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Statistical predictions from anarchic field theory landscapes

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ABSTRACT

Consistent coupling of effective field theories with a quantum theory of gravity appears to require bounds on the rank of the gauge group and the amount of matter. We consider landscapes of field theories subject to such boundedness constraints. We argue that appropriately “coarse-grained” aspects of the randomly chosen field theory in such landscapes, such as the fraction of gauge groups with ranks in a given range, can be statistically predictable. To illustrate our point we show how the uniform measures on simple classes of $\mathcal{N} = 1$ quiver gauge theories localize in the vicinity of theories with certain typical structures. Generically, this approach would predict a high energy theory with very many gauge factors, with the high rank factors largely decoupled from the low rank factors if we require asymptotic freedom for the latter.

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1. Introduction

It is commonly supposed that the huge numbers of vacua that can arise from different compactifications of string theory [1,2] imply a complete loss of predictability of low energy physics. If this is the case, the stringiness simply constrains the possible dynamics rather than the precise complement of forces and matter. Every string theory leads to some effective field theory at a high scale Λ , taken to be, say, an order of magnitude below the string scale. Predictions for low energy physics have to be made in terms of this effective field theory. Thus, the landscape of string theory vacua leads to a landscape of effective field theories at the scale Λ . Here we ask if constraints of finiteness imposed on this landscape via its origin in string theory might be sufficient to lead to a degree of predictability, at least in some statistical sense. Previous authors have discussed how continuous parameters can scan in a random landscape of effective field theories [3–9], and there has been some study of the gauge groups and matter content attainable from specific string theoretic scenarios [10–15]. For example, [14] and [15] discuss the distribution of gauge groups arising in intersecting brane models on torus orientifolds.

We will impose the weakest of the constraints arising from string theory – namely that it should be possible to couple the effective field theory consistently to a quantum theory of gravity. It has been argued [16–18] that such consistency with string theory requires that the rank of the gauge group and the number of matter fields be bounded from above.¹ Since we will not impose any constraints based on rules arising from symmetry or dynamics on the measure, we will call this an “anarchic” landscape, in recollection of the terminology in [6]. Thus we will study simple anarchic landscapes of field theories bounded in this way, and illustrate how statistics can lead to characteristic predictions for the low energy physics. These predictions are strongest for appropriately coarse-grained attributes of a theory that possess the property of *typicality* in such landscapes – i.e. they are overwhelmingly likely to lie close to certain typical values. An example of such a typical property will be the fraction of gauge groups with ranks lying within some range. We will illustrate and develop our thinking using some simple examples.

2. The set of field theories

Quiver gauge theories provide a natural, large class to consider. For simplicity, we will restrict attention to $\mathcal{N} = 1$ supersymmetric

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¹ A possible bound on the number of matter species in theories containing gravity was originally discussed by Bekenstein [19].

theories with a gauge group $G = \prod_{i=1}^L U(N_i)$, A_{ii} hypermultiplets in the adjoint of $U(N_i)$, and A_{ij} hypermultiplets in the $(\mathbf{N}_i, \bar{\mathbf{N}}_j)$ of $U(N_i) \times U(N_j)$. To specify the full gauge theory we also need a Kähler potential for the hypermultiplets, gauge kinetic terms, a superpotential and possibly Fayet–Iliopoulos terms. We will postpone discussion of these quantities and will discuss the matter and gauge group content of the $\mathcal{N} = 1$ theory.

Quiver gauge theories are ubiquitous in string theory because bifundamental matter, arising from strings with two endpoints, is common. In $\mathcal{N} = 1$ quivers constructed by wrapping D6-branes on 3-cycles inside a Calabi–Yau manifold the number of bifundamentals is related to the intersection number of the 3-cycles. By including orientifolds, one can engineer quiver theories with SO and Sp gauge factors.

2.1. Interesting classes of quiver gauge theories

Three possible restricted sets of gauge theories are:

- *Anomaly free theories.* We must impose the absence of anomalies: $\forall i, \sum_{j \neq i} (A_{ij} - A_{ji})N_j = 0$. The left-hand side has zero expectation in the unconstrained set of quiver gauge theories with the uniform measure, as the measure is invariant under $A_{ij} \leftrightarrow A_{ji}$. Therefore, “on average”, random quiver gauge theories are anomaly free, and one might be inclined to not worry about anomalies anymore. However, from a physical point of view one must not allow forbidden theories in an ensemble, as properties of the set of anomaly free theories may not be the same as the full set of random quiver gauge theories. Hence we will restrict to field theories which are anomaly free.
- *Asymptotically free theories.* Another natural constraint is asymptotic freedom, which makes a theory well-defined in the UV. Asymptotic freedom is less compelling than anomaly cancellation if we simply consider a set low-energy effective field theories obtained e.g. in string theory. Gauge group factors that are IR free and strongly coupled in the UV will typically act as global symmetries at low energies and will not directly lead to contradictions. Asymptotic freedom occurs if $\forall i, A_{ii}N_i + \sum_{j \neq i} (A_{ij} + A_{ji})N_j < 3N_i$. This constrains the A_{ij} to be of order unity.
- *Purely chiral theories.* Starting with effective field theories at a high scale M , in the absence of other dimensionful parameters, the most general superpotential will contain many mass terms of $O(M)$. Integrating these out at energies below M leaves purely chiral theories with $A_{ii} = 0$ and $A_{ij} = 0$ or $A_{ji} = 0$ for $i \neq j$. These are a natural starting point for viewing random quivers as low-energy effective field theories. Chiral theories allow for general cubic superpotentials that are marginal. Higher order terms are suppressed by a mass scale in the Lagrangian, although some quartic superpotentials can become marginal in the infrared.
- *Equal rank theories.* For simplicity, we can take all gauge group ranks to be fixed and equal. For such theories the anomaly cancellation and asymptotic freedom constraints are easier to implement. We do not have a physical motivation that would select these theories, but they helpful for developing intuition.

