_{n,k} \cap A_{n,l} = \emptyset$ for $k \neq l$ and $\bigcup_{k=1}^{M(n)} A_{n,k} = \Omega$. Choose regular versions of the conditional distributions

$$(P \otimes P')_{X_n} : \mathfrak{B}(H) \times (\Omega \times \Omega') \rightarrow [0, 1],$$

$$(P \otimes P')_{X_n}(B, (\omega, \omega')) = (P \otimes P')(X_n \in B | \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}})(\omega, \omega'),$$

$$(P \otimes P')_{Y_n} : \mathfrak{B}(H) \times (\Omega \times \Omega') \rightarrow [0, 1],$$

$$(P \otimes P')_{Y_n}(B, (\omega, \omega')) = (P \otimes P')(Y_n \in B | \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}})(\omega, \omega').$$

As $\tilde{L}(t)$ is a cylindrical Lévy process, and for each $n \in \mathbb{N}$ we have $A_{n,k} \cap A_{n,l} = \emptyset$ for $k \neq l$ and $\bigcup_{k=1}^{M(n)} A_{n,k} = \Omega$, we obtain for all $h \in H$ and $n \in \mathbb{N}$ that

$$E_{P \otimes P'} \left[e^{i \langle Y_n, h \rangle} \Big| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}} \right]$$

$$= E_{P \otimes P'} \left[e^{i \left\langle \left(\sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k} \times \Omega'} F_{n,k} \right) (\tilde{L}(t_n) - \tilde{L}(t_{n-1})), h \right\rangle} \Big| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}} \right]$$

$$= \sum_{k=1}^{M(n)} E_{P \otimes P'} \left[\mathbb{1}_{A_{n,k} \times \Omega'} e^{i \langle F_{n,k}(\tilde{L}(t_n) - \tilde{L}(t_{n-1})), h \rangle} \Big| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}} \right]$$

$$= \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k} \times \Omega'} E_{P \otimes P'} \left[e^{i \langle F_{n,k}(\tilde{L}(t_n) - \tilde{L}(t_{n-1})), h \rangle} \Big| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}} \right]$$

$$= \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k} \times \Omega'} E_{P \otimes P'} \left[e^{i \langle F_{n,k}(\tilde{L}(t_n) - \tilde{L}(t_{n-1})), h \rangle} \right]$$

$$\begin{aligned}
 &= \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k} \times \Omega'} E_{P'} \left[e^{i \langle F_{n,k}(L(t_n) - L(t_{n-1})), h \rangle} \right] \\
 &= \sum_{k=1}^{M(n)} \mathbb{1}_{A_{n,k} \times \Omega'} e^{(t_n - t_{n-1})S(F_{n,k}^* h)} \\
 &= e^{(t_n - t_{n-1})S(\Theta_n^* h)} \quad P \otimes P' - \text{almost surely,}
 \end{aligned} \tag{6.3}$$

where S denotes the cylindrical Lévy symbol of L . In the same way we obtain

$$E_{P \otimes P'} \left[e^{i \langle X_n, h \rangle} \middle| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}} \right] = e^{(t_n - t_{n-1})S(\Theta_n^* h)} \quad P \otimes P' - \text{almost surely.} \tag{6.4}$$

It follows from (6.3) and (6.4) by calculating the conditional expectation from the conditional probability, see, for example, [13, Theorem 6.4], that for $P \otimes P'$ almost all $(\omega, \omega') \in \Omega \times \Omega'$ and for all $u \in H$ we have

$$\begin{aligned}
 \varphi_{(P \otimes P')_{X_n}(\cdot, (\omega, \omega'))}(u) &= \int_H e^{i \langle h, u \rangle} (P \otimes P')_{X_n}(dh, (\omega, \omega')) \\
 &= E_{P \otimes P'} \left[e^{i \langle X_n, u \rangle} \middle| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}} \right](\omega, \omega') \\
 &= E_{P \otimes P'} \left[e^{i \langle Y_n, u \rangle} \middle| \mathcal{F}_{t_{n-1}} \otimes \mathcal{F}'_{t_{n-1}} \right](\omega, \omega') \\
 &= \int_H e^{i \langle h, u \rangle} (P \otimes P')_{X_n}(dh, (\omega, \omega')) = \varphi_{(P \otimes P')_{Y_n}(\cdot, (\omega, \omega'))}(u).
 \end{aligned}$$

As characteristic functions uniquely determine distributions on $\mathfrak{B}(H)$, we obtain

$$(P \otimes P')_{X_n}(\cdot, (\omega, \omega')) = (P \otimes P')_{Y_n}(\cdot, (\omega, \omega')) \quad P \otimes P' - \text{almost surely,}$$

establishing Condition (2) of Definition 6.2. □

7 | CHARACTERISATION OF RANDOM INTEGRANDS

The following is the main result of this paper characterising the largest space of predictable integrands which are stochastically integrable with respect to a cylindrical Lévy process L in a Hilbert space G .

