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Arno Kret & Sug Woo Shin

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H^0 OF IGUSA VARIETIES VIA AUTOMORPHIC FORMS

by Arno Kret & Sug Woo Shin

Abstract. — Our main theorem describes the degree 0 cohomology of non-basic Igusa varieties in terms of one-dimensional automorphic representations in the setup of mod p Hodge-type Shimura varieties with hyperspecial level at p. As an application we obtain a completely new approach to two geometric questions. Firstly, we deduce irreducibility of Igusa towers and its generalization to non-basic Igusa varieties in the same generality, extending previous results by Igusa, Ribet, Faltings–Chai, Hida, and others. Secondly, we verify the discrete part of the Hecke orbit conjecture, which amounts to the assertion that the irreducible components of a non-basic central leaf belong to a single prime-to-p Hecke orbit, generalizing preceding works by Chai, Oort, Yu, et al. We also show purely local criteria for irreducibility of central leaves. Our proof is based on a Langlands–Kottwitz type formula for Igusa varieties due to Mack-Crane, an asymptotic study of the trace formula, and an estimate for unitary representations and their Jacquet modules in representation theory of p-adic groups due to Howe–Moore and Casselman.

Résumé (H^0 des variétés d'Igusa via les formes automorphes). — Notre théorème principal décrit la cohomologie en degré 0 des variétés d'Igusa non basiques en termes de représentations automorphes de dimension 1 dans le cadre des réductions modulo p des variétés de Shimura de type Hodge avec niveau hyper-spécial en p. Nous obtenons comme application une approche complètement nouvelle de deux questions géométriques. Premièrement, nous déduisons l'irréductibilité de la tour d'Igusa et sa généralisation aux variétés d'Igusa non basiques dans la même généralité, ce qui étend des résultats d'Igusa, Ribet, Falting-Chai, Hida, et d'autres. Deuxièmement, nous vérifions la partie discrète de la conjecture des orbites de Hecke, qui revient à l'assertion que les composantes irréductibles d'une feuille centrale non basique appartiennent à une unique orbite sous l'action de l'algèbre de Hecke première à p, ce qui généralise des travaux de Chai, Oort, Yu, entre autres. Nous démontrons aussi des critères purement locaux d'irréductibilité de la feuille centrale. Notre preuve est basée sur une formule de type Langlands-Kottwitz pour les variétés d'Igusa due à Mack-Crane, sur une étude asymptotique de la formule des traces, et sur une estimée pour les représentations unitaires et leurs modules de Jacquet en théorie des représentations des groupes p-adiques due à Howe-Moore et à Casselman.

Mathematical subject classification (2020). — 11G18.
Keywords. — Shimura varieties, Igusa varieties, central leaves, automorphic representations, trace formula.

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1. Introduction

Igusa varieties were studied by Igusa [Igu68] and Katz–Mazur [KM85] in the case of modular curves. Harris–Taylor and Mantovan [HT01, Man05] have generalized the construction to PEL-type Shimura varieties. Recently Caraiani–Scholze [CS17] gave a slightly different definition in the PEL case which gives the same cohomology. Hamacher, Zhang, and Hamacher–Kim went further to define Igusa varieties for Hodge-type Shimura varieties [Ham17, Zha15, HK19]. In the (μ-)ordinary setting, Igusa varieties are also referred to as Igusa towers. (Often the definitions differ in a minor way.) There are versions of Igusa varieties as p-adic formal schemes or adic spaces over p-adic fields, but we concentrate on the characteristic p varieties in this paper. We mention that function-field analogues of Igusa varieties are studied in a forthcoming paper by Sempliner; see also [Zho20, Ex. 4.7.15].

The ℓ-adic cohomology of Igusa varieties (with $\ell \neq p$) has several arithmetic applications. In [HT01, Man05, HK19], the authors prove a formula computing the cohomology of Shimura varieties in terms of that of Igusa varieties and Rapoport–Zink spaces. This means that, if we understand the cohomology of Igusa varieties well enough, then our knowledge of cohomology can be propagated from Rapoport–Zink spaces to Shimura varieties or the other way around. This is the basic principle underlying [Shi11, Shi12] on the global Langlands correspondence and the Kottwitz conjecture. For another application, a description of ℓ-adic cohomology of Igusa varieties was one of the main ingredients in [CS17, CS19] to prove vanishing of cohomology of certain Shimura varieties with ℓ-torsion coefficients, which in turn supplied a critical input for a recent breakthrough on the Ramanujan and Sato–Tate conjecture for cuspidal automorphic representations of GL_2 of "weight 2" over CM fields [ACC$^+$23].

Thus an important long-term goal is to compute the ℓ-adic cohomology of Igusa varieties with a natural group action. A major first step is a Langlands–Kottwitz style trace formula for Igusa varieties, which has been obtained for Shimura varieties of Hodge type at hyperspecial level in [Shi09, Shi10, MC22] building upon [HT01, Ch. 5] in analogy with [LR87, Kot92b, KSZ]. One wishes to turn that into an expression of the cohomology via automorphic forms, but this requires a solution of various

complicated problems; some are tractable but others are out of reach in general, most notably an endoscopic classification and Arthur's multiplicity formula for the relevant groups.

The main objective of this paper is twofold. Firstly, we describe H^0 of Igusa varieties via one-dimensional automorphic representations over non-basic Newton strata of Hodge-type Shimura varieties at hyperspecial level.(1) This mirrors the well-known fact that H^0 of complex Shimura varieties is governed by one-dimensional automorphic representations. Secondly, to achieve this, we develop a method and obtain various technical results with a view towards the entire cohomology of Igusa varieties (as an alternating sum over all degrees). Our method, partly inspired by Laumon [Lau05] and also by Flicker–Kazhdan [FK88], should prove useful for studying ℓ-adic cohomology of Shimura varieties as well.

Our result on H^0 not only sets a milestone in its own right, but also reveals deep geometric information. Namely, our theorem readily implies the discrete Hecke orbit conjecture for Shimura varieties and the irreducibility of Igusa varieties in the same generality as above. (The irreducibility means that Igusa varieties are no more reducible than the underlying Shimura varieties in some precise sense.) Our work provides a completely new approach and perspective to these two problems by means of automorphic forms and representation theory.

One of our main novelties consists in a careful asymptotic argument via the trace formula to single out H^0 (or compactly supported cohomology in the top degree) without reliance on any classification. This is essential for obtaining an unconditional result. Since the "variable" for asymptotics is encoded in the test function at p, a good amount of local harmonic analysis naturally enters the picture. Another feature of our approach is to allow induction on the semisimple rank of the group; this would make little sense in a purely geometric argument as endoscopy is hard to realize in the geometry of Shimura varieties.

Roughly speaking, cohomology of Igusa varieties is closely related to that of Shimura varieties via the Jacquet module operation at p, relative to a *proper* parabolic subgroup in the *non-basic* case. To show that only one-dimensional automorphic representations contribute to H^0 of Igusa varieties, the key representation-theoretic input is an estimate for the central action on Jacquet modules due to Casselman and Howe–Moore. Though there is no direct link, it would be interesting to note that a similar situation occurs in the context of beyond endoscopy (e.g., [FLN10, §5]), where the leading term in asymptotics is accounted for by the "most non-tempered" (namely one-dimensional) representations. This is also analogous to the spectral gap, which plays a crucial role in the Hecke equidistribution theorems in characteristic zero (cf. Section 1.3 below).

(1)In the basic case, Igusa varieties are 0-dimensional, and it can be deduced from [MC22] that their H^0 is expressed as the space of algebraic automorphic forms on an inner form of G. Such a description goes back to Serre [Ser96] for modular curves, and Fargues [Far04, Ch. 5] in the PEL case.

1.1. THE MAIN THEOREM. — Let (G, X) be a Shimura datum of Hodge type with reflex field $E \subset \mathbb{C}$. Assume that the reductive group G over $\mathbb{Q}$ admits a reductive model over $\mathbb{Z}_p$, and take $K_p := G(\mathbb{Z}_p)$. (Namely G is unramified at p, and K_p is a hyperspecial subgroup.) Write G^{ad} for the adjoint group of G. We do not assume $p > 2$ as the case $p = 2$ is covered in [KSZ, MC22].

Fix field maps $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$, and $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ (which will be mostly implicit). The resulting embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$ induces a place $\mathfrak{p}$ of E above p. Let $k(\mathfrak{p})$ denote the residue field of E at $\mathfrak{p}$, which embeds into the residue field $\overline{\mathbb{F}}_p$ of $\overline{\mathbb{Q}}_p$. Thereby we identify $\overline{k(\mathfrak{p})} \simeq \overline{\mathbb{F}}_p$. Let $\mathscr{S}_{K_p}$ denote the integral canonical model over $\mathcal{O}_{E_\mathfrak{p}}$ with a $G(\mathbb{A}^{\infty,p})$-action. In the main text, it is sometimes important to pass to finite level away from p in order to apply a fixed-point formula. However, we will ignore this point and pretend that we are always at infinite level away from p to simplify exposition.

An embedding of (G, X) into a Siegel Shimura datum determines a $G(\mathbb{A}^{\infty,p})$-equivariant map from $\mathscr{S}_{K_p}$ to a suitable Siegel moduli scheme over $\mathcal{O}_{E_\mathfrak{p}}$. Via pullback, we obtain a universal abelian scheme $\mathscr{A}$ over $\mathscr{S}_{K_p}$, which can be equipped with a family of étale and crystalline tensors over geometric points $\overline{x} \to \mathscr{S}_{K_p, k(\mathfrak{p})}$. This assigns to $\overline{x}$ the p-divisible group $\mathscr{A}_{\overline{x}}[p^\infty]$ (with G-structure, encoded by the family of crystalline tensors).

Let $\mu_p : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ denote the "Hodge" cocharacter arising from (G, X) (via $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$). This cuts out a finite subset $B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ in the Kottwitz set $B(G_{\mathbb{Q}_p})$ of G-isocrystals. Fix an element $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$. (The containment ensures that the Newton stratum N_b below is nonempty.) Then $[b]$ determines a p-divisible group with G-structure over $\overline{\mathbb{F}}_p$ up to isogeny via Dieudonné theory and the embedding of Shimura data above. Choose (the isomorphism class of) a p-divisible group with G-structure Σ_b in the isogeny class, which amounts to specifying $b \in G(\breve{\mathbb{Q}}_p)$ (up to σ-conjugation under $G(\breve{\mathbb{Z}}_p)$) whose image in $B(G_{\mathbb{Q}_p})$ is $[b]$.

We obtain a Newton cocharacter ν_b from b, which may be conjugated to be dominant with respect to a suitable Borel subgroup B of $G_{\mathbb{Q}_p}$ defined over $\mathbb{Q}_p$. Write ρ for the half sum of all B-positive roots. For simplicity, assume that Σ_b is defined over $k(\mathfrak{p})$ and that $[k(\mathfrak{p}) : \mathbb{F}_p]\nu_b$ is a cocharacter, not just a fractional cocharacter. (In practice, these assumptions are unnecessary since it is sufficient to have a finite extension of $k(\mathfrak{p})$ in the last sentence.)

Write J_b for the $\mathbb{Q}_p$-group of self-quasi-isogenies of Σ_b (preserving G-structure) over $\overline{\mathbb{F}}_p$, and J_b^{int} for the subgroup of $J_b(\mathbb{Q}_p)$ consisting of automorphisms. Then J_b^{int} is an open compact subgroup of $J_b(\mathbb{Q}_p)$. As a general fact, J_b is an inner form of a $\mathbb{Q}_p$-rational Levi subgroup M_b of $G_{\mathbb{Q}_p}$. We say that b is basic if ν_b is a central in $G_{\mathbb{Q}_p}$, or equivalently if $M_b = G_{\mathbb{Q}_p}$ (namely if J_b is an inner form of $G_{\mathbb{Q}_p}$). The element b is $\mathbb{Q}$-*non-basic* if the image of b in $B(G^{\mathrm{ad}}_{\mathbb{Q}_p})$ is non-basic in every $\mathbb{Q}$-simple factor of G^{ad}, cf. Definition 5.3.2. If G^{ad} is $\mathbb{Q}$-simple then this is equivalent to the condition that b is non-basic.

The central leaf C_b (resp. Newton stratum N_b) is the locus of $x \in \mathscr{S}_{K_p}$ on which the geometric fibers of $\mathscr{A}_{\overline{x}}[p^\infty]$ are isomorphic (resp. isogenous) to Σ_b. Thus N_b depends

only on $[b]$. By construction, C_b and N_b are stable under the $G(\mathbb{A}^{\infty,p})$-action on $\mathscr{S}_{K_p}$. We also define the Igusa variety $\mathfrak{Ig}_b$ over $\mathscr{S}_{K_p,k(\mathfrak{p})}$ to be the parameter space of isomorphisms between Σ_b and $\mathscr{A}_{\overline{x}}[p^\infty]$. The obvious action of J_b^{int} on $\mathfrak{Ig}_b$ naturally extends to a $J_b(\mathbb{Q}_p)$-action. Below are some basic facts (Sections 5.3, 6.1, and 6.2). Put $q := \#k(\mathfrak{p})$.

Fact 1. C_b is (formally) smooth over $k(\mathfrak{p})$ and closed in N_b,

Fact 2. C_b is equidimensional of dimension $\langle 2\rho, \nu_b\rangle$,

Fact 3. $\mathfrak{Ig}_b$ is a pro-étale J_b^{int}-torsor over the perfection of C_b,

Fact 4. In the completely slope divisible case, the q^r-th power Frobenius on $\mathfrak{Ig}_b$ coincides with the action of $\nu_b(q^r) \in Z_{J_b}(\mathbb{Q}_p)$ for sufficiently divisible r.

Fact 5. As an $\overline{\mathbb{F}}_p$-scheme with a $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-action, $\mathfrak{Ig}_b$ (up to isomorphism) depends on b only through $[b]$.

In particular, $\dim \mathfrak{Ig}_b = \langle 2\rho, \nu_b\rangle$, and every connected component of $\mathfrak{Ig}_{b,\overline{\mathbb{F}}_p}$ (resp. $C_{b,\overline{\mathbb{F}}_p}$) is irreducible.

Our main theorem describes the connected components (= irreducible components) of Igusa varieties over $\overline{\mathbb{F}}_p$ together with the $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-action. Let us introduce some notation. Write $G(\mathbb{Q}_p)^{\mathrm{ab}}$ for the abelianization of $G(\mathbb{Q}_p)$ as a topological group. There is a canonical map $\zeta_b : J_b(\mathbb{Q}_p) \twoheadrightarrow G(\mathbb{Q}_p)^{\mathrm{ab}}$ coming from the fact that J_b is an inner form of a Levi subgroup of $G_{\mathbb{Q}_p}$, cf. Section 6.1 below. Each one-dimensional smooth representation π_p of $G(\mathbb{Q}_p)$ factors through $G(\mathbb{Q}_p)^{\mathrm{ab}}$, giving rise to a one-dimensional representation $\pi_p \circ \zeta_b$ of $J_b(\mathbb{Q}_p)$.

THEOREM A (Theorem 6.1.4). — *Assume that b is $\mathbb{Q}$-non-basic with $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$. Then there is a $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-module isomorphism*

$$H^0(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_\pi \pi^{\infty,p} \otimes (\pi_p \circ \zeta_b),$$

where the sum runs over one-dimensional automorphic representations

$$\pi = \pi^{\infty,p} \otimes \pi_p \otimes \pi_\infty$$

of $G(\mathbb{A})$ such that π_∞ is trivial on the preimage of the neutral component $G^{\mathrm{ad}}(\mathbb{R})^0$ in $G(\mathbb{R})$.

Before we sketch the idea of proof, let us discuss two geometric applications.

1.2. Application to irreducibility of Igusa towers and a generalization

In Hida theory of p-adic automorphic forms, an important role is played by Igusa varieties over the *ordinary* Newton stratum, namely when the underlying p-divisible group is ordinary. In this case, Igusa varieties (and their natural extension to p-adic formal schemes) are usually referred to as Igusa towers. Recently Eischen and Mantovan [EM21] developed Hida theory in the more general μ-ordinary PEL-type situation, where Howe [How20] (and its sequel) also shed new light on the role of Igusa varieties (à la Caraiani–Scholze). Igusa towers are also featured in Andreatta–Iovita–Pilloni's work [AIP16, AIP18] on overconvergent automorphic forms.

A key property of Igusa towers is irreducibility. This property has an application to the q-expansion principle for p-adic automorphic forms, which is a basic ingredient for the construction of p-adic L-functions. See p. 96 in [Hid04] for the relevant remark, and also refer to Th. 3.3 (Igusa), 4.21 (Ribet), 6.4.3 (Faltings–Chai), and Cor. 8.17 (Hida) therein for the known cases (elliptic modular, Hilbert, Siegel, and PEL type A/C cases, respectively, all over the ordinary stratum) and further references. Irreducibility in the μ-ordinary case of PEL type A was proved in [EM21]. Such a result was obtained for Igusa varieties of a specific PEL type A by Boyer [Boy07] without assuming μ-ordinariness.

There are various methods to show the irreducibility as explained in [Cha08] and the introduction of [Hid11], e.g., by using the automorphism group of the function fields of Shimura varieties in characteristic 0 or by showing that the family of abelian varieties has large monodromy. As an application of Theorem A, we obtain an entirely different representation-theoretic proof and also a natural generalization from the μ-ordinary case to the general $\mathbb{Q}$-non-basic case (and from the PEL case to the case of Hodge type). In the non-μ-ordinary case, Igusa varieties lie over a central leaf rather than an entire Newton stratum, but our method is insensitive to such a distinction.

Write $J_b(\mathbb{Q}_p)' := \ker(\zeta_b : J_b(\mathbb{Q}_p) \to G(\mathbb{Q}_p)^{\mathrm{ab}})$. Our result is as follows.

Theorem B. — *Assume that b is $\mathbb{Q}$-non-basic. The stabilizer subgroup in $J_b(\mathbb{Q}_p)$ of each connected component of $\mathfrak{Ig}_b$ is equal to $J_b(\mathbb{Q}_p)'$.*

Roughly speaking, the stabilizer subgroup cannot be larger than $J_b(\mathbb{Q}_p)'$, and this should be thought of as saying that Igusa varieties are at least as reducible as Shimura varieties. The point of the theorem is that, conversely, the stabilizer is as large as possible under the given constraint; so Igusa varieties are "irreducible" in the sense that they are no more reducible than Shimura varieties. (This is made precise by Corollary 8.1.2 below.) The proof is almost immediate from the $J_b(\mathbb{Q}_p)$-action on H^0 described in Theorem A. See Section 8.1 below for further details.

1.3. Application to the discrete Hecke orbit conjecture. — In 1995 [Oor19, §15] (also see [EMO01, Prob. 18]), Oort proposed the Hecke Orbit (HO) conjecture that the prime-to-p Hecke orbit of a point should be Zariski dense in the central leaf containing it, if the point lies outside the basic stratum (if G^{ad} is simple). The reader is referred to [CO19] for an excellent survey of the HO conjecture with updates. Oort drew analogy with the André–Oort conjecture for a Shimura variety in characteristic zero, which asserts that the irreducible components of the Zariski closure of a set of special points are special subvarieties. (See [Tsi18, PST21] and references therein for recent results on the André–Oort conjecture.) A common feature is that a set of points with an extraordinary structure (being a prime-to-p Hecke orbit or special points) is Zariski dense in a distinguished class of subvarieties. We can also compare the HO conjecture with the Hecke equidistribution theorems for locally symmetric spaces in characteristic zero [COU01, EO06], stating roughly that the Hecke orbit of an arbitrary point is equidistributed in the locally symmetric space in a suitable

sense. (In particular, the Hecke orbit is dense in the entire space even for the analytic topology, to be contrasted with the phenomenon in characteristic p.) It is also worth noting works to investigate Hecke orbits for the p-adic topology [GK21, HMRL20].

Chai and Oort verified the HO conjecture for Siegel modular varieties [Cha06, Th. 3.4] (details to appear in a monograph), in particular the irreducibility of leaves [Cha05, CO11]. The conjecture is also known for Hilbert modular varieties [Cha06, Th. 3.5] due to Chai and Yu. (Also see [YCO20].) The HO conjecture has seen several new results in recent years. Shankar proved the conjecture for Deligne's "strange models" (in the sense of [Del71, §6]) in an unpublished preprint. Zhou [Zho19] settled the HO conjecture in the ordinary locus of some quaternionic Shimura varieties along the way to realize a geometric level raising between Hilbert modular forms. Maulik–Shankar–Tang [MST22] proved the HO conjecture in the ordinary locus of GSpin Shimura varieties. Xiao [Xia20] proved partial results on the HO conjecture in the case of PEL type A and C.

Chai [Cha05, Cha06] proposed the strategy to divide the HO conjecture into two parts, that is, the discrete part ($\mathrm{HO}_{\mathrm{disc}}$) and the continuous part ($\mathrm{HO}_{\mathrm{cont}}$), corresponding to global and local geometry, respectively. In a nutshell, ($\mathrm{HO}_{\mathrm{disc}}$) asserts that the prime-to-p Hecke action is transitive on the set of irreducible components of each central leaf. Then ($\mathrm{HO}_{\mathrm{cont}}$) is designed to tell us that the closure of each prime-to-p Hecke orbit has the same dimension as the ambient central leaf, so that ($\mathrm{HO}_{\mathrm{disc}}$) and ($\mathrm{HO}_{\mathrm{cont}}$) together imply the HO conjecture.

We deduce the following result on ($\mathrm{HO}_{\mathrm{disc}}$) from Theorem A. (See Section 8.2 for details.)

THEOREM C. — *For Hodge-type Shimura varieties with hyperspecial level at p,* ($\mathrm{HO}_{\mathrm{disc}}$) *is true for every central leaf contained in a $\mathbb{Q}$-non-basic Newton stratum.*

To our knowledge, this is the first general theorem on ($\mathrm{HO}_{\mathrm{disc}}$). Let us remark on the proof. Since ($\mathrm{HO}_{\mathrm{disc}}$) means transitivity of the $G(\mathbb{A}^{\infty,p})$-action on $\pi_0(C_b)$, it is equivalent to the multiplicity one property of the trivial $G(\mathbb{A}^{\infty,p})$-representation in $H^0(C_b, \overline{\mathbb{Q}}_\ell) = H^0(\mathfrak{Jg}, \overline{\mathbb{Q}}_\ell)^{J_b^{\mathrm{int}}}$. To prove Theorem C, it is thus enough to observe that if $\pi^{\infty,p}$ is trivial then π_p and π_∞ must be trivial as well in the formula of Theorem A. This is an easy consequence of the weak approximation that $G(\mathbb{Q})$ is dense in $G(\mathbb{Q}_p) \times G(\mathbb{R})$. (The same approximation holds more generally, at least if G splits over an unramified extension.)

We also consider the following strengthening of ($\mathrm{HO}_{\mathrm{disc}}$):

($\mathrm{HO}^+_{\mathrm{disc}}$) The map $\pi_0(C_b) \to \pi_0(\mathrm{Sh}_{K_p})$ induced by the immersion $C_b \to \mathrm{Sh}_{K_p}$ is a bijection.

This is known as "irreducibility of central leaves", as it means that C_b is irreducible in every component of Sh_{K_p}. Since $G(\mathbb{A}^{\infty,p})$ is known to act transitively on $\pi_0(\mathrm{Sh}_{K_p})$, e.g., by weak approximation, and since $\pi_0(C_b) \to \pi_0(\mathrm{Sh}_{K_p})$ is $G(\mathbb{A}^{\infty,p})$-equivariant, it is clear that ($\mathrm{HO}^+_{\mathrm{disc}}$) implies ($\mathrm{HO}_{\mathrm{disc}}$).

We prove purely local criteria for $(\mathrm{HO}^+_{\mathrm{disc}})$, either in terms of groups at p or in terms of the stabilizers of points on some affine Deligne–Lusztig varieties (Theorem 8.2.9). Using these criteria, we deduce $(\mathrm{HO}^+_{\mathrm{disc}})$ in the μ-ordinary case, but obtain a counterexample in general with the help of Rong Zhou. See Section 8.2 below for further details. (In the earlier version of this paper `arXiv:2102.10690v1`, we incorrectly asserted that $(\mathrm{HO}^+_{\mathrm{disc}})$ was true in general. The mistake occurred during the initial reduction in the proof of Lemma 8.1.1, where changing b to a σ-conjugate element cannot be justified; we thank van Hoften for pointing it out to us.)

1.4. Some details on the proof of Theorem A. — Changing Σ_b by a quasi-isogeny, as this does not affect $\mathfrak{Ig}_b$ up to isomorphism, we may assume that Σ_b is completely slope divisible and defined over a finite field. Then $\mathfrak{Ig}_b$ can be written, up to perfection, as the projective limit of smooth varieties of finite type defined over $\mathbb{F}_{p^r}$ for a sufficiently divisible $r \in \mathbb{Z}_{>0}$. (In the main text, we use Ig_b to denote the version without perfection.) This allows us to apply a Lefschetz trace formula technique to compute the cohomology of $\mathfrak{Ig}_b$ at a finite level. Via Poincaré duality, Theorem A may be rephrased in terms of the top degree compact-support cohomology $H_c^{\langle 4\rho,\nu_b\rangle}(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell)$, which we may access by the Lang–Weil estimate.

Adapting the Langlands–Kottwitz method to Igusa varieties, as worked out in [Shi09] and Mack-Crane's thesis [MC22], one obtains a formula of the form

$$\mathrm{Tr}(\phi^{\infty,p}\phi_p \times \mathrm{Frob}_{p^r}^j | H_c(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell)) = (\text{geometric expansion}), \qquad j \in \mathbb{Z}_{\gg 1},$$

where $\phi^{\infty,p}\phi_p \in \mathcal{H}(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$ and $j \in \mathbb{Z}_{\gg 1}$. In fact, one can show that the Frob_{p^r}-action on $\mathfrak{Ig}_b$ is represented by the action of a central element of $J_b(\mathbb{Q}_p)$. Thereby $\phi_p \times \mathrm{Frob}_{p^r}^j$ in (1.4.1) may be replaced with a translate $\phi_p^{(j)} \in \mathcal{H}(J_b(\mathbb{Q}_p))$ of ϕ_p by a central element. The geometric expansion is a linear combination of orbital integrals of $\phi^{\infty,p}\phi_p^{(j)}$ on $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ over a certain set of conjugacy classes. The stabilized formula takes the form

$$\mathrm{Tr}(\phi^{\infty,p}\phi_p^{(j)} | H_c(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell)) = \sum_{\mathfrak{e}} (\text{constant}) \cdot ST_{\mathrm{ell}}^{\mathfrak{e}}(f^{\mathfrak{e},p} f_p^{\mathfrak{e},(j)}), \qquad j \in \mathbb{Z}_{\gg 1}, \tag{1.4.1}$$

where the sum runs over endoscopic data $\mathfrak{e}$ for G (Section 2.6), and $f^{\mathfrak{e},p} f_p^{\mathfrak{e},(j)}$ is a suitable function on the corresponding endoscopic group $G^{\mathfrak{e}}$. By $ST_{\mathrm{ell}}^{\mathfrak{e}}$ we mean the elliptic part of the stable trace formula for $G^{\mathfrak{e}}$. The most nontrivial point in the stabilization is the "transfer" at p. Indeed, as $G^{\mathfrak{e}}$ is not an endoscopic group of J_b, this requires a special construction as detailed in Section 3.

Ideally we would turn the right hand side of (1.4.1) into a spectral expansion and determine not only $H_c^{\langle 4\rho,\nu_b\rangle}(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell)$ but $H_c(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell)$ in the Grothendieck group of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-representations. This is the long-term goal stated earlier. On the analogous problem for Shimura varieties, a road map has been laid out in [Kot90], which can be mimicked for Igusa varieties to some extent. However there are serious obstacles: (1) An endoscopic classification for most reductive groups is out of reach; exactly the same issue occurs for Shimura varieties as well. (2) The geometric side

(stable elliptic terms) is very difficult to compare with the spectral side. One could imagine making the comparison more tractable by passing from H_c to intersection cohomology, following the strategy for Shimura varieties to "fill in" the stable non-elliptic terms, but no theory of compactification is available for Igusa varieties to allow it. (Franke's formula for H_c of locally symmetric spaces [Fra98] suggests that one should expect a similarly complicated answer for H_c of Igusa varieties.)

Our goal is to extract spectral information on $H_c^{\langle 4\rho,\nu_b\rangle}(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell)$ from the leading terms in (1.4.1) in the variable j via the Lang–Weil estimate. Thus we can get away with less by proving equalities up to error terms of lower order. To bypass (1) and (2), a key is to show that (stable) non-elliptic terms as well as endoscopic (a.k.a. unstable) terms have slower growth in j than the (stable) elliptic terms. This is the technical heart of our paper taking up Section 4. Let us provide more details.

The basic strategy is an induction on the semisimple rank, based on our observation that some key property of the function $f_p^{\mathfrak{e},(j)}$ is replicated after taking an endoscopic transfer or a constant term. (For instance, we need to pass along the Newton cocharacter through the inductive steps.) So we want to prove a bound on the trace formula for a quasi-split group over $\mathbb{Q}$, with a test function $f^p f_p^{(j)}$ satisfying such a property. The desired bound partly comes from a root-theoretic computation, involving a curious interaction between p and ∞ such as "evaluating" the Newton cocharacter (coming from p) at the infinite place (Lemma 4.1.1). The most interesting component in this part of the argument is

(∗) a spectral expansion of T_{ell}, the elliptic part of the trace formula.

The problem is actually about $ST_{\mathrm{ell}}^{\mathfrak{e}}$ in (1.4.1), but we can replace $ST_{\mathrm{ell}}^{\mathfrak{e}}$ with T_{ell} for $G^{\mathfrak{e}}$ once the difference is shown to have lower order of growth. The archimedean test function is stable cuspidal in our setting, so we have Arthur's simple trace formula [Art89] of the following shape:

$$T_{\mathrm{disc}}(f^{\mathfrak{e},p} f_p^{\mathfrak{e},(j)}) = T_{\mathrm{ell}}(f^{\mathfrak{e},p} f_p^{\mathfrak{e},(j)}) + (\text{geometric terms on proper Levi subgroups}). \tag{1.4.2}$$

The proper Levi terms at finite places look similar to the elliptic part of the trace formula for proper Levi subgroups, but a complicated behavior is seen at the infinite place due to stable discrete series characters along non-elliptic maximal tori of the ambient group. On different open Weyl chambers, we have different character formulas in terms of finite dimensional characters of the Levi subgroup, so this quickly spirals out of control in the induction. Adapting an idea of Laumon [Lau97] from the non-invariant trace formula, we overcome the difficulty by imposing a regularity condition on the test function at an auxiliary prime q ($\neq p$) and show that the $\mathbb{Q}$-conjugacy classes with nonzero contributions land in a single Weyl chamber. Then a finite dimensional character of a Levi subgroup is itself a stable discrete series character of the same Levi subgroup along elliptic maximal tori of the Levi, so that the inductive argument can continue. (No information is lost by the auxiliary hypothesis

at q, cf. Section 7.6 below.) This technique should prove useful for investigation of compactly supported cohomology of Igusa varieties and Shimura varieties alike.

Returning to our problem, the above argument turns (1.4.1) into

$$\operatorname{Tr}(\phi^{\infty,p}\phi_p^{(j)}|H_c(\mathfrak{Jg}_b,\overline{\mathbb{Q}}_\ell)) = \sum_{\pi^*} m(\pi^*)\operatorname{Tr}(f^{*,p}f_p^{*,(j)}) + (\text{error terms}),$$

where $f^{*,p}f_p^{*,(j)}$ is the test function on the quasi-split inner form G^* of G (i.e., when $G^{\mathfrak{e}} = G^*$), and the sum runs over discrete automorphic representations of $G^*(\mathbb{A})$. At this point, we apply a trace identity. Let $\phi_p^{*,(j)}$ denote a transfer of $\phi_p^{(j)}$ from J_b to its quasi-split inner form M_b. For each irreducible smooth representation π^* of $G^*(\mathbb{Q}_p)$, we have (Lemma 3.1.2)

$$\operatorname{Tr}\pi_p^*(f_p^{*,(j)}) = \operatorname{Tr} J(\pi_p^*)(\phi_p^{*,(j)}),$$

where J is the normalized Jacquet module relative to the parabolic subgroup determined by ν_b whose Levi component is M_b. Since b is non-basic, M_b is a proper Levi subgroup. Moreover the translation (j) is given by a central element satisfying a positivity condition with respect to ν_b. In these circumstances, we make a crucial use of an estimate due to Casselman and Howe–Moore (Section 2.1), showing that $J(\pi_p^*)(\phi_p^{*,(j)})$ has the highest growth if and only if $\dim\pi_p^* = 1$. A strong approximation argument (Section 2.5) promotes this to the condition that $\dim\pi^* = 1$, under a group-theoretic condition guaranteed in our setting. Moreover, it is not hard to transfer one-dimensional representations from $M_b(\mathbb{Q}_p)$ to $J_b(\mathbb{Q}_p)$ compatibly with the transfer of functions (Section 2.3). We complete the proof of Theorem A by putting this final piece of the puzzle.

1.5. A remark on the non-hyperspecial case. — This paper focuses on the case of hyperspecial level at p mainly because the trace formula for Igusa varieties [MC22] is available only in that case. Once the trace formula becomes available for Shimura varieties with parahoric level at p (cf. Section 1.7 below), the methods and results of this paper should extend to that case. To avoid group-theoretic subtleties (e.g., Remark 2.3.4 below), assume that G is quasi-split over $\mathbb{Q}_p$. Then Theorems A and B are expected to remain true (with a modified definition of J_b^{int}). As for Theorem C, a crucial group-theoretic ingredient is that the diagonal embedding $G(\mathbb{Q}) \to G(\mathbb{Q}_p)\times G(\mathbb{R})$ has dense image (weak approximation). If G does not split over an unramified extension of $\mathbb{Q}_p$, then the weak approximation can be false, in which case our argument does not apply. In fact, Oki's example [Oki23] suggests that the analogue of Theorem C is false in general, since the prime-to-p Hecke action is not even transitive on the set of connected components of the underlying Shimura variety.

1.6. The basic case. — We comment on the description of H^0 in the complementary case when b is basic, in the setting of Section 1.1. Since it is not the focus of this paper, we will be brief. In the basic case, Igusa varieties are 0-dimensional by Fact 2 above. It follows from [HZZ21, Prop. 5.2.2] and the argument of [MC22, §3.2] that

there is a $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-equivariant bijection

$$\mathfrak{Ig}_b(\overline{\mathbb{F}}_p) \simeq I(\mathbb{Q})\backslash G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p), \tag{1.6.1}$$

where I is an inner form of G over $\mathbb{Q}$ such that $I(\mathbb{A}^{\infty,p}) \simeq G(\mathbb{A}^{\infty,p})$, $I(\mathbb{Q}_p) \simeq J_b(\mathbb{Q}_p)$, and $I(\mathbb{R})$ is compact modulo center. (This I is the same as in [HZZ21].) Here $I(\mathbb{Q})$ acts by left multiplication on $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ via

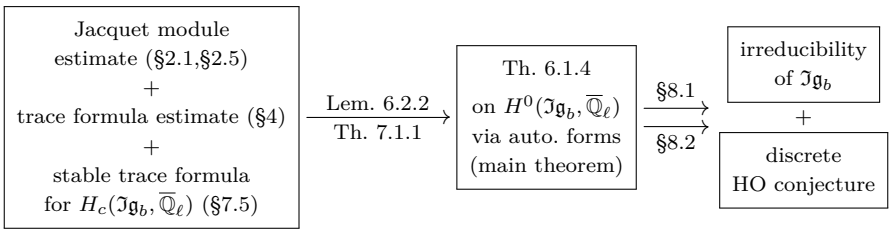
$$I(\mathbb{Q}) \longhookrightarrow I(\mathbb{A}^{\infty}) \simeq G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p).$$

(The embedding is canonical up to $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-conjugacy.) Hence the analogue of Theorem A in the basic case is that $H^0(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell)$ is the space of algebraic automorphic forms on the inner form I. In particular, H^0 is "much larger" in the basic case.

Since $C_b(\overline{\mathbb{F}}_p)$ is the quotient of $\mathfrak{Ig}_b(\overline{\mathbb{F}}_p)$ by an open compact subgroup of $J_b(\mathbb{Q}_p)$ (Lemma 6.1.1.(1) is still valid when b is basic), we can deduce from (1.6.1) and the weak approximation for I (applied as in the proof of Lemma 5.2.2) that the full Hecke orbit conjecture is true; this is equivalent to $(\mathrm{HO}_{\mathrm{disc}})$ in the case at hand. On the other hand, we see that $(\mathrm{HO}^{+}_{\mathrm{disc}})$ is generally false if b is basic.

1.7. WORK OF VAN HOFTEN AND XIAO. — Pol van Hoften and Luciena Xiao Xiao [vH10, vHX21] prove the irreducibility of Igusa varieties (but not Theorems A and C of our paper) and give a counterexample to $(\mathrm{HO}^{+}_{\mathrm{disc}})$.[(2)] Their method is more geometric and totally different from ours in that no use is made of automorphic forms. Further goals in their work and ours are disparate. For instance, [vH10] proves new results on the stratification of Shimura varieties and the Langlands–Rapoport conjecture in the parahoric case, whereas our work is a stepping stone for understanding the cohomology of Igusa varieties in all degrees. The two threads could have a future intersection though, as the Langlands–Rapoport conjecture in the parahoric case ought to be an important ingredient for deriving the analogue for Igusa varieties in that case, extending [MC22] from the hyperspecial case.

1.8. A GUIDE FOR THE READER. — The bare-bones structure of our argument is as follows.



(2) The counterexample in [vHX21, §6.3] is about $(\mathrm{HO}^{+}_{\mathrm{disc}})$, but not $(\mathrm{HO}_{\mathrm{disc}})$, cf. Section 8.2 of this paper. Note that the maps in Th. 6.2.1 and Cor. 6.2.2 therein are not asserted to be equivariant for the prime-to-p Hecke actions. In fact, our Theorem C suggests that those maps should not be equivariant in general.

On a first reading, we suggest that all complexities arising from central characters and z-extensions should be skipped, e.g., by assuming that all central character data are trivial. In fact this should be the case in many examples. The central character datum is always trivial on the level of G appearing in the Hodge-type datum, but we allow it to be nontrivial mainly because we do not know whether $\mathcal{H}$ in the endoscopic datum (Section 2.6) can always be chosen to be an L-group. Another good idea is to start reading in Section 5, especially if one's main interests lie in geometry, referring to the earlier sections only as needed and taking the results there for granted.

Sections 2 and 3 consist of mostly background materials in local harmonic analysis and representation theory. Though we claim little originality, there may be some novelty in the way we organize and present them. Some statements would be of independent interest. Section 4 is perhaps the most technical as this is where the main trace formula estimates are obtained. As such, most readers may want to take the results in Section 4.2 on faith and proceed, returning to them as needed.

Sections 5 and 6 introduce the main geometric players, namely Shimura varieties, central leaves, and Igusa varieties. Except for Section 5.1, we are always in the Hodge-type case with hyperspecial level at p. Our main theorem on Igusa varieties is stated in Section 6.1. After reduction steps in Sections 6.2–7.1 and some recollection of the trace formula setup up to Section 7.5, the proof of the theorem is completed in Section 7.6. Lastly Section 8 is devoted to the main geometric applications on irreducibility of Igusa varieties and a local criterion for the discrete Hecke orbit conjecture.

1.9. Notation

– The trivial character (of the group that is clear from the context) is denoted by $\mathbf{1}$.

– If T is a torus over a field k with algebraic closure $\overline{k}$, $X_*(T) := \mathrm{Hom}_{\overline{k}}(T, \mathbb{G}_m)$ and $X^*(T) := \mathrm{Hom}_{\overline{k}}(\mathbb{G}_m, T)$. When R is a $\mathbb{Z}$-algebra, we write $X_*(T)_R := X_*(T) \otimes_{\mathbb{Z}} R$ and $X^*(T)_R := X^*(T) \otimes_{\mathbb{Z}} R$.

– $\mathbb{D} := \varprojlim \mathbb{G}_m$ is the protorus (over an arbitrary base), where the transition maps are the n-th power maps.

– $\breve{\mathbb{Z}}_p := W(\overline{\mathbb{F}}_p)$, $\breve{\mathbb{Q}}_p := \mathrm{Frac}\, \breve{\mathbb{Z}}_p$, and $\sigma \in \mathrm{Aut}(\breve{\mathbb{Q}}_p)$ is the arithmetic Frobenius. By $\mathbb{Z}_p^{\mathrm{ur}}$ (resp. $\mathbb{Q}_p^{\mathrm{ur}}$) we mean the subring of elements in $\breve{\mathbb{Z}}_p$ (resp. $\breve{\mathbb{Q}}_p$) which are algebraic over $\mathbb{Q}_p$.

– $\mathscr{P}(S)$ is the power set of a set S.

– If H is an algebraic group over a field k, we write $H^0 \subset H$ for its neutral component.