2.2. Averages and typicality

Given a set of gauge theories with a suitable measure on them, we can compute expectation values of quantities, such as rank of a gauge group, the number of matter fields, etc. Though averages are useful, they are especially interesting when they also represent the *typical* value of a quantity. Typicality is a notion that exists in situations when a thermodynamic limit can be taken wherein some

parameter N , controlling the size of the ensemble, can be taken to infinity. Then, a quantity enjoys the property of typicality if its probability distribution is narrowly peaked around its expectation value as $N \rightarrow \infty$:

$$\lim_{N \rightarrow \infty} \frac{\langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2}{\langle \mathcal{O} \rangle^2} = 0. \quad (1)$$

“Typical” quantities equal their ensemble averages with probability one as $N \rightarrow \infty$.²

Familiar examples are pressure and free energy. Notice that for a standard Boltzmann distribution, a particular occupation number has

$$\begin{aligned} \langle N \rangle &= \frac{\sum_{k \geq 0} k e^{-\beta k}}{\sum_{k \geq 0} e^{-\beta k}} = \frac{e^{-\beta}}{1 - e^{-\beta}}, \\ \langle N^2 \rangle &= \frac{\sum_{k \geq 0} k^2 e^{-\beta k}}{\sum_{k \geq 0} e^{-\beta k}} = \frac{e^{-\beta}(1 + e^{-\beta})}{(1 - e^{-\beta})^2}. \end{aligned} \quad (2)$$

Here the variance to mean squared ratio is e^β and hence is not typical. Observables that achieve typicality are inevitably coarse-grained – e.g. the number of Boltzmann particles with energies between c/β and $(c + \epsilon)/\beta$ for constants c and ϵ will be typical. We are interested in typical “coarse-grained” structures in field theory landscapes.

2.3. Choice of measure

To discuss statistics we need a measure on the space of quiver gauge theories. Dynamics might give a complicated measure – e.g., the connection between quiver theories and D-brane moduli spaces might give field theories a weight equal to the dimension, or size, of the cohomology of their moduli spaces. Or dynamical effects might give matter fields an expectation value, breaking gauge groups to $U(1)$ – then an analysis of the distribution of gauge factors would be moot. However, in our $N = 1$ theories, the matter potential typically develops isolated minima and the gauge group is broken to a product of Abelian and non-Abelian factors (e.g., a cubic superpotential for an adjoint superfield classically breaks $U(N) \rightarrow U(p) \times U(N - p)$ for some p). Classically, in the context of Calabi–Yau compactification, one imagines some set of distinct, intersecting cycles and non-Abelian gauge factors arise from branes wrapped on each cycle. Strong dynamics might break these gauge factors further. Here we will ignore dynamics and use a uniform measure subject to various constraints of boundedness. Since we are ignoring possible rules arising dynamics, we will call our measures “anarchic”.

One might also associate Bayesian measures to field theory landscapes. For example, to predict the UV field theory, given a bound on the matter and gauge groups, we should condition our measure on known facts about IR physics. Thus, we actually want the uniform measure on a bounded space of gauge theories that, when run to the infrared, contains the standard model as a sector. Conditioning in this way is beyond our ability at present.

Directly computing averages and variances over bounded configuration spaces can be difficult. To simplify, we can use a grand canonical ensemble to constrain the total rank and the total number of matter fields. This involves summing over theories with arbitrary ranks and amounts of matter while including in the mea-

² This criterion is not very useful when $\langle \mathcal{O} \rangle = 0$. We should normalize the operator \mathcal{O} in such a way that the range of values it can take is independent of N and then require that the variance vanishes in the large N limit.

sure a Boltzmann factor for the rank of the gauge group, and a separate Boltzmann factor for the total number of matter fields

$$\rho \sim \exp\left(-\beta \sum_i N_i - \lambda \sum_{ij} A_{ij} N_i N_j\right). \quad (3)$$

One could also include Boltzmann factors for, e.g., the total number of nodes, the total number of gauge bosons, etc., but for our purposes (3) will be sufficient to illustrate the main ideas. Such an approach only works if the ensemble of theories does not grow exponentially fast in the total rank and number of matter fields. If such exponential growth occurs, the Boltzmann weight does not fall quickly enough for the microcanonical ensemble to be well approximated by the canonical ensemble.

3. Typicality in toy landscapes

3.1. Theories without matter: Coarse graining and typicality

As an example, consider a landscape of theories with no matter, where the rank of the gauge group is equal to a large number N . For simplicity, let the gauge group be a product of unitary factors $G = \prod_{i=1}^L U(N_i)$. Then the rank of G is $\sum_i N_i = N$; thus the N_i form an integer partition of N . To study the distribution of gauge factors in this landscape, we construct the canonical partition function

$$Z = \sum_{\{r_k\}} e^{-\beta \sum_k k r_k - \alpha \sum_k r_k} = \prod_k \frac{1}{1 - e^{-\beta k - \alpha}} \equiv \prod_k \frac{1}{1 - u q^k}. \quad (4)$$

Here r_k is the number of gauge factors of rank k , β is a Lagrange multiplier constraining the total rank to be N , and α is a Lagrange multiplier constraining the number of gauge factors; sometimes it is more convenient to work with $q = e^{-\beta}$ and $u = e^{-\alpha}$ instead. This measure treats gauge factor ordering as irrelevant, e.g., $U(2) \times U(3) \times U(2) \sim U(3) \times U(2) \times U(2)$. Further the $U(N_i)$ factors are not distinguished by parameters like gauge couplings. This measure will be modified if the gauge theory is realized D-branes on Calabi–Yau cycles because brane locations and cycle sizes will distinguish many different configurations that giving same gauge group. The present measure is interesting for simply counting field theories.