Theorem 7.1. *The space $\mathcal{I}_{\text{prd},L}^{\text{HS}}$ of predictable Hilbert–Schmidt operator-valued processes integrable with respect to a cylindrical Lévy process L in G coincides with the class of predictable processes in $L_P^0(\Omega, \mathcal{M}_{\text{det},L}^{\text{HS}})$.*

As in the case of deterministic integrands, the above characterisation of the space of L -integrable predictable processes strongly relies on the equivalent notion of convergences in two spaces.

Lemma 7.2. *Let L be a cylindrical Lévy process in G , and $(\Psi_n)_{n \in \mathbb{N}}$ a sequence in $S_{\text{prd}}^{\text{HS}}$. Then the following are equivalent.*

- (a) $\lim_{n \rightarrow \infty} \bar{\rho}_L(\Psi_n, 0) = 0$.
- (b) $\lim_{n \rightarrow \infty} \sup_{\Gamma \in S_{\text{prd}}^{1,\text{op}}} E \left[\left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] = 0$ and $\lim_{n \rightarrow \infty} E [m''(\Psi_n) \wedge 1] = 0$.

Proof. To prove (a) \Rightarrow (b), let $\epsilon > 0$ be fixed. Proposition 4.5 and the fact that m_L and ρ_L generate the same topology on $\mathcal{M}_{\text{det},L}^{\text{HS}}$ enables us to choose $\delta > 0$ such that we have the set inclusion:

$$\{\psi \in S_{\text{det}}^{\text{HS}} : \rho_L(\psi, 0) \leq \delta\} \subseteq \left\{ \psi \in S_{\text{det}}^{\text{HS}} : \sup_{\gamma \in S_{\text{det}}^{1,\text{op}}} P \left(\left\| \int_0^T \gamma \psi \, dL \right\| > \epsilon \right) \leq \epsilon \right\}. \tag{7.1}$$

As $\lim_{n \rightarrow \infty} \bar{\rho}_L(\Psi_n, 0) = 0$, there exists $n_0 \in \mathbb{N}$ such that the set

$$A_n := \{\omega \in \Omega : \rho_L(\Psi_n(\omega), 0) \leq \delta\}$$

satisfies $P(A_n) \geq 1 - \epsilon$ for all $n \geq n_0$. By recalling the definition of \tilde{L} and $(\Omega', \mathcal{F}', P')$ from Proposition 6.4, implication (7.1) implies for all $\omega \in A_n$ and $n \geq n_0$ that

$$\sup_{\Gamma \in S_{\text{prd}}^{1,\text{op}}} P' \left(\omega' \in \Omega' : \left\| \left(\int_0^T \Gamma(\omega) \Psi_n(\omega) \, d\tilde{L}(\omega, \cdot) \right) (\omega') \right\| > \epsilon \right) \leq \epsilon.$$

Fubini's theorem implies for all $n \geq n_0$ and $\Gamma \in S_{\text{prd}}^{1,\text{op}}$ that

$$\begin{aligned} & (P \otimes P') \left((\omega, \omega') \in \Omega \times \Omega' : \left\| \left(\int_0^T \Gamma \Psi_n \, d\tilde{L} \right) (\omega, \omega') \right\| > \epsilon \right) \\ &= \int_{\Omega} P' \left(\omega' \in \Omega' : \left\| \left(\int_0^T \Gamma(\omega) \Psi_n(\omega) \, d\tilde{L}(\omega, \cdot) \right) (\omega') \right\| > \epsilon \right) P(d\omega) \leq 2\epsilon \end{aligned}$$

As $\epsilon > 0$ is arbitrary, we obtain

$$\lim_{n \rightarrow \infty} \sup_{\Gamma \in S_{\text{prd}}^{1,\text{op}}} E_{P \otimes P'} \left[\left\| \int_0^T \Gamma \Psi_n \, d\tilde{L} \right\| \wedge 1 \right] = 0. \tag{7.2}$$

By the ideal property of $L_2(G, H)$, for each $n \in \mathbb{N}$ and $\Gamma \in S_{\text{prd}}^{1,\text{op}}$ the integrand $\Gamma \Psi_n$ lies in $S_{\text{prd}}^{\text{HS}}$ and has a representation of the form

$$\Gamma \Psi_n = \Gamma_0^n F_0^n \mathbb{1}_{\{0\}} + \sum_{i=1}^{N(n)-1} \Gamma_i^n F_i^n \mathbb{1}_{(t_i^n, t_{i+1}^n]}, \tag{7.3}$$

where $0 = t_1^n \leq \dots < t_{N(n)}^n = T$, and $\Gamma_i^n F_i^n$ is an $\mathcal{F}_{t_i^n}$ -measurable $L_2(G, H)$ -valued random variable taking only finitely many values for each $i = 0, \dots, N(n) - 1$. Proposition 6.4 guarantees for each

$n \in \mathbb{N}$ that the sequence of Radonified increments

$$(\Gamma_i^n F_i^n(L(t_{i+1}^n) - L(t_i^n)))_{i=1, \dots, N_n-1}$$

has the decoupled tangent sequence

$$(\Gamma_i^n F_i^n(\tilde{L}(t_{i+1}^n) - \tilde{L}(t_i^n)))_{i=1, \dots, N_n-1}.$$