Let G be a connected reductive group over a field k of characteristic 0.

– If k is a finite extension of k_0, then $\mathrm{Res}_{k/k_0} G$ denotes the restriction of scalars group.

– If k' is an extension field of k then $G_{k'} := G \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$.

– G_{der} is the derived subgroup, $\varrho : G_{\mathrm{sc}} \to G_{\mathrm{der}} \subset G$ the simply connected cover, Z_G the center (we also write $Z(G)$), $G^{\mathrm{ad}} := G/Z_G$ the adjoint group, and $G^{\mathrm{ab}} := G/G_{\mathrm{der}}$

the maximal commutative quotient. Write $A_G \subset Z_G$ for the maximal split subtorus over k.

– $G(k)_?$ is the set of semisimple (resp. regular semisimple, resp. strongly regular) elements in $G(k)$ for $? = \mathrm{ss}$ (resp. reg, resp. sr). We put $T(k)_? := T(k) \cap G(k)_?$ for $? \in \{\mathrm{reg}, \mathrm{sr}\}$.

– If k is a local field and G a reductive group over k, write $\mathcal{I}(G(k))$ and $\mathcal{S}(G(k))$ for the spaces of invariant and stable distributions on $G(k)$. (For more details, see Section 2.2). By $\mathrm{Irr}(G(k))$ we mean the set of isomorphism classes of irreducible admissible representations of $G(k)$.

– When $k = \mathbb{Q}_p$, two elements $\delta, \delta' \in G(\breve{\mathbb{Q}}_p)$ are $(G(\breve{\mathbb{Q}}_p), \sigma)$-*conjugate* (resp. $(G(\breve{\mathbb{Z}}_p), \sigma)$-*conjugate*) if there exists a $g \in G(\breve{\mathbb{Q}}_p)$ (resp. $g \in G(\breve{\mathbb{Z}}_p)$) such that $\delta' = \sigma(g)\delta g^{-1}$.

Let T (resp. S) be a maximal torus (resp. maximal split torus) of G over k with $T \supset S$. Let M_0 be a minimal k-rational Levi subgroup containing T.

– $\Phi(T, G)$ is the set of absolute roots, $\Phi(S, G) = \Phi_k(S, G)$ the set of k-rational roots.

– $\overline{\Omega}^G = \Omega(T, G)$ for the Weyl group over $\overline{k}$, and $\Omega_k^G = \Omega(S, G)$ for the k-rational Weyl group. We often omit k from $\Phi_k(S, G)$ and Ω_k^G when it is clear from the context.

– $\mathcal{L}(G)$ or $\mathcal{L}_k(G)$ is the set of all k-rational Levi subgroups of G containing M_0. Write $\mathcal{L}^<(G) := \mathcal{L}(G) \backslash \{G\}$.

LEMMA 1.9.1. — *If G_{der} is simply connected then every k-rational Levi subgroup of G has simply connected derived subgroup.*

Proof. — This can be checked after base change to $\overline{k}$, so assume $k = \overline{k}$. For every maximal torus $T \subset G$, the cocharacter lattice $X_*(T)$ modulo the coroot lattice is torsion free by hypothesis. Thus $X_*(T)$ modulo the lattice generated by an arbitrary subset of simple coroots is torsion free, implying that every Levi subgroup of G has simply connected derived subgroup. □

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2. Preliminaries in representation theory and endoscopy

2.1. Estimates for Jacquet modules of unitary representations. —

Here we recall some facts from work of Howe–Moore [HM79] and Casselman [Cas95] in order to bound the absolute value of central characters in the Jacquet modules of unitary representations of p-adic reductive groups.

We consider the following setup and notation.

– Let F be a non-archimedean local field of characteristic 0. We write val_F, $\mathcal{O}_F$, k, q, $\varpi = \varpi_F$ respectively for the normalized valuation of F, the ring of integers of F, the residue field of F, the cardinality of k, and an uniformizer of F so that $\mathrm{val}_F(\varpi_F) = 1$,
– G is a connected reductive group over F with center $Z = Z_G$,
– $\mathrm{Rep}(G)$ is the category of smooth representations of $G(F)$,
– $P = MN$ is a Levi decomposition of an F-rational proper parabolic subgroup of G,
– A_M is the maximal F-split torus in the center of M,
– Δ is the set of roots of A_M in N,
– $A_P^- := \{x \in A_M(F) : |\alpha(x)| \leqslant 1,\ \forall \alpha \in \Delta\}$,
– $A_P^{--} := \{x \in A_M(F) : |\alpha(x)| < 1,\ \forall \alpha \in \Delta\}$,
– $\delta_P : M(F) \to \mathbb{R}^\times_{>0}$ is the modulus character given by

$$\delta_P(m) := |\det(\mathrm{Ad}(m), \mathrm{Lie}\, N(F))|\,.$$

– $J_P \colon \mathrm{Rep}(G) \to \mathrm{Rep}(M)$ is the normalized Jacquet module functor, so $J_P(\pi)$ equals $\pi_N \otimes \delta_P^{-1/2}$ with π_N denoting the $N(F)$-coinvariants of π,
– $I_P^G \colon \mathrm{Rep}(M) \to \mathrm{Rep}(G)$ is the normalized parabolic induction functor, sending π_M to the smooth induction of $\pi_M \otimes \delta_P^{1/2}$ from $P(F)$ to $G(F)$.
– When $R \in \mathrm{Rep}(M)$ has finite length, write $\mathrm{Exp}(R)$ for the set of $A_M(F)$-characters appearing as central characters of irreducible subquotients of R.

Lemma 2.1.1. — *If G is simply connected, F-simple, and F-isotropic, then every normal subgroup of $G(F)$ is either $G(F)$ itself or contained in $Z(F)$.*

Proof. — A normal subgroup N of $G(F)$ not contained in $Z(F)$ is open of finite index in $G(F)$ by [PR94, Prop. 3.17] since G is F-simple. Since $G(F)$ is F-isotropic and simply connected, $G(F)$ is generated by the F-points of the unipotent radicals of F-rational parabolic subgroups [PR94, Th. 7.6]. Thus, by Tits' theorem proved in [Pra82], every open proper subgroup of $G(F)$ is compact. On the other hand, N is easily seen to be non-compact by considering the adjoint action of a maximal F-split torus on a root subgroup.[(3)] Therefore, $N = G(F)$. □

Proposition 2.1.2 (Howe–Moore). — *Assume that G_{sc} is F-simple. Let π be an infinite dimensional irreducible unitary representation of $G(F)$. Then there exists an integer $2 \leqslant k < \infty$ such that every matrix coefficient of π belongs to $L^k(G(F)/Z(F))$.*

Proof. — This follows from the explanation on pp. 74–75 of [HM79] below Th. 6.1, once we verify the following claim: if $\pi(g)$ is a scalar operator for $g \in G(F)$ then $g \in Z(F)$. Taking a z-extension of G, we reduce to the case when G_{der} is simply connected. Pulling back π via the multiplication map $Z(F) \times G_{\mathrm{der}}(F) \to G(F)$ and passing to one of the finitely many constituents (cf. [Xu16, Lem. 6.2]) which is infinite-dimensional, we may assume that G is itself F-simple and simply connected. Now Z'

[(3)] For instance, see the proof of Proposition 3.9 in `http://virtualmath1.stanford.edu/~conrad/JLseminar/Notes/L2.pdf` for details.

be the group of $g \in G(F)$ such that $\pi(g)$ is a scalar. Then Z' is a normal subgroup of $G(F)$, and $Z' \neq G(F)$ since $\dim \pi = \infty$. Therefore, $Z' \subset Z(F)$ by Lemma 2.1.1, proving the claim. □

Proposition 2.1.3 (Casselman). — *Let π be an irreducible unitary representation of $G(F)$. For every $\omega \in \mathrm{Exp}(\pi_N)$ and every $a \in A_P^-$, we have the inequality*

$$|\omega(a)| \leqslant 1. \tag{2.1.1}$$

If G_{sc} is F-simple and $a \in A_P^{--}$, then the equality holds if and only if $\dim \pi < \infty$.

Proof. — The inequality (2.1.1) follows from the obvious extension of [Cas95, §4.4] (where $p < \infty$ is assumed) to cover the case $p = \infty$. (For instance, [Cas95, Lem. 4.4.3, Prop. 4.4.4] have the analogues for $p = \infty$, with "bounded" in place of "summable" and "$|\chi(x)| \leqslant 1$" in place of "$|\chi(x)| < 1$".)

As for the last assertion, suppose that $\dim \pi = \infty$. In the notation of [Cas95, §2.5], Proposition 2.1.2 tells us that the matrix coefficient is L^k, i.e., $|c_{v,\widetilde{v}}|^k$ is integrable modulo center for some $2 \leqslant k < \infty$. Applying [Cas95, Cor. 4.4.5] to $p = k$, $F = c_{v,\widetilde{v}}$ and $a \in A_P^{--}$, we obtain that $|\omega(a)\delta_P^{-1/k}(a)| < 1$. Therefore, $|\omega(a)| < 1$. For the converse, suppose that $\dim \pi < \infty$. Then $\ker \pi$ is an open subgroup of $G(F)$. As the open subgroup $N(F) \cap \ker \pi$ of the unipotent subgroup $N(F)$ acts trivially on π, we see that $N(F)$ itself acts trivially on π. (Use conjugation by $A_M(F)$.) Therefore, $\mathrm{Exp}(\pi_N)$ consists of the central character ω of π (restricted to $M(F)$) only, which is unitary. In particular, $|\omega(a)| = 1$ for all $a \in A_P^{--}$ □

Remark 2.1.4. — Proposition 2.1.3 is sharp in general.

For example, consider $G = \mathrm{GL}_2(F)$ with P (resp. N) consisting of upper triangular (resp. upper triangular unipotent) matrices. The complementary series representations $\pi_\varepsilon = I_P^G(|\cdot|^\varepsilon, |\cdot|^{-\varepsilon})$ with $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 1/2$ are irreducible and unitary. We have

$$\begin{aligned}(\pi_\varepsilon)_N = J_N(\pi_\varepsilon) \otimes \delta_P^{1/2} &= \delta_P^{1/2} \otimes \big((|\cdot|^\varepsilon, |\cdot|^{-\varepsilon}) \oplus (|\cdot|^{-\varepsilon}, |\cdot|^\varepsilon)\big) \\ &= (|\cdot|^{\varepsilon+1/2}, |\cdot|^{-\varepsilon-1/2}) \oplus (|\cdot|^{-\varepsilon+1/2}, |\cdot|^{\varepsilon-1/2}).\end{aligned}$$

So in this case, $\mathrm{Exp}((\pi_\varepsilon)_N)$ contains the character $\omega = (|\cdot|^{-\varepsilon+1/2}, |\cdot|^{\varepsilon-1/2})$ of $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. Then $a = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in A_P^{--}$. We get $\omega(a) = p^{\varepsilon-1/2}$ which gets arbitrarily close to 1 as ε tends to $1/2$.

Lemma 2.1.5. — *Assume that G^{ad} has no F-anisotropic factor. Then every irreducible smooth representation of $G(F)$ is either one-dimensional or infinite-dimensional.*

Proof. — We may assume that G_{der} is simply connected via z-extensions. Suppose that π is a finite-dimensional irreducible smooth representation of $G(F)$. Then the normal subgroup $\ker \pi \cap G_{\mathrm{der}}(F)$ of $G_{\mathrm{der}}(F)$ is open. Lemma 2.1.1 implies that $\ker \pi \cap G_{\mathrm{der}}(F) = G_{\mathrm{der}}(F)$, thus π factors through the abelian quotient $G(F)/G_{\mathrm{der}}(F)$. Therefore, $\dim \pi = 1$, completing the proof. □

2.2. Local Hecke algebras and their variants. — We retain the notation from the preceding section but allow the local field F to be either non-archimedean or archimedean. A basic setup of local Hecke algebras will be introduced, partly following [Art96, §1].

Fix a Haar measure on $G(F)$ and a maximal compact subgroup $K \subset G(F)$. Let $G(F)_{\mathrm{sr}}$ denote the subset of strongly regular elements $g \in G(F)$, namely the semisimple elements whose centralizers in G are (maximal) tori. By [Ste65, 2.15], $G(F)_{\mathrm{sr}}$ is open and dense in $G(F)$ (for both the Zariski and non-archimedean topologies). Write $R(G)$ for the space of finite $\mathbb{C}$-linear combinations of irreducible characters of $G(F)$, which is a subspace in the space of functions on $G(F)_{\mathrm{sr}}$. We also identify $R(G)$ with the Grothendieck group of smooth finite-length representations of $G(F)$ with $\mathbb{C}$-coefficients. Let $\mathcal{H}(G) = \mathcal{H}(G(F))$ denote the space of smooth compactly supported bi-K-finite functions on $G(F)$. Let $\mathcal{I}(G)$ denote the invariant space of functions on $G(F)$, namely the quotient of $\mathcal{H}(G)$ by the ideal generated by functions of the form $g \mapsto f(g) - f(hgh^{-1})$ with $h \in G(F)$ and $f \in \mathcal{H}(G)$. From [Kaz86, Th. 0], we see that $f \in \mathcal{H}(G)$ has trivial image in $\mathcal{I}(G)$ if and only if its orbital integral vanishes on $G(F)_{\mathrm{sr}}$ if and only if $\operatorname{Tr}\pi(f) = 0$ for all irreducible tempered representations of $G(F)$; moreover, the same is true if $G(F)_{\mathrm{sr}}$ is replaced with $G(F)$ and if the temperedness condition is dropped. By abuse of notation, we frequently write $f \in \mathcal{I}(G)$ to mean a representative $f \in \mathcal{H}(G)$ of an element in $\mathcal{I}(G)$. The trace Paley–Wiener theorem [BDK86] describes $\mathcal{I}(G)$ as a subspace of $\mathbb{C}$-linear functionals on $R(G)$ via

$$f \longmapsto \left(\Theta \mapsto \int_{G(F)_{\mathrm{sr}}} f(g)\Theta(g)dg\right). \tag{2.2.1}$$

If $R(G)$ is thought of as a Grothendieck group, the above map is simply $f \mapsto (\pi \mapsto \operatorname{Tr}\pi(f))$.

Denote by $\mathcal{S}(G)$ the quotient of $\mathcal{H}(G)$ by the ideal generated by functions each of which has vanishing stable orbital integrals on $G(F)_{\mathrm{sr}}$. Thus we have natural surjections $\mathcal{H}(G) \twoheadrightarrow \mathcal{I}(G) \twoheadrightarrow \mathcal{S}(G)$. By $R(G)^{\mathrm{st}}$ we mean the subspace of $R(G)$ consisting of stable linear combinations (i.e., constant on each stable conjugacy class in $G(F)_{\mathrm{sr}}$). Then $\mathcal{S}(G)$ is identified with a subspace of functions on $R(G)^{\mathrm{st}}$ via (2.2.1) (since $\Theta \in R(G)^{\mathrm{st}}$ now, the image depends only on the image of f in $\mathcal{S}(G)$); the subspace is characterized by [Art96, Th. 6.1, 6.2] in the p-adic case, cf. last paragraph on p. 491 of [Xu17]. Via the obvious quotient map $\mathcal{I}(G) \to \mathcal{S}(G)$ and the restriction map from $R(G)$ to $R(G)^{\mathrm{st}}$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{I}(G) & \longrightarrow & \mathcal{S}(G) \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ \mathrm{Hom}_{\mathbb{C}\text{-linear}}(R(G), \mathbb{C}) & \longrightarrow & \mathrm{Hom}_{\mathbb{C}\text{-linear}}(R(G)^{\mathrm{st}}, \mathbb{C}). \end{array}$$

Let us extend the setup so far to allow a fixed central character. By a *local central character datum* for G, we mean a pair $(\mathfrak{X}, \chi)$, where

– $\mathfrak{X}$ is a closed subgroup of $Z(F)$ equipped with a Haar measure $\mu_{\mathfrak{X}}$ on $\mathfrak{X}$,
– $\chi : \mathfrak{X} \to \mathbb{C}^\times$ is a smooth character.

Let $\mathcal{H}(G,\chi^{-1}) = \mathcal{H}(G(F),\chi^{-1})$ denote the space of smooth bi-K-finite functions f on $G(F)$ which have compact support modulo $\mathfrak{X}$ and satisfy $f(xg) = \chi^{-1}(x)f(g)$ for $x \in \mathfrak{X}$ and $g \in G(F)$. The χ-averaging map

$$\mathcal{H}(G) \longrightarrow \mathcal{H}(G,\chi^{-1}), \qquad f \longmapsto \left(g \mapsto \int_{\mathfrak{X}} f(gz)\chi(z)d\mu_{\mathfrak{X}}\right),$$

is a surjection. We have the obvious definitions of $\mathcal{I}(G,\chi^{-1})$ and $\mathcal{S}(G,\chi^{-1})$, the χ-averaging maps $\mathcal{I}(G) \to \mathcal{I}(G,\chi^{-1})$ and $\mathcal{S}(G) \to \mathcal{S}(G,\chi^{-1})$, as well as the quotient maps

$$\mathcal{H}(G,\chi^{-1}) \twoheadrightarrow \mathcal{I}(G,\chi^{-1}) \twoheadrightarrow \mathcal{S}(G,\chi^{-1}).$$

We can think of $\mathcal{I}(G,\chi^{-1})$ as a subspace of functions on $R(G,\chi)$, the subspace of $R(G)$ generated by irreducible characters with central character χ. Analogously $\mathcal{S}(G,\chi^{-1})$ is the subspace of functions on $R(G,\chi)^{\mathrm{st}}$ defined similarly.

2.3. TRANSFER OF ONE-DIMENSIONAL REPRESENTATIONS. — Let G and G^* be connected reductive groups over a non-archimedean local field F of characteristic zero, with G^* quasi-split over F. Let $\xi : G_{\overline{F}} \xrightarrow{\sim} G^*_{\overline{F}}$ be an inner twisting, namely an $\overline{F}$-isomorphism such that $\xi^{-1}\sigma(\xi)$ is an inner automorphism of $G_{\overline{F}}$ for every $\sigma \in \mathrm{Gal}(\overline{F}/F)$. As in Section 1.9, we have canonical F-morphisms $\varrho : G_{\mathrm{sc}} \to G$ and $\varrho^* : G^*_{\mathrm{sc}} \to G^*$. Define an F-torus and two topological groups

$$G^{\mathrm{ab}} := G/G_{\mathrm{der}}, \quad G(F)^\flat := \mathrm{cok}(G_{\mathrm{sc}}(F) \xrightarrow{\varrho} G(F)), \quad G(F)^{\mathrm{ab}} := G(F)/G(F)_{\mathrm{der}},$$

where $G(F)_{\mathrm{der}}$ is the commutator subgroup of $G(F)$ as an abstract group, which is closed in $G(F)$. (This is clear if G is a torus. If not, $G(F)_{\mathrm{der}}$ is not contained in $Z_{G_{\mathrm{der}}}(F)$ so an open subgroup in $G_{\mathrm{der}}(F)$ by [PR94, Th. 3.3], after reducing to the simply connected and F-simple case via z-extensions.) Moreover, $G(F)_{\mathrm{der}}$ is contained in $\mathrm{im}(G(F)_{\mathrm{sc}} \to G(F))$ [Del79, 2.0.2], so there are natural morphisms

$$G(F) \twoheadrightarrow G(F)^{\mathrm{ab}} \twoheadrightarrow G(F)^\flat \twoheadrightarrow G(F)/G_{\mathrm{der}}(F) \hookrightarrow G^{\mathrm{ab}}(F). \tag{2.3.1}$$

In particular, $G(F)^\flat$ is an abelian group. The last two maps in (2.3.1) are isomorphisms if $G_{\mathrm{der}} = G_{\mathrm{sc}}$ by Kneser's vanishing theorem for H^1 of simply connected groups (applicable since F is non-archimedean). The definition and discussion above applies to G^* in the same way.

Let $1 \to Z_1 \to G_1 \xrightarrow{\alpha} G \to 1$ be a z-extension of G over F. Since $G_1 \to G$ induces $G_1^{\mathrm{ad}} \xrightarrow{\sim} G^{\mathrm{ad}}$, the classifying data for inner twists of G_1 and those of G are identified (up to isomorphism). Thus we may assume that there is a z-extension $1 \to Z_1 \to G_1^* \xrightarrow{\alpha^*} G^* \to 1$ with an inner twisting $\xi_1 : G_{1,\overline{F}} \xrightarrow{\sim} G^*_{1,\overline{F}}$ such that ξ_1 and ξ form a commutative square together with the maps α and α^*. The map $G_{1,\mathrm{der}} \to G_{\mathrm{der}}$ induced by α is a simply connected cover, allowing an identification $G_{1,\mathrm{der}} = G_{\mathrm{sc}}$. Likewise we have $G^*_{1,\mathrm{der}} = G^*_{\mathrm{sc}}$.

LEMMA 2.3.1. — *There is a row-exact commutative diagram where the vertical maps are isomorphisms:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z_1(F)/Z_1(F)\cap G_{1,\mathrm{der}}(F) & \longrightarrow & G_1(F)^\flat = G_1^{\mathrm{ab}}(F) & \longrightarrow & G(F)^\flat & \longrightarrow & 1 \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 1 & \longrightarrow & Z_1(F)/Z_1(F)\cap G^*_{1,\mathrm{der}}(F) & \longrightarrow & G'_1(F)^\flat = G_1^{*,\mathrm{ab}}(F) & \longrightarrow & G^*(F)^\flat & \longrightarrow & 1. \end{array}$$

Here the second vertical map is given by the isomorphism $G_1^{\mathrm{ab}} \xrightarrow{\sim} G_1^{*,\mathrm{ab}}$ induced by ξ, and the first and third vertical maps are induced by the second. Moreover the isomorphism $G(F)^\flat \xrightarrow{\sim} G^*(F)^\flat$ is canonical, i.e., independent of the choice of z-extensions.

We will write $\xi^\flat : G(F)^\flat \xrightarrow{\sim} G^*(F)^\flat$ for the canonical isomorphism.

Proof. — The row-exactness in the diagram is straightforward from the definition. The map ξ_1 induces an F-isomorphism $G_1^{\mathrm{ab}} \xrightarrow{\sim} G_1^{*,\mathrm{ab}}$ and restricts to an F-isomorphism from Z_1 onto Z_1. Thus the first two vertical maps are isomorphisms, which implies that the last vertical map is also. To check the last assertion, if $G_1 \xrightarrow{\alpha_1} G$ and $G_2 \xrightarrow{\alpha_2} G$ are two z-extensions, then the fiber product of G_1 and G_2 over G is also a z-extension. So we may assume that there is a surjection between the two z-extensions, in which case the last assertion is clear. □

LEMMA 2.3.2. — *If G_{sc} has no F-anisotropic factor, then $G(F)^\flat = G(F)^{\mathrm{ab}}$.*

Proof. — We may assume that G is not a torus. Via a z-extension, we reduce to the case when $G_{\mathrm{sc}} = G_{\mathrm{der}}$. Then $G(F)_{\mathrm{der}}$ is a noncentral normal subgroup of $G_{\mathrm{der}}(F)$. Applying Lemma 2.1.1 to each F-simple factor of G_{der}, we deduce that $G(F)_{\mathrm{der}} = G_{\mathrm{der}}(F)$, hence $G(F)^\flat = G(F)^{\mathrm{ab}}$. □

COROLLARY 2.3.3. — *If G_{sc} has no F-anisotropic factor, then the following four groups (under multiplication) are in canonical isomorphisms with each other:*

(1) *the group of smooth characters $G(F) \to \mathbb{C}^\times$,*
(2) *the group of smooth characters $G(F)^\flat \to \mathbb{C}^\times$,*
(3) *the group of smooth characters $G^*(F)^\flat \to \mathbb{C}^\times$,*
(4) *the group of smooth characters $G^*(F) \to \mathbb{C}^\times$,*

where the maps from (2) *to* (1) *and from* (3) *to* (4) *are given by pullbacks, and the map between* (2) *and* (3) *is via the isomorphism of Lemma* 2.3.1. *With no assumption on G_{sc}, we still have canonical isomorphisms between* (2), (3), *and* (4), *and a canonical embedding from* (2) *to* (1).

Proof. — Since $G(F)^{\mathrm{ab}}$ is the maximal abelian topological quotient of $G(F)$, we can replace $G(F)$ with $G(F)^{\mathrm{ab}}$ in (1), and likewise for (4). From (2.3.1) and Lemma 2.3.1, we have

$$G(F)^{\mathrm{ab}} \twoheadrightarrow G(F)^\flat \simeq G^*(F)^\flat \twoheadleftarrow G^*(F)^{\mathrm{ab}}.$$

Lemma 2.3.2 tells us that the last map is always an isomorphism (since G^* has no F-anisotropic factor); so is the first map if G has no F-anisotropic factor. The corollary follows. □

REMARK 2.3.4. — The only nontrivial F-anisotropic simply connected simple group over F is of the form $\mathrm{Res}_{F'/F}\mathrm{SL}_1(D)$ for a central division algebra D over a finite extension F' of F with $[D:F]=n^2$ and $n\geqslant 2$. So the condition in the corollary is that G_{sc} has no such factor. Two exemplary cases are (i) $G=\mathrm{GL}_1(D)$, $G^*=\mathrm{GL}_n(F)$ and (ii) $G=\mathrm{SL}_1(D)$, $G^*=\mathrm{SL}_n(F)$. In (i), it is standard (e.g., [Rie70, Intro.]) that $G(F)_{\mathrm{der}}=G_{\mathrm{der}}(F)$, and the four sets are still isomorphic. However, in (ii), $G(F)_{\mathrm{der}}$ is the group of 1-units in the maximal order of D by [Rie70, §5, Cor.]. In particular, (1) is a nontrivial group, whereas (2) and (3) are evidently trivial, thus (4) is trivial by the corollary.

REMARK 2.3.5. — One can also construct a natural map from (4) to (1) through the continuous cohomology $H^1(W_F,Z(\widehat{G}))=H^1(W_F,Z(\widehat{G}^*))$ following Langlands. (This works for archimedean local fields F as well.) Indeed, [Xu16, App. A] explains the isomorphism between $H^1(W_F,Z(\widehat{G}^*))$ and (4), and a map from $H^1(W_F,Z(\widehat{G}^*))$ to (1).[(4)]

Let $g\in G(F)_{\mathrm{ss}}$ and $g^*\in G^*(F)_{\mathrm{ss}}$. When $G_{\mathrm{der}}=G_{\mathrm{sc}}$, we say g and g^* are *matching* if their $\overline{F}$-conjugacy classes correspond via ξ. In general, *matching* is defined by lifting ξ to an inner twisting between z-extensions of G and G^* as in [Kot82, pp. 799–800] (specialized to the case $E=F$). From loc. cit. we see that the notion of matching is independent of the choice of z-extensions, and depends only on the $G(\overline{F})$-conjugacy class of ξ.

Since G^* is quasi-split, every g admits a matching element in $G^*(F)$ (again by [loc. cit.]). When g and g^* are matching, we have an inner twisting between the connected centralizers I_g, I_{g^*} in G, G^* by [Kot82, Lem. 5.8]. Fix Haar measures on the pairs of inner forms $(G(F),G^*(F))$ and $(I_g(F),I_{g^*}(F))$ compatibly in the sense of [Kot88, p. 631] to define (stable) orbital integrals at g and g^*, cf. [Kot88, pp. 637–638]. Write $e(G)\in\{\pm 1\}$ for the Kottwitz sign. Now $f\in\mathcal{H}(G(F))$ and $f^*\in\mathcal{H}(G^*(F))$ are said to be *matching* if for every $g^*\in G^*(F)_{\mathrm{sr}}$, we have the identity of stable orbital integrals

$$SO_{g^*}(f^*)=\begin{cases} SO_g(f), & \text{if there exists a matching } g\in G(F)_{\mathrm{ss}},\\ 0, & \text{if there is no such } g\in G(F)_{\mathrm{ss}}.\end{cases} \tag{2.3.2}$$

REMARK 2.3.6. — The sign convention in (2.3.2) is chosen in favor of simplicity. (See also Remark 7.4.1 below.) One could require $SO_{g^*}(f^*)=e(G)SO_g(f)$ instead, so that

(4) The latter map is asserted to be also an isomorphism in [Xu16, App. A], but this is false for $G=\mathrm{SL}_1(D)$ (in which case $Z(\widehat{G})=\{1\}$) as explained in Remark 2.3.4. In loc. cit., for a simply connected group G' over F, it is said that all continuous characters $G'(F)\to\mathbb{C}^\times$ are trivial, but this is not guaranteed unless G_{sc} has no F-anisotropic factor (e.g., this is okay for G^*). This mistake is surprisingly prevalent in the literature.

the Kottwitz sign $e(G)$ plays the role of transfer factor, but that would introduce $e(G)$ in the trace identity of Lemma 2.3.7.

A standard fact (cf. Section 2.6 below) from [Wal97] is every f admits a matching f^* as above, called a *(stable) transfer* of f. If the Harish-Chandra character Θ_{π^*} of $\pi^* \in \mathrm{Irr}(G^*(F))$ is stable, i.e., $\Theta_{\pi^*}(g_1^*) = \Theta_{\pi^*}(g_2^*)$ whenever $g_1^*, g_2^* \in G^*(F)_{\mathrm{sr}}$ are stably conjugate, then the value $\mathrm{Tr}\,\pi^*(f^*) = \int_{G^*(F)_{\mathrm{sr}}} f^*(g^*)\Theta_{\pi^*}(g^*)dg^*$ is determined by the stable orbital integrals of f^* on $G^*(F)_{\mathrm{sr}}$. This follows from the stable version of the Weyl integration formula, cf. (2.3.3) below. The analogue holds true with G and f in place of G^* and f^*. Note that Θ_{π^*} is stable if $\dim \pi^* = 1$. (This can be checked after reducing via z-extensions to the case when $G^*_{\mathrm{sc}} = G^*_{\mathrm{der}}$.)

Lemma 2.3.7. — *Let $f \in \mathcal{H}(G(F))$ and $f^* \in \mathcal{H}(G^*(F))$ be matching functions. Let $\pi^* : G^*(F) \to \mathbb{C}^\times$ be a smooth character. If $\pi : G(F) \to \mathbb{C}^\times$ is given by π^* via Corollary* 2.3.3 *then*

$$\mathrm{Tr}\,\pi(f) = \mathrm{Tr}\,\pi^*(f^*).$$

Proof. — As $\dim \pi = \dim \pi^* = 1$, we have $\Theta_\pi(g) = \pi(g)$ and $\Theta_{\pi^*}(g^*) = \pi^*(g^*)$ for $g \in G(F)$, $g^* \in G^*(F)$. Above the lemma, we observed that Θ_{π^*} is stable. This implies that Θ_π is stable. For a maximal torus T of G over F, write $W(T)$ for the associated Weyl group. By the stable Weyl integration formula,

$$\mathrm{Tr}\,\pi(f) = \int_{G(F)_{\mathrm{sr}}} f(g)\Theta_\pi(g)dg = \sum_T \frac{1}{|W(T)|} \int_{T(F)_{\mathrm{sr}}} SO_t(f)\Theta_\pi(t)dt, \tag{2.3.3}$$

where the sum runs over a set of representatives for stable conjugacy classes of maximal tori of G over F. The analogous formula holds for $G^*(F)$. From here, the proof is an easy exercise using (2.3.2) and the following fact coming from quasi-splitness of $G^*(F)$: every maximal torus of $G(F)$ is a transfer of that of $G^*(F)$ in the sense of [Kot84b, 9.5]. □

Remark 2.3.8. — The correspondence of Lemma 2.3.7 need not be the Jacquet–Langlands correspondence when $G^* = \mathrm{GL}_n$. E.g., if $G = \mathrm{GL}_1(D)$ for a central division algebra D over a p-adic field F with $n > 1$, then the trivial representation of $D^\times$ corresponds to the Steinberg representation of $\mathrm{GL}_n(F)$ under Jacquet–Langlands, but to the trivial representation of $\mathrm{GL}_n(F)$ in the lemma.

2.4. Lefschetz functions on real reductive groups. — Let G be a connected reductive group over $\mathbb{R}$ containing an elliptic maximal torus. Fix a maximal compact subgroup $K_\infty \subset G(\mathbb{R})$. Denote by $G(\mathbb{R})_+$ the preimage of the neutral component $G^{\mathrm{ad}}(\mathbb{R})^0$ (for the real topology) under the natural map $G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})$.

Lemma 2.4.1. — *We have $G(\mathbb{R})_+ = Z(\mathbb{R}) \cdot \varrho(G_{\mathrm{sc}}(\mathbb{R}))$.*

Proof. — Since $G_{\mathrm{sc}}(\mathbb{R})$ is connected, clearly $\varrho(G_{\mathrm{sc}}(\mathbb{R}))$ maps into $G^{\mathrm{ad}}(\mathbb{R})^0$. Therefore, $G(\mathbb{R})_+ \supset Z(\mathbb{R}) \cdot \varrho(G_{\mathrm{sc}}(\mathbb{R}))$. We have surjections

$$G_{\mathrm{sc}}(\mathbb{R})^0 \times Z(\mathbb{R})^0 \twoheadrightarrow G(\mathbb{R})^0 \twoheadrightarrow G^{\mathrm{ad}}(\mathbb{R})^0$$

by [Mil05, Prop. 5.1]. This implies that $G(\mathbb{R})_+ \subset Z(\mathbb{R})G(\mathbb{R})^0 = Z(\mathbb{R}) \cdot \varrho(G_{\mathrm{sc}}(\mathbb{R}))$. □

Let ξ be an irreducible algebraic representation of $G_{\mathbb{C}}$, and $\zeta : G(\mathbb{R}) \to \mathbb{C}^\times$ be a continuous character. By restriction ξ yields a continuous representation of $G(\mathbb{R})$ on a complex vector space, which we still denote by ξ. Write $\omega_\xi : Z(\mathbb{R}) \to \mathbb{C}^\times$ for the central character of ξ. By $\Pi_\infty(\xi, \zeta)$ we mean the set of isomorphism classes of irreducible discrete series representations whose central and infinitesimal characters are equal to those of the contragredient of $\xi \otimes \zeta$. This is a discrete series L-packet by the construction of [Lan89], which assigns to $\Pi_\infty(\xi, \zeta)$ an L-parameter

$$\varphi_{\xi,\zeta} \colon W_{\mathbb{R}} \longrightarrow {}^L G.$$

Thus we also write $\Pi_\infty(\varphi_{\xi,\zeta})$ for $\Pi_\infty(\xi, \zeta)$. We have $\xi \otimes \zeta \simeq \xi' \otimes \zeta'$ as representations of $G(\mathbb{R})$ if and only if there exists an algebraic character χ of $G_{\mathbb{C}}$ such that $\xi' = \xi \otimes \chi$ and $\zeta' = \zeta \otimes \chi^{-1}$. In this case $\Pi_\infty(\xi, \zeta) = \Pi_\infty(\xi', \zeta')$, and $\varphi_{\xi,\zeta} \simeq \varphi_{\xi',\zeta'}$. In fact $|\Pi_\infty(\xi, \zeta)|$ is a constant $d(G) \in \mathbb{Z}_{\geqslant 1}$ depending only on G. When $\xi = \mathbf{1}$, we also write $\Pi_\infty(\zeta)$ and f_ζ for $\Pi_\infty(\xi, \zeta)$ and $f_{\xi,\zeta}$.

Write A_G for the maximal split torus in the center of G. Let $\chi : A_G(\mathbb{R})^0 \to \mathbb{C}^\times$ be a continuous character. Let $\mathrm{Irr}_{\mathrm{temp}}(G(\mathbb{R}), \chi)$ be the set of (isomorphism classes of) irreducible tempered representations of $G(\mathbb{R})$ whose central character equals χ on $A_G(\mathbb{R})^0$. Write $\mathcal{H}(G(\mathbb{R}), \chi^{-1})$ for the space of smooth K_∞-finite functions on $G(\mathbb{R})$ with central character χ^{-1}. Following [Art89, §4], $f \in \mathcal{H}(G(\mathbb{R}), \chi^{-1})$ is said to be *stable cuspidal* if $\mathrm{Tr}\, \pi(f)$ is constant as π varies over each discrete series L-packet and if $\mathrm{Tr}\, \pi(f) = 0$ for every $\pi \in \mathrm{Irr}_{\mathrm{temp}}(G(\mathbb{R}), \chi)$ outside of discrete series.

Fix a Haar measure on $G(\mathbb{R})$ and the Lebesgue measure on $A_G(\mathbb{R})^0$, so as to determine a Haar measure on $G(\mathbb{R})/A_G(\mathbb{R})^0$. Choose a pseudo-coefficient $f_\pi \in \mathcal{H}(G(\mathbb{R}), \omega_\xi \zeta)$ for each $\pi \in \Pi_\infty(\xi, \zeta)$ à la [CD85]. Although it is not unique, the orbital integrals of f_π are uniquely determined by the property that $\mathrm{Tr}\, \pi(f_\pi) = 1$ and that $\mathrm{Tr}\, \pi'(f_\pi) = 0$ for $\pi' \in \mathrm{Irr}_{\mathrm{temp}}(G(\mathbb{R}), (\omega_\xi \zeta)^{-1})$ and $\pi' \not\simeq \pi$. An *averaged Lefschetz function* associated with (ξ, ζ) is defined as

$$f_{\xi,\zeta} := |\Pi_\infty(\xi, \zeta)|^{-1} \sum_{\pi \in \Pi_\infty(\xi,\zeta)} f_\pi \quad \in \mathcal{H}(G(\mathbb{R}), \omega_\xi \zeta). \tag{2.4.1}$$

By construction, $f_{\xi,\zeta}$ is stable cuspidal in the above sense.

For elliptic $\gamma \in G(\mathbb{R})$, let I_γ denote its connected centralizer in $G(\mathbb{R})$, and $e(I_\gamma) \in \{\pm 1\}$ its Kottwitz sign. Let I_γ^{cpt} denote an inner form of I_γ over $\mathbb{R}$ that is anisotropic modulo Z_G. From [Kot92a, p. 659], as our $O_\gamma(f_{\xi,\zeta})$ equals $d(G)^{-1}\mathrm{SO}_{\gamma_\infty}(f_\infty)$ there, we see that

$$O_\gamma(f_{\xi,\zeta}) = \begin{cases} d(G)^{-1}\mathrm{vol}(A_G(\mathbb{R})^0 \backslash I_\gamma^{\mathrm{cpt}}(\mathbb{R}))^{-1} \zeta(\gamma) e(I_\gamma) \,\mathrm{Tr}\, \xi(\gamma), & \gamma : \text{elliptic}, \\ 0, & \gamma : \text{non-elliptic}. \end{cases} \tag{2.4.2}$$

In (2.4.2), the Haar measure on $I_\gamma^{\mathrm{cpt}}(\mathbb{R})$ is chosen to be compatible (in the sense of [Kot88, p. 631]) with the measure on $I_\gamma(\mathbb{R})$ used in the orbital integral, to compute $\mathrm{vol}(A_G(\mathbb{R})^0 \backslash I_\gamma^{\mathrm{cpt}}(\mathbb{R}))$ with respect to the Lebesgue measure on $A_G(\mathbb{R})^0$. Again by

[Kot92a, p. 659],

$$SO_\gamma(f_{\xi,\zeta}) = \begin{cases} \mathrm{vol}(A_G(\mathbb{R})^0 \backslash I_\gamma^{\mathrm{cpt}}(\mathbb{R}))^{-1} \zeta(\gamma) \operatorname{Tr} \xi(\gamma), & \gamma : \text{elliptic}, \\ 0, & \gamma : \text{non-elliptic}. \end{cases} \tag{2.4.3}$$

Let G^* be a quasi-split group over $\mathbb{R}$ with inner twisting $G_{\mathbb{C}} \xrightarrow{\sim} G^*_{\mathbb{C}}$, through which ξ, ζ above are transported to G^*. Thereby we obtain an averaged Lefschetz function $f^*_{\xi,\zeta}$ on $G^*(\mathbb{R})$.

