UNIVERSITY OF AMSTERDAM

UvA-DARE (Digital Academic Repository)

Class numbers, congruent numbers and umbral moonshine

Cheng, Miranda C.N.; Duncan, John F.R.; Mertens, Michael H.

DOI
10.1016/j.jnt.2025.02.007
Publication date
2025
Document Version
Final published version
Published in
Journal of Number Theory
License
CC BY

Link to publication

Citation for published version (APA):
Cheng, M. C. N., Duncan, J. F. R., & Mertens, M. H. (2025). Class numbers, congruent numbers and umbral moonshine. *Journal of Number Theory*, *277*, 201-235. https://doi.org/10.1016/j.jnt.2025.02.007

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: https://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, P.O. Box 19185, 1000 GD Amsterdam, The Netherlands. You will be contacted as soon as possible.

UvA-DARE is a service provided by the library of the University of Amsterdam (https://dare.uva.nl)

Download date:05 Sep 2026

Journal of Number Theory 277 (2025) 201–235

Contents lists available at ScienceDirect

Journal of Number Theory

journal homepage: www.elsevier.com/locate/jnt

General Section

Class numbers, congruent numbers and umbral moonshine

Miranda C.N. Cheng [a,b], John F.R. Duncan [b], Michael H. Mertens [c,*]

[a] *Institute of Physics and Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, the Netherlands*
[b] *Institute of Mathematics, Academia Sinica, Taipei, Taiwan*
[c] *Lehrstuhl für Algebra und Zahlentheorie, RWTH Aachen University, Aachen, Germany*

ARTICLE INFO

Article history:
Received 23 April 2024
Received in revised form 18 December 2024
Accepted 27 February 2025
Available online 6 May 2025
Communicated by L. Smajlovic

MSC:
11F22
11F37
11G05
11G40
20C34

Keywords:
Hurwitz class numbers
Umbral moonshine
Congruent numbers

ABSTRACT

In earlier work we initiated a program to study relationships between finite groups and arithmetic geometric invariants of modular curves in a systematic way. In the present work we continue this program, with a focus on the two smallest sporadic simple Mathieu groups. To do this we first elucidate a connection between a special case of umbral moonshine and the imaginary quadratic class numbers. Then we use this connection to classify a distinguished set of modules for the smallest sporadic Mathieu group. Finally we establish a connection between our classification and the congruent number problem of antiquity.

© 2025 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

* Corresponding author.
E-mail addresses: mcheng@uva.nl, mcheng@gate.sinica.edu.tw (M.C.N. Cheng), jduncan@gate.sinica.edu.tw (J.F.R. Duncan), michael.helmut.mertens@rwth-aachen.de (M.H. Mertens).

https://doi.org/10.1016/j.jnt.2025.02.007
0022-314X/© 2025 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

Contents

1. Introduction . . . 202
1.1. Motivation . . . 202
1.2. Methods . . . 204
1.3. Results . . . 205
1.4. Overview . . . 206
Acknowledgments . . . 207
2. Notation . . . 207
3. Preparation . . . 209
3.1. Class numbers . . . 209
3.2. Jacobi forms . . . 211
3.3. Optimality . . . 216
3.4. Our main focus . . . 219
4. Results . . . 220
4.1. Umbral moonshine . . . 220
4.2. Modules . . . 223
4.3. Congruent numbers . . . 228
Appendix A. Characters and coefficients . . . 230
Data availability . . . 230
References . . . 230

1. Introduction

In earlier work [8] we initiated a program to systematically study interrelationships between representations of finite groups and arithmetic-geometric invariants. We also took a first step in this program by revealing families of such relationships for the cyclic groups of prime order. Since the cyclic groups of prime order are the simplest simple groups, it is natural to ask if the module structures of [8] extend to larger groups, and if so with what consequences for arithmetic. In this work we answer these questions in a special case.

1.1. Motivation

Relationships between a distinguished representation of the sporadic simple group ON of O'Nan and the arithmetic of certain elliptic curves were illuminated in [18] (see also [14,16]), and an analogous story for the sporadic simple group Th of Thompson was told in [23]. This motivated us to initiate a program to study such phenomena systematically, and our first contribution [8] to this program is an analysis of the case that we cast the cyclic simple groups in the roles played by ON and Th in [18] and [23]. As a result we obtain arithmetic relationships between distinguished infinite-dimensional virtual graded modules for cyclic groups of prime order and imaginary quadratic twists of the modular Jacobians of prime conductor.

To put this more concretely we note that Theorem 4.1.1 of [8] guarantees the existence of an infinite-dimensional virtual graded $\mathbb{Z}/N\mathbb{Z}$-module

$$W = \bigoplus_D W_D, \tag{1.1.1}$$

for each prime N, with the property that

$$\dim W_D = 12\,\mathrm{num}\left(\frac{N+1}{6}\right) h(D) \tag{1.1.2}$$

when $D < 0$ is a fundamental discriminant (i.e. the discriminant of $\mathbb{Q}(\sqrt{D})$), where $\mathrm{num}(\alpha)$ denotes the numerator of a rational number α and $h(D)$ denotes the class number of $\mathbb{Q}(\sqrt{D})$. Theorem 4.2.1 of [8] translates this property (1.1.2) into a congruence condition on $\dim W_D$ that obstructs the existence of infinite-order rational points on the D-twist of the Jacobian $J_0(N)$ of the modular curve $X_0(N)$, for suitably chosen D.

For example, $J_0(19)$ is the elliptic curve defined by the equation $y^2 = x^3 - 12096x - 544752$ (see e.g. [25]), and the D-twist of $J_0(19)$ is the elliptic curve defined by

$$y^2 = x^3 - 12096D^2x - 544752D^3. \tag{1.1.3}$$

Taking $N = 19$ in Theorem 4.2.1 of [8] we obtain that if $D < 0$ is a fundamental discriminant such that 19 is inert in the ring of integers of $\mathbb{Q}(\sqrt{D})$, and

$$h(D) \not\equiv 0 \bmod 3, \tag{1.1.4}$$

then the curve (1.1.3) has only finitely many rational points. (Cf. Corollary 4.2.2 of [8], which is a counterpart to this statement for $N = 11$.)

The reader may be interested to know how this compares to the results of the motivating works [18] and [23]. With this in mind we point out the following consequence of Theorem 1.3 of [18]: If $D < 0$ is a fundamental discriminant such that 19 is inert in the ring of integers of $\mathbb{Q}(\sqrt{D})$, and

$$\dim W_D^{\mathsf{ON}} \not\equiv -24h(D) \bmod 19, \tag{1.1.5}$$

where $W^{\mathsf{ON}} = \bigoplus_D W_D^{\mathsf{ON}}$ is the ON-module considered in [18] (with W_D^{ON} here denoted by W_{-D} there), then the curve (1.1.3) again has only finitely many rational points.

By a similar token Theorem 1.3 of [23] tells us that if $D < 0$ is a fundamental discriminant such that 19 is inert in the ring of integers of $\mathbb{Q}(\sqrt{D})$, and

$$\dim W_D^{\mathsf{Th}} \not\equiv 0 \bmod 19, \tag{1.1.6}$$

where $W^{\mathsf{Th}} = \bigoplus_D W_D^{\mathsf{Th}}$ is any of the Th-modules considered in [23] (with W_D^{Th} here denoted by W_{-D} there), then the curve (1.1.3) yet again has only finitely many rational points.

Note that the class number $h(D)$ does not appear in (1.1.6). Tracing through the arguments of [23] we see that this is because the graded dimension

$$\sum_{D \le 5} \dim W_D^{\mathsf{Th}} q^{-D} = 6q^{-5} + O(q^3) \tag{1.1.7}$$

has vanishing constant term. In (1.1.4) we have the other extreme: We can omit $\dim W_D$ from (1.1.4) because it is a fixed multiple (1.1.2) of $h(D)$ by construction.

Theorem 4.2.1 of [8] furnishes obstruction statements similar to those of (1.1.4)-(1.1.6) for each prime N. This demonstrates that the results of [18,23], involving the sporadic simple groups of O'Nan and Thompson, are part of a more expansive picture.

It is natural to ask now how expansive this picture is. To explore this general question we formulate the following more specific ones:

1. Are the cyclic group module structures on the W of (1.1.1)-(1.1.2) restrictions of module structures for more sophisticated groups?
2. If so, do these more sophisticated group-module structures on the W of (1.1.1)-(1.1.2) entail more sophisticated relationships to arithmetic geometry, than those formulated in (1.1.4)-(1.1.6) above?

In this work we answer these questions positively in the "simplest" non-trivial case, by which we mean the case that $N = 11$, since this is the smallest prime N for which $J_0(N)$ is not just a point.

1.2. Methods

As in [8] our focus is on virtual graded G-modules $W = \bigoplus_D W_D$, for G a finite group, with the property that the *McKay–Thompson series*

$$\phi_{\mathsf{g}}^W(\tau, z) := \sum_{n,s\in\mathbb{Z}} \operatorname{tr}(\mathsf{g}|W_{s^2-4n})q^n y^s \tag{1.2.1}$$

defines an optimal holomorphic mock Jacobi form of weight 2 and index 1 and level $o(\mathsf{g})$ for each $\mathsf{g} \in \mathsf{G}$, once we set $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$. We recall holomorphic mock Jacobi forms, and our notion of level, in § 3.2. The most important requirement is optimality, which we explain in detail in § 3.3. In short it is the condition that the associated McKay–Thompson series ϕ_{g}^W (1.2.1) have a common fixed constant term, and have theta-coefficients $h_{\mathsf{g},r}^W$ (see (3.2.7) and (3.3.4)-(3.3.6)) that vanish away from the infinite cusps of their invariance groups (cf. (3.3.1)). As we explain in § 1 of [8], it is a generalization of the notion of Hauptmodul—instrumental in monstrous moonshine—from modular functions to more general modular settings. In addition to [8] the reader may consult § 6.3 of [5], §§ 1.5–1.6 of [13] or [16] for expository discussions of optimality.

A key tool in this work with no counterpart in [8] is a connection between meromorphic Jacobi forms in weights 1 and 2, which, via the methods of [11], may be used to connect suitable mock Jacobi forms in weights 1 and 2. The McKay–Thompson series of umbral moonshine (see [4–6]) are naturally packaged as mock Jacobi forms of weight 1, so there is a chance that we can use this connection to translate umbral moonshine into the setting of this work. As we demonstrate in § 4.1, this works out especially well in the

$\ell = 3$ case of umbral moonshine, in that it yields for us a virtual graded module for the unique non-trivial double cover $2.\mathsf{M}_{12}$ of the sporadic simple Mathieu group M_{12}, with McKay–Thompson series that have weight 2 and index 1, and that are optimal in the sense just sketched. Moreover, by restriction we recover from this optimal $2.\mathsf{M}_{12}$-module an optimal module for $\mathbb{Z}/11\mathbb{Z}$, such as in (1.1.1)-(1.1.2).

As a result, through consideration of the specific case that $N = 11$, we obtain a positive answer to the first question we formulated at the end of § 1.1: The cyclic group module structure on the W of (1.1.1)-(1.1.2) can arise as the restriction of something more intricate.

It is a curious fact that M_{12} contains two conjugacy classes of subgroups isomorphic to the smallest sporadic simple group, M_{11}. To distinguish them, fix a non-trivial permutation representation $\mathsf{M}_{12} \to \mathsf{S}_{12}$ of M_{12} on 12 points. Then one of the classes is composed of the 12 subgroups of M_{12} that fix one of the 12 points. The 12 copies of M_{11} in the other class each act transitively on these 12 points. The group M_{11} has trivial Schur multiplier (cf. e.g. [10]), so each of the 24 copies of M_{11} in M_{12} pulls back to a direct product $2 \times \mathsf{M}_{11}$ in $2.\mathsf{M}_{12}$.

Thus there are essentially two ways to restrict the optimal module for $2.\mathsf{M}_{12}$ in weight 2 and index 1 that comes to us from umbral moonshine to an optimal module of weight 2 and index 1 for M_{11}. Since umbral moonshine naturally equips the umbral group $2.\mathsf{M}_{12}$ at $\ell = 3$ with a permutation representation $2.\mathsf{M}_{12} \to \mathsf{S}_{12}$, it is natural to refer to these two optimal modules for M_{11} as *transitive* and *intransitive*, according as the corresponding composition

$$\mathsf{M}_{11} \to 2.\mathsf{M}_{12} \to \mathsf{S}_{12} \tag{1.2.2}$$

is transitive or not. From this perspective the transitive optimal module is distinguished by associating McKay–Thompson series with non-trivial characters to elements of order 4 and 8 in M_{11}, while ϕ_{g}^{W} is a mock Jacobi form for $\Gamma_0^{\mathrm{J}}(o(\mathsf{g}))$ for all the g arising from the intransitive optimal module. Thus the smallest sporadic simple group fits naturally into our framework in two different ways. We elucidate this further by making it our focus in §§ 4.2-4.3.

1.3. Results

In § 4.1 of [8] we classified optimal virtual graded G-modules of weight 2 and index 1, for G a cyclic group of prime order. In § 4.2 of this work we carry out a directly similar classification but for G the sporadic simple Mathieu group M_{11}. In fact we establish two such classifications, corresponding to the two optimal modules for M_{11} that come to us from umbral moonshine, as we have just discussed.

More specifically, we first determine the constant $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G})$ and the lattice $\mathcal{L}_{2,1}^{\mathrm{opt}}(\mathsf{G})$ for $\mathsf{G} = \mathsf{M}_{11}$, where $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G})$ and $\mathcal{L}_{2,1}^{\mathrm{opt}}(\mathsf{G})$ are as defined in § 3.3 (this being the same as the definition in [8]). This classification includes the intransitive optimal module for M_{11}, and is the content of Theorem 4.2.1.

We then determine $\mathrm{c}^{\mathrm{opt}}_{2,1}(\mathsf{G},\alpha)$ and $\mathcal{L}^{\mathrm{opt}}_{2,1}(\mathsf{G},\alpha)$, again for $\mathsf{G}=\mathsf{M}_{11}$, but with α encoding the multiplier systems that arise from the transitive optimal module for M_{11}. As we explain in § 3.3, we define the constant $\mathrm{c}^{\mathrm{opt}}_{2,1}(\mathsf{G},\alpha)$ and the lattice $\mathcal{L}^{\mathrm{opt}}_{2,1}(\mathsf{G},\alpha)$ analogously to $\mathrm{c}^{\mathrm{opt}}_{2,1}(\mathsf{G})$ and $\mathcal{L}^{\mathrm{opt}}_{2,1}(\mathsf{G})$, respectively, but take into account the multiplier system data of α. The determination of $\mathrm{c}^{\mathrm{opt}}_{2,1}(\mathsf{G},\alpha)$ and $\mathcal{L}^{\mathrm{opt}}_{2,1}(\mathsf{G},\alpha)$ here is the content of Theorem 4.2.5, and the classification this implies includes the transitive optimal module for M_{11}.

It develops that $\mathrm{c}^{\mathrm{opt}}_{2,1}(\mathsf{G})$ and $\mathrm{c}^{\mathrm{opt}}_{2,1}(\mathsf{G},\alpha)$ coincide for $\mathsf{G}=\mathsf{M}_{11}$ and α as arising from the transitive module. It is significant that this common value also coincides with $\mathrm{c}^{\mathrm{opt}}_{2,1}(\mathsf{G})$ for $\mathsf{G}=\mathbb{Z}/11\mathbb{Z}$, since this means that every optimal module for $\mathsf{G}=\mathbb{Z}/11\mathbb{Z}$ extends to an optimal module for M_{11}, and extends in two essentially different ways, depending on the triviality or not of the multiplier systems.