To fix β and α we require that

$$N = \sum_{j=1}^{\infty} \frac{j u q^j}{1 - u q^j}; \quad L = \sum_j \frac{u q^j}{1 - u q^j}, \quad (5)$$

where N is the total rank and L is the total number of gauge factors. We will take $u \sim O(1)$; $\beta \sim 1/\sqrt{N}$, which, we will see later, implies $L \sim \sqrt{N}$. Then from (4)

$$\langle r_j \rangle = \frac{u q^j}{1 - u q^j}; \quad \text{Var}(r_j) = \frac{u q^j}{(1 - u q^j)^2} = \frac{\langle r_j \rangle}{1 - u q^j}. \quad (6)$$

The variance to mean squared ratio is

$$\frac{\text{Var}(r_j)}{\langle r_j \rangle^2} = \frac{1}{u q^j} = e^{\beta j + \alpha} \geq e^{\alpha} \geq O(1). \quad (7)$$

To last inequality used $\alpha, \beta > 0$. Thus, in such anarchic landscapes, the number of gauge factors with rank j is not typical and cannot be predicted with confidence.

Are any more coarse grained structures in such landscapes which are more predictable? Consider the number of gauge fac-

tors with ranks between $c\sqrt{N}$ and $(c + \epsilon)\sqrt{N}$ where c and ϵ are $O(1)$:

$$\langle R(c, \epsilon) \rangle \approx \int_{c\sqrt{N}}^{(c+\epsilon)\sqrt{N}} dj \langle r_j \rangle = \frac{1}{\beta} \ln \left[\frac{1 - u e^{-(c+\epsilon)\sqrt{N}\beta}}{1 - u e^{-c\sqrt{N}\beta}} \right], \quad (8)$$

where we approximated the sum as an integral. The variance is

$$\begin{aligned} \text{Var}(R(c, \epsilon)) &= \int_{c\sqrt{N}}^{(c+\epsilon)\sqrt{N}} dj \text{Var}(r_j) \\ &= \frac{u}{\beta} \left[\frac{e^{-c\sqrt{N}\beta} - e^{-(c+\epsilon)\sqrt{N}\beta}}{(1 - u e^{-c\sqrt{N}\beta})(1 - u e^{-(c+\epsilon)\sqrt{N}\beta})} \right], \end{aligned} \quad (9)$$

using the statistical independence of r_j . Thus, for $\beta \sim 1/\sqrt{N}$,

$$\begin{aligned} \langle R(c, \epsilon) \rangle &\sim O(\sqrt{N}); \\ \text{Var}(R(c, \epsilon)) &\sim O(\sqrt{N}) \Rightarrow \frac{\text{Var}(R(c, \epsilon))}{\langle R(c, \epsilon) \rangle^2} \sim O(1/\sqrt{N}). \end{aligned} \quad (10)$$

The variance to mean squared ratio vanishes at large N limit – i.e., $R(c, \epsilon)$ is a typical variable and the number of gauge factors with ranks between $c\sqrt{N}$ and $(c + \epsilon)\sqrt{N}$ can be predicted with confidence. Approximating the second equation in (5) as an integral, the total number of gauge factors is

$$L = -\frac{\ln(1 - u)}{u\beta} \sim O(\sqrt{N}). \quad (11)$$

This number is typical – thus, the total number of gauge factors is predictable. These results follow because the unordered partitions of a large integer enjoy a central limit theorem – representing partitions by Young diagrams, the boundaries of appropriately rescaled diagrams approach a limit shape encoded by $\langle r_j \rangle$ at large N [20].

3.2. Cyclic, chiral quivers

We saw how coarse-grained structures in a randomly chosen field theory in a bounded landscape might be statistically predictable. The next step is to add anomaly-free matter and implementing anomaly-freeness is one of the main challenges. Thus, we first study cyclic, chiral quiver gauge theories for which anomaly freedom is easy.

In cyclic quivers, each gauge group is connected to the next one by bifundamentals, with the circle being completed when the last group connects to the first one. Taking the i th group around the circle to be $U(N_i)$, the constraint on the total rank will be $\sum_i N_i = N$. So the N_i form a partition of N . Anomaly cancellation requires equal fundamentals antifundamentals in each group. The minimal solution is

$$A_{i(i+1)} = C^{-1} \cdot \prod_{l \neq i, (i+1)} N_l; \quad C = \text{GCD} \left(\left\{ \prod_{l \neq i, (i+1)} N_l \right\} \right). \quad (12)$$

All other solutions are integer multiples of (12). We will require matter fields to satisfy (12) in such a way that the total number of fields comes as close as possible to some bound K . Thus for this setup the matter fields are uniquely chosen once the gauge groups are selected. (More generally, we could consider an ensemble where the number of matter fields is allowed to vary, in which one would need to sum over multiples of $A_{i(i+1)}$ subject to a bound. This is difficult since the GCD of the products of integer subsets appearing in the denominator of (12) is likely sporadic.)

One key difference from the matter-free case, is that the *order* of the gauge groups is important. Different orderings will lead to different theories, except when the permutations are o symmetries of the quiver, e.g., the cyclic permutations nodes combined with reflections. These are elements of the dihedral group of symmetries of the regular polygon with vertices on the quiver nodes. Additional symmetries will arise if some N_i are equal and we will treat the exchange of groups with identical ranks as giving the same theory. This sort of measure would arise if we imagined our field theory landscape as arising from D-branes on a Calabi–Yau in which all the cycles give rise to gauge theories with the same coupling, which could happen if, e.g., we resolved an A_k singularity so that all two-cycles have equal size.