We conclude from the decoupling inequality in Equation (6.1) of Proposition 6.3 that there exists a constant $c > 0$ such that, for all $n \in \mathbb{N}$ and $\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}$, we have

$$\begin{aligned} E_{P \otimes P'} \left[\left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] &= E_{P \otimes P'} \left[\left\| \sum_{i=1}^{N(n)-1} \Gamma_i^n F_i^n(L(t_{i+1}^n) - L(t_i^n)) \right\| \wedge 1 \right] \\ &\leq c E_{P \otimes P'} \left[\left\| \sum_{i=1}^{N(n)-1} \Gamma_i^n F_i^n(\tilde{L}(t_{i+1}^n) - \tilde{L}(t_i^n)) \right\| \wedge 1 \right] \\ &= c E_{P \otimes P'} \left[\left\| \int_0^T \Gamma \Psi_n \, d\tilde{L} \right\| \wedge 1 \right]. \end{aligned}$$

We conclude from Remark 6.1 and (7.2) that

$$\lim_{n \rightarrow \infty} \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E_P \left[\left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] = \lim_{n \rightarrow \infty} \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E_{P \otimes P'} \left[\left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] = 0.$$

Seeing that m_L and ρ_L generate the same topology on $\mathcal{M}_{\text{det}, L}^{\text{HS}}$, our assumption that $\lim_{n \rightarrow \infty} \bar{\rho}_L(\Psi_n, 0) = 0$ implies $\lim_{n \rightarrow \infty} E[m''(\Psi_n) \wedge 1] = 0$, which gives (b).

For establishing (b) \Rightarrow (a), given any $\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}$ we may assume that $\Gamma \Psi_n$ has a representation of the form (7.3). We conclude from the ‘recoupling’ inequality in Equation (6.2) of Proposition 6.3 that there exists a constant $c > 0$ such that for all $\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}$ we have

$$\begin{aligned} E_{P \otimes P'} \left[\left\| \int_0^T \Gamma \Psi_n \, d\tilde{L} \right\| \wedge 1 \right] &= E_{P \otimes P'} \left[\left\| \sum_{i=1}^{N(n)-1} \Gamma_i^n F_i^n(\tilde{L}(t_{i+1}^n) - \tilde{L}(t_i^n)) \right\| \wedge 1 \right] \\ &\leq c \max_{\epsilon_i \in \{\pm 1\}} E_{P \otimes P'} \left[\left\| \sum_{i=1}^{N(n)-1} \epsilon_i \Gamma_i^n F_i^n(L(t_{i+1}^n) - L(t_i^n)) \right\| \wedge 1 \right] \\ &= c \max_{\epsilon_i \in \{\pm 1\}} E_P \left[\left\| \sum_{i=1}^{N(n)-1} \epsilon_i \Gamma_i^n F_i^n(L(t_{i+1}^n) - L(t_i^n)) \right\| \wedge 1 \right] \\ &\leq c \sup_{\Theta \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E_P \left[\left\| \sum_{i=1}^{N(n)-1} \Theta_i^n F_i^n(L(t_{i+1}^n) - L(t_i^n)) \right\| \wedge 1 \right] \\ &= c \sup_{\Theta \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E_P \left[\left\| \int_0^T \Theta \Psi_n \, dL \right\| \wedge 1 \right]. \end{aligned} \tag{7.4}$$

By choosing $\Gamma = \text{Id} \mathbb{1}_{\Omega \times (0, T]}$, the hypothesis on $(\Psi_n)_{n \in \mathbb{N}}$ implies

$$\lim_{n \rightarrow \infty} E_{P \otimes P'} \left[\left\| \int_0^T \Psi_n \, d\tilde{L} \right\| \wedge 1 \right] = 0.$$

It follows that for every subsequence $(\Psi_{n_m})_{m \in \mathbb{N}}$ of $(\Psi_n)_{n \in \mathbb{N}}$, there exists a further subsequence $(\Psi_{n_{m_j}})_{j \in \mathbb{N}}$ and a set $N \subseteq \Omega \times \Omega'$ with $(P \otimes P')(N) = 0$ satisfying

$$\lim_{j \rightarrow \infty} \left(\int_0^T \Psi_{n_{m_j}} \, d\tilde{L} \right) (\omega, \omega') = 0 \quad \text{for each } (\omega, \omega') \in N^c.$$

Define the section of the set N for each $\omega \in \Omega$ by

$$N_\omega = \left\{ \omega' \in \Omega' : \lim_{j \rightarrow \infty} \left(\int_0^T \Psi_{n_{m_j}}(\omega) \, d\tilde{L}(\omega, \cdot) \right) (\omega') \neq 0 \right\},$$

where we note that as $\Psi_{n_{m_j}}$ are step processes, it holds that

$$\left(\int_0^T \Psi_{n_{m_j}} \, d\tilde{L} \right) (\omega, \cdot) = \int_0^T \Psi_{n_{m_j}}(\omega) \, d\tilde{L}(\omega, \cdot) \quad \text{for all } \omega \in \Omega.$$