It is at least as significant that $\mathcal{L}^{\mathrm{opt}}_{2,1}(\mathsf{G})$ is the same for $\mathsf{G}=\mathsf{M}_{11}$ and $\mathsf{G}=\mathbb{Z}/11\mathbb{Z}$, but differs from $\mathcal{L}^{\mathrm{opt}}_{2,1}(\mathsf{G},\alpha)$ for $\mathsf{G}=\mathsf{M}_{11}$ and α as in the transitive optimal module. Indeed, it turns out that the latter lattice properly contains the former, because there is a non-zero cuspidal Jacobi form of weight 2 and index 1 for $\Gamma^{\mathrm{J}}_0(8)$ that is consistent with α as in the transitive optimal module. It is the involvement of this cusp form that allows us to draw out a new kind of connection between groups and geometry in this work.

To explain this recall that a positive integer n is called *congruent* if it occurs as the area of a right triangle with rational side lengths. That is, n is congruent if there exist $a,b,c\in\mathbb{Q}$ such that $a^2+b^2=c^2$ and $n=\frac{1}{2}ab$. The problem of determining which positive integers are congruent goes back to Diophantus, and remains unsolved to this day. We refer to [29] for a recent review. In § 4.3 we show that the representation theory of the transitive optimal module for M_{11} obstructs the existence of congruent numbers. Specifically, it follows from Theorem 4.3.1 that if $W=\bigoplus_D W_D$ is the transitive optimal module for M_{11}, and if n is square-free and congruent to 3 modulo 24, then n is not a congruent number whenever W_{-n} contains the unique 55-dimensional irreducible M_{11}-module (cf. Table 1) with non-zero multiplicity.

Thus we also answer in the affirmative, at least for $N=11$, the second question that we formulated the end of § 1.1: More intricate group-module structures on the W of (1.1.1)-(1.1.2) can entail more intricate relationships to arithmetic geometry, than those formulated in (1.1.4)-(1.1.6).

To conclude we mention that we obtain further positive answers to the questions of § 1.1 in the forthcoming work [7], by showing that 1) the optimal $\mathbb{Z}/N\mathbb{Z}$-modules of (1.1.1)-(1.1.2) extend to modules for the second largest sporadic simple Mathieu group M_{23} in the case that $N=23$, and 2) these modules entail interdependencies between arithmetic-geometric invariants of elliptic curves with coprime conductors.

1.4. Overview

The structure of this article is as follows. We present a guide to the specialized notation that we use in § 2. Then in § 3 we prepare for the statements and proofs of our main results. Specifically, we review generalized Hurwitz class numbers, and some related

notions, in § 3.1, and review mock and meromorphic Jacobi forms in § 3.2. We discuss the notion of optimality we use, in general weight and index, in § 3.3, and describe it in more detail in the special case of weight 2 and index 1 in § 3.4. Our main results appear in § 4. We first explain the connection between class numbers and umbral moonshine in § 4.1. We then use this to classify optimal modules of weight 2 and index 1 for M_{11}, in two different ways, in § 4.2. Finally we use the optimal modules of § 4.2 to relate the representation theory of M_{11} to the congruent number problem in § 4.3.

Acknowledgments

We are grateful to Lea Beneish, Mathew Emerton, Maryam Khaqan, Kimball Martin, Ken Ono, Preston Wake, Eric Zhu and David Zureick-Brown for helpful discussions on topics closely related to the content of this work. We are grateful to anonymous referees for helpful comments and advice. The work of M.C. was supported by the National Science and Technology Council of Taiwan (110-2115-M-001-018-MY3), and by a Vidi grant (number 016.Vidi.189.182) from the Dutch Research Council (NWO). The work of J.D. was supported in part by the U.S. National Science Foundation (DMS 1601306), the Simons Foundation (#708354), Academia Sinica (AS-IA-113-M03) and the National Science and Technology Council of Taiwan (111-2115-M-001-001-MY2, 112-2115-M-001-006-MY3).

2. Notation

$\mathrm{Av}^{(m)}$	The averaging operator of index m. See (3.2.25).
α	An assignment of discrete groups $\Gamma_{\mathsf{g}} < G^{\mathrm{J}}$ and characters $\rho_{\mathsf{g}} : \Gamma_{\mathsf{g}} \to \mathbb{C}^*$ to the elements g of a finite group G. See (3.3.3) and cf. (3.3.11).
c	The negative of the constant term of a holomorphic mock Jacobi form. See (3.3.2).
$\mathrm{c}^{\mathrm{opt}}_{k,m}(\mathsf{G})$	A shorthand for $\mathrm{c}^{\mathrm{opt}}_{k,m}(\mathsf{G},\alpha)$, for the default choice (3.3.11) of α.
$\mathrm{c}^{\mathrm{opt}}_{k,m}(\mathsf{G},\alpha)$	An invariant we attach to a finite group G and choice of α in § 3.3. See (3.3.10).
$C_N(D)$	A shorthand for $C_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}(D)$, when $o(\mathsf{g}) = N$, in the proof of Theorem 4.3.1.
$C_{\varphi_{8\|4}}(D)$	A coefficient in the Fourier expansion of $\varphi_{8\|4}$. See the proof of Theorem 4.3.1.
δ_N	The class function on a group G defined by setting $\delta_N(\mathsf{g}) := 1$ in case $o(\mathsf{g}) = N$, and $\delta_N(\mathsf{g}) = 0$ otherwise. Cf. (4.2.6) and (4.2.15).
e	The identity element of a finite group G. See § 3.3.
$\mathrm{e}(\cdot)$	We set $\mathrm{e}(x) := e^{2\pi i x}$.
E	The elliptic curve over $\mathbb{Q}$ defined by $y^2 = x^3 - x$. See the proof of Theorem 4.3.1.
$E \otimes D$	The D-twist of the elliptic curve E. See the proof of Theorem 4.3.1.
ϕ_{g}^{W}	The McKay–Thompson series associated to the action of g on W. See (3.3.5) and (3.4.3).
φ_{11}	A certain cuspidal Jacobi form of weight 2 and index 1 for $\Gamma_0^{\mathrm{J}}(11)$. See (4.2.1).

$\varphi_{8|4}$ A certain cuspidal Jacobi form of weight 2 and index 1 for $\Gamma_0^{\mathrm{J}}(32)$. See (4.2.9).

g An element in a finite group G. See § 3.3.

G A finite group. See § 3.3.

G^{J} The Jacobi Group. Cf. (3.2.2).

Γ A subgroup of G^{J}. Cf. (3.2.3).

$\Gamma_0^{\mathrm{J}}(N)$ A group of the form $\Gamma_0(N) \ltimes \mathbb{Z}^2$. Cf. (3.2.3).

h_{g}^W The vector-valued function that takes the $h_{\mathsf{g},r}^W$ as its components. See (3.3.5).

$h_{\mathsf{g},r}^W$ The theta-coefficients of ϕ_{g}^W. See (3.3.6).

$H^{\mathrm{Hur}}(D)$ The Hurwitz class number of discriminant D. See (3.1.3).

$H^{\mathrm{Hur}}{}_N(D)$ The generalized Hurwitz class number of level N and discriminant D. See (3.1.2)-(3.1.5).

$H^{\mathrm{Coh}}{}_N(D)$ A coefficient of the Cohen–Eisenstein series $\mathscr{H}_N^{\mathrm{Coh}}$. Cf. (3.1.8)–(3.1.9).

$\mathscr{H}^{\mathrm{Hur}}$ A shorthand for $\mathscr{H}_1^{\mathrm{Hur}}$. Cf. (3.1.6).

$\mathscr{H}^{\mathrm{Hur}}{}_N$ A holomorphic mock Jacobi form defined by the $H_N^{\mathrm{Hur}}(D)$. See (3.1.6).

$\mathscr{H}^{\mathrm{Coh}}{}_N$ A holomorphic Jacobi form defined by the $H_N^{\mathrm{Coh}}(D)$ for N prime. See (3.1.9).

$\iota(N)$ The index of $\Gamma_0(N)$ as a subgroup of $\Gamma_0(1) = SL_2(\mathbb{Z})$. Cf. (3.1.5).

$J_{k,m}(\Gamma, \rho)$ The holomorphic Jacobi forms of weight k and index m for Γ with character ρ. See § 3.2.

$\mathbb{J}_{k,m}(\Gamma, \rho)$ The holomorphic mock Jacobi forms of weight k and index m for Γ with character ρ. See § 3.2.

$\mathbb{J}_{k,m}^{\mathrm{wh}}(\Gamma, \rho)$ The weakly holomorphic mock Jacobi forms of weight k and index m for Γ with character ρ. See § 3.2.

$L_{k,m}(\mathsf{G})$ A certain lattice. See (3.3.9).

$\mathcal{L}_{k,m}^{\mathrm{opt}}(\mathsf{G})$ A shorthand for $\mathcal{L}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ for the default choice (3.3.11) of α.

$\mathcal{L}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ The lattice of 0-optimal virtual graded modules of weight k and index m for (G, α). Cf. Proposition 3.3.1.

ψ^F The finite part of a meromorphic Jacobi form ψ. See (3.2.22).

ψ^P The polar part of a meromorphic Jacobi form ψ. See (3.2.22).

$\psi^{(1)}$ A certain meromorphic Jacobi form of weight 2 and index 1. See (3.2.19) and (4.1.5).

$\psi_{\mathsf{g}}^{(1)}$ A certain meromorphic Jacobi form of weight 2 and index 1, generally with level and character, for $\mathsf{g} \in 2.\mathsf{M}_{12}$. See (4.1.6).

$\psi^{(3)}$ A certain meromorphic Jacobi form of weight 1 and index 3. See (4.1.3).

$\psi_{\mathsf{g}}^{(3)}$ The meromorphic Jacobi form of weight 1 and index 3 attached to $\mathsf{g} \in 2.\mathsf{M}_{12}$ by umbral moonshine at $\ell = 3$. See (4.1.6).

q We set $q := \mathrm{e}(\tau)$ for $\tau \in \mathbb{H}$.

$\mathcal{Q}_N(D)$ A set of binary quadratic forms with integer coefficients. See § 3.1.

$R(\mathsf{G})$ The Grothendieck group of finitely generated $\mathbb{C}\mathsf{G}$-modules. Cf. (3.3.9).

$R(\mathsf{G})_{(N)}$ The subgroup of $R(\mathsf{G})$ composed of virtual G-modules V such that $\mathrm{tr}(\mathsf{g}|V) = 0$ unless $o(\mathsf{g}) = N$. Cf. (4.2.5) and (4.2.14).

ρ A character of a subgroup Γ of G^{J} with level. Cf. (3.2.5).

ϱ_m A certain unitary representation of $\widetilde{SL}_2(\mathbb{Z})$. Cf. (3.2.14).

$S_{k,m}(\Gamma,\rho)$ The cuspidal Jacobi forms of weight k and index m for Γ with character ρ. See § 3.2.
$S_{k,m}(\Gamma,\rho)_{\mathbb{Z}}$ The $\phi \in S_{k,m}(\Gamma,\rho)$ with rational integer Fourier coefficients. See § 3.2.
$\widetilde{SL}_2(\mathbb{Z})$ The metaplectic double cover of $SL_2(\mathbb{Z})$. Cf. (3.2.14).
θ_m The vector-valued function that takes the $\theta_{m,r}$ as its components. Cf. (3.2.14).
$\theta_{m,r}$ The theta series defined by the positive-definite even lattices of rank 1. See (3.2.8).
$\vartheta_j(\tau,z)$ A Jacobi theta function. See (3.2.20) and (4.1.1).
$\mathcal{W}^{\mathrm{opt}}_{k,m}(\mathsf{G})$ A shorthand for $\mathcal{W}^{\mathrm{opt}}_{k,m}(\mathsf{G},\alpha)$ for the default choice (3.3.7) of α.
$\mathcal{W}^{\mathrm{opt}}_{k,m}(\mathsf{G})_{\mathrm{c}}$ A shorthand for $\mathcal{W}^{\mathrm{opt}}_{k,m}(\mathsf{G},\alpha)_{\mathrm{c}}$ for the default choice (3.3.7) of α.
$\mathcal{W}^{\mathrm{opt}}_{k,m}(\mathsf{G},\alpha)$ The set of optimal virtual graded modules of weight k and index m for (G,α). Cf. (3.3.8).
$\mathcal{W}^{\mathrm{opt}}_{k,m}(\mathsf{G},\alpha)_{\mathrm{c}}$ The set of c-optimal virtual graded modules of weight k and index m for (G,α). Cf. (3.3.8).
χ_j An irreducible character of M_{11}. See Table 1.
y We set $y := \mathrm{e}(z)$ for $z \in \mathbb{C}$.

3. Preparation

This section is a counterpart to § 3 in [8], and offers preparation for the arguments that will appear in § 4. We briefly review generalized Hurwitz class numbers in § 3.1. The Jacobi forms we consider here are more general than those that appear in [8], as we discuss in § 3.2. The notion of optimality is correspondingly more general, and we explain it in § 3.3. We work with Jacobi forms of general weight and index in §§ 3.2-3.3, for the sake of future applications. We specialize to the situation of weight 2 and index 1, being the setting of our main results, in § 3.4.

3.1. Class numbers

As in [8] we let $\mathcal{Q}_N(D)$ denote the set of integer coefficient binary quadratic forms $Q(x,y) = Ax^2 + Bxy + Cy^2$ of discriminant $D := B^2 - 4AC$ with $A \equiv 0 \bmod N$. We equip each $\mathcal{Q}_N(D)$ with a right action by the group

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\} \tag{3.1.1}$$

in the usual way, and we write $\Gamma_0(N)_Q$ for the stabilizer in $\Gamma_0(N)$ of a particular $Q \in \mathcal{Q}_N(D)$.

Note that $\mathcal{Q}_N(D)$ is non-empty if and only if D is a quadratic residue modulo $4N$. Say that $D \in \mathbb{Z}$ is a *discriminant* if $\mathcal{Q}_1(D)$ is not empty (i.e. D is congruent to 0 or 1 modulo 4), and say that D is a *fundamental discriminant* if D is the discriminant of the number field $\mathbb{Q}(\sqrt{D})$. (This is the same as saying that either $D \equiv 1 \bmod 4$ and D is square-free, or $D = 4d$ where d is square-free and not a quadratic residue modulo 4.)

We restrict to negative discriminants in this work. For D a negative fundamental discriminant write $h(D)$ for the class number of $\mathbb{Q}(\sqrt{D})$. Then $h(D)$ is just the number of orbits of $\Gamma_0(1) = SL_2(\mathbb{Z})$ in its action on $\mathcal{Q}_1(D)$. The *generalized Hurwitz class numbers*, denoted $H_N^{\mathrm{Hur}}(D)$, are defined for $D < 0$ by setting

$$H_N^{\mathrm{Hur}}(D) := \sum_{Q \in \mathcal{Q}_N(D)/\Gamma_0(N)} \frac{1}{\#\Gamma_0(N)_Q} \tag{3.1.2}$$

when $\mathcal{Q}_N(D)$ is not empty, and $H_N^{\mathrm{Hur}}(D) := 0$ otherwise. Therefore, the $H_N^{\mathrm{Hur}}(D)$ are closely related to the usual class numbers $h(D)$. Indeed, the *Hurwitz class number*

$$H^{\mathrm{Hur}}(D) := H_1^{\mathrm{Hur}}(D) \tag{3.1.3}$$

coincides with $h(D)$ if D is fundamental and less than -4, while for the remaining negative fundamental discriminants we have $H^{\mathrm{Hur}}(-3) = \frac{1}{3}$ and $H^{\mathrm{Hur}}(-4) = \frac{1}{2}$. For a more general statement we note from Lemma 3.1.2 of [8] that

$$H_N^{\mathrm{Hur}}(D) = \left(1 + \left(\frac{D}{N}\right)\right) H^{\mathrm{Hur}}(D) \tag{3.1.4}$$

for D negative and fundamental when N is prime, where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol (see e.g. p. 503 of [21] for the definition). In particular $H_N^{\mathrm{Hur}}(D) = 2h(D)$ when $D < -4$ and D is a square modulo $4N$.