3.2.1. The canonical ensemble breaks down

We will first try to analyze the statistics of cyclic, chiral quivers in a canonical ensemble. All along, as motivated above, we will assume that the gauge groups uniquely fix the matter content. Let r_k be the number of times the group $U(k)$ appears. Then, the total rank N , and the number gauge factors L , are

$$N = \sum_k k r_k; \quad L = \sum_k r_k. \tag{13}$$

We want the partition function of this ensemble of *ordered* partitions of N :

$$Z = \sum_{\{r_k\}} \frac{1}{2} \left(\sum_k r_k - 1 \right)! e^{-\beta \sum_k k r_k - \alpha \sum_k r_k} \prod_k \frac{1}{r_k!}. \tag{14}$$

The combinatorial factor is the number of ways of choosing r_1, r_2, \dots gauge factors out of $\sum_k r_k$, divided by $2(\sum_k r_k)$ to account for the cyclic and reflection symmetry of the quiver.³ Rewriting in terms of the Gamma function, and using $\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t}$, we obtain

$$Z = \frac{1}{2} \int_0^\infty dt \frac{e^{-t}}{t} \exp\left(\frac{te^{-\alpha} e^{-\beta}}{1 - e^{-\beta}}\right). \tag{15}$$

This integral is only convergent if

$$\frac{e^{-\alpha} e^{-\beta}}{1 - e^{-\beta}} < 1 \Rightarrow e^{-\beta} < \frac{1}{1 + e^{-\alpha}} \equiv e^{-\beta_H}. \tag{16}$$

This implies a limiting β above which the partition function is undefined, because the integrand diverges as $t \rightarrow \infty$. There is also always a divergence as $t \rightarrow 0$ which can be regulated by recognizing that the divergence is a constant independent of α and β . To show this, define $\gamma = \frac{e^{-\alpha} e^{-\beta}}{1 - e^{-\beta}}$, and find $\frac{dZ}{d\gamma} = \int_0^\infty dt e^{-(1-\gamma)t} = \frac{1}{1-\gamma}$ which implies that, below the limiting temperature,

$$\begin{aligned} Z &= -\log(1 - \gamma) = -\log\left(1 - \frac{e^{-\alpha} e^{-\beta}}{1 - e^{-\beta}}\right) \\ &= -\log\left(1 - \frac{uq}{1 - q}\right), \end{aligned} \tag{17}$$

where $u = e^{-\alpha}$ and $q = e^{-\beta}$.

In order to achieve large rank, β must be tuned to close to its limiting value β_H (16). Then, if we put $u = 1$, the expectation value of the total rank is

$$\langle N \rangle = q \frac{\partial}{\partial q} \log Z \sim \frac{-1}{2\epsilon \log(4\epsilon)}, \tag{18}$$

where we tuned $q = q_H - \epsilon = \frac{1}{2} - \epsilon$ to get a large rank. Similarly, in this approximation

$$\langle r_k \rangle \sim \left(\frac{1}{2}\right)^{k+1} \frac{-1}{2\epsilon \log(4\epsilon)} \sim \left(\frac{1}{2}\right)^{k+1} \langle N \rangle. \tag{19}$$

This differs from the matter-free result for the typical partition: for example, on average one quarter of the nodes will be Abelian. However, we also find that

$$\text{Var}(r_k) \sim \left(\frac{1}{2}\right)^{2r+2} \frac{-1}{(2\epsilon)^2 \log(4\epsilon)} \sim -(1 + \log(4\epsilon)) \langle r_k \rangle^2. \tag{20}$$

This is much larger (as $\epsilon \rightarrow 0$) than the expectation value squared. In other words, the number of group factors with a given rank is not typical in the sense of (1).

Would a more coarse-grained question have a more statistically predictable answer? For example, how many gauge factors appear within some range of ranks? The mean and variance can be sums over (19), (20) because the r_k are independent random variables. In the central limit theorem, summing M identically distributed random variables enhance both the mean and the variance by M ; thus the variance to mean squared ratio is *reduced* by M . In the matter-free example, this happened because, although the r_k were not identically distributed, their dependence on k was sufficiently weak. Here, the exponential dependence of (19), (20) on the rank k means that this mechanism fails – the mean and the variance are dominated by the smallest k in the sum. Thus, there is no simple statistically predictable quantity in this landscape.

Here the canonical ensemble is breaking down and does not approximate the microcanonical ensemble. The canonical ensemble will reproduce the microcanonical ensemble when the growth of configuration space with total rank is slow enough so that, multiplied by a Boltzmann factor, a localized measure results. Here the Gamma function and the exponential in the measure compete on equal footing, leading to a widely spread out measure in which the rank of the gauge group fluctuates wildly over the ensemble. This sort of behavior will occur generally in the statistics of quivers since the number of graphs increases rapidly with the number of nodes. Thus we turn to the microcanonical ensemble.

3.2.2. Microcanonical analysis

Consider again a cyclic quiver and ignore accidental symmetries. The microcanonical partition function for cyclic gauge theories of rank N and L nodes is simply the number of such theories. This is given by the coefficient of q^N in

$$\frac{1}{2L} [q + q^2 + q^3 + \dots]^L. \tag{21}$$

The $1/2L$ divides out the cyclic permutations and reflections. We find that $Z_L = (1/2L)(N - 1)! / ((N - L)!(L - 1)!)$. Summing over L , a partition function which is canonical in the number of nodes and microcanonical in the total rank N is:

$$Z(u) = \sum_{L=1}^N u^L Z_L = \frac{(1 + u)^N - 1}{2N}. \tag{22}$$

To get the unbiased landscape in which all theories of equal rank have equal weight, we can set $u = 1$. The expectation value of L is

$$\langle L \rangle = u \partial_u \log(Z(u)) = \frac{u(1 + u)^{N-1}}{(1 + u)^N - 1} N. \tag{23}$$

When $u = 1$, we get $\langle L \rangle = \frac{N}{2}$ in the large N limit. However, if $u \sim \frac{1}{\sqrt{N}}$, then $\langle L \rangle \sim \sqrt{N}$, and if $u \sim \frac{1}{N}$, then $\langle L \rangle \sim O(1)$. In fact,