Fubini's theorem implies $0 = (P \otimes P')(N) = \int_\Omega P'(N_\omega) \, dP(\omega)$, from which it follows that there exists $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) = 1$ such that $P'(N_\omega) = 0$ for all $\omega \in \Omega_1$. In other words, for each fixed $\omega \in \Omega_1$, the sequence of random variables

$$\left(\int_0^T \Psi_{n_{m_j}}(\omega) \, d\tilde{L}(\omega, \cdot) \right)_{j \in \mathbb{N}}$$

converges P' -almost surely to 0 as H -valued random variables on $(\Omega', \mathcal{F}', P')$. For each fixed $\omega \in \Omega_1$, the above sequence is infinitely divisible and has characteristics

$$\left(\int_0^T b_{\Psi_{n_{m_j}}(\omega, t)}^\theta \, dt, \int_0^T \Psi_{n_{m_j}}(\omega, t) Q \Psi_{n_{m_j}}^*(\omega, t) \, dt, (\lambda \otimes \text{Leb}) \circ \kappa_{\Psi_{n_{m_j}}(\omega)}^{-1} \right),$$

by Lemma 3.11 and [4, Lemma 3.6] where $\kappa_\psi : G \times [0, T] \rightarrow H$ is defined by $\kappa_\psi(g, t) = \psi(t)g$ for $\psi \in \mathcal{S}_{\text{det}}^{\text{HS}}$. As the cylindrical Lévy process $\tilde{L}(\omega, \cdot)$ has the same cylindrical characteristics as L for each $\omega \in \Omega$, we obtain for all $\omega \in \Omega_1$ that

$$\lim_{j \rightarrow \infty} k_L(\Psi_{n_{m_j}}(\omega)) = \lim_{j \rightarrow \infty} k_{\tilde{L}(\omega, \cdot)}(\Psi_{n_{m_j}}(\omega)) = 0.$$

As $P(\Omega_1) = 1$, the above argument proves that for all $\epsilon > 0$ we have

$$\lim_{n \rightarrow \infty} P \left(\int_0^T k_L(\Psi_n(t)) \, dt > \epsilon \right) = 0. \tag{7.5}$$

To finish the proof, it remains to show that for all $\epsilon > 0$ we have

$$\lim_{n \rightarrow \infty} P \left(\int_0^T l_L(\Psi_n(t)) \, dt > \epsilon \right) = 0. \tag{7.6}$$

Let $\epsilon \in (0, 1)$ be fixed. As stochastic integrals with deterministic integrands with respect to L are infinitely divisible, Remark 2.1 implies that there exists $\delta \in (0, \epsilon)$ such that for all $\psi \in \mathcal{M}_{\det}^{\text{HS}}$ we have the implication

$$P' \left(\left\| \int_0^T \psi \, dL \right\| > \sqrt{\delta} \right) < \sqrt{\delta} \Rightarrow \left\| \int_0^T b_{\psi(t)}^\theta \, dt \right\| < \epsilon. \tag{7.7}$$

It follows from Equation (7.4) that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} (P \otimes P') \left(\left\| \int_0^T \Gamma \Psi_n \, d\tilde{L} \right\| > \sqrt{\delta} \right) < \delta. \tag{7.8}$$

Chebyshev’s inequality, Fubini’s theorem and Equation (7.8) imply for all $n \geq N$ and $\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}$ that

$$\begin{aligned} &P \left(\omega \in \Omega : P' \left(\omega' \in \Omega' : \left\| \left(\int_0^T \Gamma(\omega) \Psi_n(\omega) \, d\tilde{L}(\omega, \cdot) \right) (\omega') \right\| > \sqrt{\delta} \right) < \sqrt{\delta} \right) \\ &= 1 - P \left(\omega \in \Omega : P' \left(\omega' \in \Omega' : \left\| \left(\int_0^T \Gamma(\omega) \Psi_n(\omega) \, d\tilde{L}(\omega, \cdot) \right) (\omega') \right\| > \sqrt{\delta} \right) \geq \sqrt{\delta} \right) \\ &\geq 1 - \frac{1}{\sqrt{\delta}} \int_{\Omega} P' \left(\omega' \in \Omega' : \left\| \left(\int_0^T \Gamma(\omega) \Psi_n(\omega) \, d\tilde{L}(\omega, \cdot) \right) (\omega') \right\| > \sqrt{\delta} \right) dP(\omega) \\ &= 1 - \frac{1}{\sqrt{\delta}} (P \otimes P') \left((\omega, \omega') \in \Omega \times \Omega' : \left\| \left(\int_0^T \Gamma \Psi_n \, d\tilde{L} \right) (\omega, \omega') \right\| > \sqrt{\delta} \right) \\ &\geq 1 - \sqrt{\delta}. \end{aligned} \tag{7.9}$$

In light of Equations (7.7) and (7.9), we have for all $n \geq N$ and $\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}$ that

$$P \left(\left\| \int_0^T b_{\Gamma \Psi_n}^\theta \, dt \right\| < \epsilon \right) \geq 1 - \sqrt{\delta},$$

or equivalently, for all $n \geq N$ we have

$$\sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} P \left(\left\| \int_0^T b_{\Gamma \Psi_n}^\theta \, dt \right\| \geq \epsilon \right) \leq \sqrt{\delta}.$$