As in [8] our main motivation for the definition (3.1.2) is that it gives rise to holomorphic mock Jacobi forms (cf. Proposition 3.4.1). With this in mind we define

$$H_N^{\mathrm{Hur}}(0) := -\frac{1}{12}\iota(N) \tag{3.1.5}$$

where $\iota(N) := [\Gamma_0(1) : \Gamma_0(N)]$, and set $H_N^{\mathrm{Hur}}(D) := 0$ when $D > 0$. We also define $q := \mathrm{e}(\tau)$ for τ in the upper half-plane, $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$, and $y := \mathrm{e}(z)$ for $z \in \mathbb{C}$, where $\mathrm{e}(x) := e^{2\pi i x}$. The generating functions

$$\mathscr{H}_N^{\mathrm{Hur}}(\tau, z) := \sum_{n,s \in \mathbb{Z}} H_N^{\mathrm{Hur}}(s^2 - 4n) q^n y^s \tag{3.1.6}$$

and $\mathscr{H}^{\mathrm{Hur}} := \mathscr{H}_1^{\mathrm{Hur}}$ will play an important role in what follows.

In § 4.3 we will also employ Cohen–Eisenstein series, whose coefficients furnish a further variation on the class numbers $h(D)$ (cf. [9]). To define these series recall that for D an arbitrary discriminant we have $D = f^2 D_0$, where D_0 is the discriminant of $\mathbb{Q}(\sqrt{D})$ and f is the *conductor* of the order

$$\mathcal{O}_D := \mathbb{Z}\left[\frac{D + \sqrt{D}}{2}\right]. \tag{3.1.7}$$

Next choose a prime N, and for $D = f^2 D_0$ as above let f' be the largest factor of f that is coprime to N, and set $D' := (f')^2 D_0$. Then we may follow § 1 of [22] in defining $H_N^{\mathrm{Coh}}(D)$ for N prime and D a negative discriminant by setting

$$H_N^{\mathrm{Coh}}(D) := \begin{cases} 0 & \text{if } N \text{ splits in } \mathcal{O}_{D'}, \\ \frac{1}{2} H^{\mathrm{Hur}}(D') & \text{if } N \text{ is ramified in } \mathcal{O}_{D'}, \\ H^{\mathrm{Hur}}(D') & \text{if } N \text{ is inert in } \mathcal{O}_{D'}. \end{cases} \tag{3.1.8}$$

For convenience we also set $H_N^{\mathrm{Coh}}(0) := \frac{N-1}{24}$, and set $H_N^{\mathrm{Coh}}(D) := 0$ when D is positive, or a negative integer that is not a discriminant. The *Cohen–Eisenstein series* of (prime) level N are now defined by setting

$$\mathscr{H}_N^{\mathrm{Coh}}(\tau, z) := \sum_{n,s \in \mathbb{Z}} H_N^{\mathrm{Coh}}(s^2 - 4n) q^n y^s, \tag{3.1.9}$$

where q and y are as in (3.1.6).

For N prime the Cohen–Eisenstein series $\mathscr{H}_N^{\mathrm{Coh}}$ is related to the generalized Hurwitz class number generating functions $\mathscr{H}^{\mathrm{Hur}}$ and $\mathscr{H}_N^{\mathrm{Hur}}$ by the formula $\mathscr{H}^{\mathrm{Hur}} = \mathscr{H}_N^{\mathrm{Coh}} + \frac{1}{2}\mathscr{H}_N^{\mathrm{Hur}}$. This is the content of Lemma 3.1.1 of [8]. Lemma 3.1.2 of [8] also gives us a counterpart

$$H_N^{\mathrm{Coh}}(D) = \frac{1}{2}\left(1 - \left(\frac{D}{N}\right)\right) H^{\mathrm{Hur}}(D), \tag{3.1.10}$$

for D negative and fundamental, to (3.1.4).

3.2. Jacobi forms

Here we explain our conventions on mock Jacobi forms. We assume some familiarity with the basic definitions, suggest § 3.1 of [3] and [19] for more on Jacobi forms, and suggest § 3.2 of [3] and § 7.2 of [11] for more on mock Jacobi forms. We will also make use of meromorphic Jacobi forms. We refer to § 2 of [2] and § 8 of [11] for more background on these.

In § 4.1 we will require a notion of level for discrete subgroups of the Jacobi Group,

$$G^{\mathrm{J}} = SL_2(\mathbb{R}) \ltimes (\mathbb{R}^2 \cdot S^1). \tag{3.2.1}$$

To explain what we mean by this we first realize G^{J} as the set of triples $(\gamma, (\lambda, \mu), \varsigma)$, where $\gamma \in SL_2(\mathbb{R})$, and $(\lambda, \mu) \in \mathbb{R}^2$, and $\varsigma \in \mathbb{C}$ satisfies $|\varsigma| = 1$. The multiplication in G^{J} is then given by

$$(\gamma, (\lambda, \mu), \varsigma)(\gamma', (\lambda', \mu'), \varsigma') = \left(\gamma\gamma', (\lambda, \mu)\gamma' + (\lambda', \mu'), \varsigma\varsigma' \mathrm{e}\left(\det\begin{pmatrix} (\lambda, \mu)\gamma' \\ (\lambda', \mu') \end{pmatrix}\right)\right). \tag{3.2.2}$$

So in particular, the subgroup $\{(I,(0,0),\varsigma)\} = S^1$ is central. We write $\overline{G}^{\mathrm{J}}$ for the quotient of G^{J} by this copy of S^1.

Next, for N a positive integer, let $\Gamma_0^{\mathrm{J}}(N)$ denote the subgroup of G^{J} composed of the triples $(\gamma,(\lambda,\mu),1)$ with $\gamma \in \Gamma_0(N)$ and $(\lambda,\mu) \in \mathbb{Z}^2$. We tacitly identify $\Gamma_0^{\mathrm{J}}(N)$ with its image in $\overline{G}^{\mathrm{J}} = SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ under the natural projection when convenient, and let $\Gamma_\infty^{\mathrm{J}}$ denote the intersection of the groups $\Gamma_0^{\mathrm{J}}(N)$, so that

$$\Gamma_\infty^{\mathrm{J}} = \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (\lambda,\mu), 1 \right) \middle| \, n,\lambda,\mu \in \mathbb{Z} \right\}. \tag{3.2.3}$$

Finally we say that $\Gamma < G^{\mathrm{J}}$ has *level* N if the image of Γ in $\overline{G}^{\mathrm{J}}$ under the natural projection contains $\Gamma_0^{\mathrm{J}}(N)$ to finite index for some N, and if N is the minimal positive integer for which this is true. We also assume that Γ contains $\Gamma_\infty^{\mathrm{J}} = \mathbb{Z} \ltimes \mathbb{Z}^2$ (3.2.3). A subgroup $\Gamma < G^{\mathrm{J}}$ with *level* is a subgroup with level N for some N.

We will require a notion of cusp for subgroups of G^{J} with level in the above sense. To formulate this we consider the usual transitive action of $SL_2(\mathbb{R})$ on the real projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, and obtain from this an action of G^{J} on $\mathbb{P}^1(\mathbb{R})$ by letting the normal subgroup $\{(I,(\lambda,\mu),\varsigma)\} = \mathbb{R}^2 \cdot S^1$ (cf. (3.2.2)) act trivially. Thus an arbitrary subgroup $\Gamma < G^{\mathrm{J}}$ acts naturally on $\mathbb{P}^1(\mathbb{R})$ by restriction. If $\Gamma = \Gamma_0^{\mathrm{J}}(N)$ for some N then this action preserves the subset $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ of rational points, and by our definitions, this statement holds true also for $\Gamma < G^{\mathrm{J}}$ a subgroup with level (cf. e.g. Proposition 1.10.2 of [15]). So given $\Gamma < G^{\mathrm{J}}$ with level we may consider the set

$$\Gamma \backslash \mathbb{P}^1(\mathbb{Q}) = \{\Gamma \cdot \alpha \mid \alpha \in \mathbb{Q} \cup \{\infty\}\} \tag{3.2.4}$$

of orbits of Γ on $\mathbb{P}^1(\mathbb{Q})$. We call these orbits (3.2.4) the *cusps* of Γ, and we refer to the orbit $\Gamma \cdot \infty$ containing ∞ as the *infinite cusp* of Γ.

For Γ a subgroup of G^{J} with level, a *character* of Γ will mean a morphism of groups $\rho : \Gamma \to \mathbb{C}^*$ that is trivial when restricted to $\Gamma_\infty^{\mathrm{J}}$ (3.2.3). For such Γ and ρ, and for integers k and m, we now define the *weight* k, *index* m and *character* ρ action of Γ on a smooth function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ by setting

$$(\phi|_{k,m,\rho}(\gamma,(\lambda,\mu),\varsigma))(\tau,z) := (\phi|_{k,m}(\gamma,(\lambda,\mu),\varsigma))(\tau,z)\rho(\gamma,(\lambda,\mu),\varsigma) \tag{3.2.5}$$

for $(\gamma,(\lambda,\mu),\varsigma) \in \Gamma$, where $\phi \mapsto \phi|_{k,m}(\gamma,(\lambda,\mu),\varsigma)$ denotes the usual weight k and index m action of G^{J} (as in e.g. Theorem 1.4 in [19]).

To formulate the notion of Jacobi form for a group $\Gamma < G^{\mathrm{J}}$ as above say that a smooth function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is *elliptic* of index m for Γ with character ρ if it is invariant for the restriction of the action (3.2.5) to the intersection of Γ with the normal subgroup $\mathbb{R}^2 \cdot S^1$ of G^{J}. This restricted action is given explicitly by

$$(\phi|_{k,m,\rho}(I,(\lambda,\mu),\varsigma))(\tau,z) := \mathrm{e}\left(m(\lambda^2\tau + 2\lambda z + \lambda\mu + \varsigma)\right)\phi(\tau, z+\lambda\tau+\mu)\rho(I,(\lambda,\mu),\varsigma). \tag{3.2.6}$$

In particular, it is independent of k.

Given our assumptions on ρ, an elliptic function of index m for Γ with character ρ admits a *theta-decomposition*

$$\phi(\tau, z) = \sum_{r \bmod 2m} h_r(\tau)\theta_{m,r}(\tau, z), \tag{3.2.7}$$

where the *theta-coefficients* h_r are smooth functions on $\mathbb{H}$, and where the theta series $\theta_{m,r}$ are defined for integers m and r, with m positive, by setting

$$\theta_{m,r}(\tau, z) := \sum_{s \equiv r \bmod 2m} q^{\frac{s^2}{4m}} y^s. \tag{3.2.8}$$

Suppose now that ϕ is a holomorphic function on $\mathbb{H} \times \mathbb{C}$ that is elliptic of index m for Γ with character ρ. In this work we say that ϕ is a *mock Jacobi form* of weight k and index m for Γ with character ρ if there exist holomorphic functions $g_r : \mathbb{H} \to \mathbb{C}$, for r mod $2m$, such that if we set

$$\widehat{h}_r(\tau) := h_r(\tau) + g_r^*(\tau), \tag{3.2.9}$$

where g_r^* is the weight $k - \frac{1}{2}$ Eichler integral of g_r (see e.g. (7.2) of [11]), then the real-analytic function $\widehat{\phi}$, given by

$$\widehat{\phi}(\tau, z) := \sum_{r \bmod 2m} \widehat{h}_r(\tau)\theta_{m,r}(\tau, z), \tag{3.2.10}$$

is invariant for the weight k, index m and character ρ action (3.2.5) of Γ. For completeness we note that if

$$g_r(\tau) = \sum_{D \geq 0} B(D, r) q^{\frac{D}{4m}} \tag{3.2.11}$$

is the Fourier expansion of g_r, then g_r^* in (3.2.9) may be expressed explicitly as

$$g_r^*(\tau) = \overline{B(0,r)} \frac{(4\pi\Im(\tau))^{-k+\frac{3}{2}}}{k - \frac{3}{2}} + \sum_{D>0} \left(\frac{D}{4m}\right)^{k-\frac{3}{2}} \overline{B(D,r)} \Gamma(\tfrac{3}{2} - k, 4\pi \tfrac{D}{4m} \Im(\tau)) q^{-\frac{D}{4m}}, \tag{3.2.12}$$

where $\Gamma(\frac{3}{2} - k, x) := \int_x^\infty t^{-k+\frac{1}{2}} e^{-t} \mathrm{d}t$ is the incomplete gamma function.

For ϕ a mock Jacobi form, with theta-coefficients h_r as in (3.2.7), we call $\widehat{h}_r$ as in (3.2.9) the *completion* of h_r, and call $\widehat{\phi}$ as in (3.2.10) the *completion* of ϕ. Under our hypotheses on ρ the theta-coefficients h_r of a mock Jacobi form for Γ with character ρ admit *Fourier series* expansions of the form

$$h_r(\tau) = \sum_{D \equiv r^2 \bmod 4m} C_\phi(D, r) q^{-\frac{D}{4m}}. \tag{3.2.13}$$

We write the theta-decomposition (3.2.7) compactly as $\phi(\tau, z) = h(\tau)^{\mathrm{t}} \theta_m(\tau, z)$ or $\phi = h^{\mathrm{t}} \theta_m$ when convenient, taking $h = (h_r)$ to be the vector-valued function with the theta-coefficients h_r (3.2.13) as its components, taking $\theta_m = (\theta_{m,r})$ to be the vector-valued function whose components are the theta series $\theta_{m,r}$ (3.2.8), and using the superscript in $h(\tau)^{\mathrm{t}}$ and h^{t} to denote matrix transposition.

We write $\widetilde{SL}_2(\mathbb{Z})$ for the metaplectic double cover of the modular group, and realize it as the set of pairs (γ, υ), where $\gamma \in SL_2(\mathbb{Z})$ and $\upsilon : \mathbb{H} \to \mathbb{C}$ is either of the two holomorphic functions such that $\upsilon(\tau)^2 = c\tau + d$ when (c, d) is the lower row of γ. We also recall the *index* m Weil representation $\varrho_m : \widetilde{SL}_2(\mathbb{Z}) \to GL_{2m}(\mathbb{C})$, defined by requiring that

$$\overline{\varrho_m(\gamma, \upsilon)} \theta_m \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) \frac{1}{\upsilon(\tau)} \mathrm{e}\left(-\frac{cmz^2}{c\tau + d} \right) = \theta_m(\tau, z) \tag{3.2.14}$$

when $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, for $(\gamma, \upsilon) \in \widetilde{SL}_2(\mathbb{Z})$, and use this to define weakly holomorphic, holomorphic, and cuspidal mock Jacobi forms as follows. For $\phi = h^{\mathrm{t}} \theta_m$ a mock Jacobi form of weight k and index m for Γ with character ρ we say that ϕ is *weakly holomorphic* if there exists a constant $C > 0$ such that

$$\varrho_m(\gamma, \upsilon)(\widehat{h}|_{k-\frac{1}{2}}(\gamma, \upsilon))(\tau) = O(e^{C\Im(\tau)}) \tag{3.2.15}$$

as $\Im(\tau) \to \infty$, for all $(\gamma, \upsilon) \in \widetilde{SL}_2(\mathbb{Z})$, where $\widehat{h} = (\widehat{h}_r)$, and where $\widehat{h} \mapsto \widehat{h}|_{k-\frac{1}{2}}(\gamma, \upsilon)$ denotes the usual weight $k - \frac{1}{2}$ action of the metaplectic group on vector-valued functions on $\mathbb{H}$ (see e.g. § 3.2 of [8]). We say that ϕ is *holomorphic* if the exponential bound (3.2.15) can be replaced with boundedness,

$$\varrho_m(\gamma, \upsilon)(\widehat{h}|_{k-\frac{1}{2}}(\gamma, \upsilon))(\tau) = O(1) \tag{3.2.16}$$

as $\Im(\tau) \to \infty$, for all $(\gamma, \upsilon) \in \widetilde{SL}_2(\mathbb{Z})$, and say that ϕ is *cuspidal* if boundedness (3.2.16) can in fact be replaced with vanishing,

$$\varrho_m(\gamma, \upsilon)(\widehat{h}|_{k-\frac{1}{2}}(\gamma, \upsilon))(\tau) \to 0 \tag{3.2.17}$$

as $\Im(\tau) \to \infty$, for all $(\gamma, \upsilon) \in \widetilde{SL}_2(\mathbb{Z})$. Note that a cuspidal mock Jacobi form necessarily has vanishing shadow, so there are no cuspidal mock Jacobi forms that are not actually Jacobi forms. This can be seen by using the pairing introduced by Bruinier–Funke in Proposition 3.5 of [1]. (See Proposition 3.2.1 of [3] for a formulation of the Bruinier–Funke pairing in terms of Jacobi forms.)