³ This counting ignores accidental symmetries. For example, in a cyclic quiver in which gauge groups $U(N_1)$ and $U(N_2)$ alternate, only one cyclic permutation gives a different quiver configuration. The complete counting can be derived using Polyá theory – we are using the leading terms.

if $u \sim N^{-a}$, $\langle L \rangle \sim N^{1-a}$. The canonical analysis gives the same expectation values. The microcanonical variance is

$$\text{Var}(L) = \left(1 - \frac{Nu}{(1+u)^N - 1}\right) \frac{1}{1+u} \langle L \rangle. \quad (24)$$

For the three scalings of u , i.e. $u \sim N^{-a}$, the variance in L is an order 1 number times the mean value of L , independent of a . Thus, when $\langle L \rangle$ is large, the variance to mean squared ratio is small, unlike the canonical analysis. This means that in such landscapes the number of gauge factors is typical predictable.

The expectation value for the number of Abelian factors is:

$$\langle r_1 \rangle = \frac{1}{Z} \sum_{L=1}^N \frac{u^L}{L} L(N - 2L - 2) = \frac{u^2(1+u)^{N-2}}{(1+u)^N - 1} N. \quad (25)$$

When $u = 1$, this is $\langle r_1 \rangle = 1/4N$ at large N . When $u \sim 1/\sqrt{N}$, $\langle r_1 \rangle \sim O(1)$. And when $u \sim 1/N$, $\langle r_1 \rangle \rightarrow 0$. In fact, for $u \sim N^{-a}$, $\langle r_1 \rangle \sim N^{1-2a}$. These expectations match the canonical ensemble, but microcanonical variance in r_1 is much smaller:

$$\langle r_1^2 \rangle = \frac{u^2(1+u)^{N-4}(u(uN+4)+1)}{(1+u)^N - 1} N. \quad (26)$$

Therefore, the ratio of the variance to the mean squared is

$$\frac{\langle r_1^2 \rangle - \langle r_1 \rangle^2}{\langle r_1 \rangle^2} = \frac{1 + u(4 - \frac{Nu}{(1+u)^N - 1})}{(1+u)^2} \times \frac{1}{\langle r_1 \rangle}. \quad (27)$$

The coefficient of $1/\langle r_1 \rangle$ in this expression is of $O(1)$ for $u \sim N^{-a}$, with $0 \leq a \leq 1$.

Pulling everything together, in the unbiased ensemble ($u = 1$), the average number of gauge factors is $N/2$ and the number of Abelian factors is $N/4$. These quantities are highly predictable in this landscape without any coarse-graining. In a biased ensemble with $u \sim 1/\sqrt{N}$, the total number of gauge factors is $O(\sqrt{N})$, and the number of Abelian factors is $O(1)$. Since variance is of the same order as the mean, the number of gauge factors is thus predictable, but the number of Abelian factors is not. In this case, we expect that a coarse-grained statistic, such the fraction of gauge groups in a given range, would be more predictable as in the matter-free case.

Higher ranks To find the expectation value of the occupation number of rank r , we can insert a “chemical potential” for that rank. So

$$Z(u, \{y_k\}) = \sum_{L=1}^N \frac{u^L}{2L} \left[\sum_{k=1}^N q^k y_k \right]_{q^N}^L, \quad (28)$$

where the left-hand side equals the coefficient of q^N in the right-hand side. The expectation value $\langle r_k \rangle$ is given by

$$\begin{aligned} \langle r_k \rangle &= \partial_{y_k} \log(Z(u, \{y_k\})) \Big|_{\{y_k\}=1} = \frac{1}{Z[u]} \sum_{L=1}^N u^L \binom{N-k-1}{L-2} \\ &= \frac{u^2(1+u)^{N-k-1}}{(1+u)^N - 1} N. \end{aligned} \quad (29)$$

In the unbiased ensemble ($u \sim 1$), $\langle r_k \rangle \sim (1/2)^{k+1} N$ canonically. Similarly,

$$\langle r_k^2 \rangle = \frac{u^2(1+u)^{N-2r-2}(2u + (N-2r+1)u^2 + (1+u)^{r+1})}{(1+u)^N - 1} N. \quad (30)$$

So the ratio of the variance to the mean squared is

$$\begin{aligned} \frac{\text{Var}(r_k)}{\langle r_k \rangle^2} &= \frac{1}{(1+u)^{k+1}} \left\{ (1+u)^{k+1} + u((1-2k)u+2) \right. \\ &\quad \left. - \frac{Nu^2}{(1+u)^N - 1} \right\} \times \frac{1}{\langle r_k \rangle}. \end{aligned} \quad (31)$$

This is always $O(1)$ times $1/\langle r_k \rangle$, and hence the number of gauge groups of a given rank is typical, and hence highly predictable, if the average is large.

Lessons In an anarchic landscape of cyclic quiver gauge theories, the number of gauge factors of a given rank is highly predictable. The distribution of ranks is exponential and low rank populations are predictable with high confidence. In a biased landscape in which the measure favors a number of gauge factors that is sufficiently smaller than the total rank, the number of factors with a fixed rank is not typical in general although the total number of factors can be. In this case, one could test whether a coarse grained quantity, like the fraction of gauge groups with ranks in some range, is more predictable.

4. Thinking about the general quiver

To extend our analysis to the general quiver gauge theory we could try to compute a partition sum of the form $Z = \sum_L \sum_{N_i, A_{ij}} \exp(-\beta \sum_i N_i - \lambda \sum_{ij} A_{ij} N_i N_j)$ where L is the number of nodes of the quiver, N_i are the ranks of the gauge groups, and A_{ij} are the numbers of bifundamentals between nodes i and j . One difficulty is that this partition sum is canonical and, as we found, it may not implement the constraints on the total rank and the amount of matter very well because of the rapid growth of the space of theories. Secondly the sum should only be over anomaly cancelled theories. Thirdly, there are discrete symmetries which tend to lead to vanishing expectation values. In view of this, below we will develop some approaches to dealing with the two latter issues.