LEMMA 2.4.2. — *The function $f^*_{\xi,\zeta}$ is a transfer of $f_{\xi,\zeta}$ (in the sense of* (2.3.2)*).*

Proof. — This is immediate from (2.4.3). □

LEMMA 2.4.3. — *Assume that $\xi = \mathbf{1}$. Let $\pi : G(\mathbb{R}) \to \mathbb{C}^\times$ be a continuous character whose central character equals ζ^{-1} when restricted to $A_G(\mathbb{R})^0$. Then $\pi|_{G(\mathbb{R})_+} = \zeta^{-1}|_{G(\mathbb{R})_+}$ if and only if $\pi|_{Z(\mathbb{R})} = \zeta^{-1}|_{Z(\mathbb{R})}$. If the equivalent conditions hold then $\mathrm{Tr}(f_\zeta|\pi) = 1$ if $\pi = \zeta^{-1}$; otherwise $\mathrm{Tr}(f_\zeta|\pi) = 0$.*

Proof. — The first assertion is clear from Lemma 2.4.1. For the second assertion, it follows from (2.4.2) via the Weyl integration formula that $\mathrm{Tr}(f_\zeta|\pi) = \mathrm{vol}(K)^{-1} \int_K \zeta(k)\, \pi(k) dk$, where K is a maximal compact-modulo-$A_G(\mathbb{R})^0$ subgroup of $G(\mathbb{R})$. The integral vanishes unless $\pi = \zeta^{-1}$ on K, in which case $\pi = \zeta^{-1}$ on the entire $G(\mathbb{R})$ (since K meets every component of $G(\mathbb{R})$) and $\mathrm{Tr}(f_\zeta|\pi) = 1$. □

2.5. ONE-DIMENSIONAL AUTOMORPHIC REPRESENTATIONS. — Now let G be a connected reductive group over a *number field* F. Let v be a place of F and set $G_v := G_{F_v}$. We have a finite decomposition of G_{sc} into F-simple factors

$$G_{\mathrm{sc}} = \prod_{i \in I} G_i, \qquad \text{with} \quad G_i = \mathrm{Res}_{F_i/F} H_i, \tag{2.5.1}$$

for a finite extension F_i/F and an absolutely F_i-simple simply connected group H_i over each F_i. Accordingly $G^{\mathrm{ad}} = \prod_{i \in I} G_i^{\mathrm{ad}}$. Note that we have a natural composite map $G \to G^{\mathrm{ad}} \to G_i^{\mathrm{ad}}$ for each $i \in I$, where the last arrow is the projection onto the i-component. Let $P_v = M_v N_v$ be a Levi decomposition of a parabolic subgroup of G_v. We consider the following assumption, where "nb" stands for non-basic (cf. Definition 5.3.2 and Lemma 5.3.7 below).

($\mathbb{Q}$-nb(P_v)) The image of P_v in $(G_i^{\mathrm{ad}})_v$ is a *proper* parabolic subgroup for every $i \in I$.

The assumption implies that G^{ad} has no nontrivial F-simple factor that is anisotropic over F_v, thus so the embedding $G_{\mathrm{sc}}(F) \hookrightarrow G_{\mathrm{sc}}(\mathbb{A}_F^v)$ has dense image by strong approximation. When G^{ad} is itself F-simple, ($\mathbb{Q}$-nb(P_v)) is simply saying that P_v is a *proper* parabolic subgroup of G_v.

LEMMA 2.5.1. — *Assume that $G_{\mathrm{der}} = G_{\mathrm{sc}}$ and that G_i is isotropic over F_v for every $i \in I$. Let π be a discrete automorphic representation of $G(\mathbb{A}_F)$, and π' an irreducible*

$G_{\mathrm{der}}(\mathbb{A}_F)$-subrepresentation of π. Decompose $\pi' = \otimes_i \pi'_i$ according to $G_{\mathrm{der}}(\mathbb{A}_F) = \prod_{i\in I} G_i(\mathbb{A}_F)$. Write

$$G_i(F_v) = H_i(F_i \otimes_F F_v) = \prod_{w|v} H_i(F_{i,w}),$$

where w runs over the set of places of F_i above v, and decompose $\pi'_{i,v} = \otimes_{w|v}\pi'_{i,w}$ accordingly. If for every $i \in I$, there exists $w|v$ such that $\pi'_{i,w}$ is trivial, then $\dim \pi = 1$.

Proof. — By strong approximation, the embedding $H_i(F_i) \hookrightarrow H_i(\mathbb{A}_{F_i}^w)$ has dense image for each $i \in I$. Since the underlying space of π'_i consists of automorphic forms which are left-invariant under $H_i(F_i)$, and since $\pi'_{i,w}$ is trivial, the argument for [KST20, Lem. 6.2] shows that π'_i is trivial on the entire $H_i(\mathbb{A}_{F_i})$. Hence π' is trivial. Since $G(\mathbb{A}_F)/G_{\mathrm{der}}(\mathbb{A}_F)$ is abelian, we deduce that $\dim \pi = 1$ as π is generated by π' as a $G(\mathbb{A}_F)$-module. □

Write A_G for the maximal $\mathbb{Q}$-split torus in the center of $\mathrm{Res}_{F/\mathbb{Q}}G$.

COROLLARY 2.5.2

Let π be an irreducible $G(\mathbb{A}_F)$-subrepresentation of $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}_F)/A_G(\mathbb{R})^0)$ and let $\omega_v \in \mathrm{Exp}(J_{P_v}(\pi_v))$. Then

$$|\omega_v(a)| \leqslant \delta_{P_v}^{-1/2}(a), \qquad a \in A_{P_v}^{--}. \tag{2.5.2}$$

Now assume ($\mathbb{Q}$-nb(P_v)). Then the equality holds for some $a \in A_{P_v}^{--}$ if and only if the equality holds for all $a \in A_{P_v}^{--}$ if and only if $\dim \pi = 1$.

Proof. — The inequality in (2.5.2) is immediate from Proposition 2.1.3 and the normalization $J_{P_v}(\pi_v) = (\pi_v)_{N_v} \otimes \delta_{P_v}^{-1/2}$. It remains to check the three conditions for the equality are equivalent. The only nontriviality is to show that $\dim \pi = 1$, assuming that $|\omega_v(a)| = \delta_{P_v}^{-1/2}(a)$ for some $a \in A_{P_v}^{--}$.

We may assume $G_{\mathrm{der}} = G_{\mathrm{sc}}$ via z-extensions. We decompose

$$P'_v := P_v \cap G_{\mathrm{der}} = \prod_{i\in I}\prod_{w|v} P_{v,i,w} \quad \text{according as} \quad (G_{\mathrm{der}})_v = \prod_{i\in I}\prod_{w|v}(H_i)_w,$$

where w runs over places of F_i above v. Similarly $A_{P'_v} = \prod_{i,w} A_{P_{v,i,w}}$. Assumption ($\mathbb{Q}$-nb($P_v$)) tells us that for every i, there exists $w|v$ such that $P_{v,i,w}$ is a *proper* parabolic subgroup of $(H_i)_w$. In particular, $G_{i,v}$ is isotropic for every i. So we can apply Lemma 2.5.1. Adopting the setup and notation from there, it suffices to show that for every i, there exists a place $w|v$ such that $\pi'_{i,w} = \mathbf{1}$. In fact, we only need to find $w|v$ such that $\dim \pi'_{i,w} < \infty$ by Lemma 2.1.5 and Corollary 2.3.3.

The central isogeny $Z\times G_{\mathrm{der}} \to G$ induces a map $A_{G_v} \times A_{P'_v} \to A_{P_v}$, which has finite kernel and cokernel on the level of F_v-points. Replacing a with a finite power, we may assume that a is the image of $(a_0, a') \in A_{G_v}(F_v) \times A_{P'_v}(F_v)$, so that $|\omega(a)| = |\omega(a')|$. (The central character of π'_v is unitary on $A_{G_v}(F_v)$, so $|\omega(a_0)| = 1$.) Write $a' = (a_{i,w})_{i,w}$ and $\omega_v|_{A_{P'_v}(F_v)} = (\omega_{v,i,w})_{i,w}$ according to the decomposition of P'_v above.

We have $|\omega_{v,i,w}(a_{i,w})| \leqslant \delta_{P_{v,i,w}}^{-1/2}(a_i)$ by Proposition 2.1.3, while $\prod_{i,w} |\omega_{v,i,w}(a_{i,w})| = \prod_{i,w} \delta_{P_{v,i,w}}^{-1/2}(a_{i,w})$ from our running assumption. Therefore

$$|\omega_{v,i,w}(a_{i,w})| = \delta_{P_{v,i,w}}^{-1/2}(a_{i,w}), \quad \forall i \in I.$$

Since $J_{P'_v}(\pi'_v) \subset J_{P_v}(\pi_v)$, we see that $\omega_v|_{A_{P'_v}(F_v)} \in \mathrm{Exp}(J_{P'_v}(\pi'_v))$. Thus we have

$$\omega_{v,i,w} \in \mathrm{Exp}(J_{P_{v,i,w}}(\pi'_{v,i,w})).$$

Finally for each i, we apply the equality criterion of Proposition 2.1.3 at a place w where $P_{v,i,w}$ is proper in $(H_i)_w$. Thereby we deduce that $\dim \pi'_{i,w} < \infty$ as desired. □

Let $\xi\colon G_{\overline{F}} \xrightarrow{\sim} G^*_{\overline{F}}$ be an inner twisting, with G^* a connected reductive group over F.

LEMMA 2.5.3. — *One-dimensional automorphic representations of $G(\mathbb{A}_F)$ are in a canonical bijection with those of $G^*(\mathbb{A}_F)$, compatibly with the bijection of Corollary* 2.3.3 *at every place of F.*

Proof. — Define $G(\mathbb{A}_F)^\flat := \mathrm{cok}(G_{\mathrm{sc}}(\mathbb{A}_F) \xrightarrow{\varrho} G(\mathbb{A}_F))$. Similarly we have $G^*(\mathbb{A}_F)^\flat$, $G(F)^\flat$, and $G^*(F)^\flat$. Adapting the arguments of Section 2.3 via z-extensions, we see that $G(\mathbb{A}_F)^\flat$ is an abelian group and that there exists a canonical isomorphism $G(\mathbb{A}_F)^\flat \simeq G^*(\mathbb{A}_F)^\flat$ compatible with the isomorphism of Lemma 2.3.1 at every place of v and that the above isomorphism carries $G(F)^\flat$ onto $G^*(F)^\flat$.

Again by taking a z-extension, we can assume that $G_{\mathrm{sc}} = G_{\mathrm{der}}$. It suffices to show that the inclusion $G_{\mathrm{der}}(F)G(\mathbb{A}_F)_{\mathrm{der}} \subset G_{\mathrm{der}}(\mathbb{A}_F)$ has dense image so that every one-dimensional automorphic representations of $G(\mathbb{A}_F)$ factors through $G(\mathbb{A}_F)^\flat$ (and likewise for G^*). Since $G(\mathbb{A}_F)_{\mathrm{der}}$ contains $G(F_v)_{\mathrm{der}} = G_{\mathrm{der}}(F_v)$ whenever G is quasi-split over F_v (Lemma 2.3.2), the desired density follows from the strong approximation for G_{der}. □

To state the next lemma, define a *(global) central character datum* to be a pair $(\mathfrak{X}, \chi)$ as follows, where $\prod'_v$ means the restricted product over all places of F.

– $\mathfrak{X} = \prod'_v \mathfrak{X}_v$ is a closed subgroup of $Z(\mathbb{A}_F)$ such that $Z(F)\mathfrak{X}$ is closed in $Z(\mathbb{A}_F)$, and

– $\chi = \prod_v \chi_v : \mathfrak{X} \cap Z(F)\backslash\mathfrak{X} \to \mathbb{C}^\times$, with $\chi_v : \mathfrak{X}_v \to \mathbb{C}^\times$ a continuous character. (Implicitly for each $x = (x_v) \in \mathfrak{X}$, we have $\chi_v(x_v) = 1$ for almost all v, so that χ is well defined on $\mathfrak{X}$.)

LEMMA 2.5.4. — *Let $(\mathfrak{X}, \chi)$ be a central character datum for G. Let v be a finite place of F, and $g_v \in G(F_v)$ such that the image of g_v in $G(F_v)^{\mathrm{ab}}$ is not contained in the image of $\mathfrak{X}_v$. Then there exists a one-dimensional automorphic representation π of $G(\mathbb{A}_F)$ with $\pi|_{\mathfrak{X}} = \chi$ such that $\pi_v(g_v) \neq 1$.*

Proof. — Replacing G with a z-extension and $(\mathfrak{X}, \chi)$ with its pullback to the z-extension, we may assume that $G_{\mathrm{der}} = G_{\mathrm{sc}}$. Then we may replace G with G^{ab} as $(\mathfrak{X}, \chi)$ factors through a central character datum for G^{ab}. Thus we assume that $G = T$ is a torus. By assumption $g_v \in T(F_v)$ lies outside $\mathfrak{X}_v$, and viewing g_v as an element of

$T(\mathbb{A}_F)$ via the obvious embedding $T(F_v) \hookrightarrow T(\mathbb{A}_F)$, we see that g_v does not belong to the subgroup $T(F)\mathfrak{X}$ of $T(\mathbb{A}_F)$. Thus the proof is complete by the fact (from Pontryagin duality) that, for every non-identity element x in a locally compact Hausdorff abelian group X, there exists a unitary character of X whose value is nontrivial at x. (Take $X = T(\mathbb{A}_F)/T(F)\mathfrak{X}$ and $x = g_v$.) □

2.6. ENDOSCOPY WITH FIXED CENTRAL CHARACTER. — Let F be a local or global field of characteristic 0. Let G be a connected reductive group over F with an inner twisting $G_{\overline{F}} \xrightarrow{\sim} G^*_{\overline{F}}$ with G^* quasi-split over F. Let $\mathcal{E}(G)$ (resp. $\mathcal{E}_{\mathrm{ell}}(G)$) denote a set of representatives for isomorphism classes of endoscopic (resp. elliptic endoscopic) data for G as defined in [LS87, KS99]. A member of $\mathcal{E}(G)$ is represented by a quadruple $\mathfrak{e} = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta^{\mathfrak{e}})$ consisting of a quasi-split group $G^{\mathfrak{e}}$, a split extension $\mathcal{G}^{\mathfrak{e}}$ of W_F by $\widehat{G}^{\mathfrak{e}}$, $s^{\mathfrak{e}} \in Z_{G^{\mathfrak{e}}}$, and $\eta^{\mathfrak{e}} : \mathcal{G}^{\mathfrak{e}} \hookrightarrow {}^L G$ satisfying the conditions detailed in loc. cit. Write $\mathrm{Out}(\mathfrak{e})$ for the outer automorphism group [KS99, p. 19]. In particular, $\mathfrak{e}^* := (G^*, {}^L G^*, 1, \mathrm{id}) \in \mathcal{E}_{\mathrm{ell}}(G)$. Write $\mathcal{E}^<_{\mathrm{ell}}(G) := \mathcal{E}_{\mathrm{ell}}(G) \backslash \{\mathfrak{e}^*\}$.

From now on, let $\mathfrak{e} \in \mathcal{E}(G)$. Set

$$\iota(G, G^{\mathfrak{e}}) := \tau(G)\tau(G^{\mathfrak{e}})^{-1}|\mathrm{Out}(\mathfrak{e})|^{-1} \in \mathbb{Q}.$$

Throughout Section 2.6, we make the following assumption, which will be removed via z-extensions in the next subsection. (The assumption is known to be true if $\mathfrak{e} = \mathfrak{e}^*$, when it is evident, or if G^{der} is simply connected, by [Lan79, Prop. 1].)

– (assumption) $\mathcal{G}^{\mathfrak{e}} = {}^L G^{\mathfrak{e}}$.

For now we restrict to the case when F is local. Let $\mathfrak{e}$ be as above. Consider a local central character datum $(\mathfrak{X}, \chi)$ for G as in Section 2.2. Let $\mathfrak{X}^{\mathfrak{e}} \subset Z_{G^{\mathfrak{e}}}(F)$ denote the image of $\mathfrak{X}$ under the canonical embedding $Z_G \hookrightarrow Z_{G^{\mathfrak{e}}}$. Thus we can identify $\mathfrak{X} = \mathfrak{X}^{\mathfrak{e}}$. We say a semisimple element $\gamma^{\mathfrak{e}} \in G^{\mathfrak{e}}(F)$ is strongly G-regular if $\gamma^{\mathfrak{e}}$ corresponds to (the $G(\overline{F})$-conjugacy class of) an element of $G(F)_{\mathrm{sr}}$ via the correspondence between semisimple conjugacy classes in $G^{\mathfrak{e}}(\overline{F})$ and those in $G(\overline{F})$ [LS87, 1.3]. Write $G^{\mathfrak{e}}(F)_{G\text{-sr}} \subset G^{\mathfrak{e}}(F)$ for the subset of strongly G-regular elements.

Thanks to the proof of the transfer conjecture and the fundamental lemma [Wal06, CL10, Ngô10], we know that each $f \in \mathcal{H}(G(F))$ admits a *transfer* $f^{\mathfrak{e}} \in \mathcal{H}(G^{\mathfrak{e}}(F))$ whose stable orbital integrals on strongly G-regular semisimple elements are determined by the following formula, where the sum runs over strongly regular $G(F)$-conjugacy classes, and $\Delta(\cdot,\cdot)$ denotes the transfer factor as in [LS87] (see the remark below on normalization).

$$SO_{\gamma^{\mathfrak{e}}}(f^{\mathfrak{e}}) = \sum_{\gamma \in G(F)_{\mathrm{sr}}/\sim} \Delta(\gamma^{\mathfrak{e}}, \gamma) O_\gamma(f), \qquad \gamma^{\mathfrak{e}} \in G^{\mathfrak{e}}(F)_{G\text{-sr}}. \tag{2.6.1}$$

The assignment of $f^{\mathfrak{e}}$ to f is not unique on the level of Hecke algebras, but (2.6.1) determines a well-defined map $\mathrm{LS}^{\mathfrak{e}} : \mathcal{I}(G) \to \mathcal{S}(G^{\mathfrak{e}})$.

The transfer satisfies an equivariance property. For each $z \in Z_G(F) \subset Z_{G^{\mathfrak{e}}}(F)$, define the translates $f_z, f^{\mathfrak{e}}_z$ of $f, f^{\mathfrak{e}}$ by $f_z(g) = f(zg)$ and $f^{\mathfrak{e}}_z(h) = f^{\mathfrak{e}}(zh)$. The equivariance of transfer factors under translation by central elements (see [LS87, Lem. 4.4.A])

implies that $f_z^{\mathfrak{e}}$ is a transfer of $\lambda^{\mathfrak{e}}(z)f_z$ for a smooth character $\lambda^{\mathfrak{e}} : Z_G(F) \to \mathbb{C}^\times$. The character $\lambda^{\mathfrak{e}}$ is independent of $f^{\mathfrak{e}}$ and f, and its restriction $\lambda^{\mathfrak{e}}|_{Z_G^0(F)}$ can be described as follows. Consider the composite map

$$W_F \longrightarrow {}^L G^{\mathfrak{e}} \xrightarrow{\eta^{\mathfrak{e}}} {}^L G \longrightarrow {}^L Z_G^0, \tag{2.6.2}$$

where the last map is dual to the embedding $Z_G^0 \hookrightarrow G$. Then $\lambda^{\mathfrak{e}}|_{Z_G^0(F)}$ is the character of $Z_G^0(F)$ corresponding to the composite map above by [KSZ, Lem. 7.4.6]. Define a smooth character $\chi^{\mathfrak{e}} : \mathfrak{X}^{\mathfrak{e}} \to \mathbb{C}^\times$ by the relation

$$\chi^{\mathfrak{e}}(z) = \lambda^{\mathfrak{e}}(z)^{-1}\chi(z), \qquad z \in \mathfrak{X} = \mathfrak{X}^{\mathfrak{e}}. \tag{2.6.3}$$

In light of the equivariance property above, the transfer map $\mathrm{LS}^{\mathfrak{e}} : \mathcal{I}(G) \to \mathcal{S}(G^{\mathfrak{e}})$ descends to

$$\mathrm{LS}^{\mathfrak{e}} : \mathcal{I}(G, \chi^{-1}) \longrightarrow \mathcal{S}(G^{\mathfrak{e}}, \chi^{\mathfrak{e},-1}) \tag{2.6.4}$$

via averaging, still denoted by $\mathrm{LS}^{\mathfrak{e}}$ for simplicity. The identity (2.6.1) still holds if $f^{\mathfrak{e}} = \mathrm{LS}^{\mathfrak{e}}(f)$ under (2.6.4). In the special case of $\mathfrak{e} = \mathfrak{e}^*$ (so that $\chi^{\mathfrak{e}} = \chi$), we write $f^* \in \mathcal{H}(G^*(F), \chi^{-1})$ for a transfer of $f \in \mathcal{H}(G(F), \chi^{-1})$. If $\mathfrak{X} = \{1\}$ then f^* here coincides with the one in Section 2.3, noting that $e(G)$ in (2.3.2) plays the role of transfer factor.

The fundamental lemma tells us the following. Assume that G and $\mathfrak{e}$ are unramified; the latter means that $G^{\mathfrak{e}}$ is an unramified group and that the L-morphism $\eta^{\mathfrak{e}}$ is inflated from a morphism of L-groups with respect to an unramified extension of F. We fix pinnings for G and $G^{\mathfrak{e}}$ defined over F, which determine hyperspecial subgroups $K \subset G(F)$ and $K^{\mathfrak{e}} \subset G^{\mathfrak{e}}(F)$ as in [Wal08, §4.1]. The Haar measures on $G(F)$ and $G^{\mathfrak{e}}(F)$ are normalized to assign volume 1 to K and $K^{\mathfrak{e}}$. We also assume that χ is unramified, i.e., χ is trivial on $\mathfrak{X} \cap K$. We normalize the transfer factors canonically as in [LS87] (which is possible as G is quasi-split). Then $\mathrm{LS}^{\mathfrak{e}}$ can be realized by a linear map of the unramified Hecke algebras (defined relative to K and $K^{\mathfrak{e}}$)

$$\xi^{\mathfrak{e},*} \colon \mathcal{H}^{\mathrm{ur}}(G(F), \chi^{-1}) \longrightarrow \mathcal{H}^{\mathrm{ur}}(G^{\mathfrak{e}}(F), \chi^{\mathfrak{e},-1}).$$

We turn to the case of global field F. Recall that Z is the center of G. Let $(\mathfrak{X}, \chi)$ be a global central character datum (Section 2.5). As in the local case, we define $\mathfrak{X}^{\mathfrak{e}} = \prod_v \mathfrak{X}_v^{\mathfrak{e}}$ to be the image of $\mathfrak{X}$ under the canonical embedding $Z_G(\mathbb{A}_F) \hookrightarrow Z_{G^{\mathfrak{e}}}(\mathbb{A}_F)$. We have $\chi^{\mathfrak{e}} := \prod_v \chi_v^{\mathfrak{e}} : \mathfrak{X}^{\mathfrak{e}} \to \mathbb{C}^\times$, where $\chi_v^{\mathfrak{e}}$ was given by the local consideration above, so that functions in $\mathcal{H}(G(\mathbb{A}_F), \chi^{-1})$ transfer to those in $\mathcal{H}(G^{\mathfrak{e}}(\mathbb{A}_F), (\chi^{\mathfrak{e}})^{-1})$. Denote by $\lambda^{\mathfrak{e}} = \prod_v \lambda_v^{\mathfrak{e}} : Z_G(F)\backslash Z_G(\mathbb{A}_F) \to \mathbb{C}^\times$ the character with $\lambda_v^{\mathfrak{e}}$ as in the local context above. (The $Z_G(F)$-invariance of $\lambda^{\mathfrak{e}}$ follows from the equivariance of transfer factors [LS87, Lem. 4.4.A] and the product formula [LS87, Cor. 6.4.B].) The restriction of λ to $Z_G^0(\mathbb{A}_F)$ corresponds to the composite map (2.6.2) (with F now global). There is an equality $\chi^{\mathfrak{e}} = \lambda^{\mathfrak{e},-1}\chi$ as characters on $\mathfrak{X} = \mathfrak{X}^{\mathfrak{e}}$ as in (2.6.3) since this holds at every place of F. In particular, $\chi^{\mathfrak{e}}$ is trivial on $Z_G(F) \cap \mathfrak{X}^{\mathfrak{e}}$, and $(\mathfrak{X}^{\mathfrak{e}}, \chi^{\mathfrak{e}})$ is a central character datum for $G^{\mathfrak{e}}$.

REMARK 2.6.1. — The local transfer factors are well defined only up to a nonzero scalar (unless G is quasi-split or $G^{\mathfrak{e}} = G^*$, if no further choices are made), so we always choose a normalization implicitly, for instance throughout Section 3. Scaling the transfer factor results in scaling the transfer map (2.6.4). However, according to [LS87, §6.4], we may and will choose a normalization at each place such that the product of local transfer factors over all places *is* the canonical global transfer factor. This will not introduce ambiguity in our main argument as it takes place in the global context.

It simplifies some later arguments if $\mathfrak{e}$ is chosen to enjoy a boundedness property. We say that a subgroup of ${}^LG = \widehat{G} \rtimes W_F$ is *bounded* if its projection to $\widehat{G} \rtimes \mathrm{Gal}(E/F)$ is contained in a compact subgroup for some (thus every) finite Galois extension E/F containing the splitting field of G.

LEMMA 2.6.2. — *In either local or global case, we can choose the representative* $\mathfrak{e} = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta^{\mathfrak{e}})$ *in its isomorphism class to satisfy the following condition:* $\eta^{\mathfrak{e}}(W_F)$ *is a bounded subgroup of* ${}^LG$. *(We restrict* $\eta^{\mathfrak{e}}$ *to* W_F *via the splitting* $W_F \to \mathcal{G}^{\mathfrak{e}}$ *built into the data.)*

REMARK 2.6.3. — Bergeron–Clozel [BC17, Lem. 3.7] proved a similar lemma when $F = \mathbb{R}$.

Proof. — Since $\eta^{\mathfrak{e}}|_{\widehat{G}^{\mathfrak{e}}}$ will be fixed throughout, we use it to identify $\widehat{G}^{\mathfrak{e}}$ with a subgroup of $\widehat{G}$. We take the convention that all cocycles/cohomology below are continuous cocycles/cohomology.

It suffices to show that there exists an L-morphism $\eta_0^{\mathfrak{e}} : \mathcal{G}^{\mathfrak{e}} \to {}^LG$ extending $\eta^{\mathfrak{e}}|_{\widehat{G}^{\mathfrak{e}}}$ such that $\eta_0^{\mathfrak{e}}(W_F)$ is bounded. Indeed, $\mathfrak{e}_0 = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta_0^{\mathfrak{e}})$ is then the desired representative.

To prove the existence of $\eta_0^{\mathfrak{e}}$ as above, we reduce to the case that $G_{\mathrm{der}} = G_{\mathrm{sc}}$ and that $\mathcal{G}^{\mathfrak{e}} = {}^LG^{\mathfrak{e}}$ via a z-extension. (In the notation of Section 2.7 below, the idea is to multiply $\eta_1^{\mathfrak{e}}$ by a suitable 1-cocycle $c : W_F \to Z(\widehat{G}_1)$ to make the image of $(c \cdot \eta_1^{\mathfrak{e}})_{W_F}$ contained in ${}^LG$ and still bounded.) In the case that $G_{\mathrm{der}} = G_{\mathrm{sc}}$ and $\mathcal{G}^{\mathfrak{e}} = {}^LG^{\mathfrak{e}}$, our approach is to refine the proof of [Lan79, Prop. 1], where Langlands shows that $\eta^{\mathfrak{e}}|_{\widehat{G}^{\mathfrak{e}}}$ extends to an L-morphism $\eta_0^{\mathfrak{e}} : {}^LG^{\mathfrak{e}} \to {}^LG$ under the hypothesis but without guaranteeing boundedness of image. To construct $\eta_0^{\mathfrak{e}}$ (denoted ξ therein), Langlands reduces to the elliptic endoscopic case, chooses a sufficiently large finite extension K/F, and then constructs $\xi' : W_{K/F} \to {}^LG$ such that $\eta_0^{\mathfrak{e}}(g \rtimes w) := \eta^{\mathfrak{e}}(g)\xi'(w)$ gives the desired L-morphism. (In the current proof, we follow Langlands to use the Weil group $W_{K/F}$ to form the L-group, i.e., ${}^LG = \widehat{G} \rtimes W_{K/F}$.) It is enough to arrange that ξ' has bounded image in Langlands's construction.

Write $\widehat{N}$ for the normalizer of $\widehat{T}$ (which is ${}^LT^0$ in loc. cit.) in $\widehat{G}$. Let $\widehat{N}_c$ (resp. $Z(\widehat{G}^{\mathfrak{e}})_c$) denote the maximal compact subgroup of $\widehat{N}$ (resp. $Z(\widehat{G}^{\mathfrak{e}})$). The starting point is a set-theoretic map $\xi' : W_{K/F} \to {}^LG$ satisfying the second displayed

formula on p. 709 therein. Such a ξ' is chosen using the Langlands–Shelstad representative of each Weyl group element ω, denoted by $n(\omega) \in \widehat{N}$ in [LS87, §2.1]. (The point is that the σ-action $\omega_{T/G}(\sigma)$ on ${}^LT^0$ and the action $\omega^1(\sigma)$ differ by the Weyl action $\omega^2(\sigma)$ in his notation. See the seventh displayed formula on p. 703.) In fact $n(\omega) \in \widehat{N}_c$ since it is a product of finite-order elements in $\widehat{N}$. Thereby ξ' has image in $\widehat{N}_c \rtimes W_{K/F}$ (thus bounded). It follows that the 2-cocycle of $W_{K/F}$ given by

$$a_{w_1,w_2} = \xi'(w_1)\xi'(w_2)\xi'(w_1w_2)^{-1}$$

has values in $Z(\widehat{G}^{\mathfrak{e}})_c$ (not just $Z(\widehat{G}^{\mathfrak{e}})$ as in [Lan79, p. 709]). We need to verify the claim that the 2-cocycle is trivial in $H^2(W_{K/F}, Z(\widehat{G}^{\mathfrak{e}})_c)$; then ξ' can be made a homomorphism after multiplying a $Z(\widehat{G}^{\mathfrak{e}})_c$-valued 1-cocycle, keeping its image bounded, so we will be done. In fact, thanks to Lemma 2 therein (stated for $Z(\widehat{G}^{\mathfrak{e}})$ but also applicable to $Z(\widehat{G}^{\mathfrak{e}})_c$ since both groups have the same group of connected components), we may assume that $a_{w_1,w_2} \in (Z(\widehat{G}^{\mathfrak{e}})_c)^0$. Then the claim follows from a variant of Lem. 4 therein, with S replaced by the maximal compact subtorus in the statement and proof. (In particular, the map (1) on p. 719 is still surjective if S_1 and S_2 are replaced with their maximal compact subtori, by considering unitary characters.) □

2.7. Endoscopy and z-extensions. — Here we explain a general endoscopic transfer with fixed central character by removing the assumption that $\mathcal{G}^{\mathfrak{e}} = {}^LG^{\mathfrak{e}}$ in Section 2.8 via z-extensions. For the time being, let the base field F of G be either local or global. Fix a z-extension over F

$$1 \longrightarrow Z_1 \longrightarrow G_1 \longrightarrow G \longrightarrow 1.$$

Let $\mathfrak{e} = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta^{\mathfrak{e}}) \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$. As explained in [LS87, §4.4] (see also [KSZ, §7.2.6]), we have a central extension

$$1 \longrightarrow Z_1 \longrightarrow G_1^{\mathfrak{e}} \longrightarrow G^{\mathfrak{e}} \longrightarrow 1,$$

and $\mathfrak{e}$ can be promoted to an endoscopic datum $\mathfrak{e}_1 = (G_1^{\mathfrak{e}}, {}^LG_1^{\mathfrak{e}}, s_1^{\mathfrak{e}}, \eta_1^{\mathfrak{e}})$ for G_1 such that $\eta_1^{\mathfrak{e}} : {}^LG_1^{\mathfrak{e}} \hookrightarrow {}^LG$ extends $\eta^{\mathfrak{e}} : \mathcal{G}^{\mathfrak{e}} \hookrightarrow {}^LG$. Moreover, changing $\mathfrak{e}_1$ and $\mathfrak{e}$ in their isomorphism classes if necessary, we may ensure that $\eta_1^{\mathfrak{e}}(W_F)$ and $\eta^{\mathfrak{e}}(W_F)$ are bounded subgroups in ${}^LG_1$ and ${}^LG$, respectively. Indeed, this is done in the course of proof of Lemma 2.6.2 in the general case. Write $\mathfrak{X}_1$ (resp. $\mathfrak{X}_1^{\mathfrak{e}}$) for the preimage of $\mathfrak{X}$ in G_1 (resp. $G_1^{\mathfrak{e}}$), and $\chi_1 : \mathfrak{X}_1 \to \mathbb{C}^\times$ for the character pulled back from χ.

To describe endoscopic transfers, it is enough to work locally, so let F be a local field. Applying Section 2.8 to G_1 and $\mathfrak{e}_1$ in place of G and $\mathfrak{e}$, we obtain an identification $\mathfrak{X}_1^{\mathfrak{e}} = \mathfrak{X}_1$ under the canonical embedding $Z_{G_1} \hookrightarrow Z_{G_1^{\mathfrak{e}}}$ as well as characters $\lambda_1^{\mathfrak{e}} : Z_{G_1}(F) \to \mathbb{C}^\times$ and $\chi_1^{\mathfrak{e}} : \mathfrak{X}_1^{\mathfrak{e}} = \mathfrak{X}_1 \to \mathbb{C}^\times$ such that $\chi_1^{\mathfrak{e}} = \lambda_1^{\mathfrak{e},-1}\chi_1$ as characters on $\mathfrak{X}_1^{\mathfrak{e}} = \mathfrak{X}_1$. Again $\lambda_1^{\mathfrak{e}}|_{Z^0_{G_1}(F)}$ corresponds to the parameter (2.6.2) (with $G_1^{\mathfrak{e}}, G_1$ replacing $G^{\mathfrak{e}}, G$). We also have a transfer

$$\mathrm{LS}^{\mathfrak{e}} : \mathcal{I}(G, \chi^{-1}) = \mathcal{I}(G_1, \chi_1^{-1}) \xrightarrow{(2.6.4)} \mathcal{S}(G_1^{\mathfrak{e}}, \chi^{\mathfrak{e},-1}),$$

where the equality is induced by $G_1(F) \twoheadrightarrow G(F)$.

2.8. THE TRACE FORMULA WITH FIXED CENTRAL CHARACTER. — In this subsection, G is a connected reductive group over $\mathbb{Q}$. Let A_G denote the maximal $\mathbb{Q}$-split torus in Z_G, and $A_{G_\mathbb{R}}$ denote the maximal $\mathbb{R}$-split torus in $Z_{G_\mathbb{R}}$. Put

$$A_{G,\infty} := A_G(\mathbb{R})^0, \qquad A_{G_\mathbb{R},\infty} := A_{G_\mathbb{R}}(\mathbb{R})^0.$$

Let $\chi_0 : A_{G,\infty} \to \mathbb{C}^\times$ denote a continuous character. By $L^2_{\mathrm{disc},\chi_0}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ we mean the discrete spectrum in the space of square-integrable functions (modulo $A_{G,\infty}$) on $G(\mathbb{Q})\backslash G(\mathbb{A})$ which transforms under $A_{G,\infty}$ by χ_0. Let $(\mathfrak{X} = \prod_v \mathfrak{X}_v, \chi = \prod_v \chi_v)$ be a central character datum as in Section 2.5. Henceforth we always assume that

$$A_{G,\infty} \subset \mathfrak{X}_\infty.$$

We can define $L^2_{\mathrm{disc},\chi}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ in the same way as $L^2_{\mathrm{disc},\chi_0}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Let $\mathcal{A}_{\mathrm{disc},\chi}(G)$ stand for the set of isomorphism classes of irreducible $G(\mathbb{A})$-subrepresentations in $L^2_{\mathrm{disc},\chi}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. The multiplicity of $\pi \in \mathcal{A}_{\mathrm{disc},\chi}(G)$ in $L^2_{\mathrm{disc},\chi}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is denoted $m(\pi)$.

Define $\mathcal{H}(G(\mathbb{A}), \chi^{-1}) := \otimes'_v \mathcal{H}(G(\mathbb{Q}_v), \chi_v^{-1})$ as a restricted tensor product. Each $f \in \mathcal{H}(G(\mathbb{A}), \chi^{-1})$ defines a trace class operator, yielding the discrete part of the trace formula:

$$T^G_{\mathrm{disc},\chi}(f) := \mathrm{Tr}\left(f \mid L^2_{\mathrm{disc},\chi}(G(\mathbb{Q})\backslash G(\mathbb{A}))\right) = \sum_{\pi \in \mathcal{A}_{\mathrm{disc},\chi}(G)} m(\pi)\,\mathrm{Tr}(f|\pi). \tag{2.8.1}$$

Fix a minimal $\mathbb{Q}$-rational Levi subgroup $M_0 \subset G$. Write $\mathcal{L}$ for the set of $\mathbb{Q}$-rational Levi subgroups of G containing M_0. Define the subset $\mathcal{L}_{\mathrm{cusp}} \subset \mathcal{L}$ of *relatively cuspidal* Levi subgroups; by definition, $M \in \mathcal{L}$ belongs to $\mathcal{L}_{\mathrm{cusp}}$ exactly when the natural map $A_{M,\infty}/A_{G,\infty} \to A_{M_\mathbb{R},\infty}/A_{G_\mathbb{R},\infty}$ is an isomorphism. Let $M \in \mathcal{L}$ and $\gamma \in M(\mathbb{Q})$ be a semisimple element. Write M_γ for the centralizer of γ in M, and $I^M_\gamma := (M_\gamma)^0$ for the identity component. Write $\iota^M(\gamma) \in \mathbb{Z}_{\geqslant 1}$ for the number of connected components of M_γ containing $\mathbb{Q}$-points. Write $|\Omega^M|$ for the order of the Weyl group of M. For $\gamma \in M(\mathbb{Q})$, let $\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)$ denote the set of $x \in \mathfrak{X}$ such that γ and $x\gamma$ are $M(\mathbb{Q})$-conjugate. Note that $\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)$ is necessarily finite (by reducing to the case of general linear groups via a faithful representation). When $M = G$, we often omit M, e.g., $I_\gamma = I^G_\gamma$ and $\iota(\gamma) = \iota^G(\gamma)$.

Fix Tamagawa measures on $M(\mathbb{A})$ and $I^M_\gamma(\mathbb{A})$ for $M \in \mathcal{L}_{\mathrm{cusp}}$ and fix their decomposition into Haar measures on $M(\mathbb{A}^\infty)$ and $M(\mathbb{R})$ (resp. $I^M_\gamma(\mathbb{A}^\infty)$ and $I^M_\gamma(\mathbb{R})$). This determines a measure on the quotient $I^M_\gamma(\mathbb{A})\backslash M(\mathbb{A})$, which is used to define the adelic orbital integral at γ in M, and similarly over finite-adelic groups. We also fix Haar measures on $\mathfrak{X}$ and $\mathfrak{X}_\infty$. We equip $I^M_\gamma(\mathbb{Q})$ and $\mathfrak{X}_\mathbb{Q} := \mathfrak{X} \cap Z(\mathbb{Q})$ with the counting measures and $A_G(\mathbb{R})^0$ with the multiplicative Lebesgue measure. Thereby we have quotient measures on $I^M_\gamma(\mathbb{Q})\backslash I^M_\gamma(\mathbb{A})/\mathfrak{X}$, $\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_G(\mathbb{R})^0$, and $\mathfrak{X}_\infty/A_G(\mathbb{R})^0$.

We define the elliptic part of the trace formula as

$$T^G_{\mathrm{ell},\chi}(f) := \sum_{\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(G)} |\mathrm{Stab}^G_{\mathfrak{X}}(\gamma)|^{-1} \iota(\gamma)^{-1} \mathrm{vol}(I_\gamma(\mathbb{Q})\backslash I_\gamma(\mathbb{A})/\mathfrak{X}) O_\gamma(f), \tag{2.8.2}$$

for $f \in \mathcal{H}(G(\mathbb{A}), \chi^{-1})$ and where $\Gamma_{\mathrm{ell},\mathfrak{X}}(G)$ is the set of $\mathfrak{X}$-orbits of elliptic conjugacy classes of G.

Now we assume that $G_{\mathbb{R}}$ contains an elliptic maximal torus. Let ξ be an irreducible algebraic representation of $G_{\mathbb{C}}$, and $\zeta : G(\mathbb{R}) \to \mathbb{C}^{\times}$ a continuous character. Let $M \in \mathcal{L}_{\mathrm{cusp}}$ and T_{∞} an $\mathbb{R}$-elliptic torus in M. Arthur introduced the function $\Phi_M(\gamma, \xi)$ in $\gamma \in T_{\infty}(\mathbb{R})$ in [Art89, (4.4), Lem. 4.2]. (See Lemma 4.4.1 below for a concrete description.)

Let $\gamma \in M(\mathbb{Q})$ and suppose that γ is elliptic in $M(\mathbb{R})$. Denote by $I_\gamma^{M,\mathrm{cpt}}$ a compact-mod-center inner form of $(I_\gamma^M)_{\mathbb{R}}$. We choose a Haar measure on $I_\gamma^{M,\mathrm{cpt}}(\mathbb{R})$ compatibly with that on $I_\gamma^M(\mathbb{R})$. Write $q(I_\gamma) \in \mathbb{Z}_{\geqslant 0}$ for the real dimension of the symmetric space associated with the adjoint group of $(I_\gamma^M)_{\mathbb{R}}$. Following [Art89, (6.3)], define

$$\chi(I_\gamma^M) := (-1)^{q(I_\gamma)} \tau(I_\gamma^M) \mathrm{vol}(A_{I_\gamma^M,\infty} \backslash I_\gamma^{M,\mathrm{cpt}}(\mathbb{R}))^{-1} d(I_\gamma^M). \tag{2.8.3}$$

For $f^\infty \in \mathcal{H}(G(\mathbb{A}^\infty), (\chi^\infty)^{-1})$, let $f_M^\infty \in \mathcal{H}(M(\mathbb{A}^\infty), (\chi^\infty)^{-1})$ denote the constant term, cf. Section 3.2 and Section 3.5 below. Dalal extended Arthur's Lefschetz number formula [Art89, Th. 6.1] to the setting with fixed central characters. It is a harmless condition that is satisfied in our main setup, but we expect it to be superfluous.