With the above definitions in place we let $\mathbb{J}^{\mathrm{wh}}_{k,m}(\Gamma, \rho)$ denote the space of weakly holomorphic (3.2.15) mock Jacobi forms of weight k and index m for Γ with character

ρ, and write $\mathbb{J}_{k,m}(\Gamma,\rho)$ for the subspace of mock Jacobi forms that are holomorphic (3.2.16). The *holomorphic Jacobi forms* of weight k and index m for Γ with character ρ are the forms $\phi = h^{\mathrm{t}}\theta_m$ in $\mathbb{J}_{k,m}(\Gamma,\rho)$ for which the g_r^* in (3.2.9) all vanish, so that $\widehat{h} = h$, &c. We denote the space they comprise by $J_{k,m}(\Gamma,\rho)$, and write $S_{k,m}(\Gamma,\rho)$ for the subspace comprised of Jacobi forms that are cuspidal. We suppress the character ρ from these notations when it is trivial.

In the remainder of this section we offer a quick review of some of the meromorphic Jacobi form theory that appears in § 8 of [11]. Briefly, a meromorphic Jacobi form of weight k and index m for Γ with character ρ is a function that is invariant under the corresponding action (3.2.5) of Γ, but is allowed to have poles in the z variable. A classic example is the Weierstrass elliptic function,

$$\wp(\tau,z) := \frac{1}{z^2} + \sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}} \frac{1}{(a\tau+b-z)^2} - \frac{1}{(a\tau+b)^2}, \tag{3.2.18}$$

which has weight 2 and index 0 for $\Gamma_0^{\mathrm{J}}(1) = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ (with trivial character), and a double pole at $z = a\tau + b$, for all $a, b \in \mathbb{Z}$. An example of particular relevance in this work is

$$\psi^{(1)}(\tau,z) := \frac{1}{12}\left(\frac{9}{4\pi^2}\wp(\tau,z)^2 - E_4(\tau)\right)\frac{\vartheta_1(\tau,z)^2}{\eta(\tau)^6}, \tag{3.2.19}$$

where E_4 is the weight 4 Eisenstein series for $SL_2(\mathbb{Z})$, normalized so that $E_4(\tau) = 1+O(q)$ as $\Im(\tau)\to\infty$, and where the *Jacobi theta function* and *Dedekind eta function* are defined by setting

$$\vartheta_1(\tau,z) := -\theta_{1,1}(\tfrac{1}{2}\tau, \tfrac{1}{2}(z+\tfrac{1}{2})), \tag{3.2.20}$$

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n>0}(1-q^n), \tag{3.2.21}$$

respectively, for $\tau\in\mathbb{H}$ and $z\in\mathbb{C}$, where $\theta_{1,1}$ is as in (3.2.8). Similar to $\wp$, the function $\psi^{(1)}$ has a double pole at every lattice point $z\in\mathbb{Z}\tau+\mathbb{Z}$, but in contrast to $\wp$ it transforms with weight 2 and index 1 under the action of $\Gamma_0^{\mathrm{J}}(1)$, rather than weight 2 and index 0.

Observe now that, because of its poles, a meromorphic Jacobi form generally does not admit a theta-decomposition as in (3.2.7). A solution to this problem is proposed in [11], following earlier work [30] of Zwegers. Under suitable conditions on the meromorphic Jacobi form ψ, the authors of [11] introduce a canonical decomposition

$$\psi = \psi^P + \psi^F, \tag{3.2.22}$$

where ψ^P and ψ^F are called the *polar* and *finite* parts of ψ, respectively. Here the polar part ψ^P may be realized as an average over the local behavior of $\psi(\tau,z)$ as z approaches

its poles (within a fundamental domain for $\mathbb{Z}\tau + \mathbb{Z}$), and the finite part ψ^F is defined by setting

$$\psi^F(\tau, z) = \sum_{r \bmod 2m} h_r(\tau)\theta_{m,r}(\tau, z) \tag{3.2.23}$$

(cf. (3.2.7)), where the $h_r(\tau)$ are computed by imposing certain choices on the (otherwise ambiguous) integral expression

$$h_r(\tau)q^{\frac{r^2}{4m}} = \int_{z_0}^{z_0+1} \psi(\tau, z)\,\mathrm{e}(-rz)\mathrm{d}z. \tag{3.2.24}$$

In particular, the finite part of a meromorphic Jacobi form admits a theta-decomposition (3.2.23) by construction.

Note that (3.2.24) unambiguously recovers the theta-coefficients of ψ, for any choice of $z_0 \in \mathbb{C}$, and any path from z_0 to $z_0 + 1$, if ψ is holomorphic in z and elliptic of index m (cf. (3.2.7)). We refer to § 8 of [11] for details on how to interpret (3.2.24) in the meromorphic case.

We conclude by detailing the decomposition (3.2.22) in the special case that $\psi = \psi^{(1)}$ is as in (3.2.19). For this we follow [11] in defining the *averaging operator* of index m, by setting

$$\mathrm{Av}^{(m)}(F(y)) := \sum_{s\in\mathbb{Z}} y^{2ms}q^{ms^2}F(yq^s) \tag{3.2.25}$$

for m a non-negative integer and $F(y)$ a function of polynomial growth. We are then able to identify the polar part $\psi^{(1),P}$ of $\psi^{(1)}$ as

$$\psi^{(1),P}(\tau, z) = -12\,\mathrm{Av}^{(1)}(y(1-y)^{-2}), \tag{3.2.26}$$

and from the discussion of Example 5 in § 8.5 of [11] we obtain that the finite part $\psi^{(1),F}$ satisfies

$$\psi^{(1),F}(\tau, z) = 24\mathscr{H}^{\mathrm{Hur}}(\tau, z), \tag{3.2.27}$$

where $\mathscr{H}^{\mathrm{Hur}} = \mathscr{H}_1^{\mathrm{Hur}}$ is the Hurwitz class number generating function, as defined in (3.1.6).

3.3. Optimality

In this section we explain optimality for holomorphic mock Jacobi forms of arbitrary integer weight and positive integer index. The exposition is similar to that of § 3.3 of [8] except that here we allow for forms that are automorphic for more general subgroups of the Jacobi group, as described in § 3.2, and also consider characters.

To begin we fix an integer k and a positive integer m. Then for Γ and ρ as in (3.2.5) we say that $\phi \in \mathbb{J}_{k,m}(\Gamma, \rho)$ is *optimal* if

$$\left(\varrho_m(\gamma, \upsilon)\widehat{h}|_{k-\frac{1}{2}}(\gamma, \upsilon)\right)(\tau) \to 0 \tag{3.3.1}$$

as $\Im(\tau) \to \infty$ (cf. (3.2.17)), whenever $(\gamma, \upsilon) \in \widetilde{SL}_2(\mathbb{Z})$ is such that $\gamma \cdot \infty$ does not belong to the infinite cusp of Γ (cf. (3.2.4)). We then refine this notion by saying that $\phi \in \mathbb{J}_{k,m}(\Gamma, \rho)$ is c-*optimal*, for a given constant c, if ϕ is optimal (3.3.1) and satisfies

$$\phi(\tau, z) = -\mathrm{c} + O(q), \tag{3.3.2}$$

for every $z \in \mathbb{C}$, as $\Im(\tau) \to \infty$.

As in [8] we are interested in situations whereby collections of optimal mock Jacobi forms are organized by finite groups. To setup for this we let G be a finite group, and consider an assignment

$$\alpha : [\mathsf{g}] \mapsto \alpha(\mathsf{g}) = (\Gamma_{\mathsf{g}}, \rho_{\mathsf{g}}) \tag{3.3.3}$$

of pairs to the conjugacy classes $[\mathsf{g}]$ of G, where for each $\mathsf{g} \in \mathsf{G}$ the first component Γ_{g} of $\alpha(\mathsf{g})$ is a subgroup of G^{J} with level $N = o(\mathsf{g})$ in the sense of § 3.2, and the second component ρ_{g} is a character of Γ_{g} as in (3.2.5). Also, given a virtual graded G-module W of the form

$$W = \bigoplus_{r \bmod 2m} \bigoplus_{D \equiv r^2 \bmod 4m} W_{r, \frac{D}{4m}} \tag{3.3.4}$$

we define the associated *McKay–Thompson series* ϕ_{g}^W, for each $\mathsf{g} \in \mathsf{G}$, by requiring that

$$\phi_{\mathsf{g}}^W(\tau, z) = \sum_{r \bmod 2m} h_{\mathsf{g},r}^W(\tau)\theta_{m,r}(\tau, z) \tag{3.3.5}$$

(cf. (3.2.10)), where $h_{\mathsf{g},r}^W$ is defined, for each integer r modulo $2m$, by setting

$$h_{\mathsf{g},r}^W(\tau) := \sum_{D \equiv r^2 \bmod 4m} \operatorname{tr}\left(g|W_{r, \frac{D}{4m}}\right) q^{-\frac{D}{4m}}. \tag{3.3.6}$$

As in [8] we use the term virtual G-module to refer to an integer combination of irreducible ordinary characters of G, and use the term virtual graded G-module to refer to an indexed collection of such things. See § 3.3 of [8] for more detail.

With this setup in place we say that W as in (3.3.4) is c-*optimal (mock Jacobi)* of weight k and index m for (G, α) if

$$\phi_{\mathsf{g}}^W \in \mathbb{J}_{k,m}(\Gamma_{\mathsf{g}}, \rho_{\mathsf{g}}), \tag{3.3.7}$$

and if ϕ_{g}^{W} also satisfies the c-optimality conditions (3.3.1)-(3.3.2), for every $\mathsf{g} \in \mathsf{G}$, and we say that W as in (3.3.4) *optimal* for (G, α) if it is c-optimal for some c.

We are interested in classifying the optimal virtual graded modules of a given weight and index for a pair (G, α), where G is a finite group and α is as in (3.3.3). With this in mind we let $\mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ denote the set of optimal virtual graded modules of weight k and index m for (G, α), and for each $\mathrm{c} \in \mathbb{Z}$ let $\mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)_{\mathrm{c}}$ denote the subset of modules that are c-optimal. As in [8] we note that $\mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ and $\mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)_0$ are naturally free abelian groups (cf. (3.3.11) of [8]), and

$$\mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha) = \sum_{\mathrm{c} \in \mathbb{Z}} \mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)_{\mathrm{c}} \tag{3.3.8}$$

is a decomposition of the former into modules for the latter. We also write $\mathcal{L}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ for the lattice structure on $\mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)_0$ that we obtain by embedding it in

$$L_{k,m}(\mathsf{G}) := S_{k,m}(\#\mathsf{G})_{\mathbb{Z}} \otimes_{\mathbb{Z}} R(\mathsf{G}) \tag{3.3.9}$$

in the natural way. (Here $R(\mathsf{G})$ denotes the Grothendieck group of the category of finitely generated G-modules, which we identify with the free $\mathbb{Z}$-module generated by the irreducible ordinary characters of G. We refer to (3.3.12-3.3.14) of [8] for the lattice structure on $L_{k,m}(\mathsf{G})$, and refer to (3.3.15-3.3.19) of [8] for the map that identifies $\mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)_0$ as a sublattice.)

Define $\mathrm{c}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ to be the minimal positive integer for which a c-optimal virtual graded module of weight k and index m for (G, α) exists,

$$\mathrm{c}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha) := \min \left\{ \mathrm{c} \in \mathbb{Z}^{+} \mid \mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)_{\mathrm{c}} \neq \emptyset \right\}. \tag{3.3.10}$$

Then we have the following counterpart to Proposition 3.3.1 of [8], by exactly the same argument.

Proposition 3.3.1. *The sets $\mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ and $\mathcal{L}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ are naturally free abelian groups of finite rank. If $\mathrm{c} \equiv 0 \bmod \mathrm{c}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ then $\mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)_{\mathrm{c}}$ is naturally a $\mathcal{L}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$-torsor. If $\mathrm{c} \not\equiv 0 \bmod \mathrm{c}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ then $\mathcal{W}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)_{\mathrm{c}}$ is empty.*

Motivated by Proposition 3.3.1 we refer to the determination of $\mathrm{c}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ and $\mathcal{L}_{k,m}^{\mathrm{opt}}(\mathsf{G}, \alpha)$, for a fixed choice of G and α, as the *classification problem* for optimal virtual graded modules of weight k and index m for (G, α).

To conclude this section we define the *default* choice for α as in (3.3.3) to be the assignment $\alpha(\mathsf{g}) = (\Gamma_{\mathsf{g}}, \rho_{\mathsf{g}})$ for which

$$\Gamma_{\mathsf{g}} = \Gamma_0^{\mathrm{J}}(o(\mathsf{g})) \tag{3.3.11}$$

and ρ_{g} is trivial, for each $\mathsf{g} \in \mathsf{G}$. We indicate this default choice of α by suppressing it from notation, writing $\mathcal{W}^{\mathrm{opt}}_{k,m}(\mathsf{G})$ for $\mathcal{W}^{\mathrm{opt}}_{k,m}(\mathsf{G}, \alpha)$, and writing $\mathcal{L}^{\mathrm{opt}}_{k,m}(\mathsf{G})$ for $\mathcal{L}^{\mathrm{opt}}_{k,m}(\mathsf{G}, \alpha)$, &c., when α is as in (3.3.11).

3.4. Our main focus

Having discussed mock Jacobi forms with arbitrary integer weight and positive integer index in §§ 3.2-3.3, we now review some of the special features that arise when the weight is 2 and the index is 1, since this is the situation upon which we mostly focus in the remainder of this work.

To begin we note that for a mock Jacobi form ϕ with index $m = 1$ we may unambiguously write $C_\phi(D)$ for the Fourier coefficient $C_\phi(D, r)$ (cf. (3.2.13)), because the latter depends only on D and the parity of r (cf. (3.2.7)), and the relation $D \equiv r^2 \bmod 4$ forces the parity of D and r to match. With this convention we have a Fourier expansion

$$\phi(\tau, z) = \sum_{n,s \in \mathbb{Z}} C_\phi(s^2 - 4n) q^n y^s \tag{3.4.1}$$

when ϕ has index 1 (cf. (3.1.6), (3.1.9)). (More generally we may drop the r from $C_\phi(D, r)$ whenever the index m of ϕ is not composite, for in this case $r^2 \equiv s^2 \bmod 4m$ implies $r \equiv \pm s \bmod 2m$.)

By a similar token, for W as in (3.3.4) with $m = 1$, there is no loss in considering $\check{W} = \bigoplus_D \check{W}_D$ in place of W, where $\check{W}_D := W_{D, \frac{D}{4}}$. As in [8] we adopt this simplification, and also drop the accents from $\check{W}$ and $\check{W}_D$ henceforth. Thus, from now on we write simply

$$W = \bigoplus_D W_D \tag{3.4.2}$$

for the grading of a virtual graded G-module W as in (3.3.4) when $m = 1$, where each W_D in (3.4.2) is taken to be $W_{D, \frac{D}{4}}$ in (3.3.4). With this convention the McKay–Thompson series ϕ^W_{g} associated to W (see (3.3.5)-(3.3.6)) may be defined by setting

$$\phi^W_{\mathsf{g}}(\tau, z) := \sum_{n,s \in \mathbb{Z}} \mathrm{tr}(\mathsf{g} | W_{s^2 - 4n}) q^n y^s \tag{3.4.3}$$

(cf. (3.4.1)).

The main significance for us of the weight $k = 2$ is that it plays host to the generalized Hurwitz class number generating functions $\mathscr{H}^{\mathrm{Hur}}_N$ of (3.1.6), and the Cohen–Eisenstein series $\mathscr{H}^{\mathrm{Coh}}_N$ of (3.1.9). We put this precisely as follows.