4.1. Implementing anomaly cancellation

A loop basis for anomaly free theories

If all the gauge groups have the same rank, the general anomaly free theory can be constructed by making sure that the bifundamental fields always form closed loops. One can always construct such matter distributions by saying that each of the possible loops in the quiver has n_i fields running around it. Where loops overlap the matter content will either add or subtract depending on the orientation of the loops (again here we are supposing that non-chiral doublets decouple; in addition, we identify negative A_{ij} with a positive A_{ji} and vice versa). Any loop in the quiver can be constructed by summation of a basis of independent 3-loops and it can be shown that this basis will have

$$N_L = \frac{(L-1)(L-2)}{2} \quad (32)$$

elements. For example, consider the case with $L = 6$ nodes, i.e. there are six gauge groups that we label from 1 to 6. Then, the following three loops form a basis for all loops: (123), (124), (125), (126), (234), (235), (236), (345), (346), (456). The basis has 10 elements which is equal to $N_6 = (6-1)(6-2)/2$. We can check that the N_L loops provide enough free parameters to parameterize the space of anomaly free theories. To see this, note that the solutions to the anomaly cancellation equations form a vector space of dimension

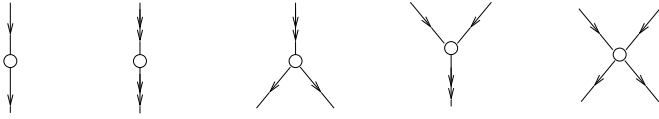


Fig. 1. The five vertices from which all anomaly-free, chiral, asymptotically free, equal rank theories are constructed.

$$\frac{L(L-1)}{2} - (L-1) = \frac{(L-1)(L-2)}{2} = N_L, \quad (33)$$

where $L(L-1)/2$ is the number of parameters A_{ij} from which we have subtracted the $(L-1)$ anomaly cancellation conditions on L groups.

Even when the ranks are unequal, anomaly free theories can be constructed from a basis of 3-loops because (32) and (33) are equal. However, the links of any given 3-loop will have to be populated with a different number of fields in a way related to the GCDs of the three groups appearing in it. For example, suppose one has the three gauge groups $SU(r_1 \cdot g) \times SU(r_2 \cdot g) \times SU(r_3 \cdot g)$ where r_i are a triple of positive integers that do not share a common factor and g is any another positive integer. Then if we take number of chiral bifundamentals between gauge group i and j to be $A_{ij} = \epsilon^{ijk} r_k$, we get an anomaly free theory.⁴

This suggests that one way to do the statistics of anomaly free theories is to select a basis of anomaly free 3-loops and then do the statistics of populations of these loops given a bound on the total number of loops.

Anomaly free, asymptotically free, chiral, equal rank gauge theories

This set of theories is very easy to analyze, as there are can be only five different types of vertices in such quivers (see Fig. 1). Therefore the most general quiver arises by combining these five vertices in various combinations. Superficially, the second vertex with two separate lines coming in and two separate lines going out allows for the largest amount of combinatorial freedom and will quite likely dominate this set of theories. It would be interesting to explore this class further. Possibly it can be mapped to an existing solvable lattice model in statistical mechanics.

Anomaly cancellation for a general quiver by using an extra node

If we drop the constraint of asymptotic freedom, the set of anomaly free, chiral, and equal rank theories is easy to parametrize. It is not difficult to see that if we take any set of edges \mathcal{S} such that the edges together form a connected tree which contain all vertices of the quiver, then the anomaly equations uniquely determine the A_{ij}, A_{ji} with $(ij) \in \mathcal{S}$ in terms of the A_{ij}, A_{ji} with $(ij) \notin \mathcal{S}$. Thus we can simply take an arbitrary set of chiral matter fields for all edges not in \mathcal{S} , after which anomaly cancellation uniquely fixes the remaining links.

An example of \mathcal{S} is the star-shaped tree consisting of all edges $(1i), i = 2 \dots L$. That is, if we remove one vertex and all its edges, and arbitrarily specify the chiral matter content in the remaining quiver with $L-1$ vertices, this uniquely determines an anomaly free, chiral, equal rank quiver gauge theory with L gauge groups. To illustrate, consider a four-node quiver. Take $A_{12} = a, A_{32} = b$ and

⁴ E.g., consider a 4-node quiver with gauge group $SU(3)_1 \times SU(5)_2 \times SU(7)_3 \times SU(8)_4$. We can get an anomaly free theory by making a loop of four with $A_{12} = 7 \cdot 8, A_{23} = 3 \cdot 8, A_{34} = 3 \cdot 5, A_{41} = 7 \cdot 5$. This is a sum of two 3 loops: $(124) + (234)$. The loop (123) corresponds to an $SU(3)_a \times SU(5)_b \times SU(8)_c$ with $A_{ab} = 8, A_{bc} = 3, A_{ca} = 5$ while the loop (234) corresponds to $SU(5)_i \times SU(7)_j \times SU(8)_k$ with $A_{ij} = 8, A_{jk} = 5, A_{ki} = 7$. To get the four loop (1234) , we must cancel the (24) link which means that need to add $7 \cdot (124) + 3 \cdot (234)$. Another anomaly free theory could be generated by adding (124) to (234) . In this case, the fields along link (24) will not cancel, but the number of fields going entering each gauge group will cancel.

$A_{13} = c$.⁵ Then anomaly cancellation uniquely fixes $A_{24} = a + b, A_{41} = a + c, A_{43} = b - c$.