Proposition 2.8.1. — *Let $(\mathfrak{X}, \chi), \xi, \zeta$ be as above. Then for each $f^\infty \in \mathcal{H}(G(\mathbb{A}^\infty), (\chi^\infty)^{-1})$,*

$$T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta} f^\infty) = \frac{1}{d(G)} \sum_{M \in \mathcal{L}_{\mathrm{cusp}}} \frac{(-1)^{\dim(A_M/A_G)}}{\mathrm{vol}(\mathfrak{X}_{\mathbb{Q}} \backslash \mathfrak{X} / A_{G,\infty})} \frac{|\Omega^M|}{|\Omega^G|} \sum_\gamma \frac{\chi(I_\gamma^M) \zeta(\gamma) \Phi_M(\gamma, \xi) O_\gamma^M(f_M^\infty)}{\iota^M(\gamma) \cdot |\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|},$$

where the second sum runs over the $\mathfrak{X}$-orbits on the set of $\mathbb{R}$-elliptic conjugacy classes of $M(\mathbb{Q})$.

Proof. — This is [Dal22, Cor. 6.5.1]. We just note the difference of notation: his $A_{G,\mathrm{rat}}, A_{G,\infty}$ are our $A_{G,\infty}, A_{G_{\mathbb{R}},\infty}$, respectively. □

2.9. The stable trace formula. — Let H be a quasi-split group over $\mathbb{Q}$. Let $(\mathfrak{X}_H, \chi_H)$ be a central character datum for H. Write $\Sigma_{\mathrm{ell},\chi_H}(H)$ for the set of stable elliptic conjugacy classes in $H(\mathbb{Q})$ modulo $\mathfrak{X}_H$, namely two stable conjugacy classes are equivalent if one is mapped to the other by multiplying an element $x \in \mathfrak{X}_H$. Following [KSZ, §8.3.7], define

$$ST^H_{\mathrm{ell},\chi_H}(h) := \tau_{\mathfrak{X}_H}(H) \sum_{\gamma_H \in \Sigma_{\mathrm{ell},\chi_H}(H)} |\mathrm{Stab}_{\mathfrak{X}_H}(\gamma_H)|^{-1} SO^{H(\mathbb{A})}_{\gamma_H}(h),$$

for $h \in \mathcal{H}(H(\mathbb{A}), \chi_H^{-1})$.

Consider a central character datum $(\mathfrak{X}, \chi)$ for G as well as $f = \otimes'_v f_v \in \mathcal{H}(G(\mathbb{A}), \chi^{-1})$. For each $\mathfrak{e} \in \mathcal{E}^<_{\mathrm{ell}}(G)$, we have $\mathfrak{e}_1$ and a central character datum $(\mathfrak{X}_1^{\mathfrak{e}}, \chi_1^{\mathfrak{e}})$ (whose v-components are given as in the preceding subsection). Write $f_{1,v}^{\mathfrak{e}} \in \mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{A}), (\chi_1^{\mathfrak{e}})^{-1})$ for a transfer of f_v at each v. Put $f_1^{\mathfrak{e}} := \otimes'_v f_{1,v}^{\mathfrak{e}}$. For $\mathfrak{e} = \mathfrak{e}^*$, we transfer f to $f^* \in \mathcal{H}(G^*(\mathbb{A}), \chi^{-1})$ as in Section 2.6.

PROPOSITION 2.9.1. — *Let $f = \otimes'_v f_v \in \mathcal{H}(G(\mathbb{A}), \chi^{-1})$. Assume that there exists a finite place q such that $O_g(f_q) = 0$ for every $g \in G(\mathbb{Q}_q)_{\mathrm{ss}}$ that is not regular. With f^* and $f_1^{\mathfrak{e}}$ as above, we have*

$$T^G_{\mathrm{ell},\chi}(f) = ST^{G^*}_{\mathrm{ell},\chi}(f^*) + \sum_{\mathfrak{e}\in\mathcal{E}^<_{\mathrm{ell}}(G)} \iota(G, G^{\mathfrak{e}}) ST^{G_1^{\mathfrak{e}}(\mathbb{A})}_{\mathrm{ell},\chi_1^{\mathfrak{e}}}(f_1^{\mathfrak{e}}).$$

Proof. — By hypothesis, the stable orbital integral of $f^{\mathfrak{e}}_{1,q}$ vanishes outside G-regular semisimple conjugacy classes. When the central character datum is trivial, the stabilization of regular elliptic terms is due to Langlands [Lan83], cf. [Kot86, Th. 9.6], [KS99, §7.4], [Lab99]. For general central character data, the argument is essentially the same if one uses the Langlands–Shelstad transfer with fixed central character as in Section 2.6. □

The following finiteness result is going to tell us that the sum in Theorem 7.5.1 (and a similar sum in Theorem 7.1.1 below) is finite for each choice of $\phi^{\infty,p}$.

LEMMA 2.9.2

(1) *Let v be a rational prime such that $G_{\mathbb{Q}_v}$ and χ_v are unramified. Let $f_v \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_v), \chi_v^{-1})$. Then f_v transfers to the zero function on $G_1^{\mathfrak{e}}(\mathbb{Q}_v)$ for each $\mathfrak{e} = (G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta^{\mathfrak{e}}) \in \mathcal{E}^<_{\mathrm{ell}}(G)$ if $G^{\mathfrak{e}}$ is ramified over $\mathbb{Q}_v$.*

(2) *Let S be a finite set of rational primes. The set of $\mathfrak{e} \in \mathcal{E}^<_{\mathrm{ell}}(G)$ such that $G^{\mathfrak{e}}_{\mathbb{Q}_v}$ is unramified at every rational prime $v \notin S$ is finite.*

Proof. — Part (1) follows from [Kot86, Prop. 7.5]. Part (2) is well known; see [Lan83, Lem. 8.12]. □

3. JACQUET MODULES, REGULAR FUNCTIONS, AND ENDOSCOPY

Throughout this section, let F be a finite extension of $\mathbb{Q}_p$ with a uniformizer ϖ and residue field cardinality q. The valuation on F is normalized such that $|\varpi| = q^{-1}$. Let G be a connected reductive group over F. We study how certain maps of invariant or stable distributions between G and its Levi subgroups interact with Jacquet modules and endoscopy, based on [Shi10, Xu17].

3.1. ν-ASCENT AND JACQUET MODULES. — Let $\nu : \mathbb{G}_m \to G$ be a cocharacter defined over F. Let M_ν denote the centralizer of ν in G, which is an F-rational Levi subgroup. The maximal F-split torus in the center of M_ν is denoted by A_{M_ν}.

Write P_ν (resp. P_ν^{op}) for the F-rational parabolic subgroup of G which contains M_ν as a Levi component and such that $\langle \alpha, \nu\rangle < 0$ (resp. $\langle \alpha, \nu\rangle > 0$) for every root α of A_{M_ν} in P_ν (resp. P_ν^{op}). The set of α as such is denoted by $\Phi^+(P_\nu)$ (resp. $\Phi^+(P_\nu^{\mathrm{op}})$). Let $N_\nu, N_\nu^{\mathrm{op}}$ denote the unipotent radical of $P_\nu, P_\nu^{\mathrm{op}}$. For every $\alpha \in \Phi^+(P_\nu^{\mathrm{op}})$, we have $|\alpha(\nu(\varpi))| = q^{-\langle\alpha,\nu\rangle} < 1$. Therefore, $\nu(\varpi) \in A^{--}_{P_\nu^{\mathrm{op}}}$. The following definition is a rephrase of [Shi10, Def. 3.1].

DEFINITION 3.1.1. — We say that $\gamma \in M_\nu(\overline{F})$ is *acceptable* (with respect to ν) if the action of $\mathrm{Ad}(\gamma)$ on $(\mathrm{Lie}\, N_\nu^{\mathrm{op}})_{\overline{F}}$ is contracting, i.e., all its eigenvalues $\lambda \in \overline{F}$ have the property that $|\lambda| < 1$.

By definition, $a \in A_{M_\nu}(F)$ is acceptable if and only if $a \in A_{P_\nu^{\mathrm{op}}}^{--}$. The subset of acceptable elements is nonempty, open, and stable under $M_\nu(\overline{F})$-conjugacy. Define $\mathcal{H}_{\mathrm{acc}}(M_\nu) \subset \mathcal{H}(M_\nu)$ as the subspace of functions supported on acceptable elements. We also write $\mathcal{H}_{\nu\text{-acc}}(M_\nu)$ to emphasize the dependence on ν. As in Section 2.2 we often omit F for simplicity.

LEMMA 3.1.2. — *Let $\phi \in \mathcal{H}_{\mathrm{acc}}(M_\nu)$. There exists $f \in \mathcal{H}(G)$ with the following properties.*

(1) *For every $g \in G(F)_{\mathrm{ss}}$,*

$$O_g^G(f) = \delta_{P_\nu}(m)^{-1/2} O_m^{M_\nu}(\phi)$$

if there exists an acceptable $m \in M_\nu(F)$ which is conjugate to g in $G(F)$ (in which case m is unique up to $M_\nu(F)$-conjugacy, and the Haar measures are chosen compatibly on the connected centralizers of m and g), and $O_g^G(f) = 0$ otherwise.

(2) $\mathrm{Tr}(f|\pi) = \mathrm{Tr}\left(\phi | J_{P_\nu^{\mathrm{op}}}(\pi)\right)$ *for $\pi \in \mathrm{Irr}(G(F))$.*

Proof. — This is [Shi10, Lem. 3.9] except that we corrected typos in the statement. The same proof still works with two remarks. Firstly, we removed the assumption in loc. cit. that orbital integrals of ϕ vanish on semisimple elements with disconnected centralizers. This is possible by reducing to the case of G with simply connected derived subgroup (then $M_{\nu,\mathrm{der}}$ is also simply connected by Lemma 1.9.1) so that the centralizers of semisimple elements are connected in both M_ν and G. Secondly, the mistake in loc. cit. occurs in line 1, p. 806, where it should read $\phi^0 := \phi \cdot \delta_{P_\nu}^{-1/2}$. □

COROLLARY 3.1.3. — *Let ϕ and f be as in Lemma 3.1.2. For every $g \in G(F)_{\mathrm{ss}}$,*

$$SO_g^G(f) = \delta_{P_\nu}(m)^{-1/2} SO_m^{M_\nu}(\phi)$$

if there exists an acceptable $m \in M_\nu(F)$ which is conjugate to g in $G(F)$. Otherwise, $SO_g^G(f) = 0$.

Proof. — This is clear from the preceding lemma, using [Shi10, Lem. 3.5]. □

DEFINITION 3.1.4. — In the setup of Lemma 3.1.2, we say that f is a *ν-ascent* of ϕ.

Recall the definition of $\mathcal{I}(\cdot)$ and the trace Paley–Wiener theorem from Section 2.2. According to [BDK86, Prop. 3.2], the Jacquet module induces the map

$$\mathscr{J}_\nu \colon \mathcal{I}(M_\nu) \longrightarrow \mathcal{I}(G), \qquad \mathcal{F} \longmapsto \left(\pi \mapsto \mathcal{F}(J_{P_\nu^{\mathrm{op}}}(\pi))\right). \tag{3.1.1}$$

Write $\mathcal{I}_{\mathrm{acc}}(M_\nu)$ for the image of $\mathcal{H}_{\mathrm{acc}}(M_\nu)$ in $\mathcal{I}(M_\nu)$. Then Lemma 3.1.2 means that, when $\phi \in \mathcal{I}_{\mathrm{acc}}(M_\nu)$, a ν-ascent of ϕ is well defined as an element of $\mathcal{I}(G)$, which is nothing but $\mathscr{J}_\nu(\phi)$. The lemma yields extra information on orbital integrals. Xu

[Xu17, Prop. C.4] shows that (3.1.1) induces a similar map for the stable analogues, which we denote by the same symbol:

$$\mathscr{J}_\nu : \mathcal{S}(M_\nu) \longrightarrow \mathcal{S}(G). \tag{3.1.2}$$

Write $X_F^*(G)$ for the group of F-rational characters of G. Define $X_F^*(G)_\mathbb{Q} := X_F^*(G) \otimes_\mathbb{Z} \mathbb{Q}$ and $\mathfrak{a}_G := \mathrm{Hom}(X_F^*(G)_\mathbb{Q}, \mathbb{R})$. We have the map

$$H^G \colon G(F) \longrightarrow \mathfrak{a}_G, \qquad g \longmapsto (\chi \mapsto \log|\chi(g)|). \tag{3.1.3}$$

It is easy to see that H^G is invariant under $G(\overline{F})$-conjugacy. Indeed, if g_1, g_2 become conjugate in $G(F')$ for a finite extension F'/F, then it boils down to the obvious fact that $H^{G'}(g_1) = H^{G'}(g_2)$, since the map H^G is functorial with respect to $G \hookrightarrow G' := \mathrm{Res}_{F'/F} G$.

For $f \in \mathcal{H}(G)$, define the following subsets of $\mathfrak{a}_G$:

$$\begin{aligned} \mathrm{supp}_{\mathfrak{a}_G}(f) &:= \{H^G(x) : x \in G(F)_{\mathrm{ss}} \text{ s.t. } f(x) \neq 0\}, \\ \mathrm{supp}^{O}_{\mathfrak{a}_G}(f) &:= \{H^G(x) : x \in G(F)_{\mathrm{ss}} \text{ s.t. } O_x(f) \neq 0\}, \\ \mathrm{supp}^{SO}_{\mathfrak{a}_G}(f) &:= \{H^G(x) : x \in G(F)_{\mathrm{ss}} \text{ s.t. } SO_x(f) \neq 0\}. \end{aligned} \tag{3.1.4}$$

Obviously $\mathrm{supp}^{SO}_{\mathfrak{a}_G}(f) \subset \mathrm{supp}^{O}_{\mathfrak{a}_G}(f) \subset \mathrm{supp}_{\mathfrak{a}_G}(f)$. Writing

$$\mathscr{P}(*) := \text{collection of subsets of } *,$$

we obtain a map $\mathrm{supp}_{\mathfrak{a}_G}$ (resp. $\mathrm{supp}^{O}_{\mathfrak{a}_G}$, $\mathrm{supp}^{SO}_{\mathfrak{a}_G}$) from $\mathcal{H}(G)$ (resp. $\mathcal{I}(G)$, $\mathcal{S}(G)$) to $\mathscr{P}(\mathfrak{a}_G)$.

We define analogous objects for M_ν in place of G. The injective restriction map $X_F^*(G)_\mathbb{Q} \to X_F^*(M_\nu)_\mathbb{Q}$ induces a canonical surjection

$$\mathrm{pr}_G : \mathfrak{a}_{M_\nu} \longrightarrow \mathfrak{a}_G. \tag{3.1.5}$$

Set $\mathfrak{a}_{P_\nu} := \mathfrak{a}_{M_\nu}$ and identify $X_*(A_{M_\nu})_\mathbb{R} \simeq \mathfrak{a}_{P_\nu}$ by $\mu \in X_*(A_{M_\nu}) \mapsto (\chi \mapsto \langle \chi, \mu \rangle)$. Then it is an easy exercise to describe pr_G as the average map along Weyl orbits: if T is a maximal torus of M_ν (thus also of G) over F, and if the Weyl group is taken relative to T, then

$$\mathrm{pr}_G(\mu) = |\Omega^G|^{-1} \sum_{\omega \in \Omega^G} \omega(\mu) = |\overline{\Omega}^G|^{-1} \sum_{\omega \in \overline{\Omega}^G} \omega(\mu), \qquad \mu \in X_*(A_{M_\nu})_\mathbb{R}. \tag{3.1.6}$$

LEMMA 3.1.5. — *The sets* $\mathrm{supp}_{\mathfrak{a}_G}(f)$, $\mathrm{supp}^{O}_{\mathfrak{a}_G}(f)$, *and* $\mathrm{supp}^{SO}_{\mathfrak{a}_G}(f)$ *remain unchanged if we restrict* x *in the definition* (3.1.4) *to a subset* $D \subset G(F)_{\mathrm{reg}}$ *that is open dense in* $G(F)$.

Proof. — Since the map H^G is continuous with discrete image, for each y in $\mathrm{supp}_{\mathfrak{a}_G}(f)$, $\mathrm{supp}^{O}_{\mathfrak{a}_G}(f)$, or $\mathrm{supp}^{SO}_{\mathfrak{a}_G}(f)$, the preimage $(H^G)^{-1}(y)$ is open and closed. If $y \in \mathrm{supp}_{\mathfrak{a}_G}(f)$ then $\mathrm{supp}_{\mathfrak{a}_G}(f) \cap (H^G)^{-1}(y)$ is nonempty open in $G(F)$ thus intersects D. This proves the assertion for $\mathrm{supp}_{\mathfrak{a}_G}(f)$.

Next let $y \in \mathrm{supp}^{O}_{\mathfrak{a}_G}(f)$. Then $(H^G)^{-1}(y) \cap D$ is open dense in $(H^G)^{-1}(y)$. If $O_x(f) = 0$ for every $x \in (H^G)^{-1}(y) \cap D$, we claim that

$$O_x(f) = 0, \qquad x \in (H^G)^{-1}(y) \cap G(F)_{\mathrm{ss}}.$$

If x is regular, this follows from local constancy of $O_x(f)$ on regular elements. A Shalika germ argument then proves $O_x(f) = 0$ for non-regular semisimple x. (Compare with the proof of Lemma 3.4.5 (1) below.) However, the claim contradicts $y \in \mathrm{supp}^O_{\mathfrak{a}_G}(f)$. The lemma for $\mathrm{supp}^O_{\mathfrak{a}_G}(f)$ follows. Finally, the stable analogue is proved likewise. □

LEMMA 3.1.6. — *The following diagrams commute.*

$$\begin{array}{ccc} \mathcal{I}_{\mathrm{acc}}(M_\nu) & \xrightarrow{\mathscr{J}_\nu} & \mathcal{I}(G) \\ \downarrow{\scriptstyle \mathrm{supp}^O_{\mathfrak{a}_{M_\nu}}} & & \downarrow{\scriptstyle \mathrm{supp}^O_{\mathfrak{a}_G}} \\ \mathscr{P}(\mathfrak{a}_{P_\nu}) & \xrightarrow{\mathrm{pr}_G} & \mathscr{P}(\mathfrak{a}_G) \end{array} \qquad \begin{array}{ccc} \mathcal{S}_{\mathrm{acc}}(M_\nu) & \xrightarrow{\mathscr{J}_\nu} & \mathcal{S}(G) \\ \downarrow{\scriptstyle \mathrm{supp}^{SO}_{\mathfrak{a}_{M_\nu}}} & & \downarrow{\scriptstyle \mathrm{supp}^{SO}_{\mathfrak{a}_G}} \\ \mathscr{P}(\mathfrak{a}_{P_\nu}) & \xrightarrow{\mathrm{pr}_G} & \mathscr{P}(\mathfrak{a}_G) \end{array}$$

Proof. — This follows from Lemma 3.1.2 and Corollary 3.1.3 since, for each $m \in M_\nu(F)$, the canonical map $\mathfrak{a}_{M_\nu} \to \mathfrak{a}_G$ sends $H^{M_\nu}(m)$ to $H^G(m)$. □

Let $k \in \mathbb{Z}$ and $\phi \in \mathcal{H}(M_\nu)$. Define $\phi^{(k)}(l) := \phi(\nu(\varpi)^{-k}l)$ for $l \in M_\nu(F)$ so that $\phi^{(k)} \in \mathcal{H}(M_\nu)$. Since ν is central in M_ν, this induces a map

$$(\cdot)^{(k)} \colon \mathcal{I}(M_\nu) \longrightarrow \mathcal{I}(M_\nu). \tag{3.1.7}$$

LEMMA 3.1.7. — *If $\phi \in \mathcal{H}_{\mathrm{acc}}(M_\nu)$ then $\phi^{(k)} \in \mathcal{H}_{\mathrm{acc}}(M_\nu)$ for all $k \geqslant 0$. Given $\phi \in \mathcal{H}(M_\nu)$, there exists $k_0 = k_0(\phi)$ such that $\phi^{(k)} \in \mathcal{H}_{\mathrm{acc}}(M_\nu)$ for all $k \geqslant k_0$. The analogue holds true with $\mathcal{I}$ in place of $\mathcal{H}$. Moreover, letting $f^{(k)}$ denote the ν-ascent of $\phi^{(k)}$ for $k \geqslant k_0$, we have*

$$\mathrm{supp}^\star_{\mathfrak{a}_G}\big(f^{(k)}\big) = \mathrm{pr}_G(\mathrm{supp}^\star_{\mathfrak{a}_{M_\nu}}(\phi^{(k)})) = k \cdot H^G(\nu(\varpi)) + \mathrm{pr}_G(\mathrm{supp}^\star_{\mathfrak{a}_{M_\nu}}(\phi)),$$

for $\star \in \{O, SO\}$ and where $\mathrm{pr}_G : \mathfrak{a}_{M_\nu} \to \mathfrak{a}_G$ is the canonical surjection.

Proof. — The assertions before "Moreover" follow from the facts that $\nu(\varpi)$ is acceptable and that ϕ has compact support. As for the last assertion, the second equality is obvious, so we check the first equality. By Lemma 3.1.5 it is enough to verify firstly that if $O_g(f^{(k)}) \neq 0$ for $g \in G(F)_{\mathrm{reg}}$ then $H^G(g) \in \mathrm{pr}_G(\mathrm{supp}^O_{\mathfrak{a}_{M_\nu}}(\phi^{(k)}))$, and secondly that if $O_m(\phi^{(k)}) \neq 0$ for $m \in M(F)_{\mathrm{reg}}$ then $\mathrm{pr}_G(H^M(m)) \in \mathrm{supp}^O_{\mathfrak{a}_G}(f^{(k)})$. This follows from Lemma 3.1.2 (1) and Lemma 3.1.6. The case of stable orbital integrals is analogous. □

Let $\mathrm{Groth}(M_\nu(F))$ denote the Grothendieck group of admissible representations of $M_\nu(F)$.

LEMMA 3.1.8. — *Let $\pi_1, \pi_2 \in \mathrm{Groth}(M_\nu(F))$. Assume that for each $\phi \in \mathcal{I}(M_\nu)$, there exists $k_0(\phi) \in \mathbb{Z}$ such that $\mathrm{Tr}\,\pi_1(\phi^{(k)}) = \mathrm{Tr}\,\pi_2(\phi^{(k)})$ for all $k \geqslant k_0(\phi)$. Then $\pi_1 = \pi_2$ in $\mathrm{Groth}(M_\nu(F))$.*

Proof. — This is proved by the argument of [Shi09, p. 536]. □

3.2. ν-ASCENT AND CONSTANT TERMS. — Fix an F-rational minimal parabolic subgroup $P_0 \subset P_\nu^{\mathrm{op}}$ of G with a Levi factor $M_0 \subset M_\nu$. Let P be another F-rational parabolic subgroup of G containing P_0, with a Levi factor M containing M_0. Henceforth we will often write $L := M_\nu$.

We have the constant term map (compare with (3.1.1))

$$\mathscr{C}_M^G : \mathcal{I}(G) \longrightarrow \mathcal{I}(M), \qquad \mathcal{F} \longmapsto \Big((\pi_M \mapsto \mathcal{F}(\text{n-ind}_M^G(\pi_M))\Big), \tag{3.2.1}$$

where $\text{n-ind}_M^G : \mathrm{Groth}(M(F)) \to \mathrm{Groth}(G(F))$ is the normalized parabolic induction (which does not change if P is replaced with a different parabolic with Levi factor M). On the level of functions, when $f \in \mathcal{H}(G)$, we can define $f_M \in \mathcal{H}(M)$ by an integral formula (e.g., [Shi11, (3.5)]) so that

$$\begin{aligned} &O_g^G(f) = 0, \quad \forall g \in G(F)_{\mathrm{reg}} \text{ not conjugate to an } m \in M(F), \\ &O_m^G(f) = D_{G/M}(m)^{1/2} O_m^M(f_M), \quad \forall\, G\text{-regular } m \in M(F), \end{aligned} \tag{3.2.2}$$

where $D_{G/M} : M(F) \to \mathbb{R}_{>0}^\times$ denotes the Weyl discriminant of G relative to M. This identity and parts (i) and (ii) of [Shi11, Lem. 3.3] tell us that $f \mapsto f_M$ descends to the map $\mathscr{C}_M^G$ above. (Even though G is a general linear group in loc. cit. everything applies to general reductive groups since that lemma is based on the general results of [vD72].)

Since n-ind_M^G induces a map $R(M)^{\mathrm{st}} \to R(G)^{\mathrm{st}}$ [KV16, Cor. 6.13], the map $\mathscr{C}_M^G$ descends to a map on the stable spaces, still denoted by the same symbol:

$$\mathscr{C}_M^G : \mathcal{S}(G) \longrightarrow \mathcal{S}(M).$$

Define the following set of representatives for $\Omega^L \backslash \Omega^G / \Omega^M$:

$$\Omega_{M,L}^G := \{\omega \in \Omega^G : \omega(M \cap P_0) \subset P_0,\ \omega^{-1}(L \cap P_0) \subset P_0\}.$$

For $\omega \in \Omega_{M,L}^G$, write $M_\omega := M \cap \omega^{-1}(L)$, $P_\omega := M \cap \omega^{-1}(P_\nu)$, and $L_\omega := \omega(M) \cap L$. Note that M_ω (resp. L_ω) is an F-rational Levi subgroup of M (resp. L) and that ω induces an isomorphism $M_\omega \xrightarrow{\sim} L_\omega$, thus also $\omega : \mathcal{I}(M_\omega) \xrightarrow{\sim} \mathcal{I}(L_\omega)$ by

$$\phi \longmapsto (g \mapsto \phi(\omega^{-1} g)).$$

Since ν is central in L, its image lies in L_ω. So $\nu_\omega := \omega^{-1}(\nu)$ is a cocharacter of M_ω. Hence we have a chain of maps

$$\mathcal{I}(L) \xrightarrow{\mathscr{C}_{L_\omega}^L} \mathcal{I}(L_\omega) \overset{\omega^{-1}}{\simeq} \mathcal{I}(M_\omega) \xrightarrow{\mathscr{J}_{\nu_\omega}} \mathcal{I}(M).$$

LEMMA 3.2.1. — *If $\phi \in \mathcal{I}_{\nu\text{-acc}}(L)$ then $\mathscr{C}_{L_\omega}^L(\phi)$ is contained in $\mathcal{I}_{\nu\text{-acc}}(L_\omega)$.*

Proof. — The proof of Lemma 3.3.5 below works verbatim: just replace stable orbital integrals there with ordinary orbital integrals, and use (3.2.2). (Since Lemma 3.3.5 is more general, we supply a detailed argument only for the latter.) □

LEMMA 3.2.2. — *We have the following commutative diagram. The exact analogue is true with $\mathcal{S}(\cdot)$ in place of $\mathcal{I}(\cdot)$.*

$$\begin{array}{ccccc} \mathcal{I}(L) & \xrightarrow{\mathscr{J}_\nu} & \mathcal{I}(G) & \xrightarrow{\mathscr{C}_M^G} & \mathcal{I}(M) \\ \downarrow{\scriptstyle \oplus \mathscr{C}_{L_\omega}^L} & & & & \uparrow{\scriptstyle \sum_\omega \mathscr{J}_{\nu_\omega}} \\ \bigoplus_{\omega \in \Omega_{M,L}^G} \mathcal{I}(L_\omega) & \xrightarrow[\sim]{\oplus \omega^{-1}} & & & \bigoplus_{\omega \in \Omega_{M,L}^G} \mathcal{I}(M_\omega) \end{array}$$

Proof. — Let $\phi \in \mathcal{I}(L)$. We check that the images of ϕ in $\mathcal{I}(M)$ given in the two different ways have the same trace against every $\pi_M \in \mathrm{Irr}(M(F))$:

$$\begin{aligned} &\mathrm{Tr}\left(\mathscr{C}_{L_\omega}^L(\mathscr{J}_\nu(\phi))|\pi_M\right) = \mathrm{Tr}\left(\mathscr{J}_\nu(\phi)|\text{n-ind}_L^G(\pi_M)\right) = \mathrm{Tr}\left(\phi|J_{P_\nu}(\text{n-ind}_L^G(\pi_M))\right) \\ &\quad = \sum_{\omega \in \Omega_{M,L}^G} \mathrm{Tr}\left(\phi|\text{n-ind}_{L_\omega}^L\left(\omega(J_{P_{\nu_\omega}}(\pi_M))\right)\right) = \sum_{\omega \in \Omega_{M,L}^G} \mathrm{Tr}\Big(\mathscr{J}_{\nu_\omega}(\omega^{-1}(\mathscr{C}_{L_\omega}^L(\phi)))|\pi_M\Big), \end{aligned}$$

where the second last equality comes from Bernstein–Zelevinsky's geometric lemma [BZ77, 2.12]. The $\mathcal{S}(\cdot)$-version is immediate from the $\mathcal{I}(\cdot)$-version proved just now, since each map in the big diagram descends to a map between the stable analogues. □

3.3. ν-ASCENT AND ENDOSCOPIC TRANSFER. — In this subsection we assume that G is quasi-split over F. Let $\mathfrak{e} = (G^\mathfrak{e}, \mathcal{G}^\mathfrak{e}, s^\mathfrak{e}, \eta^\mathfrak{e})$ be an endoscopic datum for G such that $\mathcal{G}^\mathfrak{e} = {}^L G^\mathfrak{e}$. (The last condition will be removed via z-extensions in Section 3.6.) Here we fix Γ_F-pinnings $(\mathcal{B}^\mathfrak{e}, \mathcal{T}^\mathfrak{e}, \{\mathcal{X}_{\alpha^\mathfrak{e}}\})$ and $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}_\alpha\})$ for $\widehat{G}^\mathfrak{e}$ and $\widehat{G}$, respectively. (These choices are implicit in the discussion of Section 2.6.) Conjugating $\eta^\mathfrak{e}$ we may and will assume that $\eta^\mathfrak{e}(\mathcal{T}^\mathfrak{e}) = \mathcal{T}$ and $\eta^\mathfrak{e}(\mathcal{B}^\mathfrak{e}) \subset \mathcal{B}$.

We have a standard embedding ${}^L P_\nu \hookrightarrow {}^L G$ and a Levi subgroup ${}^L M_\nu \subset {}^L P_\nu$ as in [Bor79, 3.3, 3.4]. Choose a subtorus $S \subset \mathcal{T}$ such that $\mathrm{Cent}(S, {}^L G) = {}^L M_\nu$. (This is possible by [Bor79, Lem. 3.5].) Following [Xu17, §6], define

$$\Omega^G(\mathfrak{e}, \nu) := \{\omega \in \Omega^G \mid \mathrm{Cent}(\omega(S), {}^L G^\mathfrak{e}) \to W_F \text{ is surjective}\}$$

and $\Omega_{\mathfrak{e},\nu} := \Omega^{G^\mathfrak{e}} \backslash \Omega^G(\mathfrak{e}, M_\nu)/\Omega^{M_\nu}$. For each $\omega \in \Omega_{\mathfrak{e},\nu}$, we obtain an endoscopic datum

$$\mathfrak{e}_\omega = (G_\omega^\mathfrak{e}, {}^L G_\omega^\mathfrak{e}, s_\omega^\mathfrak{e}, \eta_\omega^\mathfrak{e}) \quad \text{for } L = M_\nu$$

as follows. (Henceforth we view ${}^L G^\mathfrak{e}$ as a subgroup of ${}^L G$ via $\eta^\mathfrak{e}$.) Pick $g \in \widehat{G}$ such that $\mathrm{Int}(g)$ induces ω on S. Then $g\,{}^L P_\nu g^{-1} \cap {}^L G^\mathfrak{e}$ is a parabolic subgroup of ${}^L G^\mathfrak{e}$ with Levi subgroup $g\,{}^L M_\nu g^{-1}$, so there is a corresponding standard parabolic subgroup $P_\nu^\mathfrak{e} = M_\nu^\mathfrak{e} N_\nu^\mathfrak{e}$ such that the standard embedding ${}^L P_\nu^\mathfrak{e} \hookrightarrow {}^L G^\mathfrak{e}$ (resp. ${}^L M_\nu^\mathfrak{e} \hookrightarrow {}^L G^\mathfrak{e}$) becomes $g\,{}^L P_\nu g^{-1} \cap {}^L G^\mathfrak{e}$ (resp. $g\,{}^L M_\nu g^{-1} \cap {}^L G^\mathfrak{e}$) after composing with $\mathrm{Int}(g^\mathfrak{e})$ for some $g^\mathfrak{e} \in \widehat{G}^\mathfrak{e}$. Then there is a unique L-embedding $\eta_\omega^\mathfrak{e} : {}^L M_\nu^\mathfrak{e} \hookrightarrow {}^L M_\nu$ such that $\mathrm{Int}(g) \circ \eta_\omega^\mathfrak{e} = \eta^\mathfrak{e} \circ \mathrm{Int}(g^\mathfrak{e})$. Set $G_\omega^\mathfrak{e} := M_\nu^\mathfrak{e}$, and $s_\omega^\mathfrak{e} := g^{-1} s g \in \widehat{M}_\nu$. Then it is a routine exercise to check that $(G_\omega^\mathfrak{e}, {}^L G_\omega^\mathfrak{e}, s_\omega^\mathfrak{e}, \eta_\omega^\mathfrak{e})$ is an endoscopic datum for M_ν.

There is a canonical embedding $A_{M_\nu} \hookrightarrow A_{M_\nu^{\mathfrak{e}}} = A_{G_\omega^{\mathfrak{e}}}$ (just like $Z_H \hookrightarrow Z$ in Section 2.6). Composing with $\nu : \mathbb{G}_m \to A_{M_\nu}$, we obtain

$$\nu_\omega \colon \mathbb{G}_m \longrightarrow A_{G_\omega^{\mathfrak{e}}}.$$

By construction, $G_\omega^{\mathfrak{e}} = M_\nu^{\mathfrak{e}}$ is a Levi subgroup of $G^{\mathfrak{e}}$ that is the centralizer of ν_ω. In particular, we have a map $\mathscr{J}_{\nu_\omega} : \mathcal{S}(G_\omega^{\mathfrak{e}}) \to \mathcal{S}(G^{\mathfrak{e}})$ as in (3.1.2). Consider the following commutative diagram

$$\begin{array}{ccccccc} W_F & \longrightarrow & {}^L G_\omega^{\mathfrak{e}} & \xrightarrow{\eta_\omega^{\mathfrak{e}}} & {}^L M_\nu & \longrightarrow & {}^L Z^0_{M_\nu} \\ & & \big\downarrow{\scriptstyle \mathrm{Int}(g^{\mathfrak{e}})} & & \big\downarrow{\scriptstyle \mathrm{Int}(g)} & & \big\downarrow \\ W_F & \longrightarrow & {}^L G^{\mathfrak{e}} & \xrightarrow{\eta^{\mathfrak{e}}} & {}^L G & \longrightarrow & {}^L Z^0_G, \end{array} \tag{3.3.1}$$

where the maps out of W_F come from canonical splittings for the L-groups, the two horizontal maps on the right are induced by $Z^0_{M_\nu} \subset M_\nu$ and $Z^0_G \subset G$, the first two vertical maps correspond to the Levi embeddings (coming from $G_\omega^{\mathfrak{e}} \subset G^{\mathfrak{e}}$ and $M_\nu \subset G$) followed by $\mathrm{Int}(g^{\mathfrak{e}})$ and $\mathrm{Int}(g)$ respectively, and finally the rightmost vertical map is induced by $Z^0_G \subset Z^0_{M_\nu}$. The left square in (3.3.1) commutes by $\mathrm{Int}(g) \circ \eta_\omega^{\mathfrak{e}} = \eta^{\mathfrak{e}} \circ \mathrm{Int}(g^{\mathfrak{e}})$ above. The commutativity of the right square is obvious since $\mathrm{Int}(g)$ acts trivially on ${}^L Z^0_G$. Denote by

$$\lambda_\omega^{\mathfrak{e}} : Z^0_{M_\nu}(F) \longrightarrow \mathbb{C}^\times \qquad (\text{resp. } \lambda^{\mathfrak{e}} : Z^0_G(F) \longrightarrow \mathbb{C}^\times)$$

the smooth character corresponding to the composite morphism from W_F to ${}^L Z^0_{M_\nu}$ (resp. ${}^L Z^0_G$) in the first (resp. second) row. The character $\lambda^{\mathfrak{e}}$ is the same as in Section 2.6. The commutativity of (3.3.1) implies that $\lambda_\omega^{\mathfrak{e}}|_{Z^0_G(F)} = \lambda^{\mathfrak{e}}$. The canonical splittings from W_F to ${}^L G_\omega^{\mathfrak{e}}$ and ${}^L G^{\mathfrak{e}}$ commute with the Levi embedding ${}^L G_\omega^{\mathfrak{e}} \hookrightarrow {}^L G^{\mathfrak{e}}$ without $\mathrm{Int}(g^{\mathfrak{e}})$, but the point is that $\mathrm{Int}(g^{\mathfrak{e}})$ on ${}^L G^{\mathfrak{e}}$ is equivariant with the trivial action on ${}^L Z^0_G$ via the horizontal maps in (3.3.1).

LEMMA 3.3.1. — *Assume that $\eta^{\mathfrak{e}}(W_F)$ is a bounded subgroup of ${}^L G$ in the sense above Lemma 2.6.2. (This condition can always be ensured by that lemma.) Then $\lambda_\omega^{\mathfrak{e}}$ is a unitary character.*

Proof. — By assumption and commutativity of (3.3.1), $\eta_\omega^{\mathfrak{e}}(W_F)$ is a bounded subgroup of ${}^L M_\nu$, whose image in ${}^L Z^0_{M_\nu}$ is a bounded subgroup accordingly. Therefore, $\lambda_\omega^{\mathfrak{e}}$ is a unitary character via the Langlands correspondence for tori. □

PROPOSITION 3.3.2. — *The following diagram commutes.*

$$\begin{array}{ccccc} \mathcal{I}(M_\nu) & \xrightarrow{\mathscr{J}_\nu} & \mathcal{I}(G) & \xrightarrow{\mathrm{LS}^{\mathfrak{e}}} & \mathcal{S}(G^{\mathfrak{e}}) \\ & {\scriptstyle \oplus \mathrm{LS}^{\mathfrak{e},\omega}} \searrow & & \nearrow {\scriptstyle \sum_\omega \mathscr{J}_{\nu_\omega}} & \\ & & \bigoplus_{\omega \in \Omega_{\mathfrak{e},\nu}} \mathcal{S}(G_\omega^{\mathfrak{e}}) & & \end{array}$$

Let $\phi \in C_c^\infty(M_\nu(F))$. If $f^{(k)} = \mathscr{J}_\nu(\phi^{(k)})$ then writing $\phi_\omega^{(k)} := \mathrm{LS}^{\mathfrak{e},\omega}(\phi)^{(k)}$, we have

$$\phi_\omega^{(k)} = \lambda_\omega^{\mathfrak{e}}(\nu(\varpi))^{-k}\mathrm{LS}^{\mathfrak{e},\omega}(\phi^{(k)}), \qquad \mathrm{LS}^{\mathfrak{e}}(f^{(k)}) = \sum_{\omega\in\Omega_{\mathfrak{e},\nu}} \lambda_\omega^{\mathfrak{e}}(\nu(\varpi))^k \mathscr{J}_{\nu_\omega}\big(\phi_\omega^{(k)}\big).$$

REMARK 3.3.3. — When $\mathfrak{e}$ is given by a Levi subgroup M as in Section 3.2 (so that $G^{\mathfrak{e}} = M$), we have $\mathrm{LS}^{\mathfrak{e}} = \mathscr{C}_M^G$, $\mathrm{LS}^{\mathfrak{e},\omega} = \mathscr{C}_{L_\omega}^L$, and the meaning of ν_ω is consistent between Section 3.2 and Section 3.3. We leave it to the interested reader to compare the diagram above with that of Lemma 3.2.2.