Proposition 3.4.1. *For N a positive integer the function $\mathscr{H}^{\mathrm{Hur}}_N$ belongs to $\mathbb{J}_{2,1}(\Gamma^{\mathrm{J}}_0(N))$. For N prime the function $\mathscr{H}^{\mathrm{Coh}}_N$ belongs to $J_{2,1}(\Gamma^{\mathrm{J}}_0(N))$.*

See Proposition 3.4.1 in [8] for the proof of Proposition 3.4.1 above.

To construct optimal mock Jacobi forms we take a positive integer N and set

$$\mathscr{H}_N^{\mathrm{Rad}} := \frac{12}{\phi(N)} \sum_{M|N} \mu\left(\frac{N}{M}\right) \frac{M}{\iota(M)} \mathscr{H}_M^{\mathrm{Hur}}, \tag{3.4.4}$$

where $\phi(N)$ denotes the Euler totient function, $\mu(N)$ is the Möbius function, and $\iota(N)$ is as in (3.1.5). Then, as explained in § 3.4 of [8], the $\mathscr{H}_N^{\mathrm{Rad}}$ are optimal.

Proposition 3.4.2. *For any positive integer N the function $\mathscr{H}_N^{\mathrm{Rad}}$ is a 1-optimal element of $\mathbb{J}_{2,1}(\Gamma_0^{\mathrm{J}}(N))$.*

Note that the Fourier coefficients of $\mathscr{H}_N^{\mathrm{Rad}}$ are generally not integers. For example, $\mathrm{c} = 2$ is the smallest positive integer for which $\mathrm{c}\mathscr{H}_5^{\mathrm{Rad}}$ has integer coefficients. (Cf. the proof of Theorem 4.2.1.) Also, $\mathbb{J}_{2,1}(\Gamma_0^{\mathrm{J}}(1))$ is spanned by $\mathscr{H}_1^{\mathrm{Rad}} = 12\mathscr{H}^{\mathrm{Hur}}$ according to Proposition 3.4.4 of [8]. Thus for any finite group G, and for any choice of α in (3.3.3) such that $\Gamma_{\mathsf{e}} = \Gamma_0^{\mathrm{J}}(1)$ and ρ_{e} is trivial, a c-optimal virtual graded module $W = \bigoplus_D W_D$ of weight 2 and index 1 for (G, α) (cf. (3.4.2)) satisfies

$$\phi_{\mathsf{e}}^W(\tau, z) = \sum_{n,s\in\mathbb{Z}} \dim(W_{s^2-4n}) q^n y^s = \mathrm{c}\mathscr{H}_1^{\mathrm{Rad}}(\tau, z) = 12\mathrm{c}\mathscr{H}^{\mathrm{Hur}}(\tau, z) \tag{3.4.5}$$

(cf. (3.4.3)), where e denotes the identity element of G.

4. Results

In this section we present concrete examples of the classification problem formulated in § 3.3 (cf. Proposition 3.3.1), in the special setting described in § 3.4, and also explore some arithmetic consequences. Specifically, we first explain the connection between class numbers and the $\ell = 3$ case of umbral moonshine in § 4.1. Then we use this as a springboard to classify optimal modules for the smallest sporadic Mathieu group M_{11}, for two different assignments of characters, in § 4.2. Finally, in § 4.3 we demonstrate a connection between the M_{11}-modules of § 4.2 and the congruent number problem from antiquity.

4.1. Umbral moonshine

In this section we use the proof of the $\ell = 3$ case of umbral moonshine (see [4–6] and [17]) to establish the existence of a 2-optimal module for the unique non-trivial double cover $2.\mathsf{M}_{12}$ of the Mathieu group M_{12} (cf. [10]), and a particular assignment $\alpha : [\mathsf{g}] \to (\Gamma_{\mathsf{g}}, \rho_{\mathsf{g}})$ (cf. (3.3.3)).

To get started let us first formulate the relevant function from umbral moonshine. For this we recall the Jacobi theta function ϑ_1 (3.2.20) and Dedekind eta function η (3.2.21)

from § 3.2, and define three more Jacobi theta functions, $\vartheta_j(\tau, z)$ for $j \in \{2, 3, 4\}$, by setting

$$\begin{aligned} \vartheta_2(\tau, z) &:= \theta_{1,1}(\tfrac{1}{2}\tau, \tfrac{1}{2}z), \\ \vartheta_3(\tau, z) &:= \theta_{1,0}(\tfrac{1}{2}\tau, \tfrac{1}{2}z), \\ \vartheta_4(\tau, z) &:= \theta_{1,0}(\tfrac{1}{2}\tau, \tfrac{1}{2}(z + \tfrac{1}{2})), \end{aligned} \tag{4.1.1}$$

for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$, where $\theta_{1,0}$ and $\theta_{1,1}$ are as in (3.2.8). We use these functions to specify a weakly holomorphic Jacobi form $Z^{(3)}$ of weight 0 and index 2 for $\Gamma_0^{\mathrm{J}}(1)$ (cf. § 2.5 of [4]) by defining

$$Z^{(3)}(\tau, z) := 4\left(\frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2}\frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2} + \frac{\vartheta_3(\tau, z)^2}{\vartheta_3(\tau, 0)^2}\frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2} + \frac{\vartheta_4(\tau, z)^2}{\vartheta_4(\tau, 0)^2}\frac{\vartheta_2(\tau, z)^2}{\vartheta_2(\tau, 0)^2}\right). \tag{4.1.2}$$

Then the meromorphic Jacobi form of weight 1 and index 3 attached to the $\ell = 3$ case of umbral moonshine (cf. § 4.3 of [5]), denoted $\psi^{(3)}$, may be defined by the formula

$$\psi^{(3)}(\tau, z) := i\frac{\vartheta_1(\tau, 2z)\eta(\tau)^3}{\vartheta_1(\tau, z)^2} Z^{(3)}(\tau, z). \tag{4.1.3}$$

(We remark that the factor in front of $Z^{(3)}$ in (4.1.3) is a meromorphic Jacobi form of weight 1 and index 1 for $\Gamma_0^{\mathrm{J}}(1)$, with a simple pole at each lattice point $z \in \mathbb{Z}\tau + \mathbb{Z}$.)

The connection between (4.1.3) and umbral moonshine is that the finite part of $\psi^{(3)}$, denoted $\psi^{(3),F}$ (cf. (3.2.22)), satisfies

$$\psi^{(3),F}(\tau, z) = \sum_{r \bmod 6} H_r^{(3)}(\tau)\theta_{3,r}(\tau, z), \tag{4.1.4}$$

where $H^{(3)} = (H_r^{(3)})$ is the McKay–Thompson series attached to the identity element of the umbral group at $\ell = 3$, which is none other than $2.\mathsf{M}_{12}$.

We now consider the quotient of $Z^{(3)}$ (4.1.2) by the weakly holomorphic Jacobi form $\vartheta_1^2\eta^{-6}$ (cf. (3.2.20)-(3.2.21)), the latter having weight -2 and index 1, and a double zero at each lattice point $z \in \mathbb{Z}\tau + \mathbb{Z}$, and compare to the meromorphic Jacobi form $\psi^{(1)}$ of (3.2.19). After an elementary manipulation of the expressions involved we arrive at a precise coincidence,

$$\psi^{(1)}(\tau, z) = -i\frac{\eta(\tau)^3}{\vartheta_1(\tau, 2z)}\psi^{(3)}(\tau, z) = \frac{\eta(\tau)^6}{\theta_1(\tau, z)^2} Z^{(3)}(\tau, z). \tag{4.1.5}$$

That is to say, we recover (a rescaling of) the Hurwitz class number generating function $\mathscr{H}^{\mathrm{Hur}} = \mathscr{H}_1^{\mathrm{Hur}}$ (cf. (3.1.6)) by taking the finite part (cf. (3.2.27)) of a suitable multiple (4.1.5) of the meromorphic Jacobi form (4.1.3) whose finite part (4.1.4) captures umbral moonshine at $\ell = 3$.

Equipped with (4.1.3)-(4.1.5) we now attach a meromorphic Jacobi form of weight 2 and index 1 (with level and generally with non-trivial character) to each $\mathsf{g} \in 2.\mathsf{M}_{12}$ by setting

$$\psi_{\mathsf{g}}^{(1)}(\tau, z) := -i\frac{\eta(\tau)^3}{\vartheta_1(\tau, 2z)}\psi_{\mathsf{g}}^{(3)}(\tau, z), \tag{4.1.6}$$

where $\psi_{\mathsf{g}}^{(3)}(\tau, z)$ is the meromorphic Jacobi form of weight 1 and index 3 (with level and generally with non-trivial character) attached to $\mathsf{g} \in 2.\mathsf{M}_{12}$ by umbral moonshine. We define $\mathscr{H}_{\mathsf{g}}^{(2.\mathsf{M}_{12},\alpha)}$ for $\mathsf{g} \in 2.\mathsf{M}_{12}$ to be the finite part of $\psi_{\mathsf{g}}^{(1)}$ in (4.1.6), so that

$$\mathscr{H}_{\mathsf{g}}^{(2.\mathsf{M}_{12},\alpha)} := \psi_{\mathsf{g}}^{(1),F}. \tag{4.1.7}$$

To make (4.1.6) more explicit let $\overline{\chi}_{12}$ be the sum of the first two irreducible characters of $2.\mathsf{M}_{12}$, according to the ordering in [4,5] (which is the same as the ordering given by [20], for example), and let χ_{12} denote the unique faithful irreducible character of $2.\mathsf{M}_{12}$ of degree 12. Then we have

$$\psi_{\mathsf{g}}^{(3)}(\tau, z) = -\chi_{12}(\mathsf{g})\mu^{(3),0}(\tau, z) - \overline{\chi}_{12}(\mathsf{g})\mu^{(3),1}(\tau, z) + \sum_{r \bmod 6} H_{\mathsf{g},r}^{(3)}(\tau)\theta_{3,r}(\tau, z) \tag{4.1.8}$$

where $\mu^{(3),k}$ and the $H_{\mathsf{g},r}^{(3)}$ are as specified in § 4.3 of [5] (cf. also § B.3.2 of [12]).

Correspondingly, we now define a mock Jacobi form $\mathscr{H}_{\mathsf{g}}^{(2.\mathsf{M}_{12},\alpha)}$ for each $\mathsf{g} \in 2.\mathsf{M}_{12}$ by requiring that

$$\psi_{\mathsf{g}}^{(1)}(\tau, z) = -\chi_{12}(\mathsf{g})\mu^{(1),0}(\tau, z) - \overline{\chi}_{12}(\mathsf{g})\mu^{(1),1}(\tau, z) + \mathscr{H}_{\mathsf{g}}^{(2.\mathsf{M}_{12},\alpha)}(\tau, z) \tag{4.1.9}$$

where $\mu^{(1),k}(\tau, z) := \frac{1}{2}\left(\mathrm{Av}^{(1)}(\frac{y}{(1-y)^2}) + (-1)^k\,\mathrm{Av}^{(1)}(\frac{-y}{(1+y)^2})\right)$ (cf. (3.2.25)-(3.2.26)).

It follows from the construction (4.1.6)-(4.1.9), and the $\ell = 3$ case of the main result of [12], that the $\mathscr{H}_{\mathsf{g}}^{(2.\mathsf{M}_{12},\alpha)}$ arise as the graded trace functions attached to a virtual graded $2.\mathsf{M}_{12}$-module.

Proposition 4.1.1. *There exists a virtual graded $2.\mathsf{M}_{12}$-module $W = \bigoplus_D W_D$ such that $\phi_{\mathsf{g}}^W = \mathscr{H}_{\mathsf{g}}^{(2.\mathsf{M}_{12},\alpha)}$ for $\mathsf{g} \in 2.\mathsf{M}_{12}$.*

It also follows from (4.1.6)-(4.1.9) that the $\mathscr{H}_{\mathsf{g}}^{(2.\mathsf{M}_{12},\alpha)}$ are optimal, and have the same characters as those of the meromorphic Jacobi forms $\psi_{\mathsf{g}}^{(3)}$. We have $\mathscr{H}_{\mathsf{g}}^{(2.\mathsf{M}_{12},\alpha)} = -2 + O(q)$ as $\Im(\tau) \to \infty$ for all $\mathsf{g} \in 2.\mathsf{M}_{12}$ so Proposition 4.1.1 implies that $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G}, \alpha) \leq 2$ for $\mathsf{G} = 2.\mathsf{M}_{12}$ and α as above. On the other hand the inequality $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G}, \alpha) \geq 2$ holds because the function $\mathscr{H}_{\mathsf{g}}^{(2.\mathsf{M}_{12},\alpha)}$ belongs to $\mathbb{J}_{2,1}(\Gamma_0^{\mathrm{J}}(5))$ in case $o(\mathsf{g}) = 5$, the unique-up-to-scale optimal form in this space is $\mathscr{H}_5^{\mathrm{Rad}}$, and $\mathrm{c} = 2$ is the smallest positive integer such that $\mathrm{c}\mathscr{H}_5^{\mathrm{Rad}}$ has integer coefficients. Thus we have proved the following result.

Theorem 4.1.2. *For $\mathsf{G} = 2.\mathsf{M}_{12}$ and α as above we have $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G}, \alpha) = 2$.*

4.2. Modules

Here we consider the smallest sporadic simple group M_{11}, which has order $7920 = 2^4.3^2.5.11$ (cf. [10]). Our first objective is the following classification (cf. Proposition 3.3.1) of optimal modules for M_{11} (i.e. optimal modules for $(\mathsf{M}_{11}, \alpha)$ with the default choice (3.3.11) of α). To formulate it let φ_{11} denote the unique element of $S_{2,1}(\Gamma_0^{\mathrm{J}}(11))$ such that

$$\varphi_{11}(\tau, z) = (y^{-1} - 1 + y)q + O(q^2) \tag{4.2.1}$$

as $\Im(\tau) \to \infty$, let χ_i denote the i-th irreducible ordinary character of M_{11} (as displayed in Table 1), and recall the definition (3.3.9) of the lattice $L_{2,1}(\mathsf{G})$.

Theorem 4.2.1. *For $\mathsf{G} = \mathsf{M}_{11}$ we have $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G}) = 2$, and $\mathcal{L}_{2,1}^{\mathrm{opt}}(\mathsf{G})$ is the rank 2 sublattice of $L_{2,1}(\mathsf{G})$ generated by*

$$\begin{aligned} &\varphi_{11} \otimes (\chi_1 - \chi_2 - \chi_3 - \chi_4 - \chi_6 + \chi_9), \\ &\varphi_{11} \otimes (\chi_1 - \chi_2 - \chi_3 - \chi_4 - \chi_7 + \chi_9). \end{aligned} \tag{4.2.2}$$

Remark 4.2.2. Let N_{11} denote the norm of φ_{11} with respect to the Petersson inner product. Then from Theorem 4.2.1 we see that $\mathcal{L}_{2,1}^{\mathrm{opt}}(\mathsf{G})$ is a copy of the lattice structure on $\mathbb{Z}^2$ for which the associated quadratic form is $Q(x_1, x_2) = N_{11}^2(3x_1^2 + 5x_1x_2 + 3x_2^2)$.

Our main task in proving Theorem 4.2.1 is establishing that a 2-optimal module for M_{11} exists. To this end we define a holomorphic mock Jacobi form $\mathscr{H}_{\mathsf{g}}^{\mathsf{M}_{11}}$ for each $\mathsf{g} \in \mathsf{M}_{11}$ by setting

$$\mathscr{H}_{\mathsf{g}}^{\mathsf{M}_{11}} := \begin{cases} 2\mathscr{H}_{o(\mathsf{g})}^{\mathrm{Rad}} & \text{for } o(\mathsf{g}) < 11, \\ 2\mathscr{H}_{11}^{\mathrm{Rad}} - \frac{11}{5}\varphi_{11} & \text{for } o(\mathsf{g}) = 11. \end{cases} \tag{4.2.3}$$

Here $\mathscr{H}_N^{\mathrm{Rad}}$ is as in (3.4.4) and φ_{11} is as in (4.2.1). The first few Fourier coefficients of each of the $\mathscr{H}_{\mathsf{g}}^{\mathsf{M}_{11}}$ are given in Table 2.