The above inequality, combined with an approximation argument using functions in $\mathcal{S}_{\text{prd}}^{1,\text{op}}$ shows that for any predictable $\tilde{B}_{L(H)}$ -valued process Λ and $n \geq N$ it holds that

$$P \left(\left\| \int_0^T b_{\Lambda \Psi_n}^\theta \, dt \right\| \geq \epsilon \right) \leq \sqrt{\delta}. \tag{7.10}$$

For each fixed $n \geq N$, define a process $H_n : \Omega \times [0, T] \rightarrow \bar{B}_{L(H)}$ by

$$H_n(\omega, t) = f\left(b_{i(\Psi_n(\omega, t))\Psi_n(\omega, t)}^\theta\right)(i(\Psi_n(\omega, t))\Psi_n(\omega, t)),$$

with i and f as in the proof of Lemma 4.9. Then, H_n is predictable and, by Inequality (4.4), it satisfies for each $\omega \in \Omega$ that

$$\int_0^T \sup_{O \in \bar{B}_{L(H)}} \|b_{O\Psi_n(\omega, t)}^\theta\| dt \leq \left\| \int_0^T b_{H_n(\omega, t)\Psi_n(\omega, t)}^\theta dt \right\|.$$

Applying Equation (7.10) for $\Lambda = H_n$ shows for all $n \geq N$ that

$$P\left(\int_0^T \sup_{O \in \bar{B}_{L(H)}} \|b_{O\Psi_n(t)}^\theta\| dt \geq 2\epsilon\right) \leq P\left(\left\| \int_0^T b_{H_n(t)\Psi_n(t)}^\theta dt \right\| \geq \epsilon\right) \leq \sqrt{\delta}.$$

As we have that $\delta < \epsilon$, this finishes the proof of the claim in Equation (7.6). Finally, by Equations (7.5), (7.6), and the assumption that $\lim_{n \rightarrow \infty} E[m''(\Psi_n) \wedge 1] = 0$, we obtain that $\lim_{n \rightarrow \infty} E[m_L(\Psi_n) \wedge 1] = 0$. This completes the proof, as m_L and ρ_L generate the same topology. \square

Proof of Theorem 7.1. If $\Psi \in \mathcal{I}_{\text{prd}, L}^{\text{HS}}$ then Definition 5.2 guarantees the existence of a sequence $(\Psi_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{S}_{\text{prd}}^{\text{HS}}$ converging P_T -almost everywhere to Ψ and satisfying

$$\lim_{m, n \rightarrow \infty} \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E\left[\left\| \int_0^T \Gamma(\Psi_m - \Psi_n) dL \right\| \wedge 1\right] = 0.$$

Lemma 7.2 implies that $\lim_{m, n \rightarrow \infty} \bar{\rho}_L(\Psi_m, \Psi_n) = 0$. Completeness of the metric space $(L_P^0(\Omega, \mathcal{M}_{\text{det}, L}^{\text{HS}}), \bar{\rho}_L)$ and the fact that $(\Psi_n)_{n \in \mathbb{N}}$ converges P_T -almost everywhere to Ψ together yield that the sequence $(\Psi_n)_{n \in \mathbb{N}}$ has a limit in $L_P^0(\Omega, \mathcal{M}_{\text{det}, L}^{\text{HS}})$ and that this limit necessarily coincides with Ψ . Thus, $\Psi \in L_P^0(\Omega, \mathcal{M}_{\text{det}, L}^{\text{HS}})$.

To establish the reverse inclusion, let Ψ be a predictable process in the space $L_P^0(\Omega, \mathcal{M}_{\text{det}, L}^{\text{HS}})$. Lemma 5.4 guarantees that there exists a sequence $(\Psi_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{S}_{\text{prd}}^{\text{HS}}$ converging to Ψ in $\bar{\rho}_L$ and P_T -almost everywhere. Then, $(\Psi_m - \Psi_n)$ converges to 0 both in $\bar{\rho}_L$ and P_T -almost everywhere as $m, n \rightarrow \infty$. This implies by Lemma 7.2 that

$$\lim_{m, n \rightarrow \infty} \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E\left[\left\| \int_0^T \Gamma(\Psi_m - \Psi_n) dL \right\| \wedge 1\right] = 0.$$

Thus, Ψ satisfies the conditions of Definition 5.2, which means that $\Psi \in \mathcal{I}_{\text{prd}, L}^{\text{HS}}$. \square

Lemma 7.2 is crucial to characterise the space of integrable predictable processes in Theorem 7.1, as it describes convergence of predictable step processes in the space of integrands in terms of convergence in the randomised modular space. Having identified the space of integrable predictable processes, we can extend Lemma 7.2 to the whole space of integrable predictable processes.

Corollary 7.3. *Let L be a cylindrical Lévy process in G , and $(\Psi_n)_{n \in \mathbb{N}}$ a sequence in $\mathcal{I}_{\text{prd}, L}^{\text{HS}}$. Then the following are equivalent.*

- (a) $\lim_{n \rightarrow \infty} \bar{\rho}_L(\Psi_n, 0) = 0$.
- (b) $\lim_{n \rightarrow \infty} \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E \left[\left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] = 0$ and $\lim_{n \rightarrow \infty} E [m''(\Psi_n) \wedge 1] = 0$.