Proof. — The first equality follows from the equivariance property of transfer as discussed in the paragraph containing (2.6.3) (applied to $z = \nu(\varpi)^{-k}$, $G = M_\nu$, $G^{\mathfrak{e}} = G_\omega^{\mathfrak{e}}$, and $f = \phi$). The commutative diagram comes from (C.4) in [Xu17] (when θ is trivial). This, together with the first equality, implies the last equality. □

COROLLARY 3.3.4. — *Let $\phi^{(k)}$, $\phi_\omega^{(k)}$, and $f^{(k)}$ be as in Proposition 3.3.2. Then*

$$\mathrm{supp}^{SO}_{\mathfrak{a}_{G^{\mathfrak{e}}}}\Big(\mathscr{J}_{\nu_\omega}(\phi_\omega^{(k)})\Big) = k\cdot H^{G^{\mathfrak{e}}}(\nu_\omega(\varpi)) + \mathrm{pr}_{G^{\mathfrak{e}}}\Big(\mathrm{supp}^{SO}_{\mathfrak{a}_{L_\omega}}(\mathrm{LS}^{\mathfrak{e},\omega}(\phi))\Big), \qquad \omega\in\Omega_{\mathfrak{e},\nu},$$

where $\mathrm{pr}_{G^{\mathfrak{e}}} : \mathfrak{a}_{G_\omega^{\mathfrak{e}}} \twoheadrightarrow \mathfrak{a}_{G^{\mathfrak{e}}}$ is the natural projection.

Proof. — By Lemma 3.1.7 and Proposition 3.3.2,

$$\begin{aligned}\mathrm{supp}^{SO}_{\mathfrak{a}_{G^{\mathfrak{e}}}}\Big(\mathscr{J}_{\nu_\omega}(\phi_\omega^{(k)})\Big) &= \mathrm{pr}_{G^{\mathfrak{e}}}\Big(\mathrm{supp}^{SO}_{\mathfrak{a}_{G_\omega^{\mathfrak{e}}}}(\phi_\omega^{(k)})\Big) = \mathrm{pr}_{G^{\mathfrak{e}}}\Big(\mathrm{supp}^{SO}_{\mathfrak{a}_{G_\omega^{\mathfrak{e}}}}(\mathrm{LS}^{\mathfrak{e},\omega}(\phi^{(k)}))\Big)\\ &= \mathrm{pr}_{G^{\mathfrak{e}}}\Big(k\cdot H^{G_\omega^{\mathfrak{e}}}(\nu_\omega(\varpi)) + \mathrm{supp}^{SO}_{\mathfrak{a}_{G_\omega^{\mathfrak{e}}}}(\mathrm{LS}^{\mathfrak{e},\omega}(\phi))\Big).\end{aligned}$$

We finish by observing that $\mathrm{pr}_{G^{\mathfrak{e}}}(H^{G_\omega^{\mathfrak{e}}}(\nu_\omega(\varpi))) = H^{G^{\mathfrak{e}}}(\nu_\omega(\varpi))$. □

It is useful to know preservation of acceptability in the setting of Proposition 3.3.2 as this will allow an inductive argument in the proof of Corollary 4.2.3 below.

LEMMA 3.3.5. — *If $\phi \in \mathfrak{I}_{\mathrm{acc}}(M_\nu)$ then $\phi_\omega := \mathrm{LS}^{\mathfrak{e},\omega}(\phi)$ is contained in $\mathfrak{S}_{\mathrm{acc}}(G_\omega^{\mathfrak{e}})$.*

Proof. — Suppose that $SO_{\gamma_\omega}(\phi_\omega) \neq 0$ for a strongly M_ν-regular element $\gamma_\omega \in G_\omega^{\mathfrak{e}}(F)$. We need to check that γ_ω is ν_ω-acceptable. (This is enough thanks to Lemma 3.1.5.)

From the orbital integral identity for $SO_{\gamma_\omega}(\phi_\omega)$ (cf. (2.6.1)), we see the existence of $\gamma \in M_\nu(F)_{\mathrm{sr}}$ whose stable conjugacy class matches that of γ_ω such that $O_\gamma(\phi) \neq 0$. The latter implies that γ is ν-acceptable. Write T, T_ω for the centralizers of γ, γ_ω in M_ν, $G_\omega^{\mathfrak{e}}$, respectively. The matching of conjugacy classes tells us that there is a canonical F-isomorphism $i : T \simeq T_\omega$ which carries γ to γ_ω, cf. [Kot86, §3.1]. (A priori, i sends the stable conjugacy class of γ to that of γ_ω and is canonical up to a Weyl group orbit. But i is determined if required to send γ to γ_ω.) Since ν is central in M_ν, the map i necessarily carries ν to ν_ω. Regarding T and T_ω as maximal tori of G and $G^{\mathfrak{e}}$, respectively, we have an injection $i^* : R(G_\omega^{\mathfrak{e}}, T_\omega) \hookrightarrow R(G,T)$ between the sets of roots induced by i (again [Kot86, §3.1]) such that

$$\langle \alpha, \nu_\omega\rangle = \langle i^*(\alpha), \nu\rangle, \qquad \alpha \in R(G_\omega^{\mathfrak{e}}, T_\omega). \tag{3.3.2}$$

We are ready to show that γ_ω is ν_ω-acceptable. Let $\alpha \in R(G^{\mathfrak{e}}_\omega, T_\omega)$ such that $\langle \alpha, \nu_\omega \rangle > 0$. We need to verify that $|\alpha(\gamma_\omega)| < 1$, cf. Definition 3.1.1. But $\langle i^*(\alpha), \nu \rangle > 0$ by (3.3.2), so the ν-acceptability of γ implies that $|i^*(\alpha)(\gamma)| < 1$. Since $i^*(\alpha)(\gamma) = \alpha(\gamma_\omega)$, the proof is finished. □

3.4. *C-regular functions and constant terms.* — Assume that G is *split* over F and fix a reductive model over $\mathcal{O}_F$, still denoted by G. Let T be a split maximal torus of G over $\mathcal{O}_F$. Let $C \in \mathbb{R}_{>0}$.

Definition 3.4.1. — A cocharacter $\mu : \mathbb{G}_m \to T$ is *C-regular* if the following two conditions hold.

(1) $|\langle \alpha, \mu \rangle| > C$ for every $\alpha \in \Phi(T, G)$,

(2) $|\langle \alpha|_{A_M}, \mathrm{pr}_M(\omega\mu) \rangle| > C$ for every proper Levi subgroup M of G containing T, every $\omega \in \Omega^G$, and every $\alpha \in \Phi(T, G) \backslash \Phi(T, M)$.

Write $X_*(T)_{C\text{-reg}}$ for the set of C-regular cocharacters.

Lemma 3.4.2. — *The following are true.*

(1) *The subset $X_*(T)_{C\text{-reg}}$ of $X_*(T)$ is nonempty, and stable under both nonzero $\mathbb{Z}$-multiples and the Ω^G-action.*

(2) *Let $\mu, \mu_0 \in X_*(T)$. If μ is C-regular, then there exists $k_0 \in \mathbb{Z}_{>0}$ such that $\mu_0 + k\mu$ is C-regular for all $k \geqslant k_0$.*

Proof

(1) Let $X_*(T)_{\mathbb{R},C\text{-reg}}$ denote the subset of $X_*(T)_{\mathbb{R}}$ defined by the same inequalities as in Definition 3.4.1. We choose an inner product on $X_*(T)_{\mathbb{R}}$ invariant under the Weyl group action. Clearly $X_*(T)_{C\text{-reg}}$ and $X_*(T)_{\mathbb{R},C\text{-reg}}$ are stable under nonzero $\mathbb{Z}$-multiples and the Weyl group action, and the latter is open. It suffices to verify the claim that $X_*(T)_{\mathbb{R},C\text{-reg}}$ is nonempty. Indeed, if the claim is true, we choose an open ball $U \subset X_*(T)_{\mathbb{R},C\text{-reg}}$. For $k \in \mathbb{Z}_{>0}$ large enough, $k \cdot U$ contains a point of $X_*(T)$, which then also lies in $X_*(T)_{C\text{-reg}}$.

Let us prove the claim. Identify $X_*(T)_{\mathbb{R}}$ with the standard inner product space $\mathbb{R}^n$ via a linear isomorphism. Say that a measurable subset $A \subset \mathbb{R}^n$ has density 0 if $\mathrm{vol}(A \cap B(0,r))/\mathrm{vol}(B(0,r)) \to 0$ as $r \to \infty$, where $B(0,r)$ denotes the ball of radius r centered at 0. We will show that the complement of $X_*(T)_{\mathbb{R},C\text{-reg}}$ in $X_*(T)_{\mathbb{R}}$ is a density 0 set. Since a finite union of density 0 sets still has density 0, it is enough to check that each of the conditions $|\langle \alpha, \mu \rangle| \leqslant C$ and $|\langle \alpha|_{A_M}, \mathrm{pr}_M(\omega\mu) \rangle| \leqslant C$ defines a density 0 subset in $X_*(T)_{\mathbb{R}}$. Either condition defines a subset of $\mathbb{R}^n$ of the form

$$|a_1 x_1 + \cdots + a_n x_n| \leqslant C \tag{3.4.1}$$

in the standard coordinates $(x_1, ..., x_n)$, with $a_1, ..., a_n \in \mathbb{R}$. Moreover, not all a_i's are zero, since neither $\langle \alpha, \mu \rangle$ nor $\langle \alpha|_{A_M}, \mathrm{pr}_M(\omega\mu) \rangle$ is identically zero on all $\mu \in X_*(T)_{\mathbb{R}}$. (In the case of $\langle \alpha|_{A_M}, \mathrm{pr}_M(\omega\mu) \rangle$, the reason is that $\mathrm{pr}_M : X_*(T)_{\mathbb{R}} \to X_*(A_M)$ is surjective, and that $\alpha|_{A_M} \in X^*(A_M)$ is nontrivial since $\alpha \notin \Phi(T, M)$.) Now it is elementary to see that (3.4.1) determines a density 0 subset. This proves the claim.

(2) Since the pairings in Definition 3.4.1 are linear in μ, it is enough to choose k_0 such that $(k_0-1)C$ is greater than $|\langle\alpha,\mu\rangle|$ and $|\langle\alpha|_{A_M},\mathrm{pr}_M(\omega\mu)\rangle|$ for all α, M, ω as in that definition. □

Define $T(F)_{C\text{-reg}}$ to be the union of $\mu(\varpi_F)T(\mathcal{O}_F)$ as μ runs over the set of C-regular cocharacters. For each $M \in \mathcal{L}(G)$ containing T, set

$$\mathfrak{a}_{M,C} := \{a \in \mathfrak{a}_M : |\langle\alpha,a\rangle| > C|\log|\varpi||,\ \forall\alpha \in \Phi^+(T,G)\backslash\Phi^+(T,M)\}. \tag{3.4.2}$$

Here we use the pairing $X^*(T)_{\mathbb{R}}\times X_*(T)_{\mathbb{R}} \to \mathbb{R}$ to compute $\langle\alpha,a\rangle$, viewing a in $X_*(T)_{\mathbb{R}}$ via $\mathfrak{a}_M = X_*(A_M)_{\mathbb{R}} \subset X_*(T)_{\mathbb{R}}$. Recall that $X_*(A_M)_{\mathbb{R}} \simeq \mathrm{Hom}(X^*(M)_{\mathbb{R}},\mathbb{R})$ via $a \mapsto (\chi \mapsto \langle\chi,a\rangle)$. Analogously $X_*(A_T)_{\mathbb{R}} \simeq \mathrm{Hom}(X^*(T)_{\mathbb{R}},\mathbb{R})$. Write $\mathrm{pr}_M : X_*(A_T)_{\mathbb{R}} \to \to X_*(A_M)_{\mathbb{R}}$ for the map induced by the restriction $X_F^*(M) \to X_F^*(T)$. (This is the analogue of pr_G in Section 3.1.)

Lemma 3.4.3. — *Let $M \subsetneq G$ be a Levi subgroup containing T over F. Then the image of $T(F)_{C\text{-reg}}$ under H^M is contained in $\mathfrak{a}_{M,C}$.*

Proof. — Consider $t := \mu(\varpi)$ with $\mu \in X_*(T)_{C\text{-reg}}$. Then $H^M(t) \in \mathrm{Hom}(X^*(M)_{\mathbb{R}},\mathbb{R})$ is identified with the unique element $a \in X_*(A_M)_{\mathbb{R}}$ such that

$$\langle\chi,a\rangle = \log|\chi(\mu(\varpi))| = \langle\chi,\mu\rangle\log|\varpi|, \qquad \chi \in X^*(M)_{\mathbb{R}}.$$

Let $\alpha \in \Phi(T,G)\backslash\Phi(T,M)$. Since the composite of the restriction maps $X_F^*(M)_{\mathbb{R}} \to X_F^*(T)_{\mathbb{R}} \to X^*(A_M)_{\mathbb{R}}$ is an isomorphism, we can find $\chi \in X_F^*(M)_{\mathbb{R}}$ such that $\chi|_{A_M} = \alpha|_{A_M}$. Hence

$$\begin{aligned}\langle\alpha,a\rangle &= \langle\alpha|_{A_M},a\rangle = \langle\chi|_{A_M},a\rangle = \langle\chi,a\rangle = \langle\chi,\mu\rangle\log|\varpi| = \langle\chi|_{A_T},\mu\rangle\log|\varpi| \\ &= \langle\chi,\mathrm{pr}_M(\mu)\rangle\log|\varpi| = \langle\alpha|_{A_M},\mathrm{pr}_M(\mu)\rangle\log|\varpi|.\end{aligned}$$

Since μ is C-regular, $|\langle\alpha|_{A_M},\mathrm{pr}_M(\mu)\rangle| > C$. Hence $|\langle\alpha,a\rangle| > C|\log|\varpi||$. □

The following definition is motivated by [FK88, p. 195].

Definition 3.4.4. — Let $C > 0$. We say $f \in \mathcal{H}(G)$ is *C-regular* if $\mathrm{supp}(f)$ is contained in the $G(F)$-conjugacy orbit of $T(F)_{C\text{-reg}}$.[(5)] Write $\mathcal{H}(G)_{C\text{-reg}}$ for the space of C-regular functions.

Lemma 3.4.5. — *Let $f' \in \mathcal{H}(G)$. Assume that every $g \in G(F)_{\mathrm{reg}}$ such that $O_g(f') \neq 0$ (resp. $SO_g(f') \neq 0$) is $G(F)$-conjugate (resp. stably conjugate) to an element of $T(F)_{C\text{-reg}}$. Then*

(1) *$O_g(f') = 0$ (resp. $SO_g(f') = 0$) if $g \in G(F)_{\mathrm{ss}}$ is not regular, and*

(2) *there exists $f \in \mathcal{H}(G)_{C\text{-reg}}$ such that f and f' have the same image in $\mathcal{I}(G)$ (resp. $\mathcal{S}(G)$).*

(5) In practice it seems enough to impose the condition on $\mathrm{supp}^O(f)$. However when producing examples of C-regular f, often we have this condition satisfied.

Proof. — (1) If $g \in G(F)_{\mathrm{ss}}$ is not regular then no regular element in a sufficiently small neighborhood of g intersects the $G(F)$-orbit of $T(F)_{C\text{-reg}}$. (Since every $t \in T(F)_{C\text{-reg}}$ satisfies $|1-\alpha(t)|=1$ for $\alpha \in \Phi(T,G)$, no $\alpha(t)$ approaches 1.) Thus f' has vanishing regular orbital integrals in a neighborhood of g. This implies that $O_g(f')=0$, by an argument as in the proof of [Rog83, Lem. 2.6] via the Shalika germ expansion around g. The case of stable orbital integrals is analogous.

(2) The point is that the $G(F)$-conjugacy orbit of $T(F)_{C\text{-reg}}$ is open and closed in $G(F)$. (Since $T(F)_{C\text{-reg}}$ is open and closed in $T(F)$, and the map

$$G(F)/T(F) \times T(F)_{C\text{-reg}} \longrightarrow G(F)$$

induced by $(g,t) \mapsto gtg^{-1}$ is a local isomorphism.) Thus the product of f' and the characteristic function on the latter orbit is smooth and compactly supported, and thus belongs to $\mathcal{H}(G)_{C\text{-reg}}$. Denoting the product by f, we see that f and f' have equal orbital integrals (resp. stable orbital integrals) on regular semisimple elements. Therefore have the same image in $\mathcal{I}(G)$ (resp. $\mathcal{S}(G)$). □

COROLLARY 3.4.6. — *Fix a C-regular cocharacter $\mu : \mathbb{G}_m \to T$. Let $\phi \in \mathcal{H}(T)$. Then there exists an integer $k_0 = k_0(\phi)$ such that for every integer $k \geqslant k_0$, the μ-ascent of $\phi^{(k)}$ is represented by a C-regular function on $G(F)$.*

Proof. — There is a finite subset $X \subset X_*(T)$ such that $\mathrm{supp}(\phi) \subset \bigcup_{\mu_0 \in X} \mu_0(\varpi_F) T(\mathcal{O}_F)$. Applying Lemma 3.4.2 to each $\mu_0 \in X$ and also Lemma 3.1.7, we can find $k_0 = k_0(\phi) \in \mathbb{Z}_{\geqslant 0}$ such that $\phi^{(k)}$ is μ-acceptable and $\mathrm{supp}(\phi^{(k)}) \subset T(F)_{C\text{-reg}}$ for all $k \geqslant k_0$. Write $f^{(k)}$ for a μ-ascent of $\phi^{(k)}$. By Lemma 3.4.5 it suffices to check for each $k \geqslant k_0$ and $g \in G(F)_{\mathrm{reg}}$ that if $O_g(f^{(k)}) \neq 0$ then g is in the $G(F)$-orbit of $T(F)_{C\text{-reg}}$. This follows from the observed properties of $\phi^{(k)}$ by Lemma 3.1.2. □

LEMMA 3.4.7. — *Let $f \in \mathcal{H}(G)_{C\text{-reg}}$, $M \in \mathcal{L}^<(G)$, and $\mathfrak{e} \in \mathcal{E}^<(G)$. The following are true.*

(1) *$\mathscr{C}_M^G(f) \in \mathcal{I}(M)$ is represented by a function $f_M \in \mathcal{H}(M)$ whose support is contained in the $M(F)$-conjugacy orbit of $T(F)_{C\text{-reg}}$. (In particular, f_M is a C-regular function on $M(F)$.)*

(2) *$\mathrm{LS}^{\mathfrak{e}}(f) \in \mathcal{S}(G^{\mathfrak{e}})$ vanishes unless $G^{\mathfrak{e}}$ is split over F. If $G^{\mathfrak{e}}$ is split over F then $\mathrm{LS}^{\mathfrak{e}}(f)$ is represented by a C-regular function on $G^{\mathfrak{e}}(F)$.*

Proof. — (1) We keep writing $T(F)_{C\text{-reg}}$ for the set of C-regular elements relative to G, which contain C-regular elements relative to M. Since $T(F)_{C\text{-reg}}$ is invariant under the Weyl group of G, an element $\gamma \in M(F)_{\mathrm{ss}}$ is conjugate to an element of $T(F)_{C\text{-reg}}$ in $G(F)$ if and only if it is so in $M(F)$. In light of Lemma 3.4.5, it suffices to show the following: if $O_\gamma(\mathscr{C}_M^G(f)) \neq 0$ for regular semisimple $\gamma \in M(F)$ then γ is $M(F)$-conjugate to an element of $T(F)_{C\text{-reg}}$.

If γ is G-regular then we have from Section 3.2 that $O_\gamma(f) = D_{G/M}(\gamma) O_\gamma(\mathscr{C}_M^G(f))$, which is nonzero only if γ is conjugate to an element of $T(F)_{C\text{-reg}}$. If γ is regular but outside the $M(F)$-orbit of $T(F)_{C\text{-reg}}$, then a sufficiently small neighborhood V of γ

does not intersect the $M(F)$-orbit of $T(F)_{C\text{-reg}}$. On the other hand, G-regular elements are dense in V. Since an orbital integral is locally constant on the regular semisimple set, it follows that $O_\gamma(\mathscr{C}_M^G(f)) = 0$.

(2) If T transfers to a maximal torus in $G^{\mathfrak{e}}$ then $G^{\mathfrak{e}}$ is split over F since T is a split torus. Thus $\mathrm{LS}^{\mathfrak{e}}(f) = 0$ unless $G^{\mathfrak{e}}$ is split over F. Now we assume that T transfers to a maximal torus $T^{\mathfrak{e}} \subset G^{\mathfrak{e}}$, equipped with an F-isomorphism $T \simeq T^{\mathfrak{e}}$ (canonical up to the Weyl group action). Via the isomorphism we transport λ to $\lambda^{\mathfrak{e}} : \mathbb{G}_m \to T^{\mathfrak{e}}$ and identify $\Phi(T^{\mathfrak{e}}, G^{\mathfrak{e}})$ as a subset of $\Phi(T, G)$. By abuse of notation, keep writing $T(F)_{C\text{-reg}}$ for its image in $T^{\mathfrak{e}}(F)$. Then C-regular elements of $T^{\mathfrak{e}}(F)$ are contained in $T(F)_{C\text{-reg}}$.

Now the rest of the proof of (2) similar to that of (1), based on Lemma 3.4.5. It suffices to check that if $SO_{\gamma^{\mathfrak{e}}}(\mathrm{LS}^{\mathfrak{e}}(f)) \neq 0$ for G-regular semisimple $\gamma^{\mathfrak{e}} \in G^{\mathfrak{e}}(F)$ then $\gamma^{\mathfrak{e}}$ is stably conjugate to an element of $T(F)_{C\text{-reg}}$. This is evident from the transfer of orbital integral identity. □

COROLLARY 3.4.8. — *For $f \in \mathcal{H}(G)_{C\text{-reg}}$ and $M \in \mathcal{L}^<(G)$, we have* $\mathrm{supp}^O_{\mathfrak{a}_M}(\mathscr{C}_M^G(f)) \subset \mathfrak{a}_{M,C}$.

Proof. — Let f_M be as in the preceding lemma. Then

$$\mathrm{supp}^O_{\mathfrak{a}_M}(\mathscr{C}_M^G(f)) = \mathrm{supp}^O_{\mathfrak{a}_M}(f_M) \subset \mathrm{supp}_{\mathfrak{a}_M}(f_M) \subset H^M(T(F)_{C\text{-reg}}) \subset \mathfrak{a}_{M,C},$$

where the last inclusion comes from Lemma 3.4.3. □

The following lemma, to be invoked in the proof of Corollary 7.6.2, sheds light on how much C-regular functions detect.

LEMMA 3.4.9. — *Let I be a finite set and let $C > 0$. Let $\pi_i \in \mathrm{Irr}(G(F))$ and $c_i \in \mathbb{C}$ for $i \in I$. If*

$$\sum_{i \in I} c_i \operatorname{Tr} \pi_i(f) = 0 \qquad \forall f \in \mathcal{H}(G)_{C\text{-reg}}$$

then $\sum_{i \in I} c_i \cdot J_{P_0}(\pi_i) = 0$ in $\mathrm{Groth}(G(F)) \otimes_{\mathbb{Z}} \mathbb{C}$.

Proof. — Fix a regular cocharacter $\mu : \mathbb{G}_m \to T$ over F such that $P_0 = P_\mu^{\mathrm{op}}$. For each $\phi \in \mathcal{H}(T)$, we have some integer k_0 such that $\phi^{(k)}$ are μ-acceptable for all $k \geqslant k_0$ and their μ-ascent $f^{(k)}$ are represented by C-regular functions by Corollary 3.4.6. Thanks to Lemma 3.1.2,

$$0 = \sum_{i \in I} c_i \operatorname{Tr}(f^{(k)} | \pi_i) = \sum_{i \in I} c_i \operatorname{Tr}(\phi^{(k)} | J_{P_0}(\pi_i)), \quad \forall k \geqslant k_0.$$

We conclude by Lemma 3.1.8. □

3.5. FIXED CENTRAL CHARACTER. — We explain that the facts thus far in Section 3 hold in the setup with fixed central character. Let $\nu : \mathbb{G}_m \to G$ be a cocharacter over F and $(G^{\mathfrak{e}}, \mathcal{G}^{\mathfrak{e}}, s^{\mathfrak{e}}, \eta^{\mathfrak{e}})$ an endoscopic datum for G with $\mathcal{G}^{\mathfrak{e}} = {}^L G^{\mathfrak{e}}$. We can view $\mathfrak{X}$ as a closed subgroup of $M_\nu(F)$, $G^{\mathfrak{e}}(F)$, and $G^{\mathfrak{e}}_\omega(F)$ of the preceding sections via the canonical embeddings of $Z(F)$ into their centers.

As before, $\mathcal{H}_{\mathrm{acc}}(M_\nu, \chi^{-1}) \subset \mathcal{H}(M_\nu, \chi^{-1})$ is the subspace of functions which are supported on acceptable elements. Taking the image, we also have $\mathcal{I}_{\mathrm{acc}}(M_\nu, \chi^{-1})$ and $\mathcal{S}_{\mathrm{acc}}(M_\nu, \chi^{-1})$. Since acceptability is invariant under the translation by central elements, the χ-averaging map induces a surjection $\mathcal{H}_{\mathrm{acc}}(M_\nu) \to \mathcal{H}_{\mathrm{acc}}(M_\nu, \chi^{-1})$. The analogous surjectivity holds for $\mathcal{I}_{\mathrm{acc}}$ and $\mathcal{S}_{\mathrm{acc}}$.

The earlier results are valid in the setting of fixed central characters, with the following minor modifications. The proofs are omitted as no new ideas are required.

To adapt Section 3.1. — Averaging the ν-ascent map, we obtain

$$\mathscr{J}_\nu : \mathcal{I}(M_\nu, \chi^{-1}) \longrightarrow \mathcal{I}(G, \chi^{-1}), \qquad \mathscr{J}_\nu : \mathcal{S}(M_\nu, \chi^{-1}) \longrightarrow \mathcal{S}(G, \chi^{-1})$$

satisfying the orbital integral and trace identities in Lemma 3.1.2 (with central character of π equal to χ) and Corollary 3.1.3. The obvious analogues of Lemmas 3.1.5 and 3.1.6 hold true (with no changes to the bottom rows in the latter lemma). The map $(\cdot)^{(k)}$ in (3.1.7) induces linear automorphisms on $\mathcal{I}(M_\nu, \chi^{-1})$ and $\mathcal{I}(G, \chi^{-1})$. With this, Lemmas 3.1.7 and 3.1.8 imply their natural analogues, restricting π_1, π_2 in the latter lemma to those with central character χ.

To adapt Section 3.2. — Averaging the map $\mathcal{H}(G) \to \mathcal{H}(M)$ given by $f \mapsto f_M$, we obtain a map $\mathcal{H}(G, \chi^{-1}) \to \mathcal{H}(M, \chi^{-1})$, which induces

$$\mathscr{C}_M^G : \mathcal{I}(G, \chi^{-1}) \longrightarrow \mathcal{I}(M, \chi^{-1})$$

satisfying the same orbital integral identity as in Section 3.2. We can also describe $\mathscr{C}_M^G$ by the same formula (3.2.1) from the space of linear functionals on $R(G, \chi)$ to that on $R(M, \chi)$. Lemmas 3.2.2 and 3.2.1 carry over as written, with χ^{-1}-equivariance imposed everywhere.

To adapt Section 3.3. — The Langlands–Shelstad transfer with fixed central character was already considered in Section 2.6 by averaging the transfer without fixed central character. With this in mind, we deduce the obvious analogues of Proposition 3.3.2, Corollary 3.3.4, and Lemma 3.3.5. In particular, the diagram in that proposition yields the following analogue.

$$\begin{array}{ccccc} \mathcal{I}(M_\nu, \chi^{-1}) & \xrightarrow{\mathscr{J}_\nu} & \mathcal{I}(G, \chi^{-1}) & \xrightarrow{\mathrm{LS}^{\mathfrak{e}}} & \mathcal{S}(G^{\mathfrak{e}}, \chi^{\mathfrak{e},-1}) \\ & \searrow^{\oplus \mathrm{LS}^{\mathfrak{e},\omega}} & & \nearrow_{\sum_\omega \mathscr{J}_{\nu_\omega}} & \\ & & \bigoplus_{\omega \in \Omega_{\mathfrak{e},\nu}} \mathcal{S}(G^{\mathfrak{e}}_\omega, \chi^{\mathfrak{e},-1}) & & \end{array}$$

To adapt Section 3.4. — Definition 3.4.4 extends obviously to $\mathcal{H}(G, \chi^{-1})$ by the same support condition. A key observation is that the notion of C-regularity is invariant under $Z(F)$-translation, so that the latter definition behaves well. More precisely, the χ-averaging map from $\mathcal{H}(G) \twoheadrightarrow \mathcal{H}(G, \chi^{-1})$ is still surjective when restricted to the respective subspaces of C-regular functions. Using this, we carry over all results in

Section 3.4 to the setup with fixed central character, restricting to χ^{-1}-equivariant functions and representations with central character χ.

3.6. *z*-EXTENSIONS. — Throughout this section up to now, we assumed $\mathcal{G}^{\mathfrak{e}} = {}^L G^{\mathfrak{e}}$ on the endoscopic datum $\mathfrak{e}$. When the assumption is not guaranteed, we pass from $\mathfrak{e}$ and G to $\mathfrak{e}_1$ and G_1 via z-extensions and pull back the central character datum from $(\mathfrak{X}, \chi)$ to $(\mathfrak{X}_1, \chi_1)$ as explained in Section 2.7.

Let $\nu_1 : \mathbb{G}_m \to G_1$ be a cocharacter lifting ν. (Such a ν_1 is going to be chosen in practice; see Section 7.4 below.) By Definition 3.1.1, $\gamma_1 \in M_{\nu_1}(F)$ is ν_1-acceptable if and only if its image in $M_\nu(F)$ is ν-acceptable. Everything in this section goes through with $\mathfrak{e}_1, G_1, (\mathfrak{X}_1, \chi_1), \nu_1$ playing the roles of $\mathfrak{e}, G, (\mathfrak{X}, \chi), \nu$. We write $\lambda^{\mathfrak{e}}_1, \lambda^{\mathfrak{e}}_{1,\omega}$ for the characters $\lambda^{\mathfrak{e}}, \lambda^{\mathfrak{e}}_\omega$ of Section 3.3 in the setup for $\mathfrak{e}_1$ and G_1.

4. Asymptotic analysis of the trace formula

We prove key trace formula estimates in this section, to be applied to identify leading terms in the trace formula for Igusa varieties in Section 7. The main estimate is Theorem 4.2.2, whose lengthy proof is presented in Section 4.4. We work in a purely group-theoretic setup, with no reference to Shimura or Igusa varieties in order to enable an inductive argument on $\mathbb{Q}$-semisimple rank. The point is that the trace formula appearing in the intermediate steps need not arise from geometry.

4.1. SETUP AND SOME BASIC LEMMAS. — Throughout Section 4, G is a connected *quasi-split* reductive group over $\mathbb{Q}$.

Let $(\mathfrak{X}, \chi)$ be a central character datum as in Section 2.8. Let ξ be an irreducible algebraic representation of $G_{\mathbb{C}}$ and $\zeta\colon G(\mathbb{R}) \to \mathbb{C}^\times$ be a continuous character such that $\xi \otimes \zeta$ has central character χ_∞^{-1} on $\mathfrak{X}_\infty$. The restriction $\chi_\infty|_{A_{G,\infty}}$ via $A_{G,\infty} \subset \mathfrak{X}_\infty$ can be viewed as an element of $X^*(A_G)_{\mathbb{C}}$, which is again denoted χ_∞ by abuse of notation.

We will write $H_v^G\colon G(\mathbb{Q}_v) \to \mathfrak{a}_{G_{\mathbb{Q}_v}}$ for the function defined by Equation (3.1.3) (which also makes sense for $F = \mathbb{R}$). We have a canonical identification

$$\mathfrak{a}_G = X_*(A_G)_{\mathbb{R}} = \mathrm{Hom}(X^*_{\mathbb{Q}}(G), \mathbb{R}), \qquad a \longmapsto (\chi \mapsto \langle \chi|_{A_G}, a\rangle). \tag{4.1.1}$$

A similar map induces identifications $\mathfrak{a}_{G_{\mathbb{Q}_v}} = X_*(A_{G_{\mathbb{Q}_v}})_{\mathbb{R}} = \mathrm{Hom}(X^*_{\mathbb{Q}_v}(G), \mathbb{R})$. In particular, there is a natural surjection $\mathfrak{a}_{G_{\mathbb{Q}_v}} \twoheadrightarrow \mathfrak{a}_G$ at each place v. Thereby we often view the image of H_v^G in $\mathfrak{a}_G$.

Let λ_{χ_∞} denote the unique character making the following diagram commute. (The existence is obvious since the composition $A_{G,\infty} \to \mathfrak{a}_G$ is an isomorphism.)

$$\begin{array}{ccccccccc} A_{G,\infty} & \hookrightarrow & G(\mathbb{R}) & \xrightarrow{H_\infty^G} & \mathfrak{a}_{G_{\mathbb{R}}} & \twoheadrightarrow & \mathfrak{a}_G = X_*(A_G)_{\mathbb{R}} & \xrightarrow{\lambda_{\chi_\infty}} & \mathbb{C}^\times \\ & & & & & \searrow & & \nearrow_{\chi_\infty} & \end{array}$$

Fix distinct primes p, q. Let $\nu : \mathbb{G}_m \to G_{\mathbb{Q}_p}$ be a cocharacter over $\mathbb{Q}_p$. Let $\overline{\nu} \in \mathrm{Hom}(X^*_{\mathbb{Q}}(G), \mathbb{Q})$ denote the image of $\nu \in X_*(A_{M_\nu})_{\mathbb{Q}} = \mathrm{Hom}(X^*_{\mathbb{Q}}(M_\nu), \mathbb{Q})$ induced by

$M_\nu \hookrightarrow G$.[6] By definition, $\overline{\nu}(\chi) = \nu(\chi)$ for $\chi \in X^*_{\mathbb{Q}}(G)$. Viewing $\overline{\nu}$ as a member of $\mathfrak{a}_G$, we can compute $\langle \chi_\infty, \overline{\nu}\rangle \in \mathbb{C}$ via the canonical pairing $X^*(A_G)_{\mathbb{C}} \times X_*(A_G)_{\mathbb{C}} \to \mathbb{C}$.

LEMMA 4.1.1. — $\lambda_{\chi_\infty}(H_p^G(\nu(p))) = p^{-\langle \chi_\infty, \overline{\nu}\rangle}$.

Proof. — By definition $H_p^G(\nu(p))$ sends $\chi \in X^*_{\mathbb{Q}}(G)$ to $\log|\chi(\nu(p))|_p$. Similarly for $a \in A_{G,\infty}$, we have $H_\infty^G(a) = (\chi \mapsto \log|\chi(a)|_\infty)$. We claim that $H_p^G(\nu(p)) \in \mathfrak{a}_{G_{\mathbb{Q}_p}}$ and $H_\infty^G(\overline{\nu}(p)^{-1}) \in \mathfrak{a}_{G_{\mathbb{R}}}$ have the same image in $\mathfrak{a}_G$. To show this, choose $r \in \mathbb{Z}_{\geqslant 1}$ such that $r\overline{\nu} \in X_*(A_G)$. Since $\mathfrak{a}_G$ is torsion-free it suffices to check that $H_p^G((r\nu)(p)) = H_\infty^G((r\overline{\nu})(p)^{-1})$, or equivalently that

$$|\chi((r\nu)(p))|_p = |\chi((r\overline{\nu})(p))|_\infty^{-1}, \qquad \chi \in X^*_{\mathbb{Q}}(G).$$

Since $\chi((r\overline{\nu})(p)) \in \mathbb{Q}$ is an integral power of p (as both χ and $r\overline{\nu}$ are algebraic), we have $|\chi((r\overline{\nu})(p))|_\infty = |\chi((r\overline{\nu})(p))|_p^{-1} = |\chi((r\nu)(p))|_p^{-1}$. This proves the claim. Now the claim implies that

$$\lambda_{\chi_\infty}(H_p^G(\nu(p))) = \lambda_{\chi_\infty}(H_\infty^G(\overline{\nu}(p)^{-1})) = \chi_\infty(\overline{\nu}(p)^{-1}) = p^{-\langle \chi_\infty, \overline{\nu}\rangle}. \qquad \square$$

If G is a connected reductive group over $\mathbb{Q}$ and S is a set of $\mathbb{Q}$-places, we write

$$H_S^G(\gamma) := \sum_{v \in S} H_v^G(\gamma) \in \mathfrak{a}_G.$$

If S^c is the complement of S, we write $H^{G,S^c} := H_S^G$.

LEMMA 4.1.2. — *Let G be a connected reductive group over $\mathbb{Q}$.*

(i) *Let S be a set of $\mathbb{Q}$-places. Let $\gamma, \gamma' \in G(\mathbb{Q})$. If γ and γ' are conjugate in $G(\overline{\mathbb{Q}})$ then we have $H_S^G(\gamma) = H_S^G(\gamma') \in \mathfrak{a}_G$.*

(ii) *Let S be the set of all $\mathbb{Q}$-places. Let $\gamma \in G(\mathbb{Q})$, then $H_S^G(\gamma) = 0 \in \mathfrak{a}_G$.*

Proof. — (i) Let $F/\mathbb{Q}$ be a finite extension such that γ and γ' are conjugate in $G(F)$. Set $G' := \mathrm{Res}_{F/\mathbb{Q}} M$. The natural embedding $i : G \to G'$ allows to view γ, γ' as elements of $G'(\mathbb{Q})$, and induces an injection $\mathfrak{a}_G \hookrightarrow \mathfrak{a}_{G'}$. Thus it suffices to prove that $H_S^{G'}(\gamma) = H_S^{G'}(\gamma')$, since the map $H_S^G \colon G(\mathbb{A}_S) \to \mathfrak{a}_G$ is functorial with respect to i. By the reduction in the preceding paragraph, we may assume that γ and γ' are conjugate in $G(\mathbb{Q})$. Then the proof is trivial since H_S^G is a homomorphism into an abelian group.

(ii) Using the functoriality for $G \to Z'_G$ from step 1, we may replace G by its cocenter Z'_G, then Z'_G by the maximally split torus A inside Z'_G, and finally we may replace A by $\mathbb{G}_m$, in which case the statement boils down to the usual product formula. $\square$

If $M \in \mathcal{L}_{\mathrm{cusp}}(G)$, we write $\Gamma_{\mathbb{R}\text{-ell},\mathfrak{x}}(M)$ for the set of $\gamma \in M(\mathbb{Q})$ such that $\gamma \in M(\mathbb{R})$ is elliptic, and γ is taken up to $M(\mathbb{Q})$-conjugacy. The following will be useful when studying Levi terms in the geometric side of the trace formula.

[6] In the notation of the preceding section, $\overline{\nu} = \mathrm{pr}_G(\nu)$.

LEMMA 4.1.3. — *Let $M \in \mathcal{L}_{\mathrm{cusp}}(G)$ and let $\gamma \in \Gamma_{\mathbb{R}\text{-ell},\mathfrak{x}}(M)$ be a regular element. Let $P \subset G$ be a parabolic subgroup with Levi component M. Let ξ be an irreducible representation of $M_{\mathbb{C}}$, and $\zeta\colon M(\mathbb{R}) \to \mathbb{C}^\times$ a continuous character. Write $f^M_{\zeta,\xi}$ for the function on $M(\mathbb{R})$ given by* (2.4.1)*. Then*

$$\mathrm{vol}(\mathfrak{x}_{\mathbb{Q}}\backslash\mathfrak{x}/A_{G,\infty})^{-1}\chi(I^M_\gamma)\zeta(\gamma)\,\mathrm{Tr}(\gamma;\xi) = d(M)\mathrm{vol}(I^M_\gamma(\mathbb{Q})A_{I^M_\gamma,\infty}\backslash I^M_\gamma(\mathbb{A})/\mathfrak{x})O^M_\gamma(f^M_{\xi,\zeta}).$$

Proof. — By Equation (2.8.3) we have

$$\chi(I^M_\gamma) = (-1)^{q(I^M_\gamma)}\tau(I^M_\gamma)\mathrm{vol}(A_{I^M_{\gamma,\infty}}\backslash I^{M,\mathrm{cmpt}}_\gamma(\mathbb{R}))^{-1}d(I^M_\gamma) = 1.$$

As γ is regular, I^M_γ is a torus and $d(I^M_\gamma) = 1$. Additionally $q(I^M_\gamma) = 0$ (as γ is elliptic) and

$$\tau(I^M_\gamma) = \mathrm{vol}(I^M_\gamma(\mathbb{Q})A_{I^M_\gamma,\infty}\backslash I^M_\gamma(\mathbb{A})).$$

Thus we obtain

$$\chi(I^M_\gamma) = \mathrm{vol}(I^M_\gamma(\mathbb{Q})A_{I^M_\gamma,\infty}\backslash I^M_\gamma(\mathbb{A}))\mathrm{vol}(A_{I^M_{\gamma,\infty}}\backslash I^{M,\mathrm{cmpt}}_\gamma(\mathbb{R}))^{-1}.$$

Since γ is $\mathbb{R}$-elliptic, we obtain from (2.4.2) that

$$\zeta(\gamma)\,\mathrm{Tr}(\gamma;\xi) = d(M)\mathrm{vol}(A_{M,\infty}\backslash I^{M,\mathrm{cmpt}}_\gamma(\mathbb{R}))O^M_\gamma(f^M_{\xi,\zeta}).$$

(we also used $e(I^M_\gamma) = 1$; recall $A_{M,\infty} := A_M(\mathbb{R})^0$). We obtain

$$\begin{aligned}
&\mathrm{vol}(\mathfrak{x}_{\mathbb{Q}}\backslash\mathfrak{x}/A_{G,\infty})^{-1}\chi(I^M_\gamma)\zeta(\gamma)\,\mathrm{Tr}(\gamma;\xi)\\
&\quad= d(M)\frac{\mathrm{vol}(I^M_\gamma(\mathbb{Q})A_{I^M_\gamma,\infty}\backslash I^M_\gamma(\mathbb{A}))}{\mathrm{vol}(\mathfrak{x}_{\mathbb{Q}}\backslash\mathfrak{x}/A_{G,\infty})}\frac{\mathrm{vol}(A_{M,\infty}\backslash I^{M,\mathrm{cmpt}}_\gamma(\mathbb{R}))}{\mathrm{vol}(A_{I^M_\gamma,\infty}\backslash I^{M,\mathrm{cmpt}}_\gamma(\mathbb{R}))}O^M_\gamma(f^M_{\xi,\zeta})\\
&\quad= d(M)\mathrm{vol}(I^M_\gamma(\mathbb{Q})A_{I^M_\gamma,\infty}\backslash I^M_\gamma(\mathbb{A})/\mathfrak{x})O^M_\gamma(f^M_{\xi,\zeta}),
\end{aligned}$$

where we used that A_M equals $A_{I^M_\gamma}$ because γ is elliptic. □

4.2. THE MAIN ESTIMATE AND ITS CONSEQUENCES. — We prove the following bounds for elliptic endoscopic groups and Levi subgroups of G, to be applied in Section 7.