The $\mathscr{H}_N^{\mathrm{Rad}}$ are optimal for all N according to Proposition 3.4.2, so the $\mathscr{H}_{\mathsf{g}}^{\mathsf{M}_{11}}$ (4.2.3) are all optimal by construction. By the 1-optimality of $\mathscr{H}_N^{\mathrm{Rad}}$ (see Proposition 3.4.2) we have $\mathscr{H}_{\mathsf{g}}^{\mathsf{M}_{11}}(\tau, z) = -2 + O(q)$ for all $\mathsf{g} \in \mathsf{M}_{11}$. In light of this the next result amounts to the statement that $\mathcal{W}_{2,1}^{\mathrm{opt}}(\mathsf{G})_2$ is not empty when $\mathsf{G} = \mathsf{M}_{11}$.

Proposition 4.2.3. *There exists a virtual graded M_{11}-module $W = \bigoplus_D W_D$ such that $\phi_{\mathsf{g}}^W = \mathscr{H}_{\mathsf{g}}^{\mathsf{M}_{11}}$ for $\mathsf{g} \in \mathsf{M}_{11}$.*

Proof. Recall from § 1.2 that we say that a subgroup M_{11} in $2.\mathsf{M}_{12}$ is intransitive if its image under the composition (1.2.2) fixes a point. We verify the existence of the M_{11}-module of the statement of the proposition by restricting the $2.\mathsf{M}_{12}$-module structure

of Proposition 4.1.1 to an intransitive copy of M_{11} in $2.\mathsf{M}_{12}$. We require to determine how the assignment $\alpha : [\mathsf{g}] \to \rho_{\mathsf{g}}$ of characters to $2.\mathsf{M}_{12}$ restricts to such a subgroup. These characters are described explicitly in § 4.8 of [5]. From this description we check that ρ_{g} is trivial for every g in an intransitive copy of M_{11}. So W as in Proposition 4.1.1 restricts along an intransitive copy of M_{11} to an M_{11}-module with the same characters as in Proposition 4.2.3. This completes the proof. □

Remark 4.2.4. Since the coefficients of the $\mathscr{H}_{\mathsf{g}}^{\mathsf{M}_{11}}$ are rational numbers by construction, they must in fact be rational integers for all $\mathsf{g} \in \mathsf{M}_{11}$, and in particular for $o(\mathsf{g}) = 11$, in order for Proposition 4.2.3 to be true. This particular consequence of Proposition 4.2.3 may be compared to the $N = 11$ case of Theorem 4.1.1 in [8]. (See also Corollary 4.2.2 in [8].)

The character table of M_{11} is given in Table 1. For χ an irreducible character of M_{11} define the multiplicity generating function

$$\mathscr{H}_{\chi}^{\mathsf{M}_{11}}(\tau) := \sum_{n,s\in\mathbb{Z}} m_{\chi}(W_{s^2-4n})q^n y^s \tag{4.2.4}$$

where W_D is as in Proposition 4.2.3, and $m_{\chi}(W_D)$ is the multiplicity of χ in the M_{11}-module W_D. The first few Fourier coefficients of each of the resulting holomorphic mock Jacobi forms $\mathscr{H}_{\chi}^{\mathsf{M}_{11}}$ are given in Table 4.

Proof of Theorem 4.2.1. We have $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G}) \leq 2$ according to Proposition 4.2.3. To see that $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G}) \geq 2$ we note, for example, that the space of optimal holomorphic mock Jacobi forms of weight 2 and index 1 for $\Gamma_0^{\mathrm{J}}(5)$ is one-dimensional, spanned by $\mathscr{H}_5^{\mathrm{Rad}}(\tau, z) = -1 + O(q)$ (cf. (3.4.4)), and $\mathrm{c} = 2$ is the smallest positive integer such that $\mathrm{c}\mathscr{H}_5^{\mathrm{Rad}}$ has integer coefficients. So $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G}) = 2$.

It remains to compute $\mathcal{L}_{2,1}^{\mathrm{opt}}(\mathsf{G})$. For this we note that if $W \in \mathcal{W}_{2,1}^{\mathrm{opt}}(\mathsf{G})_0$ then, being optimal and having vanishing constant term, ϕ_{g}^W must belong to $S_{2,1}(\Gamma_0^{\mathrm{J}}(o(\mathsf{g})))$ for every $\mathsf{g} \in \mathsf{G}$. We have $S_{2,1}(\Gamma_0^{\mathrm{J}}(o(\mathsf{g}))) = \{0\}$ for $\mathsf{g} \in \mathsf{M}_{11}$ except when $o(\mathsf{g}) = 11$, and $S_{2,1}(\Gamma_0^{\mathrm{J}}(11))_{\mathbb{Z}}$ is generated by φ_{11} (as in (4.2.1)), so $\mathcal{W}_{2,1}^{\mathrm{opt}}(\mathsf{G})_0$ is identified with the sublattice

$$\mathbb{Z}\varphi_{11} \otimes_{\mathbb{Z}} R(\mathsf{G})_{(11)} \tag{4.2.5}$$

of $L_{2,1}(\mathsf{G})$, where $R(\mathsf{G})_{(11)}$ denotes the $\mathbb{Z}$-module of virtual characters of M_{11} that are supported on elements of order 11.

To compute $R(\mathsf{G})_{(11)}$ let δ_{11} be the class function on G such that $\delta_{11}(\mathsf{g}) = 1$ when $o(\mathsf{g}) = 11$, and $\delta_{11}(\mathsf{g}) = 0$ when $o(\mathsf{g}) \neq 11$. Then by applying Thompson's reformulation of Brauer's theorem on virtual characters (see e.g. [26]) to the character table of G (see Table 1) we may deduce that if x is a class function in $R(\mathsf{G})_{(11)}$ with rational values then x is an integer multiple of $11\delta_{11}$. To understand the rest of $R(\mathsf{G})_{(11)}$ we observe that

χ_6 and χ_7 (as in Table 1) are the only characters of G that have non-rational entries on elements of order 11. Also, $\chi_6 - \chi_7$ vanishes away from elements of order 11 so is contained in $R(\mathsf{G})_{(11)}$. Thus $R(\mathsf{G})_{(11)}$ is composed of the (possibly rational) combinations of $11\delta_{11}$ and $\chi_6 - \chi_7$ that are integer combinations of the irreducible characters χ_j. A computation with Table 1 reveals that

$$11\delta_{11} = 2\chi_1 - 2\chi_2 - 2\chi_3 - 2\chi_4 - \chi_6 - \chi_7 + 2\chi_9, \tag{4.2.6}$$

so $R(\mathsf{G})_{(11)}$ is generated, as a $\mathbb{Z}$-module, by

$$\begin{aligned} \frac{11}{2}\delta_{11} - \frac{1}{2}\chi_6 + \frac{1}{2}\chi_7 &= \chi_1 - \chi_2 - \chi_3 - \chi_4 - \chi_6 + \chi_9, \\ \frac{11}{2}\delta_{11} + \frac{1}{2}\chi_6 - \frac{1}{2}\chi_7 &= \chi_1 - \chi_2 - \chi_3 - \chi_4 - \chi_7 + \chi_9. \end{aligned} \tag{4.2.7}$$

Thus we have confirmed the description (4.2.2) of $\mathcal{L}^{\mathrm{opt}}_{2,1}(\mathsf{G})$ that appears in the statement of the theorem. This completes the proof. □

Our next objective is a classification of optimal M_{11}-modules for a particular non-trivial assignment of characters. For this define $\alpha : [\mathsf{g}] \mapsto (\Gamma_{\mathsf{g}}, \rho_{\mathsf{g}})$ by setting $\Gamma_{\mathsf{g}} = \Gamma^{\mathrm{J}}_0(o(\mathsf{g}))$, and

$$\rho_{\mathsf{g}}(\gamma, v, \varsigma) := \begin{cases} 1 & \text{if } o(\mathsf{g}) \notin \{4, 8\}, \\ \mathrm{e}\left(-\frac{2cd}{o(\mathsf{g})^2}\right) & \text{if } o(\mathsf{g}) \in \{4, 8\}, \end{cases} \tag{4.2.8}$$

for $(\gamma, (\lambda, \mu), \varsigma) \in \Gamma^{\mathrm{J}}_0(o(\mathsf{g}))$ (cf. (3.2.2)), when $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Also let $\varphi_{8|4}$ denote the unique element of $S_{2,1}(\Gamma^{\mathrm{J}}_0(32))$ such that

$$\varphi_{8|4}(\tau, z) = (y^{-1} + y)q + O(q^2) \tag{4.2.9}$$

as $\Im(\tau) \to \infty$. Then actually $\varphi_{8|4}$ belongs to $S_{2,1}(\Gamma^{\mathrm{J}}_0(8), \rho_{\mathsf{g}})$ for $\mathsf{g} \in \mathsf{M}_{11}$ an element of order 8 (cf. (4.2.8)), and the statement of the classification of optimal modules for (G, α) is as follows.

Theorem 4.2.5. *For $\mathsf{G} = \mathsf{M}_{11}$ and α as in (4.2.8) we have $\mathrm{c}^{\mathrm{opt}}_{2,1}(\mathsf{G}, \alpha) = 2$, and $\mathcal{L}^{\mathrm{opt}}_{2,1}(\mathsf{G}, \alpha)$ is the sublattice of $L_{2,1}(\mathsf{G})$ generated by*

$$\begin{aligned} &\varphi_{8|4} \otimes (\chi_3 - \chi_4), \\ &\varphi_{8|4} \otimes (\chi_1 - \chi_5 - \chi_9 + \chi_{10}), \\ &\varphi_{11} \otimes (\chi_1 - \chi_2 - \chi_3 - \chi_4 - \chi_6 + \chi_9), \\ &\varphi_{11} \otimes (\chi_1 - \chi_2 - \chi_3 - \chi_4 - \chi_7 + \chi_9). \end{aligned} \tag{4.2.10}$$

Remark 4.2.6. Let $N_{8|4}$ denote the norm of $\varphi_{8|4}$ with respect to the Petersson inner product, and let N_{11} be as in Remark 4.2.2. Then from Theorem 4.2.5 we see that $\mathcal{L}_{2,1}^{\mathrm{opt}}(\mathsf{G},\alpha)$ is a copy of the lattice structure on $\mathbb{Z}^4$ for which the associated quadratic form is $Q(x_1,x_2,x_3,x_4) = N_{8|4}^2(x_1^2+2x_2^2) + N_{11}^2(3x_3^2+5x_3x_4+3x_4^2)$.

To compute $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G},\alpha)$ we first seek a suitable counterpart to Proposition 4.2.3. For this we define

$$\mathscr{H}_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)} := \begin{cases} 2\mathscr{H}_{o(\mathsf{g})}^{\mathrm{Rad}} & \text{for } o(\mathsf{g}) \notin \{4,8,11\}, \\ -2\theta_{1,0}^0\left(\theta_{4,0}^0-\theta_{4,4}^0\right)^2\theta_{1,0} & \text{for } o(\mathsf{g})=4, \\ -2\theta_{1,0}^0\left(\theta_{16,0}^0-\theta_{16,16}^0\right)^2\theta_{1,0}-4\varphi_{8|4} & \text{for } o(\mathsf{g})=8, \\ 2\mathscr{H}_{11}^{\mathrm{Rad}}-\frac{11}{5}\varphi_{11} & \text{for } o(\mathsf{g})=11, \end{cases} \tag{4.2.11}$$

where $\mathscr{H}_N^{\mathrm{Rad}}$ is as in (3.4.4), and $\theta_{m,r}^0(\tau) := \theta_{m,r}(\tau,0)$ for $\theta_{m,r}$ as in (3.2.8), and $\varphi_{8|4}$ and φ_{11} are as in (4.2.9) and (4.2.1), respectively. The first few Fourier coefficients of each of the $\mathscr{H}_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}$ are given in Table 3.

We may check directly that the $\mathscr{H}_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}$ of (4.2.11) are all optimal. So the next result states that $\mathcal{W}_{2,1}^{\mathrm{opt}}(\mathsf{G},\alpha)_2$ is not empty, for $\mathsf{G}=\mathsf{M}_{11}$ and α as in (4.2.8).

Proposition 4.2.7. *There exists a virtual graded* M_{11}*-module* $W = \bigoplus_D W_D$ *such that* $\phi_{\mathsf{g}}^W = \mathscr{H}_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}$ *for each* $\mathsf{g}\in\mathsf{M}_{11}$.

Proof. As in the proof of Proposition 4.2.3 we verify the existence of the M_{11}-module in question by restricting the $2.\mathsf{M}_{12}$-module structure of Proposition 4.1.1 to a suitable subgroup $\mathsf{M}_{11} < 2.\mathsf{M}_{12}$, but in this case we choose a copy of M_{11} that maps to a transitive subgroup of S_{12} under the composition (1.2.2). (Such a copy of M_{11} is called transitive in § 1.2.) Again using the explicit description of the characters ρ_{g} for $\mathsf{g}\in 2.\mathsf{M}_{12}$ that appears in § 4.8 of [5] we find that ρ_{g} is trivial for every g in a transitive copy of M_{11} except for when $o(\mathsf{g})\in\{4,8\}$. Moreover, the ρ_{g} for g in a transitive copy agree with the specification (4.2.8). So W as in Proposition 4.1.1 restricts along a transitive copy of M_{11} to an M_{11}-module with the same characters as in Proposition 4.2.7. Inspecting further we find that the trace functions obtained by restriction agree with those in (4.2.11) except when $o(\mathsf{g})=8$, in which case the difference is $4\varphi_{8|4}$ (cf. (4.2.9)). We finish the proof of Proposition 4.2.7 by noting that the coefficients of $\varphi_{8|4}$ are rational integers, and observing that

$$\chi_1 - \chi_5 - \chi_9 + \chi_{10} \tag{4.2.12}$$

is a virtual character that takes the value 4 on elements of M_{11} of order 8, and vanishes on all other elements. (Cf. Table 1.) □

For χ an irreducible character of M_{11} define the multiplicity generating function $\mathscr{H}_{\chi}^{(\mathsf{M}_{11},\alpha)}$ by the right-hand side of (4.2.4) but with W_D as in Proposition 4.2.7. The

first few Fourier coefficients of each of the resulting holomorphic mock Jacobi forms $\mathscr{H}_\chi^{(\mathsf{M}_{11},\alpha)}$ are given in Table 5.

Proof of Theorem 4.2.5. Comparing (4.2.11) with (4.2.3) we see that $\mathscr{H}_{\mathsf{g}}^{\mathsf{M}_{11}} = \mathscr{H}_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}$ for $o(\mathsf{g}) \notin \{4,8\}$. We have already seen that the $\mathscr{H}_{\mathsf{g}}^{\mathsf{M}_{11}}$ are 2-optimal, so the $\mathscr{H}_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}$ are 2-optimal for $o(\mathsf{g}) \notin \{4,8\}$. The 2-optimality of $\mathscr{H}_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}$ for $o(\mathsf{g}) \notin \{4,8\}$ follows from an explicit calculation using the known modular transformation properties of the theta series $\theta_{m,r}$ (cf. (3.2.8)). We also check directly that $\mathscr{H}_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}$ transforms according to the characters specified in (4.2.8) for $o(\mathsf{g}) \in \{4,8\}$. So we have $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G},\alpha) \leq 2$ according to Proposition 4.2.7. The inequality $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G},\alpha) \geq 2$ holds for the same reason we gave that $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G}) \geq 2$ in the proof of Theorem 4.2.1, so $\mathrm{c}_{2,1}^{\mathrm{opt}}(\mathsf{G},\alpha) = 2$.