Proof. To establish the implication (a) \Rightarrow (b), first note that it follows from the definition of $\bar{\rho}_L$ and the fact that ρ_L generate the same topology as m_L that $m''(\Psi_n) \rightarrow 0$ in probability. Let $\epsilon > 0$ be fixed. Lemma 7.2 implies that there exists a $\delta(\epsilon) > 0$ such that we have for all $\Psi \in \mathcal{S}_{\text{prd}}^{\text{HS}}$ the implication:

$$\bar{\rho}_L(\Psi, 0) < \delta(\epsilon) \quad \Rightarrow \quad \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E \left[\left\| \int_0^T \Gamma \Psi \, dL \right\| \wedge 1 \right] < \epsilon. \tag{7.11}$$

As $\lim_{n \rightarrow \infty} \bar{\rho}_L(\Psi_n, 0) = 0$, there exists an $n_0 \in \mathbb{N}$ such that $\bar{\rho}_L(\Psi_n, 0) < \frac{\delta(\epsilon)}{2}$ for all $n \geq n_0$. By Theorem 7.1, we have that $(\Psi_n)_{n \in \mathbb{N}} \subseteq L_P^0(\Omega, \mathcal{M}_{\text{det}, L}^{\text{HS}})$, hence Lemma 5.4 guarantees for each $n \in \mathbb{N}$ the existence of a sequence $(\Psi_n^m)_{m \in \mathbb{N}} \subseteq \mathcal{S}_{\text{prd}}^{\text{HS}}$ converging to Ψ_n in $\bar{\rho}_L$ and P_T -almost everywhere. Consequently, we can find $m_0(n, \epsilon) \in \mathbb{N}$ for each $n \in \mathbb{N}$ such that for all $m \geq m_0(n, \epsilon)$ we have $\bar{\rho}_L(\Psi_n^m, \Psi_n) < \frac{\delta(\epsilon)}{2}$. We obtain for each $n \geq n_0$ and $m \geq m_0(n, \epsilon)$ that

$$\bar{\rho}_L(\Psi_n^m, 0) \leq \bar{\rho}_L(\Psi_n^m - \Psi_n) + \bar{\rho}_L(\Psi_n, 0) < \delta(\epsilon),$$

which implies by (7.11) that

$$\sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E \left[\left\| \int_0^T \Gamma \Psi_n^m \, dL \right\| \wedge 1 \right] < \epsilon. \tag{7.12}$$

Thus, if we fix an $n \geq n_0$ and recall that the integral of Ψ_n is defined to be the limit in probability of the integrals of Ψ_n^m as $m \rightarrow \infty$, we obtain from Equation (7.12) that

$$\begin{aligned} \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E \left[\left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] &= \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} \lim_{m \rightarrow \infty} E \left[\left\| \int_0^T \Gamma \Psi_n^m \, dL \right\| \wedge 1 \right] \\ &\leq \limsup_{m \rightarrow \infty} \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E \left[\left\| \int_0^T \Gamma \Psi_n^m \, dL \right\| \wedge 1 \right] < \epsilon. \end{aligned}$$

To establish the reverse implication (b) \Rightarrow (a), let $\epsilon > 0$ be fixed. Lemma 7.2 implies that there exists a $\delta(\epsilon) > 0$ such that we have for all $\Psi \in \mathcal{S}_{\text{prd}}^{\text{HS}}$ the implication:

$$\sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1, \text{op}}} E \left[\left\| \int_0^T \Gamma \Psi \, dL \right\| \wedge 1 \right] + E [m''(\Psi) \wedge 1] < \delta(\epsilon) \quad \Rightarrow \quad \bar{\rho}_L(\Psi, 0) < \frac{\epsilon}{2}. \tag{7.13}$$

By assumption, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E \left[\left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] + E[m''(\Psi_n) \wedge 1] < \frac{\delta(\epsilon)}{4}. \tag{7.14}$$

As $(\Psi_n)_{n \in \mathbb{N}} \subseteq \mathcal{I}_{\text{prd},L}^{\text{HS}}$, it follows from Theorem 7.1 and Lemma 5.4 that for each $n \in \mathbb{N}$ there exists a sequence $(\Psi_n^m)_{m \in \mathbb{N}}$ of elements of $\mathcal{S}_{\text{prd}}^{\text{HS}}$ converging to Ψ_n in $\bar{\rho}_L$ and P_T -almost everywhere. Consequently, we can find $m_0(n, \epsilon) \in \mathbb{N}$ for each $n \in \mathbb{N}$, such that for all $m \geq m_0(n, \epsilon)$ we have

$$\bar{\rho}_L(\Psi_n^m, \Psi_n) < \epsilon/2. \tag{7.15}$$

As for each $n \in \mathbb{N}$ we have that $\lim_{m \rightarrow \infty} \bar{\rho}_L(\Psi_n^m, \Psi_n) = 0$, the first part of this corollary and the reverse triangle inequality shows that for each $n \in \mathbb{N}$ there exists an $m_1(n, \epsilon) \in \mathbb{N}$ such that for all $m \geq m_1(n, \epsilon)$ we have

$$\left| \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E \left[\left\| \int_0^T \Gamma \Psi_n \, dL \right\| \wedge 1 \right] - \sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E \left[\left\| \int_0^T \Gamma \Psi_n^m \, dL \right\| \wedge 1 \right] \right| < \frac{\delta(\epsilon)}{4}. \tag{7.16}$$