The notation $O(f(k))$ (resp. $o(k)$) for a nonzero $\mathbb{C}$-valued function $f(k)$ on $k \in \mathbb{Z}$ means that the quantity divided by $|f(k)|$ has bounded absolute value (resp. tends to 0) as $k \to +\infty$. In practice we take $f(k)$ to be complex powers of p (so we take absolute values). In our argument, every instance of $o(f(k))$ turns out to represent a power-saving, namely it is bounded by a power of p with (the real part of) exponent strictly smaller than the exponent for $f(k)$.

Let us fix a $\mathbb{Q}$-rational Borel subgroup B with Levi component $T \subset B$ (which is a maximal torus in G). We fix a Levi decomposition $B = TN_0$. As before we write $A_T \subset T$ for the maximal $\mathbb{Q}$-split subtorus. Additionally we write $S_p \subset T_{\mathbb{Q}_p}$ for the maximal $\mathbb{Q}_p$-split subtorus.

Part of our setup is a cocharacter $\nu\colon \mathbb{G}_m \to G$ over $\mathbb{Q}_p$. By conjugating, we may and do assume that ν has image in T and that ν is B-dominant. Write $\rho \in X^*(T)_{\mathbb{Q}}$ for the half sum of all B-positive roots of T in G over $\overline{\mathbb{Q}}_p$. Thus we have $\langle\rho,\nu\rangle \in \frac{1}{2}\mathbb{Z}_{>0}$. We transport various data over $\mathbb{Q}_p$ or $\overline{\mathbb{Q}}_p$ to ones over $\mathbb{C}$ via an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$. (We will fix ι_p in Section 5.2.)

We also fix a prime q such that $G_{\mathbb{Q}_q}$ is a split group. (For the existence, choose a number field F over which G splits. Then any prime q that splits completely in F will do.) If needed for endoscopy, an auxiliary z-extension G_1 of G over $\mathbb{Q}$ is always chosen to be split over $\mathbb{Q}_q$; this is possible because $G_{\mathbb{Q}_q}$ is split. Thus the contents of Section 3.4 and their adaptation to z-extensions apply to G and G_1 over $\mathbb{Q}_q$. Since all endoscopic groups appearing in the argument will be split over $\mathbb{Q}_q$ (to be ensured by Lemma 3.4.7 in the proof of Corollary 4.2.3), whenever choosing their z-extensions, we take them to be also split over $\mathbb{Q}_q$ without further comments.

PROPOSITION 4.2.1. — *Let $f^{\infty,p} = \prod_{v\neq\infty,p} f_v \in \mathcal{H}(G(\mathbb{A}^{\infty,p}), (\chi^{\infty,p})^{-1})$ and $\phi_p \in \mathcal{H}_{\mathrm{acc}}(M_\nu(\mathbb{Q}_p), \chi_p^{-1})$. For $k \in \mathbb{Z}$, write $f_p^{(k)} \in \mathcal{H}(G(\mathbb{Q}_p), \chi_p^{-1})$ for a ν-ascent of $\phi_p^{(k)}$ as in Section 3.2. Then*

$$T^G_{\mathrm{disc},\chi}(f_p^{(k)} f^{\infty,p} f_{\xi,\zeta}) = O\big(p^{k(\langle\rho,\nu\rangle - \langle\chi_\infty,\overline{\nu}\rangle)}\big).$$

Proof. — The left hand side equals

$$\sum_{\pi\in\mathcal{A}_{\mathrm{disc},\chi}(G)} m(\pi)\,\mathrm{Tr}(f_p^{(k)}|\pi_p)\,\mathrm{Tr}(f^{\infty,p}|\pi^p)\,\mathrm{Tr}(f_{\xi,\zeta}|\pi_\infty).$$

Write $J_{P_\nu^{\mathrm{op}}}(\pi_p) = \sum_i c_i\tau_i$ in $\mathrm{Groth}(M_\nu(\mathbb{Q}_p))$ with $\tau_i \in \mathrm{Irr}(M_\nu(\mathbb{Q}_p))$. Let ω_{τ_i} denote the central character of τ_i. Then

$$\mathrm{Tr}(f_p^{(k)}|\pi_p) = \mathrm{Tr}\big(\phi_p^{(k)}|J_{P_\nu^{\mathrm{op}}}(\pi_p)\big) = \sum_i c_i\,\mathrm{Tr}(\phi_p^{(k)}|\tau_i) = \sum_i c_i\omega_{\tau_i}(\nu(p))^k\,\mathrm{Tr}(\phi_p|\tau_i).$$

We define a character $\lambda_{\mathbb{A}} : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{R}^\times_{>0}$ as the composite

$$\lambda_{\mathbb{A}} \colon G(\mathbb{Q})\backslash G(\mathbb{A}) \xrightarrow{H^G} \mathfrak{a}_G \xrightarrow{\lambda_{\chi_\infty}} \mathbb{R}^\times_{>0}.$$

Write λ_v for the restriction of $\lambda_{\mathbb{A}}$ to $G(\mathbb{Q}_v)$ for a place v of $\mathbb{Q}$. For each $\pi \in \mathcal{A}_{\mathrm{disc},\chi}(G)$ contributing to the sum, we see that $\pi\otimes\lambda_{\mathbb{A}}^{-1}$ is a unitary automorphic representation of $G(\mathbb{A})$ since $\pi_\infty\otimes\lambda_\infty^{-1}$ is unitary (by construction, $\pi_\infty\otimes\lambda_\infty^{-1}$ has trivial central character on $A_G(\mathbb{R})^0$). Thus $\pi_p\otimes\lambda_p^{-1}$ is unitary. Applying Corollary 2.5.2 to $\pi\otimes\lambda^{-1}$ at p, we have

$$\big|\omega_{\tau_i}(\nu(p))\lambda_p^{-1}(\nu(p))\big| \leqslant \delta_{P_\nu^{\mathrm{op}}}^{-1/2}(\nu(p)) = p^{\langle\rho,\nu\rangle},$$

noting that $\nu(p) \in A^{--}_{P_\nu}$. We deduce via Lemma 4.1.1 that

$$|\omega_{\tau_i}(\nu(p))| \leqslant p^{\langle\rho,\nu\rangle}|\lambda_p(\nu(p))| = p^{\langle\rho,\nu\rangle - \langle\chi_\infty,\overline{\nu}\rangle}.$$

By [BZ77, Cor. 2.13] the length of $J_{P_\nu^{\mathrm{op}}}(\pi_p)$, namely $\sum_i c_i$, can be bounded only in terms of G. This completes the proof. □

We state the main trace formula estimate of this paper. The proof will be given in Section 4.4 below.

THEOREM 4.2.2 (main estimate). — *Let $G, (\mathfrak{X},\chi), p, q, \xi, \zeta, \nu$ be as defined in the beginning of Section 4. Let*

- *$f^{\infty,p,q} = \prod_{v\neq\infty,p,q} f_v \in \mathcal{H}(G(\mathbb{A}^{\infty,p,q}), (\chi^{\infty,p,q})^{-1})$,*
- *$\phi_p \in \mathcal{H}_{\mathrm{acc}}(M_\nu(\mathbb{Q}_p), \chi_p^{-1})$, and*
- *$f_p^{(k)} \in \mathcal{H}(G(\mathbb{Q}_p), \chi_p^{-1})$ be a ν-ascent of $\phi_p^{(k)}$, for $k \in \mathbb{Z}_{\geqslant 0}$.*

Then there exists a constant $C = C(f^{\infty,q}, \phi_p) \in \mathbb{R}_{>0}$ *such that for each* $f_q \in \mathcal{H}(G(\mathbb{Q}_q), \chi_q^{-1})_{C\text{-reg}}$,

$$T^G_{\mathrm{ell},\chi}(f_p^{(k)} f_q f^{\infty,p,q} f_{\xi,\zeta}) = T^G_{\mathrm{disc},\chi}(f_p^{(k)} f_q f^{\infty,p,q} f_{\xi,\zeta}) + o\big(p^{k(\langle\rho,\nu\rangle - \langle\chi_\infty,\overline{\nu}\rangle)}\big).$$

As a corollary, we derive the stable analogue of Theorem 4.2.2. We keep the setup of Theorem 4.2.2 and let $f^{\infty,p,q}, \phi_p, f_p^{(k)}$ be as in that theorem. For each $\mathfrak{e} \in \mathcal{E}^<_{\mathrm{ell}}(G)$, we have $\mathfrak{e}_1 = (G_1^{\mathfrak{e}}, {}^L G_1^{\mathfrak{e}}, s_1^{\mathfrak{e}}, \eta_1^{\mathfrak{e}})$ and a central character datum $(\mathfrak{X}_1^{\mathfrak{e}}, \chi_1^{\mathfrak{e}})$ as in Section 2.7. Moreover we choose the representatives $\mathfrak{e}, \mathfrak{e}_1$ such that $\eta^{\mathfrak{e}}(W_F)$ and $\eta_1^{\mathfrak{e}}(W_F)$ have bounded images, as explained in Lemma 2.6.2 and Section 2.7. Let

$$f_1^{(k),\mathfrak{e}} = \prod_v f_{1,v}^{(k),\mathfrak{e}} \in \mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{A}), (\chi_1^{\mathfrak{e}})^{-1})$$

be a transfer of $f_p^{(k)} f_q f^{\infty,p,q} f_{\xi,\zeta}$. Then we have the following bound.

COROLLARY 4.2.3. — *In the setup of Theorem* 4.2.2*, there exists a constant* $C \in \mathbb{R}_{>0}$*, depending on* $f^{\infty,q}, \phi_p, \xi, \zeta$*, such that for every* $f_q \in \mathcal{H}(G(\mathbb{Q}_q))_{C\text{-reg}}$*, firstly*

$$ST^G_{\mathrm{ell},\chi}(f_p^{(k)} f_q f^{\infty,p,q} f_{\xi,\zeta}) = \begin{cases} T^G_{\mathrm{disc},\chi}(f_p^{(k)} f^{\infty,p,q} f_q f_{\xi,\zeta}) + o\left(p^{k(\langle\rho,\nu\rangle - \langle\chi_\infty,\overline{\nu}\rangle)}\right), \\ O\left(p^{k(\langle\rho,\nu\rangle - \langle\chi_\infty,\overline{\nu}\rangle)}\right), \end{cases}$$

and secondly for each $\mathfrak{e} \in \mathcal{E}^<_{\mathrm{ell}}(G)$ *(note that* $f_{1,q}^{(k),\mathfrak{e}}$ *inherits* C*-regularity from* f_q*),*

$$\mathrm{ST}^{G_1^{\mathfrak{e}}}_{\mathrm{ell},\chi_1^{\mathfrak{e}}}\big(f_1^{(k),\mathfrak{e}}\big) = o\big(p^{k(\langle\rho,\nu\rangle - \langle\chi_\infty,\overline{\nu}\rangle)}\big).$$

REMARK 4.2.4. — The inductive proof of the last bound only uses the fact that its q-component is C-regular, the ∞-component is a Lefschetz function, and most importantly the p-component is an ascent for a suitable cocharacter. We do not rely on the fact that $f_1^{(k),\mathfrak{e}}$ is a transfer of a function on $G(\mathbb{A})$.

Proof. — The second estimate is immediate from the first via Proposition 4.2.1. Let us prove the first and third asymptotic formulas, by reducing the former to the latter.

We induct on the semisimple rank of G. (For each G, we prove the corollary for all central character data and all ν.) When G is a torus, the estimate is trivial as $\mathrm{ST}^G_{\mathrm{ell},\chi} = T^G_{\mathrm{ell},\chi} = T^G_{\mathrm{disc},\chi}$. We assume that G is not a torus and that Corollary 4.2.3 is true for all groups which have lower semisimple rank than G. Put $f^{(k)} := f_p^{(k)} f_q f^{\infty,p,q} f_{\xi,\zeta}$. Proposition 2.9.1 tells us that

$$\mathrm{ST}^G_{\mathrm{ell},\chi}(f^{(k)}) = T^G_{\mathrm{ell},\chi}(f) - \sum_{\mathfrak{e} \in \mathcal{E}^<_{\mathrm{ell}}(G)} \iota(G, G^{\mathfrak{e}}) \mathrm{ST}^{G_1^{\mathfrak{e}}}_{\mathrm{ell},\chi_1^{\mathfrak{e}}}\big(f_1^{(k),\mathfrak{e}}\big).$$

In light of Theorem 4.2.2, since the summand is nonzero only for a finite set of $\mathfrak{e}$ by Lemma 2.9.2 (depending only on the finite set of primes v where either $G_{\mathbb{Q}_v}$ or f_v is ramified), it suffices to establish the last bound of the corollary. This task takes up the rest of the proof.

If $G^{\mathfrak{e}}_{\mathbb{R}}$ contains no elliptic maximal torus or if $A_{G^{\mathfrak{e}}} \neq A_G$ (equivalently if $A_{G_1^{\mathfrak{e}}} \neq A_{G_1}$), then $f^{\mathfrak{e}}_{1,\infty}$ is trivial as observed in [Kot90, p. 182, p. 189] so the desired estimate is trivially true. Henceforth, suppose that $G^{\mathfrak{e}}_{\mathbb{R}}$ contains an elliptic maximal torus.

Then $f_{1,\infty}^{(k),\mathfrak{e}}$ is a finite linear combination of $f_{\eta_1^{\mathfrak{e}},\zeta_1^{\mathfrak{e}}}$ over the set of $(\eta_1^{\mathfrak{e}},\zeta_1^{\mathfrak{e}})$ such that $\eta_1^{\mathfrak{e}}\circ\varpi_{\eta_1^{\mathfrak{e}},\zeta_1^{\mathfrak{e}}}\simeq\varpi_{\xi,\zeta}$. Proposition 3.3.2 and its adaptation to z-extensions according to Sections 3.5 and 3.6 tell us that

$$f_{1,p}^{(k),\mathfrak{e}}=\sum_{\omega}\lambda_{1,\omega}^{\mathfrak{e}}(\nu_1(p))^k\,\mathscr{J}_{\nu_{1,\omega}}\big(\phi_{1,p,\omega}^{(k),\mathfrak{e}}\big)=\sum_{\omega}\lambda_{1,\omega}^{\mathfrak{e}}(\nu_1(p))^k f_{1,p,\omega}^{(k),\mathfrak{e}},$$

where we have put $f_{1,p,\omega}^{(k),\mathfrak{e}}:=\mathscr{J}_{\nu_{1,\omega}}\big(\phi_{1,p,\omega}^{(k),\mathfrak{e}}\big)$ for a $\nu_{1,\omega}$-ascent of the operator $\phi_{1,p,\omega}^{(k),\mathfrak{e}}\in\mathcal{H}_{\mathrm{acc}}(G_{1,\nu}^{\mathfrak{e}}(\mathbb{Q}_p),(\chi_{1,p}^{\mathfrak{e}})^{-1})$. Here we applied Lemma 3.3.5 (keeping Sections 3.5 and 3.6 in mind) to have the transfer $\phi_{1,p,\omega}^{(k),\mathfrak{e}}$ of $\phi_{1,p}^{(k)}$ supported on $\nu_{1,\omega}$-acceptable elements.

Recalling that $\eta_1^{\mathfrak{e}}(W_F)\subset{}^LG_1$ is a bounded subgroup, we see from Lemma 3.3.1 that $\lambda_{1,\omega}^{\mathfrak{e}}$ is a unitary character. Thus we are reduced to showing the existence of some $C_{\mathfrak{e}}>0$ such that the following estimate holds for ω and $(\eta_1^{\mathfrak{e}},\zeta_1^{\mathfrak{e}})$ as above whenever $f_{1,q}^{\mathfrak{e}}$ is $C_{\mathfrak{e}}$-regular:

$$\mathrm{ST}_{\mathrm{ell},\chi_1^{\mathfrak{e}}}^{G_1^{\mathfrak{e}}}\big(f_1^{\mathfrak{e},\infty,q,p}f_{1,q}^{\mathfrak{e}}f_{1,p,\omega}^{(k),\mathfrak{e}}f_{\eta_1^{\mathfrak{e}},\zeta_1^{\mathfrak{e}}}\big)\overset{?}{=}o\big(p^{k(\langle\rho,\nu\rangle-\langle\chi_\infty,\overline{\nu}\rangle)}\big),\qquad k\in\mathbb{Z}_{\geqslant 0}.\tag{4.2.1}$$

Indeed, take C to be the maximum of all $C_{\mathfrak{e}}$ over the finite set of $\mathfrak{e}$ contributing to the sum. Then for each C-regular f_q, Lemma 3.4.7 tells us either that $G_1^{\mathfrak{e}}$ is split over $\mathbb{Q}_q$ and $f_{1,q}^{\mathfrak{e}}$ is C-regular (thus also $C_{\mathfrak{e}}$-regular), or that $G_1^{\mathfrak{e}}$ is non-split over $\mathbb{Q}_q$ and $f_{1,q}^{\mathfrak{e}}$ vanishes. Thus the bound (4.2.1) applies, and we will be done.

By the induction hypothesis, there exists $C_{\mathfrak{e}}>0$ such that whenever $f_{1,q}^{\mathfrak{e}}$ is $C_{\mathfrak{e}}$-regular, the left hand side of (4.2.1) is $O\big(p^{k(\langle\rho_1^{\mathfrak{e}},\nu_{1,\omega}\rangle-\langle\chi_{1,\infty}^{\mathfrak{e}},\overline{\nu}_{1,\omega}\rangle)}\big)$, with $\overline{\nu}_{1,\omega}\in X_*(A_{G_1^{\mathfrak{e}}})$ defined from $\nu_{1,\omega}$ in the same way $\overline{\nu}$ from ν, and where $\rho_1^{\mathfrak{e}}$ is the half sum of positive roots of $G_1^{\mathfrak{e}}$ for which $\nu_{1,\omega}$ is a dominant cocharacter. (In other words, $\rho_1^{\mathfrak{e}}$ is to $\nu_{1,\omega}$ as ρ is to ν.) Therefore, it is enough to check that

(a) $\langle\rho_1^{\mathfrak{e}},\nu_{1,\omega}\rangle<\langle\rho,\nu\rangle$ (in $\mathbb{Q}$).
(b) $\mathrm{Re}\langle\chi_{1,\infty}^{\mathfrak{e}},\overline{\nu}_{1,\omega}\rangle=\mathrm{Re}\langle\chi_\infty,\overline{\nu}\rangle$.

Let us begin with (a). Since $\langle\rho,\nu\rangle=\langle\rho_1,\nu_1\rangle$, with ρ_1 defined for G_1 as ρ is for G (recall that $\nu_1:\mathbb{G}_m\to G_1$ is a lift of ν), the proof of (a) is reduced to the case when $G_1=G$ and $\nu_1=\nu$. We have an embedding $\widehat{G}^{\mathfrak{e}}\hookrightarrow\widehat{G}$ coming from $\eta^{\mathfrak{e}}$, which restricts to $\widehat{G}_\omega^{\mathfrak{e}}\hookrightarrow\widehat{M}_\nu$. Here we have chosen Γ_F-invariant pinnings for the dual groups such that the restriction works as stated. We may and will arrange that the Borel subgroup of $\widehat{G}$ restricts to that of $\widehat{G}^{\mathfrak{e}}$. Fix a maximal torus $\widehat{T}\subset\widehat{G}_\omega^{\mathfrak{e}}$ that is part of the pinning for $\widehat{G}_\omega^{\mathfrak{e}}$. Viewing $\widehat{T}$ also as a maximal torus in each of $\widehat{G}^{\mathfrak{e}}$ and $\widehat{G}$, we write $\Phi^\vee(\widehat{T},\widehat{G})$ and $\Phi^\vee(\widehat{T},\widehat{G}^{\mathfrak{e}})$ for the corresponding sets of coroots. Then

$$\langle\rho^{\mathfrak{e}},\nu_\omega\rangle=\sum_{\substack{\alpha^\vee\in\Phi^\vee(\widehat{T},\widehat{G}_\omega^{\mathfrak{e}})\\ \langle\alpha^\vee,\nu\rangle>0}}\langle\alpha^\vee,\nu\rangle,\qquad\langle\rho,\nu\rangle=\sum_{\substack{\alpha^\vee\in\Phi^\vee(\widehat{T},\widehat{G})\\ \langle\alpha^\vee,\nu\rangle>0}}\langle\alpha^\vee,\nu\rangle.\tag{4.2.2}$$

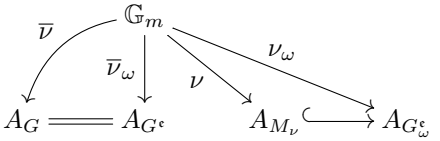
Thus it suffices to verify that there exists a coroot $\alpha^\vee\in\Phi^\vee(\widehat{T},\widehat{G})$ outside $\widehat{G}^{\mathfrak{e}}$ such that $\langle\alpha^\vee,\nu\rangle>0$. Write $\widehat{\nu}\in X^*(\widehat{T})$ for the dominant member in the Weyl orbit of characters determined by ν. The centralizer of $\widehat{\nu}$ in $\widehat{G}$ is identified with the dual group $\widehat{M_\nu}$ (namely $\langle\alpha^\vee,\nu\rangle=0$ if and only if $\alpha^\vee$ is a coroot of $\widehat{M_\nu}$), so we will be done if $\mathrm{Lie}\,\widehat{M_\nu}+\mathrm{Lie}\,\widehat{G}^{\mathfrak{e}}$ is a proper subspace of $\mathrm{Lie}\,\widehat{G}$. This is exactly proved in [KST20,

Lem. 4.5(ii)] applied to $\mathsf{G} = \widehat{G}$, $\mathsf{M} = \widehat{M_\nu}$, and $\delta = s^{\mathfrak{e}}$. (The proof of loc. cit. greatly simplifies. One reduces to the case when the Dynkin diagram of G is connected as in the first paragraph in the proof of that lemma. Then argue as in the fourth paragraph of that lemma, with $X_n = 0$ and with the role of X_{ss} played by the semisimple element $s^{\mathfrak{e}}$.)

Now we prove (b). Since $\langle \chi_\infty, \overline{\nu}\rangle = \langle \chi_{1,\infty}, \overline{\nu}_1\rangle$, we reduce to showing (b) when $G_1 = G$ and $\mathfrak{e}_1 = \mathfrak{e}$ (with possibly nontrivial central character data). Thus we drop the 1's from the subscripts and check that

$$\mathrm{Re}\langle \chi_\infty^{\mathfrak{e}}, \overline{\nu}\rangle = \mathrm{Re}\langle \chi_\infty, \overline{\nu}_\omega\rangle.$$

We claim that $\overline{\nu} = \overline{\nu}_\omega$ in $X_*(A_G)_{\mathbb{R}} = X_*(A_{G^{\mathfrak{e}}})_{\mathbb{R}}$. In the diagram below, the triangle on the right commutes, and we want that the triangle on the left commutes as well.



We choose maximal tori $T \subset M_\nu \subset G$ and $T^{\mathfrak{e}} \subset G_\omega^{\mathfrak{e}} \subset G^{\mathfrak{e}}$ with an isomorphism $T_{\overline{F}} \simeq T^{\mathfrak{e}}_{\overline{F}}$ to identify the absolute Weyl group $\overline{\Omega}^{G^{\mathfrak{e}}}$ as a subgroup of $\overline{\Omega}^G$. (This is done as in [Kot86, §3].) The isomorphism also identifies $\nu = \nu_\omega$. By (3.1.6), we have the equalities

$$\overline{\nu} = \left|\overline{\Omega}^G\right|^{-1} \sum_{\omega \in \overline{\Omega}^G} \omega(\nu), \qquad \overline{\nu}_\omega = \left|\overline{\Omega}^{G^{\mathfrak{e}}}\right|^{-1} \sum_{\omega \in \overline{\Omega}^{G^{\mathfrak{e}}}} \omega(\nu).$$

Hence $\overline{\nu} = \left|\overline{\Omega}^G / \overline{\Omega}^{G^{\mathfrak{e}}}\right|^{-1} \sum_{\omega \in \overline{\Omega}^G/\overline{\Omega}^{G^{\mathfrak{e}}}} \omega(\overline{\nu}_\omega) = \overline{\nu}_\omega$. Indeed, the last equality follows since $\overline{\nu}_\omega \in X_*(A_{G^{\mathfrak{e}}})_{\mathbb{R}} = X_*(A_G)_{\mathbb{R}}$, which tells us that $\omega(\overline{\nu}_\omega) = \overline{\nu}_\omega$ for $\omega \in \overline{\Omega}^G$.

Applying (2.6.3) at the archimedean place, we have $\chi_\infty = \lambda_\infty^{\mathfrak{e}} \chi_\infty^{\mathfrak{e}}$ as characters of $A_G(\mathbb{R})$. Since $\lambda_\infty^{\mathfrak{e}}$ is unitary, $|\chi_\infty| = |\chi_\infty^{\mathfrak{e}}|$. Since $\overline{\nu} \in X_*(A_G)_{\mathbb{R}}$ (not just in $X_*(A_G)_{\mathbb{C}}$), we conclude that $\mathrm{Re}\langle \chi_\infty^{\mathfrak{e}}, \overline{\nu}\rangle = \mathrm{Re}\langle \chi_\infty, \overline{\nu}\rangle$ as desired. This verifies (b). □

4.3. Some facts and notation on Weyl groups and Weyl chambers. — In this subsection we fix some additional notation on Weyl groups, Weyl chambers, which will be needed in the proof of the main estimate in the next subsection.

Let $P = MN \subset G$ be a parabolic subgroup such that $B \subset P$ and $T \subset M$. Write Z_M^0 for the identity component of the center of M and $S_{M,p}$ for the maximal $\mathbb{Q}_p$-split subtorus in Z_M^0. If $M = T$, we will write more simply $S_p := S_{M,p}$. Thus $A_{T,\mathbb{Q}_p} \subset S_p \subset T_{\mathbb{Q}_p}$. We will write

$$\Omega^G \subset \Omega_p^G \subset \overline{\Omega}^G$$

for the Weyl groups of A_T, S_p, and T in G. Similar notation will be used for other objects related to Weyl groups, for instance we write $\Omega_{M,p}^G \subset \Omega_p^G$ for the set of Kostant representatives for Ω_p^G / Ω_p^M.

Write $\Phi_M^G = \Phi_M^G(A_M; G)$ for the set of roots of A_M in $\mathrm{Lie}(G)$. Write $\Phi(A_M; B)$, $\Phi(S_{M,p}; B_{\mathbb{Q}_p})$, $\Phi(Z_M; B)$ for the sets of positive roots attached to A_M, $S_{M,p}$ and Z_M.

We write $\mathfrak{a}_M^{\mathrm{reg}} \subset \mathfrak{a}_M$ for the subset of all regular elements, i.e., $x \in \mathfrak{a}_M$ such that $\langle \alpha, x\rangle \neq 0$ for all $\alpha \in \Phi_M^G$. The connected components of $\mathfrak{a}_M^{\mathrm{reg}}$ are said to be the (open) *Weyl chambers* of $\mathfrak{a}_M$. The subset

$$\mathcal{C}_M^+ := \{x \in \mathfrak{a}_M^{\mathrm{reg}} \mid \forall \alpha \in \Phi(A_M, B) : \langle \alpha, x\rangle > 0\} \subset \mathfrak{a}_M^{\mathrm{reg}},$$

is the *dominant Weyl chamber*. Let $\Omega_M^G \subset \Omega^G$ be the set of Kostant representatives for the quotient Ω^G/Ω^M. The Weyl chambers $\mathcal{C} \in \pi_0(\mathfrak{a}_M^{\mathrm{reg}})$ are parametrized via the bijection

$$\Omega_M^G \longrightarrow \pi_0(\mathfrak{a}_M^{\mathrm{reg}}), \quad \omega \mapsto \mathcal{C}_\omega := \omega^{-1}(\mathcal{C}_M^+) \in \pi_0(\mathfrak{a}_M^{\mathrm{reg}})$$

If $\mathcal{C} \subset \mathfrak{a}_M^{\mathrm{reg}}$ is a Weyl chamber, we write $\mathcal{C}^\vee \subset \mathfrak{a}_M^*$ for the *dual chamber*, i.e., the set of $t \in \mathfrak{a}_M^*$ such that $t(x) > 0$ for all $x \in \mathcal{C}$.

LEMMA 4.3.1. — *The following statements are true:*

(1) *The inclusions $A_{M,\overline{\mathbb{Q}}_p} \subset S_{M,p,\overline{\mathbb{Q}}_p} \subset Z^0_{M,\overline{\mathbb{Q}}_p}$ induce (by restriction) a sequence of maps*

$$\Phi(Z_M^0; B) \longrightarrow \Phi(S_{M,p}; B) \longrightarrow \Phi(A_M; B)$$

which are all surjective.

(2) *The following three subsets of $\mathfrak{a}_M$ are equal:*

(a) *the set of $x \in \mathfrak{a}_M$ such that for all $\alpha \in \Phi(Z_M^0; B)$ we have $\langle \alpha, x\rangle > 0$;*

(b) *the set of $x \in \mathfrak{a}_M$ such that for all $\alpha \in \Phi(S_{M,p}; B)$ we have $\langle \alpha, x\rangle > 0$;*

(c) *the set of $x \in \mathfrak{a}_M$ such that for all $\alpha \in \Phi(A_M; B)$ we have $\langle \alpha, x\rangle > 0$.*

(3) *The natural maps $\pi_0(\mathfrak{a}_M^{\mathrm{reg}}) \to \pi_0(X_*(S_{M,p})^{\mathrm{reg}}_{\mathbb{R}}) \to \pi_0(X_*(Z_M^0)^{\mathrm{reg}}_{\mathbb{R}})$ are injections.*

Proof. — We have inclusions of the centralizer groups

$$M_{\overline{\mathbb{Q}}_p} = \mathrm{Cent}(A_{M,\overline{\mathbb{Q}}_p}, G_{\overline{\mathbb{Q}}_p}) \supset \mathrm{Cent}(S_{M,p,\overline{\mathbb{Q}}_p}, G_{\overline{\mathbb{Q}}_p}) \supset \mathrm{Cent}(Z^0_{M,\overline{\mathbb{Q}}_p}, G_{\overline{\mathbb{Q}}_p}) \supset M_{\overline{\mathbb{Q}}_p},$$

where the first equality is well known [Bor91, Prop. 20.6(i)]. Hence the equality holds everywhere. So $\Phi(A_M; B)$, $\Phi(S_{M,p}; B)$, and $\Phi(Z_M^0; B)$ consist of eigen-characters for the adjoint actions of $A_{M,\overline{\mathbb{Q}}_p} \subset S_{M,p,\overline{\mathbb{Q}}_p} \subset Z^0_{M,\overline{\mathbb{Q}}_p}$ on the same space $\mathrm{Lie}\,(B)/\mathrm{Lie}\,(B \cap M)$, respectively. Therefore, the maps in (1) are surjections. Statements (2) and (3) are directly deduced from (1). □

4.4. PROOF OF THEOREM 4.2.2. — The rest of this section is devoted to establishing the main estimate in Theorem 4.2.2 over several pages. Lemma 4.4.1 (which is technical) could be taken for granted at a first reading. Before diving into the details we recommend the reader to review the outline that we sketched below (1.4.2) in the introduction.

Proof of 4.2.2. — We argue by induction on the $\mathbb{Q}$-semisimple rank r_G of G. If $r_G = 0$, then we have $T^G_{\mathrm{ell},\chi} = T^G_{\mathrm{disc},\chi}$, and the statement follows. Assume now that the theorem

is established for all groups of lower $\mathbb{Q}$-semisimple rank and all accompanying data (i.e., $(\mathfrak{X}, \chi), p, q, \xi, \zeta$ and ν).

We introduce a constant to control regularity at q:

$$C = C(f^{\infty,q}, \phi_p) := \frac{1}{\log q} \cdot \max_{M, x^{p,q,\infty}, \varepsilon_p, \alpha} |\langle \alpha, x^{p,q,\infty} + \varepsilon_p \rangle|, \tag{4.4.1}$$

where $M, x^{p,q,\infty}, \varepsilon_p, \alpha$ range over the sets

$$\mathcal{L}_{\mathrm{cusp}}(G), \mathrm{supp}^O_{\mathfrak{a}_M}(f_M^{\infty,p,q}), \mathrm{pr}_M \mathrm{supp}^O_{\mathfrak{a}_{\omega_p(M)\cap M_\nu}}(\phi_{p,\omega_p(M)\cap M_\nu}), \alpha \in \Phi_M^G$$

respectively. Define the constants

$$v_{\mathfrak{X}} := \mathrm{vol}(\mathfrak{X}_{\mathbb{Q}} \backslash \mathfrak{X} / A_{G,\infty}) \quad \text{and} \quad c_M := (-1)^{\dim(A_M/A_G)} \frac{|\Omega^M|}{|\Omega^G|}, \quad M \in \mathcal{L}_{\mathrm{cusp}}(G).$$

Write $f^{\infty,(k)} := f^{\infty,p,q} f_p^{(k)} f_q$, to indicate the dependence on k at p. The running hypothesis on f_q is that it is C-regular for (4.4.1). By Proposition 2.8.1 we have

$$\begin{aligned} &T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) \\ &\quad = d(G)^{-1} \sum_{M \in \mathcal{L}_{\mathrm{cusp}}} c_M v_{\mathfrak{X}}^{-1} \sum_{\gamma \in \Gamma_{\mathbb{R}\text{-ell},\mathfrak{X}}(M)} \frac{\chi(I_\gamma^M) \zeta(\gamma) \Phi_M(\gamma, \xi) O_\gamma^M(f_M^{\infty,(k)})}{|\iota^M(\gamma)||\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|}. \end{aligned} \tag{4.4.2}$$

We first compare the term corresponding to $M = G \in \mathcal{L}_{\mathrm{cusp}}$ on the right hand side of Equation (4.4.2) with $T^G_{\mathrm{ell},\chi}$. Only regular $\mathbb{R}$-elliptic conjugacy classes contribute to (4.4.2): For $\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(G)$ non-regular, we have $O_\gamma(f_q) = 0$ since f_q is C-regular. Additionally, the orbital integrals $O_\gamma^{G(\mathbb{R})}(f_{\xi,\zeta})$ vanish for non $\mathbb{R}$-elliptic $\gamma \in G(\mathbb{R})$. In Lemma 4.1.3 we checked that for $\gamma \in \Gamma_{\mathbb{R}\text{-ell},\mathfrak{X}}(G)$ we have

$$\begin{aligned} &d(G)^{-1} c_G v_{\mathfrak{X}}^{-1} \frac{\chi(I_\gamma^G) \zeta(\gamma) \Phi_G(\gamma, \xi) O_\gamma^G(f^{\infty,(k)})}{|\iota^G(\gamma)||\mathrm{Stab}^G_{\mathfrak{X}}(\gamma)|} \\ &\qquad = \frac{\mathrm{vol}(I_\gamma(\mathbb{Q}) \backslash I_\gamma(\mathbb{A}) / \mathfrak{X}) O_\gamma^G(f_{\xi,\zeta} f^{\infty,(k)})}{\iota(\gamma)^{-1} |\mathrm{Stab}^G_{\mathfrak{X}}(\gamma)|^{-1}} \end{aligned} \tag{4.4.3}$$

(this uses $\Phi_G(\gamma, \xi) = \mathrm{Tr}(\gamma, \xi)$, cf. [Art89, below eq. (4.4)]). Therefore, $T^G_{\mathrm{ell},\chi}(f_{\xi,\zeta} f^{\infty,(k)})$ appears on the right hand side of (4.4.2) as the summand for $M = G$ (see also (2.8.2)). Thus (4.4.2) can be rearranged as

$$\begin{aligned} T^G_{\mathrm{ell},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) &= T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) \\ &\quad - d(G)^{-1} \sum_{M \in \mathcal{L}^<_{\mathrm{cusp}}} c_M v_{\mathfrak{X}}^{-1} \sum_{\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(M)} \frac{\chi(I_\gamma^M) \zeta(\gamma) \Phi_M(\gamma, \xi) O_\gamma^M(f_M^{\infty,(k)})}{|\iota^M(\gamma)||\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|}. \end{aligned} \tag{4.4.4}$$

As $f_p^{(k)}$ is a ν-ascent of $\phi_p^{(k)}$, we have by Lemma 3.2.2

$$f_{p,M}^{(k)} = \sum_{\omega_p \in \Omega^{G_{\mathbb{Q}_p}}_{M,M_\nu}} f^{(k)}_{p,M,\omega_p} \in \mathcal{H}(M(\mathbb{Q}_p), \chi_p^{-1}), \tag{4.4.5}$$

where

$$f^{(k)}_{p,M,\omega_p} := \mathscr{J}_{\nu_{\omega_p}}(\omega_p^{-1} \phi^{(k)}_{p,M_{\omega_p}}), \qquad M_{\omega_p} = \omega_p(M) \cap M_\nu. \tag{4.4.6}$$

At the prime q, we may arrange by Lemma 3.4.7(1) and Lemma 4.3.1 that the constant term $f_{q,M}$ is supported on C-regular elements. Thus $f_{q,M}$ is decomposed according to the various chambers $\mathcal{C}$ of $\mathfrak{a}_M^{\mathrm{reg}}$:

$$f_{q,M} = \sum_{\omega_q \in \Omega_M^G} f_{q,M,\omega_q} \in \mathcal{H}(M(\mathbb{Q}_q), \chi_q^{-1})_{C\text{-reg}},$$

where f_{q,M,ω_q} satisfies $\mathrm{supp}^O_{\mathfrak{a}_M}(f_{q,M,\omega_q}) \subset \mathcal{C}_{\omega_q}$. We define

$$f^{\infty,(k)}_{M,\omega_p,\omega_q} := f_M^{\infty,p,q} f^{(k)}_{p,M,\omega_p} f_{q,M,\omega_q} \in \mathcal{H}(M(\mathbb{A}), \chi^{-1}),$$

so that

$$f_M^{\infty,(k)} = \sum_{\omega_p \in \Omega_{M,M_\nu}^{G_{\mathbb{Q}_p}}, \omega_q \in \Omega_M^G} f^{\infty,(k)}_{M,\omega_p,\omega_q} \in \mathcal{H}(M(\mathbb{A}), \chi^{-1}).$$

Changing the order of summation (each sum is finite), Equation (4.4.4) becomes

$$\begin{aligned} T^G_{\mathrm{ell},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) = {} & T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) \\ & - d(G)^{-1} \sum_{M,\omega_p,\omega_q} \frac{c_M}{v_{\mathfrak{X}}} \sum_{\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(M)} \frac{\chi(I_\gamma^M)\zeta(\gamma)\Phi_M(\gamma,\xi) O_\gamma^M(f^{\infty,(k)}_{M,\omega_p,\omega_q})}{|\iota^M(\gamma)||\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|}, \end{aligned} \tag{4.4.7}$$

where the sum is over $M \in \mathcal{L}^<_{\mathrm{cusp}}$, $\omega_p \in \Omega_{M,M_\nu}^{G_{\mathbb{Q}_p}}$, $\omega_q \in \Omega_M^G$.