To extend to theories with unequal ranks, first consider an arbitrary chiral, quiver with $L-1$ nodes. Let the rank of the group at the i th node be N_i . For anomaly cancellation, the net number of fundamentals minus antifundamentals at each node must be zero. Let K_i be the net excess matter (number of fundamentals minus antifundamentals) at each node. We can add an additional $U(1)$ gauge group with $N_i K_i$ bifundamental fields under this $U(1)$ and the $U(N_i)$ of the i th node. This will give an anomaly free theory. This extra node can be non-Abelian, but its rank is restricted to be a divisor of the set $\{N_i K_i\}$. In this way, the statistics of general anomaly free quivers on L nodes can be studied by first constructing arbitrary $L-1$ node quivers and then adding a extra node according to the above algorithm.

4.2. Dealing with discrete quiver symmetries: An example

From above, the set of anomaly free, chiral and equal rank theories with four nodes is parametrized by the rank N of the gauge groups and three integers a, b, c . The measure (3) becomes $\rho = \exp(-4\beta N - \lambda N^2 (|a| + |b| + |c| + |a+b| + |a+c| + |b-c|))$. In the remainder, we will fix the value of N and look only at the distribution of a, b, c .

By symmetry, the expectation values of a, b, c are all zero. This happens because there are a number of discrete symmetries of the quiver due to which averages vanish. For example, for every chiral quiver there is the anti-chiral quiver in which the orientations of all fields are reversed. Averaging these two will formally give $a = b = c = 0$. Similar phenomena will always happen whenever we consider sets of quivers with symmetries. More structure appears once we break the symmetries and look at the average quiver in an ensemble with some symmetry breaking conditions imposed. Suppose for example that we impose $a > 0$. This leaves a \mathbb{Z}_2 symmetry that exchanges vertices 3 and 4. Therefore, the expectation value of A_{34} will be zero. Symmetry considerations further show that $\langle \frac{1}{2} A_{12} \rangle = \langle A_{23} \rangle = \langle A_{24} \rangle = \langle A_{31} \rangle = \langle A_{41} \rangle$. Furthermore, each of these expectation values is proportional to $1/\lambda N^2$.

A boundary condition that completely breaks the symmetry is to impose that $a \geq b \geq 0$. We can always achieve this up to a permutation of the vertices so there is no loss of generality. The analysis of the expectation values of the number of matter fields in this ensemble is more tedious but can still be done explicitly. To leading order in $\epsilon = \lambda N^2$ we obtain⁶ $\langle A_{12} \rangle = \frac{47}{84\epsilon}, \langle A_{32} \rangle = \frac{4}{27\epsilon}, \langle A_{31} \rangle = \frac{61}{588\epsilon}, \langle A_{24} \rangle = \frac{3}{4\epsilon}, \langle A_{41} \rangle = \frac{67}{147\epsilon}, \langle A_{43} \rangle = \frac{173}{588\epsilon}$. Thus we see that after modding out the \mathbb{Z}_2 symmetries of the quiver we are able find an interesting average quiver. Of course, since there are only four nodes here, we do not expect any notion of statistical typicality. To study whether general large quivers have some typical structure, we will have to proceed as above, by parameterizing the space of anomaly cancelled theories and then imposing symmetry breaking conditions.

4.3. Towards dynamics

While we have been focusing on the structure of those field theories in which anomalies cancel, we should also be paying attention to dynamics. Since we are dealing with $\mathcal{N} = 1$ field theories, if $N_f > 3N_c$ for any gauge group then it will be infrared free. If $N_f < 3N_c$ it will be asymptotically free. If $N_f = 3N_c$ the one-

⁵ If we say $A_{12} = a$, then we mean that $A_{12} = a$ for $a \geq 0$ and $A_{21} = -a$ for $a \leq 0$. This will guarantee that the theory is chiral.

⁶ Here, by $\langle A_{ij} \rangle$ we really mean $\langle A_{ij} - A_{ji} \rangle$.

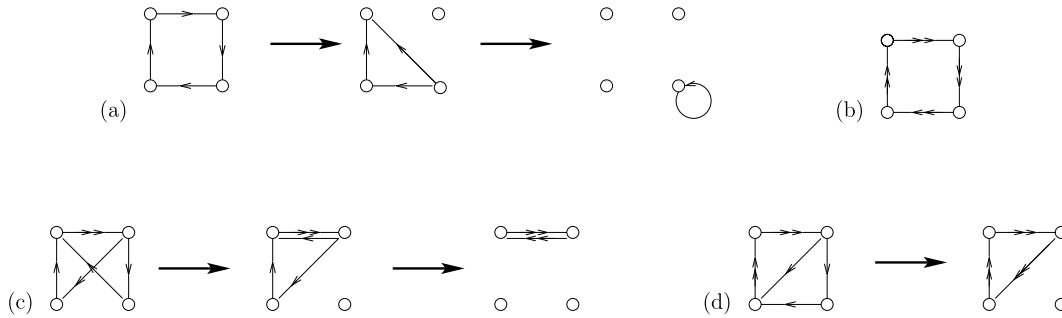


Fig. 2. Examples of RG flows in asymptotically free, four-node quiver theories with equal rank groups. The trivial tree level superpotential is assumed. (a) Flow to a Coulomb branch at low energies. (b) Flow to a CFT at low energies. (c) Assuming a higher scale for the groups with N flavors, this flows to a pair of confined groups, with the massless mesons of the two groups participating in an interacting CFT. (d) Assume that the group that confines has a higher dynamical scale than the other groups, and that the confinement is on the baryonic branch. The massless mesons of this confining factor drive a flow to an interacting CFT.

loop Beta function vanishes. If we distribute fields into a quiver, the bound of the total number of fields will tend to cause the low rank gauge groups to contain more fields. Thus they will tend to be infrared free. What is more, because, as we have seen above, anomaly cancellation including high rank gauge groups tends to require lots of fields, if a high rank group is connected to the rest of the quiver it would tend to push groups in the quiver towards infrared freedom. In general, studying RG flow requires us to know the superpotential or at least to scan statistically over them. Minimally, we should include all cubic and quartic terms in the superpotential with $O(1)$ coefficients multiplied by the appropriate scale. (The cubic terms are classically marginal, and some quartic terms are known to become marginal under RG flow.) Doing such a dynamical analysis of general quiver gauge theories is beyond the scope of this Letter, but as an initial step to gain some experience with how this works we will study some examples without a superpotential.