Moreover, as for each $n \in \mathbb{N}$ we have that $\Psi_n^m \rightarrow \Psi_n$ P_T -almost everywhere as $m \rightarrow \infty$, there exists $m_2(n, \epsilon) \in \mathbb{N}$ such that for all $m \geq m_2(n, \epsilon)$ it holds that

$$\left| E[m''(\Psi_n) \wedge 1] - E[m''(\Psi_n^m) \wedge 1] \right| < \frac{\delta(\epsilon)}{4}. \tag{7.17}$$

By combining Equations (7.14), (7.16) and (7.17), we obtain for all $n \geq n_0$ and $m \geq \max\{m_0(n, \epsilon), m_1(n, \epsilon), m_2(n, \epsilon)\}$ that

$$\sup_{\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}} E \left[\left\| \int_0^T \Gamma \Psi_n^m \, dL \right\| \wedge 1 \right] + E[m''(\Psi_n^m) \wedge 1] < \delta(\epsilon),$$

which implies by (7.13) and (7.15) that

$$\bar{\rho}_L(\Psi_n, 0) \leq \bar{\rho}_L(\Psi_n, \Psi_n^m) + \bar{\rho}_L(\Psi_n^m, 0) < \epsilon.$$

As $\epsilon > 0$ was arbitrary, this concludes the proof. □

Having introduced the notion of the stochastic integral, we now show that stochastic integral processes, obtained by fixing an integrand and varying the upper limit of the stochastic integral, are in fact semimartingales.

Theorem 7.4. *If $\Psi \in \mathcal{I}_{\text{prd},L}^{\text{HS}}$, then the integral process $(I(\Psi)(t) : t \in [0, T])$ defined by*

$$I(\Psi)(t) := \int_0^T \mathbb{1}_{[0,t]}(s) \Psi(s) L(ds) \quad \text{for } t \in [0, T],$$

is a semimartingale.

Proof. Let $\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}$ be of the form

$$\Gamma(t) = \Gamma_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{N-1} \Gamma_k \mathbb{1}_{(s_k, s_{k+1}]}(t),$$

where $0 = s_1 < \dots < s_N = T$ are deterministic times, $\Gamma_0 : \Omega \rightarrow \bar{B}_{L(H)}$ is \mathcal{F}_0 -measurable, and each $\Gamma_k : \Omega \rightarrow \bar{B}_{L(H)}$ is an \mathcal{F}_{s_k} -measurable random variable taking only finitely many values for $k = 1, \dots, N - 1$. Then we define the stochastic integral

$$\int_0^T \Gamma dI(\Psi) := \sum_{k=1}^{N-1} \Gamma_k (I(\Psi)(s_{k+1}) - I(\Psi)(s_k)).$$

To prove the claim it suffices to show that the set $\{ \int_0^T \Gamma dI(\Psi) : \Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}} \}$ is bounded in probability according to [11, Theorem 2.1]. Suppose, aiming for a contradiction, that it is not the case. Then there exists an $\epsilon > 0$ and a sequence $(\Gamma_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}_{\text{prd}}^{1,\text{op}}$ satisfying for all $n \in \mathbb{N}$ that

$$P\left(\left\| \int_0^T \Gamma_n dI(\Psi) \right\| > n\right) \geq \epsilon. \tag{7.18}$$

For each $\Psi \in \mathcal{S}_{\text{prd}}^{\text{HS}}$ and $\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}$, the very definitions of stochastic integrals show

$$\int_0^T \Gamma dI(\Psi) = \int_0^T \Gamma \Psi dL.$$

This equality can be generalised to arbitrary $\Psi \in \mathcal{I}_{\text{prd},L}^{\text{HS}}$ and $\Gamma \in \mathcal{S}_{\text{prd}}^{1,\text{op}}$ by a standard approximation argument. Using this to rewrite Equation (7.18), we obtain for all $n \in \mathbb{N}$ that

$$\epsilon \leq P\left(\left\| \int_0^T \Gamma_n dI(\Psi) \right\| > n\right) = P\left(\left\| \int_0^T \frac{1}{n} \Gamma_n \Psi dL \right\| > 1\right). \tag{7.19}$$

On the other hand, as $\bar{\rho}_L(\frac{1}{n} \Gamma_n \Psi, 0) \rightarrow 0$ as $n \rightarrow \infty$, Corollary 7.3 implies

$$\lim_{n \rightarrow \infty} E \left[\left\| \int_0^T \frac{1}{n} \Gamma_n \Psi dL \right\| \wedge 1 \right] = 0,$$

which contradicts (7.19) because of the equivalent characterisation of the topology in $L_P^0(\Omega, H)$. □

We finish this section with a stochastic dominated convergence theorem.