The computation of $\mathcal{L}_{2,1}^{\mathrm{opt}}(\mathsf{G},\alpha)$ is similar to the computation of $\mathcal{L}_{2,1}^{\mathrm{opt}}(\mathsf{G})$ that appears in the proof of Theorem 4.2.1, with the main difference being that $S_{2,1}(\Gamma_{\mathsf{g}},\rho_{\mathsf{g}})_{\mathbb{Z}}$ is now non-zero also when $o(\mathsf{g}) = 8$. Specifically, we have

$$S_{2,1}(\Gamma_{\mathsf{g}},\rho_{\mathsf{g}})_{\mathbb{Z}} = \begin{cases} \mathbb{Z}\varphi_{8|4} & \text{for } o(\mathsf{g}) = 8, \\ \mathbb{Z}\varphi_{11} & \text{for } o(\mathsf{g}) = 11, \\ 0 & \text{else}, \end{cases} \tag{4.2.13}$$

where $\varphi_{8|4}$ and φ_{11} are as in (4.2.9) and (4.2.1), respectively.

To proceed with the determination of $\mathcal{L}_{2,1}^{\mathrm{opt}}(\mathsf{G},\alpha)$ suppose that x is a class function on M_{11} that vanishes on g unless $o(\mathsf{g}) \in \{8,11\}$. In this case Thompson's result on virtual characters (cf. the proof of Theorem 4.2.1) tells us that x has integer values if and only if $x \in \mathbb{Z}4\delta_8 + \mathbb{Z}11\delta_{11}$, where δ_{11} is as in (4.2.6) and δ_8 is defined analogously. Now the only irreducible characters of G that are non-rational on elements of order 8 or 11 are χ_3 and χ_4, being non-rational only on elements of order 8, and χ_6 and χ_7, being non-rational only on elements of order 11. From this much we may conclude that $\mathcal{W}_{2,1}^{\mathrm{opt}}(\mathsf{G},\alpha)_0$ is identified with the sublattice

$$\mathbb{Z}\varphi_{8|4} \otimes R(\mathsf{G})_{(8)} + \mathbb{Z}\varphi_{11} \otimes R(\mathsf{G})_{(11)} \tag{4.2.14}$$

of $L_{2,1}(\mathsf{G})$, where $R(\mathsf{G})_{(11)}$ is as in (4.2.5), and $R(\mathsf{G})_{(8)}$ is defined analogously.

We have already computed $R(\mathsf{G})_{(11)}$ in the proof of Theorem 4.2.1. To compute $R(\mathsf{G})_{(8)}$ we use Table 1 to compute

$$4\delta_8 = \chi_1 - \chi_5 - \chi_9 + \chi_{10} \tag{4.2.15}$$

(cf. (4.2.12)) and to see that $\chi_3 - \chi_4$ vanishes away from elements of order 8. Thus $R(\mathsf{G})_{(8)}$ is generated (as a $\mathbb{Z}$-module) by $4\delta_8$ and $\chi_3 - \chi_4$, and thus we have verified the description of $\mathcal{L}_{2,1}^{\mathrm{opt}}(\mathsf{G},\alpha)$ that appears in the statement of the theorem (see (4.2.10)). Thus the theorem is proved. □

4.3. Congruent numbers

Our objective in this section is to establish a result that relates congruent numbers to the M_{11}-modules of § 4.2. For the statement we note from Table 1 that M_{11} has a unique irreducible representation of dimension 55. Also, we take $\mathsf{G} = \mathsf{M}_{11}$ in what follows, and we call to mind the optimal M_{11}-modules $\mathcal{W}_{2,1}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ that are classified, according to Proposition 3.3.1, by Theorem 4.2.5.

Theorem 4.3.1. *Let $D < 0$ be square-free and satisfy $D \equiv 21 \bmod 24$. If there exists an M_{11}-module $W = \bigoplus_D W_D$ in $\mathcal{W}_{2,1}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ such that W_D contains an irreducible 55-dimensional submodule with non-zero multiplicity then $|D|$ is not a congruent number.*

Proof. Recall the mock Jacobi forms $\mathscr{H}_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}$ of (4.2.11). Define $C_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}(D)$ for $\mathsf{g} \in \mathsf{M}_{11}$ by requiring that $\mathscr{H}_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}(\tau, z) = \sum_{n,\ell} C_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}(\ell^2 - 4n) q^n y^\ell$, and to ease notation set $C_N(D) := C_{\mathsf{g}}^{(\mathsf{M}_{11},\alpha)}(D)$ when $o(\mathsf{g}) = N$. Also define $C_{8|4}(D)$ so that $\varphi_{8|4}(\tau, z) = \sum_{n,\ell} C_{8|4}(\ell^2 - 4n) q^n y^\ell$ where $\varphi_{8|4}$ is as in (4.2.9).

Let $W \in \mathcal{W}_{2,1}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ and $D < 0$, and write $m_{55}(W_D)$ for the multiplicity of the 55-dimensional irreducible representation in the M_{11}-module W_D. Then the description (4.2.10) of $\mathcal{L}_{2,1}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ together with a computation with Table 1 yields that $m_{55}(W_D)$ is given explicitly by

$$\frac{1}{144}C_1(D) - \frac{1}{48}C_2(D) + \frac{1}{18}C_3(D) - \frac{1}{8}C_4(D) - \frac{1}{6}C_6(D) + \frac{1}{4}C_8(D) + \frac{1}{4}\Re(\lambda_{8|4})C_{8|4}(D) \tag{4.3.1}$$

for some $\lambda_{8|4} \in 4\mathbb{Z} + 2a_2\mathbb{Z}$ (where a_2 is as in Table 1).

Now $C_4(D)$ and $C_8(D)$ vanish for odd D, and the remaining $C_N(D)$ in (4.3.1) may be expressed explicitly in terms of Hurwitz class numbers. For example, $C_1(D) = 24H^{\mathrm{Hur}}(D)$ according to (4.2.11), because $\mathscr{H}_1^{\mathrm{Rad}} = 12\mathscr{H}^{\mathrm{Hur}}$ (cf. (3.4.4)). For g of order $N = o(\mathsf{g}) \in \{2, 3\}$ the mock Jacobi form $\mathscr{H}_{\mathsf{g}}^{\mathsf{M}_{11}}$ is a linear combination of $\mathscr{H}^{\mathrm{Hur}}$ and the Cohen–Eisenstein series $\mathscr{H}_{o(\mathsf{g})}^{\mathrm{Coh}}$ (cf. (3.1.9)), because these two forms span the space $\mathbb{J}_{2,1}(\Gamma_0^{\mathrm{J}}(N))$ in each case. So the formulas (3.1.4) and (3.1.10) allow us to write $C_N(D)$ in terms of class numbers, for $N \in \{2, 3\}$. For $N = 6$ we achieve an expression for $C_6(D)$ in terms of class numbers by observing that $\mathbb{J}_{2,1}(\Gamma_0^{\mathrm{J}}(6))$ is spanned by $\mathscr{H}^{\mathrm{Hur}}$, $\mathscr{H}_2^{\mathrm{Coh}}$, $\mathscr{H}_3^{\mathrm{Coh}}$, and the Jacobi form $\sum_{n,\ell} C_6'(\ell^2 - 4n) q^n y^\ell$, where $C_6'(D) := H^{\mathrm{Hur}}(36D) - H^{\mathrm{Hur}}(D)$. Noting that

$$C_6(D) = \left(1 - 2\left(\frac{D}{8}\right) - 3\left(\frac{D}{3}\right) + 6\left(\frac{D}{24}\right)\right) H^{\mathrm{Hur}}(D) \tag{4.3.2}$$

for D fundamental, we ultimately obtain from (4.3.1) that

$$m_{55}(W_D) = \left(\frac{D}{3}\right)\left(1 - \left(\frac{D}{8}\right)\right) H^{\mathrm{Hur}}(D) + \frac{1}{4}\Re(\lambda_{8|4})C_{8|4}(D), \tag{4.3.3}$$

which simplifies to $m_{55}(W_D) = \frac{1}{4}\Re(\lambda_{8|4})C_{8|4}(D)$ because $D \equiv 0 \bmod 3$ by hypothesis. So for the discriminants D in question, the multiplicity $m_{55}(W_D)$ depends only on the contribution from $\varphi_{8|4}$.

Let E denote the elliptic curve over $\mathbb{Q}$ defined by $y^2 = x^3 - x$, and let $E \otimes D$ denote the D-th quadratic twist of E, defined by $y^2 = x^3 - D^2x$. Then, as explained in the introduction of [27], for example, $|D|$ is a congruent number if and only if $E \otimes D$ has positive rank (i.e. infinitely many rational points). Define $a(n) \in \mathbb{Z}$ for $n \geq 1$ by requiring that

$$\frac{\eta(4\tau)^5\eta(16\tau)}{\eta(2\tau)^2\eta(8\tau)} = \sum_n a(n)q^n, \tag{4.3.4}$$

where η is as in (3.2.21). Then, by Theorem 3 of [27] we have that $L(E\otimes D, 1) = Ca(|D|)^2$ for some non-zero constant C. Also, $a(|D|) = 2C_{8|4}(D)$ for $D \equiv 5 \bmod 8$, because $a(3) = 2C_{8|4}(-3) = 2$ (cf. (4.2.9)), and because the plus-space projection $\sum_{n\equiv 3 \bmod 8} a(n)q^n$ is a cusp form of weight $\frac{3}{2}$ with the same level and character as $\sum_D C_{8|4}(D)q^{|D|}$, and there is a unique such form up to scale. So for the discriminants D under consideration, if $m_{55}(W_D)$ is non-zero then $L(E \otimes D, 1)$ is non-zero, and this implies that $E \otimes D$ has finitely many rational points according to [24]. So $|D|$ is not a congruent number in this case. This completes the proof. □

Remark 4.3.2. The proof of Theorem 4.3.1 shows that, for $D \equiv 21 \bmod 24$, the multiplicity $m_{55}(W_D)$ is proportional to the Jacobi form coefficient $C_{8|4}(D)$ for any $W \in \mathcal{W}_{2,1}^{\mathrm{opt}}(\mathsf{G}, \alpha)$, where the constant of proportionality depends only on W. Inspecting Table 5 we see that $m_{55}(W_{-3}) \neq 0$ for W as in Proposition 4.2.7, so the constant of proportionality is non-zero for this choice of W, and thus there is no loss in applying Theorem 4.3.1 only to the module W of Proposition 4.2.7. We note also that there are infinitely many D satisfying the hypotheses of the theorem for which $m_{55}(W_D)$ is non-zero, because the $C_{8|4}(D)$ with $D \equiv 21 \bmod 24$ are the coefficients of a non-zero Jacobi form.

Remark 4.3.3. If the Birch–Swinnerton-Dyer conjecture were known to be true (cf. [28]) we would be able to prove a converse to Theorem 4.3.1, and use the M_{11}-module W to identify congruent numbers. Indeed, the proof of Theorem 4.3.1 (together with Remark 4.3.2) shows that, for D as in the statement of the theorem and W as in Proposition 4.2.7, the multiplicity $m_{55}(W_D)$ is zero if and only if $L(E \otimes D, 1)$ is zero, where $E \otimes D$ (defined by $y^2 = x^3 - D^2x$) is the elliptic curve whose rank controls whether $|D|$ is congruent or not. For example, $D = -219$ satisfies the conditions of Theorem 4.3.1, and one can compute that the 55-dimensional irreducible representation has vanishing multiplicity in W_{-219}. This is consistent with the fact that 219 is congruent. (We have $219 = \frac{ab}{2}$ when $a = \frac{264}{13}$ and $b = \frac{949}{44}$, for example.)

Theorem 4.3.1 begs the question of what methods are available to compute the multiplicities $m_{55}(W_D)$, for $W \in \mathcal{W}_{2,1}^{\mathrm{opt}}(\mathsf{G}, \alpha)$. It is also natural to seek out counterparts to

Theorem 4.3.1 that involve other irreducible M_{11}-modules. In this work we have only implicitly identified optimal M_{11}-modules in terms of (mock) Jacobi forms, so we can only compute multiplicities like $m_{55}(W_D)$ in terms of Jacobi form coefficients. The upshot of this is that we have not, in this work, produced a practical improvement upon the congruent number criteria that appear in Tunnell's work [27]. However, it is possible that an explicit algebraic construction of one or another $W \in \mathcal{W}_{2,1}^{\mathrm{opt}}(\mathsf{G}, \alpha)$ could furnish algebraic control, of some kind, over $m_{55}(W_D)$ and other multiplicities. We could then, via Theorem 4.3.1, or one of its possible counterparts, obtain a new practical approach to determining which numbers are congruent.

The task of extending Theorem 4.3.1 to other irreducible M_{11}-modules, and the challenge of finding explicit algebraic constructions of optimal M_{11}-modules, remain problems for future work.

Appendix A. Characters and coefficients

In Table 1 we give the character table of the sporadic group M_{11}. In Tables 2-5 we give some coefficients of the McKay–Thompson series and multiplicity generating functions for the graded infinite-dimensional virtual modules for this group that feature in § 4. In Table 1 we use the notation $a_n := \sqrt{-n}$ and $b_n := -\frac{1}{2} + \frac{1}{2}\sqrt{-n}$.

Table 1
Character table of M_{11}.

[g]	1A	2A	3A	4A	5A	6A	8A	8B	11A	11B
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	10	2	1	2	0	-1	0	0	-1	-1
χ_3	10	-2	1	0	0	1	a_2	$-a_2$	-1	-1
χ_4	10	-2	1	0	0	1	$-a_2$	a_2	-1	-1
χ_5	11	3	2	-1	1	0	-1	-1	0	0
χ_6	16	0	-2	0	1	0	0	0	b_{11}	$\overline{b_{11}}$
χ_7	16	0	-2	0	1	0	0	0	$\overline{b_{11}}$	b_{11}
χ_8	44	4	-1	0	-1	1	0	0	0	0
χ_9	45	-3	0	1	0	0	-1	-1	1	1
χ_{10}	55	-1	1	-1	0	-1	1	1	0	0

Data availability

No data was used for the research described in the article.

References

[1] J. Bruinier, J. Funke, On two geometric theta lifts, Duke Math. J. 125 (1) (2004) 45–90.
[2] M. Cheng, J. Duncan, Meromorphic Jacobi forms of half-integral index and umbral moonshine modules, Commun. Math. Phys. 370 (3) (2019) 759–780.
[3] M. Cheng, J. Duncan, Optimal mock Jacobi theta functions, Adv. Math. 372 (2020) 107284.
[4] M. Cheng, J. Duncan, J. Harvey, Umbral moonshine, Commun. Number Theory Phys. 8 (2) (2014) 101–242.

Table 2
McKay–Thompson series $\mathscr{H}_{\mathsf{g}}^{\mathsf{M}_{11}}$.

$\lvert D\rvert$	1A	2A	3A	4A	5A	6A	8AB	11AB
0	−2	−2	−2	−2	−2	−2	−2	−2
3	8	−8	−1	0	−2	1	0	−3
4	12	−4	−6	−4	2	2	0	1
7	24	8	−12	0	−6	−4	0	2
8	24	−8	6	−8	−6	−2	0	2
11	24	−24	6	0	4	−6	0	2
12	32	0	−4	0	−8	0	−8	−1
15	48	16	−6	0	−2	−2	0	−7
16	36	4	−18	4	6	−2	−4	−8
19	24	−24	−12	0	4	12	0	2
20	48	−16	12	−16	−2	−4	0	−7
23	72	24	18	0	−18	6	0	−5
24	48	−16	−6	−16	8	2	0	4
27	32	−32	5	0	−8	−5	0	−1
28	48	16	−24	16	−12	−8	8	4
31	72	24	−36	0	12	−12	0	−5
32	72	8	18	8	−18	2	−8	6
35	48	−48	12	0	−2	−12	0	4
36	60	−20	6	−20	10	−2	0	−6
39	96	32	−12	0	16	−4	0	8
40	48	−16	−24	−16	−2	8	0	4
43	24	−24	−12	0	−6	12	0	2
44	96	0	24	0	16	0	−24	−3
47	120	40	30	0	−30	10	0	−12
48	80	16	−10	16	−20	−2	0	−8
51	48	−48	−6	0	8	6	0	4
52	48	−16	−24	−16	−12	8	0	4
55	96	32	−48	0	−4	−16	0	−3
56	96	−32	24	−32	16	−8	0	−14
59	72	−72	18	0	12	−18	0	−5
60	96	32	−12	32	−4	−4	16	−3
63	120	40	12	0	−30	4	0	10
64	84	20	−42	20	14	−10	4	−4
67	24	−24	−12	0	−6	12	0	−9
68	96	−32	24	−32	−24	−8	0	8
71	168	56	42	0	28	14	0	−19
72	72	−24	18	−24	−18	−6	0	6
75	56	−56	−7	0	6	7	0	−10
76	96	0	−48	0	16	0	−24	8
79	120	40	−60	0	20	−20	0	10
80	144	16	36	16	−6	4	−16	−10
83	72	−72	18	0	−18	−18	0	6
84	96	−32	−12	−32	16	4	0	8
87	144	48	−18	0	−36	−6	0	12
88	48	−16	−24	−16	−12	8	0	4
91	48	−48	−24	0	8	24	0	4
92	144	48	36	48	−36	12	24	−21
95	192	64	48	0	−8	16	0	16
96	144	16	−18	16	24	−2	−16	12
99	72	−72	18	0	12	−18	0	−5
100	60	−20	−30	−20	10	10	0	−6
103	120	40	−60	0	−30	−20	0	−12
104	144	−48	36	−48	24	−12	0	−10
107	72	−72	18	0	−18	−18	0	6
108	128	0	20	0	−32	0	−32	−15

Table 3
McKay–Thompson series $\mathscr{H}_g^{(\mathsf{M}_{11},\alpha)}$.