4.3.1. Four-node, asymptotically free quivers

Recall that $SU(N)$ gauge theory with N flavors confines at energies below its dynamical scale, while $SU(N)$ theory with $2N$ flavors flows to an interacting conformal fixed point. We will assume that the confining $SU(N)$ theory is on the baryonic branch. We can then naively take a quiver and simply allow individual gauge factors to confine, Seiberg dualize [21], etc., as their dynamics becomes strong. A cursory analysis of four-node, asymptotically free quivers (see some examples with equal ranks N in Fig. 2, constructed from the vertices in Fig. 1) suggests that one will tend to get interacting conformal field theories in which the mesons of the confining factors participate. This suggests that unparticles [22] might be generic in these settings.

4.3.2. General quiver with unequal gauge groups

First consider the case of a loop of three gauge groups, $SU(N_1) \times SU(N_2) \times SU(N_3)$ which cancels anomalies by itself. This can happen if the 3-loop is isolated within a larger quiver. Such primitive 3-loops can be used to generate larger anomaly free quiver gauge theories. To cancel anomalies, the (12), (23), (31) links will generically contain N_3 , N_1 , N_2 bifundamentals, respectively.⁷ Thus for group i to be asymptotically free, $3N_i > N_j N_k$, $i \neq j \neq k$. Taking all $N_i > 3$ and $N_1 < N_2 < N_3$, $SU(N_3)$ is the only gauge group that can be being asymptotically free. So, for any anomaly-free, chiral connected quiver with three nodes with

⁷ The minimal solution to the anomaly cancellation equations will actually be that the number of bifundamentals connecting i and j is $N_k/\text{GCD}(\{N_i, N_j, N_k\})$ as in (12). Generically the GCD = 1.

ranks ≥ 3 , either all three groups are IR free, or only the largest one is asymptotically free if it has sufficiently large rank.

This argument fails for connected quivers with more than three gauge groups, but generically high rank gauge groups with links to smaller rank gauge groups have a chance to be asymptotically free, whereas low rank gauge groups connected to higher rank gauge groups tend to be IR free. Consider cases for quiver dynamics with unequal gauge groups. (i) The number of fields K is very large. If so, it is likely that in a randomly chosen field theory all possible links in the quiver will be populated with some multiplicity, although the links between low rank groups will be enhanced. Then our arguments suggests that the entire theory will be infrared free. (ii) The number of fields K is small. The lowest rank gauge groups will tend to have matter and the quiver will typically consist of several disconnected smaller clusters that each form a connected quiver gauge theory. The high rank gauge groups with little matter would confine at their dynamical scales. (iii) For an intermediate number of fields the clusters will percolate and we expect an interesting phase structure.

5. Conclusion

It is unsettling to make *statistical* predictions for the structure of the theory describing nature because, ever since Galileo, we have been fortunate that observations and symmetries have constrained possibilities sufficiently to essentially give a unique theory. But we are trying to make predictions for the fundamental theory up to the Planck scale given observations below the TeV scale, subject to only very general constraints such as consistent coupling to quantum gravity. In such a situation, the best one can do is to predict the likelihood of possible high energy theories, conditioned on the known facts, known constraints, and our best guess regarding the measure on the space of theories. This is literally all that we can know. While this sort of Bayesian approach is unfamiliar in particle physics, it is much less unusual in cosmology where one does conceive of ensembles of possible universes or ensembles of domains with different low-energy physics in a single universe. Of course, consistency requirements plus experimental input might eventually yield a unique theory – we are merely entertaining the possibility that this will turn out otherwise.

We have used the uniform measure on specific effective field theory landscapes, but it could be dynamics can play a role in determining the appropriate measure because there can be transitions between vacua with different properties. Also, renormalization group flows can modify the measure in the infrared as theories flow towards their fixed points. Given the correct measure, our analysis could be repeated to find typical predictions. However, because the uniform measure leads to typicality for some

coarse-grained properties, an alternative measure would have to concentrate on an exponentially sparse part of the configuration space in order to change the typical predictions of the uniform measure.

These considerations do not suggest the usual desert with a high scale GUT. Instead one statistically expects a plethora of gauge factors leading to interesting structures at all scales up to the string scale. Some gauge factors will have high ranks and others will have low ranks. With a bound on the total number of matter fields, statistically, higher rank groups will tend to have fewer fundamentals (since this eats up matter). Thus they will tend towards confinement at a relatively high dynamical scale if all couplings are unified at the string scale. On the other hand if too much matter in any group will lead to infrared triviality. Thus low rank groups, to have IR dynamics, will tend to be largely decoupled from the high rank groups. Thus if we study the statistics of anarchic landscapes of field theories, conditioned on having interesting low energy dynamics, we will tend towards a structure with dynamical low rank groups largely decoupled from a complex, interacting higher rank sector.

The explicit examples that we studied do not much dynamics. The matter-free case confines. The ring quivers are generically infrared free since anomaly cancellation imposes the need for lots of matter unless individual gauge group ranks conspire to make the GCD in (12) large. Thus, conditioning on having interesting low energy dynamics, along with anomaly cancellation, will be a major constraint, and will modify the measure on the space of theories. Number theoretic properties like the appearance of large GCDs might need more weight. The results in [14,15] also suggest measures that weigh rank k gauge group factors with an extra factor of $1/k^2$.

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