To state the next lemma, we define a constant

$$k_1 = k_1(f^{\infty,q}, \phi_p) := \max_{M,\omega_p,\omega_q,\alpha,\varepsilon_p,x^{p,\infty}} \left| \frac{\langle \alpha, \varepsilon_p + x^{p,\infty} \rangle}{\log(p)\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle} \right| \in \mathbb{R}_{>0}, \tag{4.4.8}$$

where the maximum is taken over

$$\begin{gathered} M \in \mathcal{L}^<_{\mathrm{cusp}}(G), \quad \omega_p \in \Omega_{M,M_\nu}^{G_{\mathbb{Q}_p}}, \quad \omega_q \in \Omega_M^G, \\ \varepsilon_p \in \mathrm{pr}_M \mathrm{supp}^O_{\mathfrak{a}_{M_{\omega_p}}}(\omega_p^{-1}\phi_{p,M_{\omega_p}}), \ x^{p,\infty} \in \mathrm{supp}^O_{\mathfrak{a}_M}(f_M^{\infty,p}), \end{gathered}$$

and α ranges over those $\alpha \in \Phi_M^G$ such that $\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu))\rangle \neq 0$.

We have fixed a maximal torus T in $G_{\mathbb{C}}$ (we have $G_{\mathbb{C}} \simeq G_{\overline{\mathbb{Q}}_p}$ via ι_p), along with a Borel subgroup B. We write $\overline{\rho} = \overline{\rho}_G$ for the half sum of the B-positive roots of T in $\mathrm{Lie}(B)$. Note that we have $\overline{\rho}|_{A_T} = \rho$. We use similar definitions for $\overline{\rho}_M$ and ρ_M if $M \subset G$ is a Levi subgroup. Let $\lambda = \lambda_B, \lambda_B^* \in X^*(T)$ denote the highest weight of ξ and its dual representation ξ^*, respectively, relative to B.

For each $M \in \mathcal{L}_{\mathrm{cusp}}(G)$ we introduce the following notation. Denote by $\mathcal{P}(M)$ the set of parabolic subgroups P of G of which M is a Levi component. We remark that in [GKM97] the set $\mathcal{P}(M_{\mathbb{R}})$ is used, meaning parabolic subgroups P of $G_{\mathbb{R}}$ such that $M_{\mathbb{R}}$ is a Levi component of P. As $M \in \mathcal{L}_{\mathrm{cusp}}(G)$, we know that $(A_M)_{\mathbb{R}} A_{G_{\mathbb{R}}} = A_{M_{\mathbb{R}}}$, and therefore any parabolic subgroup of $G_{\mathbb{R}}$ that contains M is defined over $\mathbb{Q}$. In particular, $\mathcal{P}(M_{\mathbb{R}}) = \mathcal{P}(M)$.

For each $\lambda_0 \in X^*(T)^+$ we write $\xi^M_{\lambda_0}$ for the irreducible $M_{\mathbb{C}}$-representation with highest weight λ_0. We define $\omega_\infty \star \lambda_0 := \omega_\infty(\lambda_0 + \overline{\rho}) - \overline{\rho}$ for each $\omega_\infty \in \overline{\Omega}_M^G$ and

$\lambda_0 \in X^*(T)^+$. Let $\omega_\infty \in \overline{\Omega}_M^G$. Write $\omega_0^M \in \overline{\Omega}^M$ for the longest Weyl group element, and

$$\lambda_B(\omega_\infty) := -\omega_0^M(\omega_\infty \star \lambda_B^*) = \omega_0^M \omega_\infty \omega_0^M \lambda_B - \omega_0^M \omega_\infty \overline{\rho} - \omega_0^M \overline{\rho},$$

so that we have

$$\xi^M_{\lambda_B(\omega_\infty)} = (\xi^M_{\omega_\infty \star \lambda_B^*})^*.$$

LEMMA 4.4.1. — *Assume that $k > k_1$. Consider $M, \omega_p, \omega_q, \gamma$ as in* (4.4.7)*, and assume that*

$$O^M_\gamma(f^{\infty,(k)}_{M,\omega_p,\omega_q}) \neq 0. \tag{4.4.9}$$

Let $x_\mathbb{R} := H^M_\infty(\gamma) \in \mathfrak{a}_{M_\mathbb{R}}$ and write x_∞ for the image of $x_\mathbb{R}$ under $\mathfrak{a}_{M_\mathbb{R}} \to \mathfrak{a}_M$. The following are true.

(i) *The element $x_\infty \in \mathfrak{a}_M$ is regular and lies in the chamber $\mathcal{C}_0 = \mathcal{C}_0(M, \omega_p, \omega_q) \subset \mathfrak{a}_M^{\mathrm{reg}}$ which has the following set of positive roots*

$$\{\alpha \in \Phi_M^G \mid \langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle > 0\} \cup \{\alpha \in \Phi_M^G \mid \langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle = 0 \text{ and } \alpha \in -\mathcal{C}^\vee_{\omega_q}\}. \tag{4.4.10}$$

(ii) *There exists an explicit subset $\overline{\Omega}_M^{G\diamond} = \overline{\Omega}_M^{G\diamond}(M, \omega_p, \omega_q) \subset \overline{\Omega}_M^G$ (see* (4.4.22)*) and an explicit sign $\varepsilon^\diamond = \varepsilon^\diamond(M, \omega_p, \omega_q)$ (see* (4.4.23)*) such that we have*

$$\Phi_M(\gamma, \xi) = \varepsilon^\diamond \sum_{P \in \mathcal{P}(M)} \delta_P^{-1/2}(\gamma) \sum_{\omega_\infty \in \overline{\Omega}_M^{G\diamond}} \varepsilon(\omega_\infty) \operatorname{Tr}(\gamma; \xi^M_{\lambda_B(\omega_\infty)}),$$

where $\varepsilon(w_\infty) \in \{\pm 1\}$ denotes the sign as an element of the Weyl group $\overline{\Omega}^G$.

Proof. — (i) If S is a set of places of $\mathbb{Q}$, we write in this proof

$$x_S := H_S^M(\gamma) \in \mathfrak{a}_M, \quad x^S := H^{M,S}(\gamma) \in \mathfrak{a}_M.$$

We check that $\langle \alpha, x_\infty\rangle \neq 0$ for all $\alpha \in \Phi_M^G$ (i.e., x_∞ is regular). By the product formula in Lemma 4.1.2 we have

$$-\langle \alpha, x_\infty\rangle = \langle \alpha, x^{p,\infty}\rangle + \langle \alpha, x_p\rangle. \tag{4.4.11}$$

At p, $O_\gamma(f^{(k)}_{p,M,\omega_p}) \neq 0$ implies that $x_p \in \mathrm{supp}^O_{\mathfrak{a}_M}(f^{(k)}_{p,M,\omega_p})$. By Lemma 3.1.7 (and Equation (4.4.6))

$$\mathrm{supp}^O_{\mathfrak{a}_M}(f^{(k)}_{p,M,\omega_p}) = k \cdot H_p^M(\omega_p^{-1}\nu(p)) + \mathrm{pr}_M(\mathrm{supp}^O_{\mathfrak{a}_{M_{\omega_p}}}(\omega_p^{-1}\phi_{p,M_{\omega_p}})).$$

Therefore

$$x_p = k \cdot H_p^M(\omega_p^{-1}\nu(p)) + \varepsilon_p \tag{4.4.12}$$

for some $\varepsilon_p \in \mathrm{pr}_M \mathrm{supp}^O_{\mathfrak{a}_{M_{\omega_p}}}(\omega_p^{-1}\phi_{p,M_{\omega_p}})$. Thus

$$\langle \alpha, x_p\rangle = k \cdot \langle \alpha, H_p^M(\omega_p^{-1}\nu(p))\rangle + \langle \alpha, \varepsilon_p\rangle = -k(\log p) \cdot \langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu))\rangle + \langle \alpha, \varepsilon_p\rangle. \tag{4.4.13}$$

We now distinguish cases. First consider $\alpha \in \Phi_M^G$ such that $\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle \neq 0$. By (4.4.11) and (4.4.12),

$$-\langle \alpha, x_\infty\rangle = \langle \alpha, x^{p,\infty}\rangle + \langle \alpha, \varepsilon_p\rangle - k(\log p)\cdot \langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle. \tag{4.4.14}$$

As $k > k_1$ (see (4.4.8)) we have

$$k(\log p)\cdot |\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle| > |\langle \alpha, \varepsilon_p + x^{p,\infty}\rangle|.$$

Thus from (4.4.14) we get $\langle \alpha, x_\infty\rangle \neq 0$.

Consider $\alpha \in \Phi_M^G$ such that $\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle = 0$. Again by (4.4.11) and (4.4.12),

$$-\langle \alpha, x_\infty\rangle = \langle \alpha, x^{p,\infty}\rangle + \langle \alpha, \varepsilon_p\rangle. \tag{4.4.15}$$

As f_q is C-regular, we have from (3.4.2), (4.4.1), and Lemma 4.3.1 that

$$|\langle \alpha, x_q\rangle| > C\log q \geqslant |\langle \alpha, x^{p,q,\infty} + \varepsilon_p\rangle|. \tag{4.4.16}$$

for all $\alpha \in \Phi_M^G$. In particular

$$\langle \alpha, x_q\rangle + \langle \alpha, x^{p,q,\infty} + \varepsilon_p\rangle \neq 0.$$

Therefore, each side of (4.4.15) does not vanish. Hence $\langle \alpha, x_\infty\rangle \neq 0$ for all $\alpha \in \Phi_M^G$.

We now determine for which $\alpha \in \Phi_M^G$ we have $\langle \alpha, x_\infty\rangle > 0$. If $\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle \neq 0$, then

$$\mathrm{sign}(\langle \alpha, x_\infty\rangle) = \mathrm{sign}(\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle)$$

by the arguments following (4.4.13). If $\langle \alpha, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle = 0$, then

$$\mathrm{sign}(\langle \alpha, x_\infty\rangle) = -\mathrm{sign}(\langle \alpha, x_q\rangle)$$

by C-regularity (see (4.4.16)). We have $x_q \in \mathrm{supp}_{\mathfrak{a}_M}(f_{q,M,\omega_q})$. Statement (i) follows.

(ii) Let us start by recalling a result of Goresky, Kottwitz and MacPherson in [GKM97]. We write $\mathrm{pr}_{M_\mathbb{R}}^*\colon X^*(T)_\mathbb{R} \to X^*(A_{M_\mathbb{R}})_\mathbb{R}$ for the restriction map. Let $P = MN \in \mathcal{P}(M)$ Write $\rho_{N_\mathbb{R}}$ (resp. $\overline{\rho}_{N_\mathbb{R}}$) for the half sum of the positive roots of $A_{M_\mathbb{R}}$ (resp. $\overline{T}$) that occur in the Lie algebra of the unipotent radical $N_\mathbb{R}$ of $P_\mathbb{R}$. Write ω_ξ for the central character of ξ. Write $\alpha_1, \ldots, \alpha_n \in \mathfrak{a}_{M_\mathbb{R}}^*$ for the simple roots of $A_{M_\mathbb{R}}$ in $\mathrm{Lie}\,(N_\mathbb{R})$, which form a basis of $(\mathfrak{a}_{M_\mathbb{R}}/\mathfrak{a}_{G_\mathbb{R}})^*$. This determines the dual basis consisting of $t_1, \ldots, t_n \in \mathfrak{a}_{M_\mathbb{R}}/\mathfrak{a}_{G_\mathbb{R}}$. Put $I := \{1, 2, \ldots, n\}$. Define the following subsets of I (cf. [GKM97, p. 534])

$$\begin{aligned} I(\gamma) &:= \{i \in I \mid \langle \alpha_i, x_\mathbb{R}\rangle < 0\}, \\ I(\omega_\infty) &:= \{i \in I \mid \langle \mathrm{pr}_{M_\mathbb{R}}^*(-\omega_\infty \star \lambda_B^*) - \overline{\rho}_{N_\mathbb{R}} - \omega_\xi, t_i\rangle > 0\}. \end{aligned} \tag{4.4.17}$$

Since $M \in \mathcal{L}_{\mathrm{cusp}}$ the map

$$\mathfrak{a}_{M_\mathbb{R}}/\mathfrak{a}_{G_\mathbb{R}} \longrightarrow \mathfrak{a}_M/\mathfrak{a}_G \tag{4.4.18}$$

is an isomorphism. In particular, $\alpha_1, \ldots, \alpha_n$ is also a basis for $(\mathfrak{a}_M/\mathfrak{a}_G)^*$, and we can replace $x_\mathbb{R}$ by x_∞ in the definition of $I(\gamma)$, without changing the set.

By the discussion above Th. 7.14.B in [GKM97] we have

$$\varphi_P(-x_{\mathbb{R}}, \mathrm{pr}^*_{M_{\mathbb{R}}}(\omega_\infty \star \lambda_B^*) + \overline{\rho}_{N_{\mathbb{R}}} + \omega_\xi) = \begin{cases} (-1)^{\dim(A_{G_{\mathbb{R}}})}(-1)^{\dim(A_{M_{\mathbb{R}}}/A_{G_{\mathbb{R}}})-|I(\gamma)|}, & \text{if } I(\omega_\infty) = I(\gamma), \\ 0, & \text{otherwise.} \end{cases} \tag{4.4.19}$$

By (i), $x_\infty \in \mathfrak{a}_M$ is regular, and as $M \in \mathcal{L}_{\mathrm{cusp}}(G)$ we have $\mathfrak{a}_{M_{\mathbb{R}}}/\mathfrak{a}_{G_{\mathbb{R}}} = \mathfrak{a}_M/\mathfrak{a}_G$, so the element $x_{\mathbb{R}} \in \mathfrak{a}_{M_{\mathbb{R}}}$ is also regular.

We define $L_M(\gamma) \in \mathbb{C}$ following [GKM97, p. 511],[7] when $x_{\mathbb{R}}$ is regular[8]:

$$L_M(\gamma) := (-1)^{\dim(A_{G_{\mathbb{R}}})} \sum_{P \in \mathcal{P}(M)} \delta_P^{-1/2}(\gamma) \cdot \sum_{\omega_\infty \in \overline{\Omega}_M^G} \varepsilon(\omega_\infty) \operatorname{Tr}(\gamma^{-1}; \xi^M_{\omega_\infty \star \lambda_B^*}) \varphi_P(-x_\infty, \mathrm{pr}^*_{M_{\mathbb{R}}}(\omega_\infty \star \lambda_B^*) + \overline{\rho}_N + \omega_\xi). \tag{4.4.20}$$

Theorems 5.1 and 5.2 of [GKM97] imply the following identity[9]

$$\Phi_M(\gamma, \xi) = L_M(\gamma). \tag{4.4.21}$$

Thus the right hand side of (4.4.20) is an expression for $\Phi_M(\gamma, \xi)$.

By assumption (4.4.9), $x_\infty = H_\infty^M(\gamma)$ lies in the chamber $\mathcal{C}_0(M, \omega_p, \omega_q)$ by part (i) of this lemma. So the set $I(\gamma)$ does not depend on x_∞. Write $\mathcal{I}_0 = \mathcal{I}_0(M, \omega_p, \omega_q)$ for $I(\gamma)$, and

$$\overline{\Omega}_M^{G\diamond} = \overline{\Omega}_M^{G\diamond}(M, \omega_p, \omega_q) := \{\omega_\infty \in \overline{\Omega}_M^G \mid I(\omega_\infty) = \mathcal{I}_0\}, \tag{4.4.22}$$

in terms of (4.4.17). Then (4.4.20) simplifies thanks to (4.4.19):

$$L_M(\gamma) = (-1)^{\dim(A_{M_{\mathbb{R}}}/A_{G_{\mathbb{R}}})-|\mathcal{I}_0|} \sum_{P \in \mathcal{P}(M)} \delta_P^{-1/2}(\gamma) \sum_{\omega_\infty \in \overline{\Omega}_M^{G\diamond}} \varepsilon(\omega_\infty) \operatorname{Tr}(\gamma^{-1}; \xi^M_{\omega_\infty \star \lambda_B^*}).$$

We obtain (ii) by using $\operatorname{Tr}(\gamma^{-1}; \xi^M_{\omega_\infty \star \lambda_B^*}) = \operatorname{Tr}(\gamma, \xi^M_{\lambda_B(\omega_\infty)})$ and taking

$$\varepsilon^\diamond := (-1)^{\dim(A_{M_{\mathbb{R}}}/A_{G_{\mathbb{R}}})-|\mathcal{I}_0|}. \tag{4.4.23}$$

□

We keep on assuming $k > k_1$ and write $c'_M := \varepsilon^\diamond \varepsilon(\omega_\infty) c_M d(G)^{-1}$ from now. We apply Lemma 4.4.1 (*ii*) to Equation (4.4.7) and change the order of summation

[7] We write $L_M(\gamma)$ where the authors of [GKM97] write $L^\nu_M(\gamma)$. This is because we only need to use the "middle weight profile", so there is no need to distinguish between different profiles ν in our notation. Since we use the middle weight profile, we have $\nu = -\overline{\rho}_N - \omega_\xi$.

[8] On p. 504 the authors give a definition of $L_M(\gamma)$ without the requirement that x_∞ is regular (but we don't need it here). Under this more general definition, Equation (4.4.21) also holds for non-regular x_∞.

[9] In [GKM97], they write $\Phi_M(\gamma, \Theta^{\xi^*})$ for $\Phi_M(\gamma, \xi^*)$. Their E corresponds to our ξ^*.

to obtain

$$T^G_{\mathrm{ell},\chi}(f_{\xi,\zeta}f^{\infty,(k)}) = T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta}f^{\infty,(k)}) - \sum_{M,P,\omega_p,\omega_q,\omega_\infty} c'_M v_{\mathfrak{X}}^{-1} \sum_{\gamma\in\Gamma_{\mathrm{ell},\mathfrak{X}}(M)} \frac{\chi(I^M_\gamma)\,\mathrm{Tr}(\gamma;\xi^M_{\lambda_B(\omega_\infty)}\otimes\zeta\delta_P^{-1/2})O^M_\gamma(f^{\infty,(k)}_{M,\omega_p,\omega_q})}{|\iota^M(\gamma)||\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|}, \tag{4.4.24}$$

where M,ω_p,ω_q run over the same sets as before and P,ω_∞ range over $\mathcal{P}(M),\overline{\Omega}^{G\diamond}_M$, respectively. We apply Lemma 4.1.3 to equalize

$$\begin{aligned} v_{\mathfrak{X}}^{-1} \sum_{\gamma\in\Gamma_{\mathrm{ell},\mathfrak{X}}(M)} \frac{\chi(I^M_\gamma)\,\mathrm{Tr}(\gamma;\xi^M_{\lambda_B(\omega_\infty)}\otimes\zeta\delta_P^{-1/2})O^M_\gamma(f^{\infty,(k)}_{M,\omega_p,\omega_q})}{|\iota^M(\gamma)||\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|} \\ = d(M)\sum_{\gamma\in\Gamma_{\mathrm{ell},\mathfrak{X}}(M)} \frac{\mathrm{vol}(I^M_\gamma(\mathbb{Q})A_{I^M_\gamma,\infty}\backslash I^M_\gamma(\mathbb{A})/\mathfrak{X})O^M_\gamma(f_{\lambda_B(\omega_\infty),\zeta\delta_P^{-1/2}}f^{\infty,(k)}_{M,\omega_p,\omega_q})}{|\iota^M(\gamma)||\mathrm{Stab}^M_{\mathfrak{X}}(\gamma)|}, \end{aligned} \tag{4.4.25}$$

using that every γ with $O^M_\gamma(f^{\infty,(k)}_{M,\omega_p,\omega_q})\neq 0$ in (4.4.25) is regular since f_{q,M,ω_q} is supported on regular elements. Define

$$\mathfrak{X}_M := \mathfrak{X}\cdot A_{M,\infty}, \qquad v_M := \mathrm{vol}(\mathfrak{X}_{\mathbb{Q}}\backslash\mathfrak{X}/A_{G,\infty})^{-1}\mathrm{vol}(\mathfrak{X}_{M,\mathbb{Q}}\backslash\mathfrak{X}_M/A_{M,\infty}).$$

The restriction of the central character of $\xi^M_{\lambda_B(\omega_\infty)}\otimes\zeta\delta_P^{-1/2}$ to $A_{M,\infty}$ is denoted by

$$z^M_{\omega_\infty}: A_{M,\infty}\longrightarrow\mathbb{C}^\times.$$

Since the central character of $\xi^M_{\lambda_B(\omega_\infty)}$ restricts to the central character of ξ on Z_G, we have

$$z^M_{\omega_\infty}|_{A_{G,\infty}} = \chi_0^{-1}.$$

On the other hand, $Z_G(\mathbb{R})\cap A_{M,\infty} = A_{G,\infty}\subset Z_M(\mathbb{R})$. Therefore

$$\mathfrak{X}\cap A_{M,\infty} = A_{G,\infty}.$$

Consequently, there exists a unique character

$$\chi^M_{\omega_\infty}: \mathfrak{X}_M\longrightarrow\mathbb{C}^\times$$

such that

$$\chi^M_{\omega_\infty}|_{A_{M,\infty}} = (z^M_{\omega_\infty})^{-1} \quad\text{and}\quad \chi^M_{\omega_\infty}|_{\mathfrak{X}} = \chi.$$

The pair $(\mathfrak{X}_M,\chi^M_{\omega_\infty})$ is a central character datum for M as in Section 2.8. Moreover,

$$f_{\lambda_B(\omega_\infty),\zeta\delta_P^{-1/2}}f^{\infty,(k)}_{M,\omega_p,\omega_q}\in\mathcal{H}(M(\mathbb{A}),\chi^{M,-1}_{\omega_\infty}).$$

The expression in (4.4.25) can be rewritten as

$$\begin{aligned} d(M)\sum_{\gamma\in\Gamma_{\mathrm{ell},\mathfrak{X}_M}(M)} \frac{v_M\cdot\mathrm{vol}(I^M_\gamma(\mathbb{Q})A_{I^M_\gamma,\infty}\backslash I^M_\gamma(\mathbb{A})/\mathfrak{X}_M)O^M_\gamma(f_{\lambda_B(\omega_\infty),\zeta\delta_P^{-1/2}}f^{\infty,(k)}_{M,\omega_p,\omega_q})}{|\iota^M(\gamma)||\mathrm{Stab}^M_{\mathfrak{X}_M}(\gamma)|} \\ = d(M)v_M\cdot T^M_{\mathrm{ell},\chi^M_{\omega_\infty}}(f_{\lambda_B(\omega_\infty)},\zeta\delta_P^{-1/2}f^{\infty,(k)}_{M,\omega_p,\omega_q}). \end{aligned} \tag{4.4.26}$$

Put $c''_M := c'_M v_M d(M)$. Combining (4.4.24) and (4.4.26), we obtain

$$T^G_{\mathrm{ell},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) = T^G_{\mathrm{disc},\chi}(f_{\xi,\zeta} f^{\infty,(k)}) - \sum_{M,P,\omega_p,\omega_q,\omega_\infty} c''_M \cdot T^M_{\mathrm{ell},\chi^M_{\omega_\infty}}(f_{\lambda_B(\omega_\infty),\zeta\delta_P^{-1/2}} f^{\infty,(k)}_{M,\omega_p,\omega_q}). \tag{4.4.27}$$

Let ω_p be as in the sum. Since $\omega_p \in \Omega^{G_{\mathbb{Q}_p}}_{M,M_\nu}$, we have

$$\omega_p(M \cap B) \subset B \quad \text{and} \quad \omega_p^{-1}(M_\nu \cap B) \subset B.$$

In particular, for each root α in $\mathrm{Lie}(M \cap N_0)$, the root $\omega_p \alpha$ also appears in $\mathrm{Lie}(M \cap N_0)$. So

$$\langle \alpha, w_p^{-1}\nu \rangle = \langle w_p \alpha, \nu \rangle \geqslant 0.$$

Hence $w_p^{-1}\nu$ is dominant for $M \cap B$. (See the paragraph above Proposition 4.2.1 for dominance of ν relative to B.) By Proposition 4.2.1 and the induction hypothesis for $M \in \mathcal{L}^<_{\mathrm{cusp}}$, we have

$$T^M_{\mathrm{ell},\chi^M_{\omega_\infty}}(f_{\lambda_B(\omega_\infty),\zeta\delta_P^{-1/2}} f^{\infty,(k)}_{M,\omega_p,\omega_q}) = O\big(p^{k(\langle \rho_M, \omega_p^{-1}\nu\rangle - \langle (\chi^M_{\omega_\infty})_\infty, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle)}\big). \tag{4.4.28}$$

(To apply the induction hypothesis, we need to ensure that the setup of Theorem 4.2.2 applies to the left hand side. The point is that the conditions at p and q are satisfied. At p, this is a consequence of (4.4.6) and Lemma 3.2.1; thus each $f^{(k)}_{p,M,\omega_p}$ is an ascent from an acceptable function. At q, this follows from Lemma 3.4.7 (1).) In the special case $M = G$ we obtain

$$T^G_{\mathrm{disc},\chi}(f^{(k)}_p f^{\infty,p} f_{\xi,\zeta}) = O\big(p^{k(\langle \rho,\nu\rangle - \langle \chi_\infty, \mathrm{pr}_G \nu\rangle)}\big). \tag{4.4.29}$$

Now assume that the datum $(M, P, \omega_p, \omega_q, \omega_\infty)$ contributes to (4.4.27), in particular $M \in \mathcal{L}^<_{\mathrm{cusp}}$, and also assume that

$$O^M_\gamma(f_{\lambda_B(\omega_\infty),\zeta\delta_P^{-1/2}} f^{\infty,(k)}_{M,\omega_p,\omega_q}) \neq 0$$

for some $\gamma \in \Gamma_{\mathrm{ell},\mathfrak{x}_M}(M)$. Then we claim that

$$\mathrm{Re}(\langle \rho,\nu\rangle - \langle \chi_\infty, \mathrm{pr}_G\nu\rangle) > \mathrm{Re}(\langle \rho_M, \omega_p^{-1}\nu\rangle - \langle (\chi^M_{\omega_\infty})_\infty, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle). \tag{4.4.30}$$

This claim, together with (4.4.28) and (4.4.29), tells us that the main term for G dominates the proper Levi terms in (4.4.27), thereby implies the theorem.

It remains to verify the claim (4.4.30). Clearly it is sufficient to show that

(a) $\langle \rho,\nu\rangle > \langle \rho_M, \omega_p^{-1}\nu\rangle$,
(b) $\mathrm{Re}\langle \chi_\infty, \mathrm{pr}_G\nu\rangle \leqslant \mathrm{Re}\langle (\chi^M_{\omega_\infty})_\infty, \mathrm{pr}_M(\omega_p^{-1}\nu)\rangle$.

Moreover, it is enough to prove (a) and (b) for sufficiently large k (note that the set $\Omega^{G\diamond}_M$ and thus ω_∞ depends on k). To prove (a), we start from the equality $\langle \rho_M, \omega_p^{-1}\nu\rangle = \langle \rho_{\omega_p M}, \nu\rangle$. Since ω_p is a Kostant representative (cf. (4.4.5)),

$$\langle \rho_{\omega_p M}, \nu\rangle \leqslant \langle \rho, \nu\rangle.$$

To check that

$$\langle \rho_{\omega_p M}, \nu\rangle \neq \langle \rho, \nu\rangle,$$

we argue as in the paragraph below Equation (4.2.2): As ν is not central, the argument for Lemma 4.5(ii) of [KST20] shows that $\mathrm{Lie}(M_\nu) + \mathrm{Lie}(M) \neq \mathrm{Lie}(G)$. Hence we can find a root α in $\mathrm{Lie}(G)$ which in either $\mathrm{Lie}(M)$ or $\mathrm{Lie}(M_\nu)$, i.e., $\langle \alpha, \nu \rangle \neq 0$. The proof of (a) is finished.

Now we prove (b). Recall that $x_{\mathbb{R}} = H^G_\infty(\gamma)$, and that x_∞ is the image of $x_{\mathbb{R}}$ in $\mathfrak{a}_G$. Write shorthand

$$X = \mathrm{pr}_M^*(-\omega_\infty \star \lambda_B^*) - \overline{\rho}_N - \omega_\xi \quad \text{and} \quad Y = \mathrm{pr}_M(w_p^{-1}\nu).$$

The equality $I(\gamma) = I(\omega_\infty)$ from (4.4.22) implies $\langle \alpha_i, x_\infty \rangle < 0 \Leftrightarrow \langle X, t_i \rangle > 0$ (cf. (4.4.17), and the isomorphism in (4.4.18)), and thus also

$$\langle \alpha_i, x_\infty \rangle \geqslant 0 \iff \langle X, t_i \rangle \leqslant 0.$$

By Equation (4.4.10) we have

$$\langle \alpha_i, x_\infty \rangle > 0 \iff \langle \alpha_i, Y \rangle > 0 \text{ or } [\langle \alpha_i, Y \rangle = 0 \text{ and } \alpha_i \in -\mathcal{C}^\vee_{\omega_q}].$$

Since x_∞ is regular, we have $\langle \alpha_i, x_\infty \rangle \neq 0$, thus $\langle \alpha_i, x_\infty \rangle > 0 \Leftrightarrow \langle \alpha_i, x_\infty \rangle \geqslant 0$, and so by combining the above

$$\langle X, t_i \rangle \leqslant 0 \iff \langle \alpha_i, Y \rangle > 0 \text{ or } [\langle \alpha_i, Y \rangle = 0 \text{ and } \alpha_i \in -\mathcal{C}^\vee_{\omega_w}]. \tag{4.4.31}$$

Write $X = \sum_i c_i \alpha_i$ and $Y = \sum_i d_i t_i$. Then (4.4.31) implies

$$\langle X, Y \rangle = \sum_i c_i d_i \leqslant 0.$$

We now conclude:

$$\begin{aligned}
\mathrm{Re}\langle (\chi^M_{\omega_\infty})_\infty, Y \rangle &= -\mathrm{Re}\langle z^M_{\omega_\infty}, Y \rangle \\
&= -\mathrm{Re}\langle \mathrm{pr}_M^*(\lambda_B(\omega_\infty)), Y \rangle - \mathrm{Re}\langle \zeta \delta_P^{-1/2}, Y \rangle \\
&= -\mathrm{Re}\langle \mathrm{pr}_M^*(-w_0(\omega_\infty \star \lambda_B^*)), Y \rangle - \mathrm{Re}\langle \zeta - \rho_N, Y \rangle \\
&= \underbrace{-\mathrm{Re}\langle X, Y \rangle}_{\geqslant 0} + \mathrm{Re}\underbrace{\langle -\omega_\xi - \zeta, Y \rangle}_{= \langle \chi_\infty, Y \rangle}
\end{aligned}$$

We are now done by observing that $\langle \chi_\infty, Y \rangle = \langle \chi_\infty, \mathrm{pr}_G \nu \rangle$. □

5. Shimura varieties of Hodge type

The goal of this section is to set up the scene for the mod p geometry of Shimura varieties and central leaves, paving the way for introducing Igusa varieties in the next section. We pay special attention to the connected components and H^0.

5.1. Connected components in characteristic zero. — From this point onward, let (G, X) be a Shimura datum as in [Del79] satisfying axioms (2.1.1.1), (2.1.1.2), and (2.1.1.3) therein. Write $E = E(G, X)$ for the reflex field [Del79, 2.2.1], which is a finite extension of $\mathbb{Q}$ in $\mathbb{C}$. We have the algebraic closure $\overline{E} \subset \mathbb{C}$. Let K be a neat open compact subgroup of $G(\mathbb{A}^\infty)$. (See [Lan13, p. 82] for the definition of neatness in an adelic group following Pink.) We write $\mathrm{Sh}_K = \mathrm{Sh}_K(G, X)$ for the canonical model over E, which forms a projective system of quasi-projective varieties with finite

étale transition maps as K varies. We have the E-scheme $\mathrm{Sh} := \varprojlim_K \mathrm{Sh}_K$. Put $d := \dim \mathrm{Sh}_K$ (which does not depend on K). Write $G(\mathbb{Q})_+$ for the preimage of $G(\mathbb{R})_+$ (defined in Section 2.4) in $G(\mathbb{Q})$. The closure of $G(\mathbb{Q})_+$ in $G(\mathbb{A}^\infty)$ is denoted by $G(\mathbb{Q})_+^-$.

Recall some facts about connected components from [Del79, 2.1]. We have a bijection

$$\pi_0(\mathrm{Sh}_{K,\overline{E}}) \xrightarrow{\sim} G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})_+K, \tag{5.1.1}$$

which yields a $G(\mathbb{A}^\infty)$-equivariant bijection $\pi_0(\mathrm{Sh}_{\overline{E}}) \xrightarrow{\sim} G(\mathbb{A})/G(\mathbb{Q})\varrho(G_{\mathrm{sc}}(\mathbb{A}))G(\mathbb{R})_+$ upon taking limit over all K. Note that $G(\mathbb{A})/G(\mathbb{Q})\varrho(G_{\mathrm{sc}}(\mathbb{A}))G(\mathbb{R})_+$ is an abelian group quotient of $G(\mathbb{A})$, and $G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})_+K$ is a finite abelian group quotient.

Fix a prime ℓ and a field isomorphism $\iota : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. When V is a $\overline{\mathbb{Q}}_\ell$-vector space, write $\iota V := V \otimes_{\overline{\mathbb{Q}}_\ell,\iota} \mathbb{C}$. By convention, all instances of cohomology in this paper are étale cohomology. The description of $\pi_0(\mathrm{Sh}_{\overline{E}})$ translates into a $G(\mathbb{A}^\infty)$-module isomorphism

$$\iota H^0(\mathrm{Sh}_{\overline{E}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_\pi \pi^\infty, \tag{5.1.2}$$

where the sum runs over one-dimensional automorphic representations π such that π_∞ is trivial when restricted to $G(\mathbb{R})_+$. Indeed, at each prime p, we have $\dim \pi_p = 1$ since π_p factors through $G(\mathbb{Q}_p) \to G(\mathbb{Q}_p)^\flat = G(\mathbb{Q}_p)/\varrho(G_{\mathrm{sc}}(\mathbb{Q}_p))$, cf. Corollary 2.3.3. Since one-dimensional automorphic representations have automorphic multiplicity one, there is no multiplicity factor in (5.1.2).

Now fix a prime $p \neq \ell$ and an open compact subgroup $K_p \subset G(\mathbb{Q}_p)$. By taking limit of (5.1.1) over neat open compact subgroups $K^p \subset G(\mathbb{A}^{\infty,p})$, writing $\mathrm{Sh}_{K_p} := \varprojlim_{K^p} \mathrm{Sh}_{K_pK^p}$,

$$\pi_0(\mathrm{Sh}_{K_p,\overline{E}}) \xrightarrow{\sim} G(\mathbb{Q})_+^-\backslash G(\mathbb{A}^\infty)/K_p. \tag{5.1.3}$$

We have a $G(\mathbb{A}^{\infty,p})$-module[(10)]

$$H^i(\mathrm{Sh}_{K_p,\overline{E}}, \overline{\mathbb{Q}}_\ell) = \varinjlim_{K^p} H^i(\mathrm{Sh}_{K_pK^p,\overline{E}}, \overline{\mathbb{Q}}_\ell), \qquad i \geqslant 0,$$

where K^p runs over sufficiently small open compact subgroups of $G(\mathbb{A}^{\infty,p})$.

LEMMA 5.1.1. — *There is a $G(\mathbb{A}^{\infty,p})$-module isomorphism*

$$\iota H^0(\mathrm{Sh}_{K_p,\overline{E}}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_\pi \pi^{\infty,p},$$

where the sum runs over discrete automorphic representations π of $G(\mathbb{A})$ such that (i) $\dim \pi = 1$, (ii) *π_p is trivial on K_p, and* (iii) *π_∞ is trivial on $G(\mathbb{R})_+$.*

Proof. — This is clear from (5.1.2) by taking K_p-invariants. □

(10) See [Sta21, Tag 03Q4] for the canonical isomorphism, which is $G(\mathbb{A}^{\infty,p})$-equivariant by a routine check. Alternatively, it is harmless to think of the identity as a definition for the left hand side.

5.2. INTEGRAL CANONICAL MODELS. — Let (G, X) be a Shimura datum of *Hodge type*. This means that there exists an embedding into the Siegel Shimura datum

$$i_{V,\psi} : (G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^{\pm}_{V,\psi}),$$

where (V, ψ) is a symplectic space over $\mathbb{Q}$, and $S^{\pm}_{V,\psi}$ denotes the associated Siegel half spaces. For simplicity we write $\mathrm{GSp} = \mathrm{GSp}(V, \psi)$ and $S^{\pm} = S^{\pm}_{V,\psi}$.

DEFINITION 5.2.1. — An *unramified Shimura datum* is a quadruple $(G, X, p, \mathcal{G})$, where (G, X) is a Shimura datum, p is a prime, and $\mathcal{G}$ is a reductive model for G over $\mathbb{Z}_{(p)}$. (In particular, G is unramified over $\mathbb{Q}_p$.) Write $\mathcal{SD}^{\mathrm{ur}}_{\mathrm{Hodge}}$ for the collection of unramified Shimura data whose underlying Shimura data are of Hodge type.

For the rest of this paper, we fix $(G, X, p, \mathcal{G}) \in \mathcal{SD}^{\mathrm{ur}}_{\mathrm{Hodge}}$ and $i_{V,\psi}$, thus also a hyperspecial subgroup $K_p := \mathcal{G}(\mathbb{Z}_p)$ of $G(\mathbb{Q}_p)$. Since G is unramified over $\mathbb{Q}_p$, the prime p is unramified in the reflex field E. We fix an isomorphism $\iota_p : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$, which induces an embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$ as well as a p-adic place $\mathfrak{p}$ of E. Thereby we identify $\overline{E}_{\mathfrak{p}} \simeq \overline{\mathbb{Q}}_p$. The integer ring $\mathcal{O}_E$ localized at $\mathfrak{p}$ is denoted by $\mathcal{O}_{E,(\mathfrak{p})}$, and its residue field by $k(\mathfrak{p})$. Identify the residue field of $\overline{\mathbb{Q}}_p$ with $\overline{\mathbb{F}}_p$, thus fixing an embedding $k(\mathfrak{p}) \hookrightarrow \overline{\mathbb{F}}_p$.

We follow [Kis17, (1.3.3)] to review integral canonical models for $\mathrm{Sh} = \mathrm{Sh}(G, X)$ over $\mathcal{O}_{E,(\mathfrak{p})}$, leaving the details to *loc. cit.* We may assume that $i_{V,\psi}$ is induced by an embedding $\mathcal{G} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_{(p)}})$ for a $\mathbb{Z}_{(p)}$-lattice $V_{\mathbb{Z}_{(p)}} \subset V$ and that ψ induces a perfect pairing on $V_{\mathbb{Z}_{(p)}}$. There exists a finite set of tensors $(s_\alpha) \subset V^{\otimes}_{\mathbb{Z}_{(p)}}$ such that $\mathcal{G}$ is the scheme-theoretic stabilizer of (s_α) in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$. We may assume that one of the tensors is given by $\psi \otimes \psi^\vee \in (V^\vee_{\mathbb{Z}_{(p)}})^{\otimes 2} \otimes V^{\otimes 2}_{\mathbb{Z}_{(p)}}$, whose stabilizer is $\mathrm{GSp}(V_{\mathbb{Z}_{(p)}}, \psi)$.[(11)] We fix the set (s_α). There is a hyperspecial subgroup $K'_p \subset \mathrm{GSp}(V, \psi)(\mathbb{Q}_p)$ extending K_p (i.e., $K'_p \cap G(\mathbb{Q}_p) = K_p$) such that $i_{V,\psi}$ induces an E-embedding of Shimura varieties

$$\mathrm{Sh}_{K_p}(G, X) \hookrightarrow \mathrm{Sh}_{K'_p}(\mathrm{GSp}, S^{\pm}) \otimes_{\mathbb{Q}} E. \tag{5.2.1}$$

Kisin [Kis10, Th. 2.3.8] (for $p > 2$) and Kim–Madapusi Pera [KMP16, Th. 4.11] (for $p = 2$) constructed integral canonical models, as a projective system of smooth quasi-projective schemes $\mathscr{S}_{K_p K^p}$ over $\mathcal{O}_{E,(\mathfrak{p})}$ for all sufficiently small open compact subgroups $K^p \subset G(\mathbb{A}^{\infty,p})$ with finite étale transition maps $\mathscr{S}_{K_p K^{p,\prime}} \to \mathscr{S}_{K_p K^p}$ for $K^{p,\prime} \subset K^p$. The projective system is equipped with an action of $G(\mathbb{A}^{\infty,p})$, given by the isomorphism

$$\mathscr{S}_{K_p K^p} \xrightarrow{\sim} \mathscr{S}_{K_p g^{-1} K^p g}, \qquad g \in G(\mathbb{A}^{\infty,p}), \quad K^p \subset G(\mathbb{A}^{\infty,p}),$$

extending the isomorphism $\mathrm{Sh}_{K_p K^p} \xrightarrow{\sim} \mathrm{Sh}_{K_p g^{-1} K^p g}$ giving the action of g on the generic fiber. The inverse limit $\mathscr{S}_{K_p} := \varprojlim_{K^p} \mathscr{S}_{K_p K^p}$ is a scheme over $\mathcal{O}_{E,(\mathfrak{p})}$ with

(11) This way the weak polarization in the sense of [Kis17] is remembered by (s_α). So we need not keep track of polarizations on abelian varieties separately.

a $G(\mathbb{A}^{\infty,p})$-action, uniquely characterized by an extension property [Kis10, Th. 2.3.8]. The construction yields a map of $\mathcal{O}_{E,(\mathfrak{p})}$-schemes

$$\mathscr{S}_{K_p} \longrightarrow \mathscr{S}_{K_p'}(\mathrm{GSp}, S^{\pm}) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E,(\mathfrak{p})}, \tag{5.2.2}$$

whose base change to E is identified with (5.2.1), where $\mathscr{S}_{K_p'}(\mathrm{GSp}, S^{\pm})$ is the integral model over $\mathbb{Z}_{(p)}$ for $\mathrm{Sh}(\mathrm{GSp}(V,\psi), S^{\pm}_{V,\psi})$ parametrizing polarized abelian schemes up to prime-to-p isogenies with prime-to-p level structure, as in [Kis10, (2.3.3)]. By pulling back the universal polarized abelian scheme over the Siegel Shimura varieties, we obtain polarized abelian schemes $h : \mathcal{A}_{K_pK^p} \to \mathscr{S}_{K_pK^p}$ compatible with the transition maps in the projective system.