$\lvert D\rvert$	1A	2A	3A	4A	5A	6A	8AB	11AB
0	−2	−2	−2	−2	−2	−2	−2	−2
3	8	−8	−1	0	−2	1	−4	−3
4	12	−4	−6	4	2	2	−4	1
7	24	8	−12	0	−6	−4	0	2
8	24	−8	6	8	−6	−2	0	2
11	24	−24	6	0	4	−6	4	2
12	32	0	−4	−16	−8	0	0	−1
15	48	16	−6	0	−2	−2	0	−7
16	36	4	−18	−12	6	−2	4	−8
19	24	−24	−12	0	4	12	4	2
20	48	−16	12	16	−2	−4	16	−7
23	72	24	18	0	−18	6	0	−5
24	48	−16	−6	16	8	2	0	4
27	32	−32	5	0	−8	−5	0	−1
28	48	16	−24	0	−12	−8	0	4
31	72	24	−36	0	12	−12	0	−5
32	72	8	18	−24	−18	2	8	6
35	48	−48	12	0	−2	−12	8	4
36	60	−20	6	20	10	−2	−20	−6
39	96	32	−12	0	16	−4	0	8
40	48	−16	−24	16	−2	8	0	4
43	24	−24	−12	0	−6	12	−12	2
44	96	0	24	−48	16	0	0	−3
47	120	40	30	0	−30	10	0	−12
48	80	16	−10	−16	−20	−2	−16	−8
51	48	−48	−6	0	8	6	−8	4
52	48	−16	−24	16	−12	8	16	4
55	96	32	−48	0	−4	−16	0	−3
56	96	−32	24	32	16	−8	0	−14
59	72	−72	18	0	12	−18	−4	−5
60	96	32	−12	0	−4	−4	0	−3
63	120	40	12	0	−30	4	0	10
64	84	20	−42	−12	14	−10	−12	−4
67	24	−24	−12	0	−6	12	4	−9
68	96	−32	24	32	−24	−8	−32	8
71	168	56	42	0	28	14	0	−19
72	72	−24	18	24	−18	−6	0	6
75	56	−56	−7	0	6	7	4	−10
76	96	0	−48	−48	16	0	0	8
79	120	40	−60	0	20	−20	0	10
80	144	16	36	−48	−6	4	16	−10
83	72	−72	18	0	−18	−18	−4	6
84	96	−32	−12	32	16	4	32	8
87	144	48	−18	0	−36	−6	0	12
88	48	−16	−24	16	−12	8	0	4
91	48	−48	−24	0	8	24	8	4
92	144	48	36	0	−36	12	0	−21
95	192	64	48	0	−8	16	0	16
96	144	16	−18	−48	24	−2	16	12
99	72	−72	18	0	12	−18	−4	−5
100	60	−20	−30	20	10	10	−20	−6
103	120	40	−60	0	−30	−20	0	−12
104	144	−48	36	48	24	−12	0	−10
107	72	−72	18	0	−18	−18	12	6
108	128	0	20	−64	−32	0	0	−15

Table 4
Multiplicity generating functions $\mathscr{H}_{\chi}^{\mathsf{M}_{11}}$.

$\lvert D\rvert$	1	10	$10'$	$\overline{10'}$	11	16	$\overline{16}$	44	45	55
0	−2	0	0	0	0	0	0	0	0	0
3	−1	0	1	1	−1	0	0	0	0	0
4	0	−2	0	0	0	1	1	0	0	0
7	−2	0	−2	−2	−2	0	0	2	0	0
8	−2	−2	0	0	0	−2	−2	0	0	2
11	0	0	0	0	0	0	0	−4	2	2
12	−4	0	0	0	0	−1	−1	2	2	−2
15	−2	2	0	0	0	1	1	2	−2	0
16	−2	2	0	0	0	4	4	0	0	−2
19	2	−4	2	2	−2	2	2	0	2	−2
20	−4	−2	2	2	2	−1	−1	−2	−2	4
23	−2	2	2	2	0	−5	−5	6	−2	0
24	0	−6	0	0	2	2	2	−2	0	2
27	−3	0	1	1	−3	−2	−2	−2	2	2
28	0	4	−4	−4	−8	0	0	4	0	0
31	−2	2	−4	−4	0	7	7	0	−2	0
32	−2	2	0	0	0	−6	−6	4	4	−2
35	−2	0	0	0	−2	−2	−2	−6	4	4
36	−2	−4	2	2	4	2	2	−4	−2	4
39	4	0	−4	−4	4	4	4	0	0	0
40	−2	−8	0	0	−2	2	2	2	0	0
43	0	−4	2	2	−4	0	0	2	2	−2
44	−2	2	2	2	12	1	1	−4	6	−4
47	−4	4	4	4	0	−8	−8	10	−4	0
48	−4	6	0	0	−6	−2	−2	6	0	−2
51	2	−4	2	2	−2	2	2	−4	4	0
52	−4	−8	0	0	−4	0	0	4	0	0
55	−6	2	−6	−6	−4	5	5	4	−2	0
56	−4	−4	4	4	8	2	2	−8	−4	8
59	−2	2	2	2	0	1	1	−12	4	6
60	6	10	−2	−2	−8	1	1	4	−2	0
63	−2	0	−2	−2	−2	−8	−8	10	0	0
64	2	6	−4	−4	−4	8	8	0	0	−2
67	−2	−2	4	4	−4	1	1	2	0	−2
68	−8	−8	0	0	0	−8	−8	0	0	8
71	8	6	6	6	14	3	3	0	−6	0
72	−6	−6	0	0	0	−6	−6	0	0	6
75	−1	−2	5	5	−3	3	3	−4	2	0
76	−4	−4	−4	−4	4	8	8	0	8	−8
79	0	0	−10	−10	0	10	10	0	0	0
80	−2	8	4	4	6	−4	−4	2	4	−4
83	−6	0	0	0	−6	−6	−6	−6	6	6
84	0	−12	0	0	4	4	4	−4	0	4
87	−6	0	−6	−6	−6	−6	−6	12	0	0
88	−4	−8	0	0	−4	0	0	4	0	0
91	4	−8	4	4	−4	4	4	0	4	−4
92	6	18	6	6	−12	−9	−9	12	−6	0
95	8	0	0	0	8	−8	−8	8	0	0
96	4	2	−4	−4	6	6	6	−2	8	−6
99	−2	2	2	2	0	1	1	−12	4	6
100	−2	−8	2	2	0	6	6	0	−2	0
103	−14	4	−6	−6	−10	2	2	10	−4	0
104	−4	−8	4	4	12	2	2	−12	−4	12
107	−6	0	0	0	−6	−6	−6	−6	6	6
108	−16	4	4	4	4	−7	−7	6	6	−6

Table 5
Multiplicity generating functions $\mathscr{H}_{\chi}^{(\mathsf{M}_{11},\alpha)}$.

$\lvert D\rvert$	1	10	$10'$	$\overline{10'}$	11	16	$\overline{16}$	44	45	55
0	−2	0	0	0	0	0	0	0	0	0
3	−2	0	1	1	0	0	0	0	1	−1
4	0	0	0	0	0	1	1	0	2	−2
7	−2	0	−2	−2	−2	0	0	2	0	0
8	0	2	0	0	−2	−2	−2	0	2	0
11	1	0	0	0	−1	0	0	−4	1	3
12	−4	−4	0	0	0	−1	−1	2	−2	2
15	−2	2	0	0	0	1	1	2	−2	0
16	−2	−2	0	0	0	4	4	0	−4	2
19	3	−4	2	2	−3	2	2	0	1	−1
20	4	6	2	2	−6	−1	−1	−2	−2	4
23	−2	2	2	2	0	−5	−5	6	−2	0
24	4	2	0	0	−2	2	2	−2	4	−2
27	−3	0	1	1	−3	−2	−2	−2	2	2
28	−4	0	−4	−4	−4	0	0	4	0	0
31	−2	2	−4	−4	0	7	7	0	−2	0
32	−2	−6	0	0	0	−6	−6	4	−4	6
35	0	0	0	0	−4	−2	−2	−6	2	6
36	−2	6	2	2	4	2	2	−4	8	−6
39	4	0	−4	−4	4	4	4	0	0	0
40	2	0	0	0	−6	2	2	2	4	−4
43	−3	−4	2	2	−1	0	0	2	5	−5
44	−2	−10	2	2	12	1	1	−4	−6	8
47	−4	4	4	4	0	−8	−8	10	−4	0
48	−12	−2	0	0	2	−2	−2	6	0	−2
51	0	−4	2	2	0	2	2	−4	6	−2
52	4	0	0	0	−12	0	0	4	0	0
55	−6	2	−6	−6	−4	5	5	4	−2	0
56	4	12	4	4	0	2	2	−8	4	0
59	−3	2	2	2	1	1	1	−12	5	5
60	−2	2	−2	−2	0	1	1	4	−2	0
63	−2	0	−2	−2	−2	−8	−8	10	0	0
64	−6	−2	−4	−4	4	8	8	0	0	−2
67	−1	−2	4	4	−5	1	1	2	−1	−1
68	−8	8	0	0	0	−8	−8	0	16	−8
71	8	6	6	6	14	3	3	0	−6	0
72	0	6	0	0	−6	−6	−6	0	6	0
75	0	−2	5	5	−4	3	3	−4	1	1
76	−4	−16	−4	−4	4	8	8	0	−4	4
79	0	0	−10	−10	0	10	10	0	0	0
80	−2	−8	4	4	6	−4	−4	2	−12	12
83	−7	0	0	0	−5	−6	−6	−6	7	5
84	16	4	0	0	−12	4	4	−4	0	4
87	−6	0	−6	−6	−6	−6	−6	12	0	0
88	0	0	0	0	−8	0	0	4	4	−4
91	6	−8	4	4	−6	4	4	0	2	−2
92	−6	6	6	6	0	−9	−9	12	−6	0
95	8	0	0	0	8	−8	−8	8	0	0
96	4	−14	−4	−4	6	6	6	−2	−8	10
99	−3	2	2	2	1	1	1	−12	5	5
100	−2	2	2	2	0	6	6	0	8	−10
103	−14	4	−6	−6	−10	2	2	10	−4	0
104	8	16	4	4	0	2	2	−12	8	0
107	−3	0	0	0	−9	−6	−6	−6	3	9
108	−16	−12	4	4	4	−7	−7	6	−10	10

[5] M. Cheng, J. Duncan, J. Harvey, Umbral moonshine and the Niemeier lattices, Res. Math. Sci. 1 (3) (2014) 1–81.
[6] M. Cheng, J. Duncan, J. Harvey, Weight one Jacobi forms and umbral moonshine, J. Phys. A 51 (10) (2018) 104002.
[7] M. Cheng, J. Duncan, M. Mertens, Class numbers, elliptic curves and the second largest Mathieu group, in preparation.
[8] M. Cheng, J. Duncan, M. Mertens, Class numbers, cyclic simple groups, and arithmetic, J. Lond. Math. Soc. 2 (2023) 1–35.
[9] H. Cohen, Sums involving the values at negative integers of L-functions of quadratic characters, Math. Ann. 217 (3) (1975) 271–285.
[10] J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson, Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups, With comput. assist. from J.G. Thackray, Clarendon Press, Oxford, 1985.
[11] A. Dabholkar, S. Murthy, D. Zagier, Quantum Black Holes, Wall Crossing, and Mock Modular Forms, Cambridge Monogr. Math. Phys., 2025, in press, arXiv:1208.4074.
[12] J. Duncan, M. Griffin, K. Ono, Proof of the umbral moonshine conjecture, Res. Math. Sci. 2 (26) (March 2015).
[13] J. Duncan, J. Harvey, B. Rayhaun, An overview of penumbral moonshine, arXiv e-prints, arXiv: 2109.09756, 2021.
[14] J. Duncan, M. Mertens, K. Ono, Pariah moonshine, Nat. Commun. 8 (September 2017) 670.
[15] J. Duncan, Arithmetic groups and the affine E_8 Dynkin diagram, in: Groups and Symmetries, in: CRM Proc. Lecture Notes, vol. 47, Amer. Math. Soc., Providence, RI, 2009, pp. 135–163.
[16] J. Duncan, From the monster to Thompson to O'Nan, in: Vertex Operator Algebras, Number Theory and Related Topics, in: Contemp. Math., vol. 753, Amer. Math. Soc., Providence, RI, 2020, pp. 73–93.
[17] J. Duncan, M. Griffin, K. Ono, Proof of the umbral moonshine conjecture, Res. Math. Sci. 2 (2015) 26.
[18] J. Duncan, M. Mertens, K. Ono, O'Nan moonshine and arithmetic, Am. J. Math. 143 (4) (2021) 1115–1159.
[19] M. Eichler, D. Zagier, The Theory of Jacobi Forms, Progress in Mathematics, vol. 55, Birkhäuser Boston, Inc., Boston, MA, 1985.
[20] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.12.2, 2023.
[21] B. Gross, W. Kohnen, D. Zagier, Heegner points and derivatives of L-series. II, Math. Ann. 278 (1–4) (1987) 497–562.
[22] B. Gross, Heights and the special values of L-series, in: Number Theory, Montreal, Que., 1985, in: CMS Conf. Proc., vol. 7, Amer. Math. Soc., Providence, RI, 1987, pp. 115–187.
[23] M. Khaqan, Elliptic curves and Thompson's sporadic simple group, J. Number Theory 224 (2021) 274–306.
[24] V. Kolyvagin, Finiteness of $E(\mathbf{Q})$ and $\mathrm{SH}(E, \mathbf{Q})$ for a subclass of Weil curves, Izv. Akad. Nauk SSSR, Ser. Mat. 52 (3) (1988) 522–540, 670–671.
[25] The LMFDB Collaboration, The L-functions and modular forms database, http://www.lmfdb.org.
[26] S. Smith, On the head characters of the monster simple group, in: Finite Groups–Coming of Age, Montreal, Que., 1982, in: Contemp. Math., vol. 45, Amer. Math. Soc., Providence, RI, 1985, pp. 303–313.
[27] J. Tunnell, A classical Diophantine problem and modular forms of weight 3/2, Invent. Math. 72 (2) (1983) 323–334.
[28] A. Wiles, The Birch and Swinnerton-Dyer conjecture, in: The Millennium Prize Problems, Clay Math. Inst, Cambridge, MA, 2006, pp. 31–41.
[29] Y. Tian, Congruent number problem, ICCM Not. 5 (1) (2017) 51–61.
[30] S. Zwegers, Mock Theta Functions, Doctoral thesis, Utrecht University, 2002, arXiv:0807.4834.