Theorem 7.5. *Let $(\Psi_n)_{n \in \mathbb{N}}$ be a sequence of processes in $\mathcal{I}_{\text{prd},L}^{\text{HS}}$ such that*

- (1) $(\Psi_n)_{n \in \mathbb{N}}$ converges P_T -almost everywhere to an $L_2(G, H)$ -valued predictable process Ψ ;
- (2) there exists a process $Y \in \mathcal{I}_{\text{prd},L}^{\text{HS}}$ satisfying for all $n \in \mathbb{N}$ that

$$(k_L + l_L)(\Psi_n(\omega, t)) \leq (k_L + l_L)(Y(\omega, t)) \quad \text{for } P_T\text{-almost all } (\omega, t) \in \Omega \times [0, T].$$

Then it follows that $\Psi \in \mathcal{I}_{\text{prd},L}^{\text{HS}}$ and

$$\lim_{n \rightarrow \infty} \sup_{\Gamma \in S_{\text{prd}}^{\text{I,op}}} E \left[\left\| \int_0^T \Gamma(\Psi_n - \Psi) \, dL \right\| \wedge 1 \right] = 0.$$

Proof. By assumption, there exists a set $N \subseteq \Omega \times [0, T]$ with $P_T(N) = 0$ such that $\lim_{n \rightarrow \infty} \Psi_n(\omega, t) = \Psi(\omega, t)$ and $(k_L + l_L)(\Psi_n(\omega, t)) \leq (k_L + l_L)(Y(\omega, t))$ for all $(\omega, t) \in N^c$ and $n \in \mathbb{N}$. Fubini's theorem yields that

$$0 = P_T(N) = \int_{\Omega} \text{Leb}|_{[0,T]}(N_{\omega}) P(d\omega),$$

where

$$N_{\omega} := \left\{ t \in [0, T] : \lim_{n \rightarrow \infty} \Psi_n(\omega, t) \neq \Psi(\omega, t) \text{ or } (k_L + l_L)(\Psi_n(\omega, t)) > (k_L + l_L)(Y(\omega, t)) \right\}.$$

It follows that there exists an $\Omega_1 \subseteq \Omega$ with $P(\Omega_1) = 1$ such that $\text{Leb}|_{[0,T]}(N_{\omega}) = 0$ for all $\omega \in \Omega_1$. Consequently, for each $\omega \in \Omega_1$ we have $(k_L + l_L)(\Psi_n(\omega, t)) \leq (k_L + l_L)(Y(\omega, t))$ and $\lim_{n \rightarrow \infty} \Psi_n(\omega, t) = \Psi(\omega, t)$ for Lebesgue almost every $t \in [0, T]$. Theorem 7.1 guarantees that there exists $\Omega_2 \subseteq \Omega$ with $P(\Omega_2) = 1$ such that $m_L(Y(\omega, \cdot)) < \infty$ for all $\omega \in \Omega_2$. Continuity of k_L and l_L at 0, see Lemma 3.12, and the classical version of Lebesgue's dominated convergence theorem implies that for all $\omega \in \Omega_1 \cap \Omega_2$ we have

$$\begin{aligned} & \lim_{m, n \rightarrow \infty} m_L(\Psi_m(\omega, \cdot) - \Psi_n(\omega, \cdot)) \\ &= \lim_{m, n \rightarrow \infty} \left(\int_0^T k_L(\Psi_m(\omega, t) - \Psi_n(\omega, t)) + l_L(\Psi_m(\omega, t) - \Psi_n(\omega, t)) \, dt \right. \\ & \quad \left. + \int_0^T \|\Psi_m(\omega, t) - \Psi_n(\omega, t)\|_{L_2(G,H)}^2 \wedge 1 \, dt \right) = 0. \end{aligned}$$

Hence, for each $\omega \in \Omega_1 \cap \Omega_2$ the sequence $(\Psi_n)_{n \in \mathbb{N}}$ is Cauchy in the modular topology, which by Lemma 3.20 and the fact that $\Psi_n(\omega) \rightarrow \Psi(\omega)$ for Lebesgue almost all $t \in [0, T]$ allows us to conclude that $\Psi(\omega) \in \mathcal{M}_{\text{det},L}^{\text{HS}}$. As $P(\Omega_1 \cap \Omega_2) = 1$, Theorem 7.1 shows $\Psi \in \mathcal{I}_{\text{prd},L}^{\text{HS}}$. Another application of Lebesgue's dominated convergence theorem establishes that $\lim_{n \rightarrow \infty} \bar{\rho}_L(\Psi_n, \Psi) = 0$, which completes the proof by an application of Corollary 7.3. \square

Remark 7.6. Returning to our motivational example of Equation (1.1), it turns out that Theorem 7.5 plays a key role in proving existence and uniqueness of a solution to Equation (1.1). For further details, see the forthcoming paper by Bodó, Cox and Riedle.

Remark 7.7. Our approach to stochastic integration theory relies on the algebraic characterisation of Lévy measures in the underlying Hilbert space and on decoupling inequalities. A characterisation of Lévy measures in UMD Banach spaces is given in [28], although it is not as straightforwardly algebraic as in the Hilbert space setting. Decoupling inequalities are known to hold in UMD spaces; indeed, they characterise them. This suggests that our approach could potentially be extended to UMD Banach spaces, once a suitable integration theory for deterministic functions is developed.

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ORCID

Markus Riedle  <https://orcid.org/0000-0003-1408-8050>

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