Let $\mathscr{S}_{K_pK^p,k(\mathfrak{p})} := \mathscr{S}_{K_pK^p} \otimes_{\mathcal{O}_{E,(\mathfrak{p})}} k(\mathfrak{p})$ denote the special fiber. Write Sh_{K_p} (resp. $\mathscr{S}_{K_p,k(\mathfrak{p})}$) for the inverse limit of $\mathrm{Sh}_{K_pK^p}$ (resp. $\mathscr{S}_{K_pK^p,k(\mathfrak{p})}$) over K^p. By base change to $\overline{E}_{\mathfrak{p}}$, $\mathcal{O}_{\overline{E}_{\mathfrak{p}}}$, and $\overline{k(\mathfrak{p})}$, respectively, we obtain $\mathrm{Sh}_{K_p,\overline{E}_{\mathfrak{p}}}$, $\mathscr{S}_{K_p,\mathcal{O}_{\overline{E}_{\mathfrak{p}}}}$, and $\mathscr{S}_{K_p,\overline{k(\mathfrak{p})}}$ from Sh_{K_p}, $\mathscr{S}_{K_p}$, and $\mathscr{S}_{K_p,k(\mathfrak{p})}$. There are canonical $G(\mathbb{A}^{\infty,p})$-equivariant embeddings of generic and special fibers

$$\mathrm{Sh}_{K_p,\overline{E}_{\mathfrak{p}}} \hookrightarrow \mathscr{S}_{K_p,\mathcal{O}_{\overline{E}_{\mathfrak{p}}}} \hookleftarrow \mathscr{S}_{K_p,\overline{k(\mathfrak{p})}}.$$

These embeddings induce $G(\mathbb{A}^{\infty,p})$-equivariant bijections by means of arithmetic compactification as implied by [MP19, Cor. 4.1.11]:

$$\pi_0(\mathrm{Sh}_{K_p,\overline{E}_{\mathfrak{p}}}) \xrightarrow{\sim} \pi_0(\mathscr{S}_{K_p,\mathcal{O}_{\overline{E}_{\mathfrak{p}}}}) \xleftarrow{\sim} \pi_0(\mathscr{S}_{K_p,\overline{k(\mathfrak{p})}}).$$

Lemma 5.2.2. — *The $G(\mathbb{A}^{\infty,p})$-action is transitive on $\pi_0(\mathrm{Sh}_{K_p,\overline{E}_{\mathfrak{p}}})$ and $\pi_0(\mathscr{S}_{K_p,\overline{k(\mathfrak{p})}})$.*

Remark 5.2.3. — Oki [Oki23] showed that the analogous transitivity is false if $G_{\mathbb{Q}_p}$ is ramified.

Proof. — By the bijections above the lemma, it is enough to check the transitivity on $\pi_0(\mathrm{Sh}_{K_p,\overline{E}_{\mathfrak{p}}})$, which is [Kis10, Lem. 2.2.5] (applicable since K_p is hyperspecial). Alternatively, this also follows from weak approximation, which tells us that the diagonal embedding $G(\mathbb{Q}) \hookrightarrow G(\mathbb{Q}_p) \times G(\mathbb{R})$ has dense image. For this, apply [PR94, Th. 7.7] and notice that the set S_0 of the theorem can be taken away from p and ∞ from the discussion in §7.3 of loc. cit. since G is unramified at p. (In the argument on p. 421 of [PR94] the torus T can be chosen to be unramified by examining the proof of Prop. 2.10 in loc. cit. Thus it suffices to check that the conclusion of Prop. 7.10 therein holds for $K = \mathbb{Q}$ and $S = \{p, \infty\}$ when the diagonalizable group F in that proposition is unramified at p. This follows immediately from Cor. 2 on p. 418.) □

Let T be a $k(\mathfrak{p})$-scheme. At each point $x \in \mathscr{S}_{K_pK^p}(T)$ we have an abelian variety $\mathcal{A}_x$ over T (up to a prime-to-p isogeny) pulled back from $\mathcal{A}_{K_pK^p}$. As in [Kis17, (1.3.6)] and [KSZ, §5.1.5, Rem. 5.1.6], we have $(\mathbf{s}_{\alpha,\ell}) \subset (R^1 h_{\text{ét}*}\mathbb{Q}_\ell)^{\otimes}$ for each prime $\ell \neq p$ and also adelically away from p, ∞. By pullback, we equip the prime-to-p rational Tate module $V^p(\mathcal{A}_x)$ of $\mathcal{A}_x$ with $(\mathbf{s}_{\alpha,\ell,x})_{\ell \neq p}$.

When $T = \operatorname{Spec} k$ with $k/k(\mathfrak{p})$ an extension in $\overline{k(\mathfrak{p})}$, write $\mathbb{D}(\mathcal{A}_x[p^\infty])$ for the (integral) Dieudonné module of $\mathcal{A}_x[p^\infty]$, and Φ_x for the Frobenius operator acting on it.

Following [Kis17, (1.3.10)] we have crystalline Tate tensors $(s_{\alpha,0,x}) \subset \mathbb{D}(\mathcal{A}_x[p^\infty])^\otimes$ coming from (s_α). Lovering [Lov17], and also Hamacher [Ham19, §2.2], have globalized $(s_{\alpha,0,x})$. Namely there exist crystalline Tate tensors $(\mathbf{s}_{\alpha,0})$ on the Dieudonné crystal $\mathbb{D}(\mathcal{A}_{K_pK^p}[p^\infty])$ associated with $\mathcal{A}_{K_pK^p}[p^\infty]$ over $\mathscr{S}_{K_pK^p,\overline{k(\mathfrak{p})}}$ such that $(\mathbf{s}_{\alpha,0})$ specializes to $(s_{\alpha,0,x})$ at every $x \in \mathscr{S}_{K_pK^p}(\overline{k(\mathfrak{p})})$.

5.3. CENTRAL LEAVES. — Continuing from Section 5.2, we review central leaves in the special fiber of a Shimura variety of Hodge type. Let $B(G_{\mathbb{Q}_p})$ denote the set of $(G(\breve{\mathbb{Q}}_p), \sigma)$-conjugacy classes in $G(\breve{\mathbb{Q}}_p)$. Fix a Borel subgroup $B \subset \mathcal{G}_{\mathbb{Z}_p}$ and a maximal torus $T \subset B$ over $\mathbb{Z}_p$. We have the set of dominant coweights $X_*(T_{\overline{\mathbb{Q}}_p})^+$ and $X_*(T_{\overline{\mathbb{Q}}_p})^+_{\mathbb{Q}}$. Via the fixed isomorphism $\iota_p : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$, we obtain $T_{\mathbb{C}} \subset B_{\mathbb{C}} \subset G_{\mathbb{C}}$ as well as $X_*(T_{\mathbb{C}})^+$ and $X_*(T_{\mathbb{C}})^+_{\mathbb{Q}}$. Since the conjugacy class $\{\mu_X\}$ is defined over E and since $G_{\mathbb{Q}_p}$ is quasi-split, we have a cocharacter

$$\mu_p \in X_*(T_{\overline{\mathbb{Q}}_p})^+ \quad \text{defined over} \quad E_{\mathfrak{p}}.$$

in the conjugacy class $\{\iota_p\mu_X\}$. When there is no danger of confusion, we omit the subscripts $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$. Write $\rho \in X^*(T)_{\mathbb{Q}}$ for the half sum of all positive roots, and $\langle \cdot, \cdot \rangle$ for the canonical pairing $X^*(T)_{\mathbb{Q}} \times X_*(T)_{\mathbb{Q}} \to \mathbb{Q}$ or its extension to $\mathbb{C}$-coefficients.

Each $b \in G(\breve{\mathbb{Q}}_p)$ gives rise to a Newton cocharacter $\nu_b \colon \mathbb{D} \to G_{\breve{\mathbb{Q}}_p}$ (so it is a "fractional" cocharacter of $G_{\breve{\mathbb{Q}}_p}$) and a connected reductive group J_b over $\mathbb{Q}_p$ given by

$$J_b(R) := \{g \in G(R \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p) : g^{-1}b\sigma(g) = b\}, \quad R : \mathbb{Q}_p\text{-algebra}. \tag{5.3.1}$$

Recall that A_{J_b} denotes the maximal $\mathbb{Q}_p$-split torus in the center of J_b.

LEMMA 5.3.1. — *The Newton cocharacter ν_b factors through the center of J_b. The induced cocharacter $\mathbb{D} \to A_{J_b}$ is $\mathbb{Q}_p$-rational.*

Proof. — The centrality follows from [Kot85, (4.4.2)]. The cocharacter $\mathbb{D} \to A_{J_b}$ is σ-invariant by the definition of J_b, thus $\mathbb{Q}_p$-rational. □

We define an open compact subgroup of $J_b(\mathbb{Q}_p)$ (where "int" stands for integral):

$$J_b^{\mathrm{int}} := J_b(\mathbb{Q}_p) \cap \mathcal{G}(\breve{\mathbb{Z}}_p) = \{g \in \mathcal{G}(\breve{\mathbb{Z}}_p) : g^{-1}b\sigma(g) = b\}.$$

Given $b \in G(\breve{\mathbb{Q}}_p)$, we denote its $(G(\breve{\mathbb{Q}}_p), \sigma)$-conjugacy class by $[b]$ and $(\mathcal{G}(\breve{\mathbb{Z}}_p), \sigma)$-conjugacy class by $[[b]]$. Recall that $b \in G(\breve{\mathbb{Q}}_p)$, or $[b] \in B(G_{\mathbb{Q}_p})$, is *basic* if $\nu_b : \mathbb{D} \to G_{\breve{\mathbb{Q}}_p}$ has image in $Z(G_{\breve{\mathbb{Q}}_p})$, or equivalently if J_b is an inner form of G [RR96, Prop. 1.12]. The following condition will appear in our irreducibility results later. The definition makes a difference only when G^{ad} is not $\mathbb{Q}$-simple. See Lemma 5.3.7 below for a relation to Section 2.5.

DEFINITION 5.3.2. — Let $G^{\mathrm{ad}} = \prod_{i \in I} G_i^{\mathrm{ad}}$ be a decomposition into $\mathbb{Q}$-simple factors. An element $b \in G(\breve{\mathbb{Q}}_p)$, or $[b] \in B(G_{\mathbb{Q}_p})$, is said to be $\mathbb{Q}$-*non-basic* if its image in $B(G_{i,\mathbb{Q}_p})$ via the natural composite map $G \to G^{\mathrm{ad}} \to G_i$ is non-basic for every $i \in I$.

REMARK 5.3.3. — The definition is not purely local in that it depends on not only $G_{\mathbb{Q}_p}$ but also G. Compare $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ with $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, where F is a real quadratic field in which p splits.

Let $x \in \mathscr{S}_{K_pK^p,k(\mathfrak{p})}$ be a closed point. Then there exists a sufficiently divisible $r \in \mathbb{Z}_{\geqslant 1}$ such that $\mathbb{F}_{p^r} \supset k(\mathfrak{p})$ and there exists a $\mathbb{Z}_{p^r}$-linear isomorphism (cf. [Kis17, (1.4.1)])

$$V^*_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{p^r} \simeq \mathbb{D}(\mathcal{A}_x[p^\infty]) \otimes_{W(k(\mathfrak{p}))} \mathbb{Z}_{p^r} \tag{5.3.2}$$

carrying (s_α) to $(s_{\alpha,0,x})$. The Frobenius operator Φ_x on the right hand side is transported to an σ-semilinear operator on the left hand side of the form $b_x(1 \otimes \sigma)$ for a unique $b_x \in G(\mathbb{Q}_{p^r})$. Then $[[b_x]]$ (thus also $[b_x]$) is independent of the choice of r and the isomorphism. Now let $\overline{x} : \operatorname{Spec} \overline{\mathbb{F}}_p \to \mathscr{S}_{K_pK^p,k(\mathfrak{p})}$ be a geometric point supported at $x \in \mathscr{S}_{K_pK^p,k(\mathfrak{p})}$. Then we can define $b_{\overline{x}}$, $[b_{\overline{x}}]$, and $[[b_{\overline{x}}]]$ similarly.

Write $|\mathscr{S}_{K_pK^p,k(\mathfrak{p})}|$ for the set of closed points on $\mathscr{S}_{K_pK^p,k(\mathfrak{p})}$. As a subset of $|\mathscr{S}_{K_pK^p,k(\mathfrak{p})}|$, the central leaf associated with b is defined as

$$C_{b,K^p} := \{x \in |\mathscr{S}_{K_pK^p,k(\mathfrak{p})}| \,:\, [[b_x]] = [[b]]\},$$

Clearly the definition depends only on $[[b]]$. By [HK19, Cor. 4.12], C_{b,K^p} is a locally closed subset of $|\mathscr{S}_{K_pK^p,k(\mathfrak{p})}|$. (The result is stated for $\overline{\mathbb{F}}_p$-points there, but the same proof applies to the underlying set of closed points.) We promote C_{b,K^p} to a locally closed $k(\mathfrak{p})$-subscheme of $\mathscr{S}_{K_pK^p,k(\mathfrak{p})}$ equipped with reduced subscheme structure. We still write C_{b,K^p} for the scheme and call it the *central leaf* associated with b. As K^p varies, the transition maps for $\mathscr{S}_{K_pK^p,k(\mathfrak{p})}$, which are finite étale (Kis10, Th. 2.3.8]), induce finite étale transition maps between C_{b,K^p}. Put $C_b := \varprojlim_{K^p} C_{b,K^p}$. We say either C_{b,K^p} or C_b is $\mathbb{Q}$-non-basic if b is $\mathbb{Q}$-non-basic.

PROPOSITION 5.3.4. — *The $k(\mathfrak{p})$-scheme C_{b,K^p} is smooth. If nonempty, its dimension is $\langle 2\rho, \nu_b \rangle$.*

Proof. — These properties can be checked after extending base to $\overline{k(\mathfrak{p})}$. Since C_{b,K^p} is reduced, it is still reduced over $\overline{k}(\mathfrak{p})$. Thus the proposition follows from [Ham19, Prop. 2.6]. □

A finite subset $B(G_{\mathbb{Q}_p}, \mu_p^{-1}) \subset B(G_{\mathbb{Q}_p})$ is defined in [Kot97, §6] by a group-theoretic generalization of Mazur's inequality. The set $B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ contains exactly one basic element, but may contain several elements that are not $\mathbb{Q}$-non-basic. Set $\mathbb{Z}_p^{\mathrm{ur}} := \cup_{r \geqslant 1} \mathbb{Z}_{p^r}$ as a subring of $\breve{\mathbb{Q}}_p$.

PROPOSITION 5.3.5. — *The central leaf C_{b,K^p} is nonempty if and only if the $(\mathcal{G}(\breve{\mathbb{Z}}_p), \sigma)$-conjugacy class $[[b]]$ intersects $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\sigma\mu_p^{-1}(p)\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ nontrivially.*

Proof. — By [KMPS22, Prop. 1.3.9], the Newton stratum for b is nonempty if and only if $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$. On the other hand, if $b \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\sigma\mu_p^{-1}(p)\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ then $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ by [RZ96, §4].

To prove the "only if" part of the proposition, we assume $C_{b,K^p} \neq \varnothing$. Then $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ by the preceding paragraph. Since C_{b,K^p} is of finite type over $k(\mathfrak{p})$, a closed point $x \in C_{b,K^p}$ has finite residue field, and there exists an isomorphism (5.3.2) for some r. Then $[[b_x]] = [[b]]$, and $b_x \in \mathcal{G}(\mathbb{Z}_{p^r})\sigma\mu_p^{-1}(p)\mathcal{G}(\mathbb{Z}_{p^r})$ by [Kis17, 1.4.1].

In the "if" direction, the condition on b implies that $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$, so $N_{b,K^p} \neq \varnothing$ for neat subgroups K^p. Pick a closed point $x \in N_{b,K^p}$. Then b_x lies in the double coset $\mathcal{G}(\mathbb{Z}_{p^r})\sigma\mu_p^{-1}(p)\mathcal{G}(\mathbb{Z}_{p^r})$ by [Kis17, 1.4.1], and $[b_x] = [b]$. Writing $b = g^{-1}b_x\sigma(g)$ for some $g \in G(\breve{\mathbb{Q}}_p)$, we see that g lies in the affine Deligne–Lusztig variety for $(b_x, \sigma\mu_p^{-1})$. Using x as a base point, we can apply the p-power isogeny corresponding to g to find a closed point $y \in N_{b,K^p}$, thanks to [Kis17, Prop. 1.4.4]. By construction $[[b_y]] = [[b]]$, so C_{b,K^p} is nonempty as desired. □

For $r \in \mathbb{Z}_{\geqslant 1}$ define a subset $G(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}} \subset G(\mathbb{Q}_p^{\mathrm{ur}})$ consisting of b such that

(br1) $b \in \mathcal{G}(\mathbb{Z}_{p^r})\sigma\mu_p^{-1}(p)\mathcal{G}(\mathbb{Z}_{p^r})$,
(br2) r is divisible by $[E_{\mathfrak{p}} : \mathbb{Q}_p]$ (equivalently $\mathbb{F}_{p^r} \supset k(\mathfrak{p})$),
(br3) $r\nu_b : \mathbb{G}_m \to G_{\breve{\mathbb{Q}}_p}$ is a cocharacter (not just a fractional cocharacter).

Clearly $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\sigma\mu_p^{-1}(p)\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ is the union of $G(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}}$ over all r, and $G(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}}$ is contained in $G(\mathbb{Q}_p^{\mathrm{ur}})_{r'\text{-good}}$ if r divides r'. Proposition 5.3.5 tells us that C_{b,K^p} is nonempty if and only if b belongs to $G(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}}$ up to $(\mathcal{G}(\breve{\mathbb{Z}}_p), \sigma)$-conjugacy for a sufficiently divisible r, where r can be chosen independently of K^p. For the purpose of studying central leaves, we may and will always assume from now that

$$b \in G(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}} \quad \text{for a sufficiently divisible } r.$$

Conditions (br1)–(br3) imply the following.

(br1)' $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ and ν_b is defined over $\mathbb{Q}_{p^r}$, by (br1).
(br2)' μ_p is defined over $\mathbb{Q}_{p^r}$, by (br2).

Since μ_p is defined over $E_{\mathfrak{p}}$, which is unramified over $\mathbb{Q}_p$, (br2)' is easy to see. In (br1)', $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ comes from [RR96, Th. 4.2]. we already explained above that ν_b is defined over $\mathbb{Q}_{p^r}$ if $b \in G(\mathbb{Q}_{p^r})$. Since $b_x \in G(\mathbb{Q}_{p^r})$, [Kot85, (4.4.1)] tells us that ν_{b_x} is defined over $\mathbb{Q}_{p^r}$.

Since the $G(\mathbb{Q}_{p^r})$-conjugacy class of $r\nu_b$ is defined over $\mathbb{Q}_p$ [Kot85, (4.4.3)], but $r\nu_b$ itself need not be defined over $\mathbb{Q}_p$. To apply harmonic analysis results of Sections 3 and 4 let us introduce a σ-conjugate element b° such that $r\nu_{b^\circ}$ is a cocharacter over $\mathbb{Q}_p$. Since $G_{\mathbb{Q}_p}$ is quasi-split, there exists $h \in G(\mathbb{Q}_{p^r})$ such that $h^{-1}(r\nu_b)h$ is defined over $\mathbb{Q}_p$. Multiplying h on the right by an element of $G(\mathbb{Q}_p)$, we can ensure that $h^{-1}(r\nu_b)h$ factors through $\mathbb{G}_m \to T$ and is B-dominant, namely $h^{-1}(r\nu_b)h \in X_*(T)^+$. Fix such a h and put $b^\circ := h^{-1}b\sigma(h)$ so that $\nu_{b^\circ} = h^{-1}(\nu_b)h$ from [Kot85, (4.4.2)]. We also have a $\mathbb{Q}_p$-isomorphism

$$J_b \xrightarrow{\sim} J_{b^\circ}, \qquad g \longmapsto h^{-1}gh$$

determined by h, which carries $r\nu_b$ to $r\nu_{b^\circ}$.

Starting from $\nu_{b^\circ} \in X_*(T)^+_{\mathbb{Q}}$ defined over $\mathbb{Q}_p$ as above, we put $P_{b^\circ} := P_{\nu_{b^\circ}}$ in the notation of Section 3.1, and similarly define $P^{\mathrm{op}}_{b^\circ}$, N_{b°, $N^{\mathrm{op}}_{b^\circ}$, and M_{b°. In particular, $P^{\mathrm{op}}_{b^\circ}$ (resp. M_{b°) is a standard $\mathbb{Q}_p$-rational parabolic (resp. Levi) subgroup of $G_{\mathbb{Q}_p}$, and M_{b° is the centralizer of ν_{b° in $G_{\mathbb{Q}_p}$. There is an inner twist [RZ96, Cor. 1.14]

$$J_{b^\circ} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \simeq M_{b^\circ} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \tag{5.3.3}$$

given by the cocycle $\mathrm{Gal}(\mathbb{Q}_{p^n}/\mathbb{Q}_p) \to M_{b^\circ}(\mathbb{Q}_{p^r})$, $\sigma \mapsto b^\circ$. Thus M_{b° is also an inner twist of J_b over $\mathbb{Q}_p$ (which is independent of the choice of b° up to isomorphism of inner twists by routine check). Under the canonical $\mathbb{Q}_p$-isomorphisms $Z(M_{b^\circ}) \simeq Z(J_b)$ and $A_{M_{b^\circ}} = A_{J_b}$, it is readily checked that ν_{b° is carried to ν_b.

Example 5.3.6. — We have the following for the ordinary strata of modular curves, when $G_{\mathbb{Q}_p} = \mathrm{GL}_2$. Take B and T to the subgroup of upper triangular (resp. diagonal) matrices. Then μ is the cocharacter $z \mapsto \mathrm{diag}(z, 1)$ up to conjugacy. We can take $b = b^\circ$ such that $\nu_b(z) = \mathrm{diag}(1, z^{-1})$, which is visibly B-dominant. Then $P^{\mathrm{op}}_b = B = P_{-\nu_b}$, $M_b = T$, and $\delta_{P_b}(\nu_b(p)) = |p^{-1}| = p$.

Lemma 5.3.7. — *The element $b \in G(\breve{\mathbb{Q}}_p)$ as above is $\mathbb{Q}$-non-basic if and only if ($\mathbb{Q}$-nb(P_b)) of Section 2.5 holds.*

Proof. — Write $G^{\mathrm{ad}} = \prod_{i \in I} G^{\mathrm{ad}}_i$ as in Definition 5.3.2 and $b_i \in G^{\mathrm{ad}}_i(\breve{\mathbb{Q}}_p)$ for the image of b. By functoriality of Newton cocharacters, the composition of ν_b with the natural map $G \to G^{\mathrm{ad}}_i$ is ν_{b_i}, which is $\mathbb{Q}_p$-rational since ν_b is. This implies that the image of P_b in G^{ad}_i is P_{b_i}, where $P_{b_i} \subset G^{\mathrm{ad}}_i$ is defined analogously as P_b in G over $\mathbb{Q}_p$. Each $b_i \in G^{\mathrm{ad}}_i(\breve{\mathbb{Q}}_p)$ is basic if and only if ν_{b_i} is central in G^{ad}_i (i.e. trivial) if and only if $P_{b_i} = G^{\mathrm{ad}}_i$. Therefore, ($\mathbb{Q}$-nb($P_b$)) holds if and only if b_i is non-basic for every $i \in I$, and the latter is the definition for b to be $\mathbb{Q}$-non-basic. □

Let $1 \to Z_1 \to G_1 \to G \to 1$ be a z-extension over $\mathbb{Q}_p$ that is unramified over $\mathbb{Q}_p$. Let $\mu_{p,1} : \mathbb{G}_m \to G_{1,\mathbb{Q}_{p^r}}$ be a cocharacter lifting $\mu_p : \mathbb{G}_m \to G_{\mathbb{Q}_{p^r}}$. (Such a $\mu_{p,1}$ always exists since Z_1 is connected, but we will make a choice of $\mu_{p,1}$ coming from a lift of Shimura data, cf. Section 7.2 below.)

Lemma 5.3.8. — *Assume that $b \in G(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}}$. Then there exists an element $b_1 \in G_1(\mathbb{Q}_{p^r})$ lifting b, as well as an element $b_1^\circ \in G_1(\mathbb{Q}_{p^r})$ in the σ-conjugacy class of b_1, such that*

– the analogues of (br1), (br1)', and (br2)' hold true with G, μ_p, b replaced by $G_1, \mu_{p,1}, b_1$,

– ν_{b° is defined over $\mathbb{Q}_p$ and lifts ν_b°.

Moreover, we can make r more divisible (without changing μ_p, b, $\mu_{p,1}$, b_1, b_1°) such that $r\nu_{b_1}$ is a cocharacter of G_1.[12] *(So $b_1 \in G_1(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}}$ for the new r.)*

[12] A priori we only know that $r\nu_{b_1}$ is a fractional cocharacter, even though $r\nu_b$ is an (integral) cocharacter of G.

Proof. — Since $G_1(\mathbb{Z}_{p^r}) \to G(\mathbb{Z}_{p^r})$ is onto (by the surjectivity on $\mathbb{F}_{p^r}$-points and the smoothness of $G_1 \to G$), the map $G_1(\mathbb{Q}_{p^r}) \to G(\mathbb{Q}_{p^r})$ induces a surjection

$$G_1(\mathbb{Z}_{p^r})\sigma\mu_{p,1}^{-1}(p)G_1(\mathbb{Z}_{p^r}) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{G}(\mathbb{Z}_{p^r})\sigma\mu_p^{-1}(p)\mathcal{G}(\mathbb{Z}_{p^r}).$$

Take $b_1 \in G_1(\mathbb{Q}_{p^r})$ to be any preimage of b under this map. This takes care of the first bullet point. As for the second point, since G_1 is quasi-split over $\mathbb{Q}_p$, there exists $b_1^\circ \in G_1(\mathbb{Q}_{p^r})$ σ-conjugate to b_1 such that $\nu_{b_1^\circ}$ is defined over $\mathbb{Q}_p$, and also such that $\nu_{b_1^\circ}$ factors through $T_1 \subset G_1$, where T_1 is the preimage of T. Then the composite of $\nu_{b_1^\circ}$ with $G_1 \twoheadrightarrow G$ is conjugate to ν_{b° in G, so differs from ν_{b° by an element of the $\mathbb{Q}_p$-rational Weyl group of G [Kot84a, Lem. 1.1.3 (a)]. Identifying the latter with the $\mathbb{Q}_p$-rational Weyl group of G_1, we can use the same element to modify $\nu_{b_1^\circ}$ so that $\nu_{b_1^\circ}$ maps to ν_{b° under $G_1 \twoheadrightarrow G$. Finally, the last point on r in the lemma is obvious. □

In the setting of the lemma, we introduce $\mathbb{Q}_p$-algebraic groups J_{b_1}, $J_{b_1^\circ}$, $P_{b_1^\circ}$, $M_{b_1^\circ}$, etc. for G_1 by mimicking the definition for G. Let T_1, B_1 denote the preimages of T, B in G_1. Since $\nu_{b_1^\circ}$ maps to ν_{b°, it is clear that $\nu_{b_1}^\circ \in X_*(T_1)^+$, where $+$ means B_1-dominance, and that $P_{b_1^\circ}$, $M_{b_1^\circ}$ map to P_{b°, M_{b°. As before, we can identify $Z(M_{b_1^\circ}) = Z(J_{b_1^\circ})$, which carries $\nu_{b_1^\circ}$ to ν_{b_1}. The point of the discussion about b° and b_1° is that it is usually harmless to work with b° and b_1° in place of b and b_1 regarding harmonic analysis questions. With this understanding, we will abuse notation to write $M_b, P_b, M_{b_1}, P_{b_1}$ etc. for $M_{b^\circ}, P_{b^\circ}, M_{b_1^\circ}, P_{b_1^\circ}$ etc. to simplify notation, and write ν_b, ν_{b_1} for $\nu_{b^\circ}, \nu_{b_1^\circ}$ if there is little danger of confusion.

6. Igusa varieties

Here we state the main theorem on H^0 of Igusa varieties and carry out the initial reduction to the completely slope divisible case, where we have a tower of finite-type Igusa varieties over a fixed finite field. This prepares us to apply a fixed-point formula in the next section.

6.1. Infinite-level Igusa varieties. — We continue in the setting of Section 5.3, with b an element of $G(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}}$. Let $b' \in \mathrm{GSp}(\breve{\mathbb{Q}}_p)$ denote the image of b. By Dieudonné theory, we have a polarized p-divisible group $\Sigma_{b'}$ over $\mathbb{F}_{p^r}$ such that $\mathbb{D}(\Sigma_{b'}) = V^*_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{p^r}$ with Frobenius operator $b'(1 \otimes \sigma)$. By Σ_b we mean the p-divisible group $\Sigma_{b'}$ equipped with crystalline Tate tensors (t_α) on $\mathbb{D}(\Sigma_{b'})$ corresponding to (s_α) on $V_{\mathbb{Z}_{(p)}}$. When there is no danger of confusion, we still write Σ_b and $\Sigma_{b'}$ for their base changes to $\overline{\mathbb{F}}_p$.

Applying the construction of Section 5.3 to $\mathscr{S}_{K_p'K'^{,p}}(\mathrm{GSp}, S^\pm)$ and b', we obtain a central leaf $C_{b',K'^{,p}} \subset \mathscr{S}_{K_p'K'^{,p}}(\mathrm{GSp}, S^\pm)$. Let R be an $\overline{\mathbb{F}}_p$-algebra. Following [CS17, §4.3] we have the Igusa variety $\mathfrak{Ig}_{b',K'^{,p}} \to C_{b',K'^{,p},\overline{\mathbb{F}}_p}$ whose R-points parametrize isomorphisms

$$\Sigma_{b'} \times_{\overline{\mathbb{F}}_p} R \simeq \mathcal{A}_R[p^\infty] \tag{6.1.1}$$

compatible with polarizations up to $\mathbb{Z}_p^\times$-multiples, where $\mathcal{A}_R$ denotes the pullback of the universal abelian scheme via $\operatorname{Spec} R \to C_{b',K'^p\overline{\mathbb{F}}_p}$. Then $\mathfrak{Ig}_{b',K'^p}$ is a perfect scheme, which is an $\underline{\mathrm{Aut}}(\Sigma_{b'})$-torsor over $C_{b',K'^p\overline{\mathbb{F}}_p}$ by [CS19, Cor. 2.3.2], where $\underline{\mathrm{Aut}}(\Sigma_{b'})$ denotes the group scheme of automorphisms of $\Sigma_{b'}$ (preserving the polarization up to $\mathbb{Z}_p^\times$-multiples).

The map $\mathscr{S}_{K_pK^p,\overline{\mathbb{F}}_p} \to \mathscr{S}_{K'_pK'^p,\overline{\mathbb{F}}_p}$ clearly induces a map $C_{b,K^p,\overline{\mathbb{F}}_p} \to C_{b',K'^p,\overline{\mathbb{F}}_p}$. We define the subscheme

$$\mathfrak{Ig}_{b,K^p} \subset (\mathfrak{Ig}_{b',K'^p} \times_{C_{b',K'^p,\overline{\mathbb{F}}_p}} C_{b,K^p,\overline{\mathbb{F}}_p})^{\mathrm{perf}} = \mathfrak{Ig}_{b',K'^p} \times_{C^{\mathrm{perf}}_{b',K'^p},\overline{\mathbb{F}}_p} C^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p} \tag{6.1.2}$$

to be the locus where (6.1.1) carries (s_α) to $(\mathbf{s}_{\alpha,0})$ on the Dieudonné modules. Composing with the projection maps, we have $\overline{\mathbb{F}}_p$-morphisms $\mathfrak{Ig}_{b,K^p} \to \mathfrak{Ig}_{b',K'^p}$ and $\mathfrak{Ig}_{b,K^p} \to C^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$. The latter gives rise to the composite map

$$\mathfrak{Ig}_{b,K^p} \longrightarrow C^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p} \longrightarrow C_{b,K^p,\overline{\mathbb{F}}_p} \longrightarrow \mathscr{S}_{K_pK^p,\overline{\mathbb{F}}_p}.$$

As K^p varies, the Hecke action of $G(\mathbb{A}^{\infty,p})$ on $\mathscr{S}_{K_pK^p,\overline{\mathbb{F}}_p}$ restricts to an action on $C_{b,K^p,\overline{\mathbb{F}}_p}$ and extends to an action on $\mathfrak{Ig}_{b,K^p}$ by [HK19, Lem. 6.4]. (The point is that the central leaves and Igusa varieties are defined in terms of p-adic invariants, which are preserved under the prime-to-p Hecke action.)

Lemma 6.1.1. — *The following are true.*

(1) *The $\overline{\mathbb{F}}_p$-scheme $\mathfrak{Ig}_{b,K^p}$ is perfect and a pro-étale J_b^{int}-torsor over $C^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$.*[13]

(2) *The map $\mathfrak{Ig}_{b,K^p} \to \mathfrak{Ig}_{b',K'^p}$ is a closed embedding, under which the $J_{b'}(\mathbb{Q}_p)$-action on $\mathfrak{Ig}_{b',K'^p}$ restricts to an action of $J_b(\mathbb{Q}_p)$ on $\mathfrak{Ig}_{b,K^p}$ (via the embedding $J_b(\mathbb{Q}_p) \hookrightarrow J_{b'}(\mathbb{Q}_p)$).*

Proof. — This follows from [Ham19, Prop. 4.1, 4.10], noting that our $\mathfrak{Ig}_{b,K^p}$ is his $\mathscr{J}_\infty^{(p^{-\infty})}$ (the perfection of his $\mathscr{J}_\infty$) and that our J_b^{int} is his Γ_b. Two points require some further explanation. Firstly, we see that $J_b^{\mathrm{int}} = \Gamma_b$ as follows. Observe that $J_b^{\mathrm{int}} \subset J_b(\mathbb{Q}_p) \subset J_{b'}(\mathbb{Q}_p)$ and $J_b^{\mathrm{int}} \subset G(\breve{\mathbb{Z}}_p) \subset \mathrm{GSp}(\breve{\mathbb{Z}}_p)$. Thus J_b^{int} consists of automorphisms of $\Sigma_{b'}$ which are exactly the stabilizers of (t_α) via Dieudonné theory. Secondly, [Ham19, Prop. 4.1] tells us that $\mathscr{J}_\infty \to C_{b,K^p,\overline{\mathbb{F}}_p}$ is a pro-étale J_b^{int}-torsor. Since every perfection map (as a limit of absolute Frobenius) is a universal homeomorphism, which preserves the pro-étale topology [BS15, Lem. 5.4.2], it follows that the perfection $\mathscr{J}_\infty^{(p^{-\infty})} \to C^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$ is also a pro-étale J_b^{int}-torsor. □

Lemma 6.1.2. — *Let R be a perfect $\overline{\mathbb{F}}_p$-algebra. Then $\mathfrak{Ig}_{b,K^p}(R)$ is identified with the set of equivalence classes of (x, j), where*

- *$x \in \mathscr{S}_{K_pK^p}(R)$ is an abelian scheme over $\operatorname{Spec} R$ and*
- *$j : \Sigma_b \times_{\overline{\mathbb{F}}_p} R \to \mathcal{A}_x[p^\infty]$ is a quasi-isogeny carrying (s_α) to $(\mathbf{s}_{\alpha,0,x})$,*

[13] It can be shown that $\mathfrak{Ig}_{b,K^p} \to C_{b,K^p}$ is an $\underline{\mathrm{Aut}}(\Sigma_b)$-torsor by [CS19, Cor. 2.3.2] and adapting the argument there, but we do not need it.

and $\mathcal{A}_x$ denotes the pullback of the universal abelian scheme along x. Here (x, j) and (x', j') are considered equivalent if, in the notation of Section 5.2*, there exists a p-power isogeny $i : \mathcal{A}_x \to \mathcal{A}_{x'}$ carrying $(\mathbf{s}_{\alpha,\ell,x})_{\ell\neq p}$ to $(\mathbf{s}_{\alpha,\ell,x'})_{\ell\neq p}$ and $(\mathbf{s}_{\alpha,0,x})$ to $(\mathbf{s}_{\alpha,0,x'})$ such that $i \circ j = j'$. Each $\rho \in J_b(\mathbb{Q}_p)$ acts on the R-points of $\mathfrak{Ig}_{b,K^p}$ by sending j to $j \circ \rho$.*[14]

Proof. — This is the Hodge-type analogue of [CS17, Lem. 4.3.4] proved in the PEL case. By loc. cit. $\mathfrak{Ig}_{b',K'^{,p}}(R)$ is the set of p-power isogeny classes of (A, j) with $A \in \mathscr{S}_{K'_pK'^{,p}}(R)$ and $j : \Sigma_b \times_{\overline{\mathbb{F}}_p} R \to A[p^\infty]$ a quasi-isogeny compatible with polarizations up to $\mathbb{Q}_p^\times$. Now we have a commutative diagram from the construction of central leaves and Igusa varieties:

$$\begin{array}{ccccc}
\mathfrak{Ig}_{b,K^p} & \longrightarrow & C_{b,K^p,\overline{\mathbb{F}}_p} & \xhookrightarrow{\text{loc. closed}} & \mathscr{S}_{K_pK^p,\overline{\mathbb{F}}_p} \\
\text{closed}\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathfrak{Ig}_{b',K'^{,p}} & \longrightarrow & C_{b',K'^{,p},\overline{\mathbb{F}}_p} & \xhookrightarrow{\text{loc. closed}} & \mathscr{S}_{K'_pK'^{,p},\overline{\mathbb{F}}_p}
\end{array}$$

Now we prove the first assertion by constructing the maps in both directions, which are easily seen to be inverses of each other. Given $y \in \mathfrak{Ig}_{b,K^p}(R)$, its image gives $x \in \mathscr{S}_{K_pK^p}(R)$. The j comes from the image of y in $\mathfrak{Ig}_{b',K'^{,p}}(R)$. The compatibility of j with crystalline Tate tensors follows from the very definition of $\mathfrak{Ig}_{b,K^p}$. Conversely, let (x, j) be as in the lemma. Modifying by a quasi-isogeny, we may assume that j is an isomorphism. Then (x, j) comes from a point $y' \in \mathfrak{Ig}_{b',K'^{,p}}(R)$ as observed above. Since $\operatorname{Spec} R$ and C_{b,K^p} are reduced, $x \in \mathscr{S}_{K_pK^p}(R)$ comes from a point in $x \in C_{b,K^p}(R)$. Then y' and x have the same image in $C_{b',K'^{,p},\overline{\mathbb{F}}_p}(R)$, so determine a point

$$y \in \big(\mathfrak{Ig}_{b',K'^{,p}} \times_{C_{b',K'^{,p},\overline{\mathbb{F}}_p}} C_{b,K^p,\overline{\mathbb{F}}_p}\big)^{\mathrm{perf}}(R) = \big(\mathfrak{Ig}_{b',K'^{,p}} \times_{C_{b',K'^{,p},\overline{\mathbb{F}}_p}} C_{b,K^p,\overline{\mathbb{F}}_p}\big)(R).$$

The compatibility of j with crystalline Tate tensors exactly tells us that $y \in \mathfrak{Ig}_{b,K^p}(R)$.

It remains to show the last assertion. In light of Lemma 6.1.1 (2), the assertion on the $J_b(\mathbb{Q}_p)$-action follows from the analogue description for $J_{b'}(\mathbb{Q}_p)$-action on $\mathfrak{Ig}_{b',K'^{,p}}$ as in [CS17, Lem. 4.3.4, Cor. 4.3.5]. □

Now we compare Igusa varieties arising from two central leaves C_b and C_{b_0} in the same Newton stratum. Thus we assume that $b, b_0 \in G(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}}$ for some r and that b is σ-conjugate to b_0 in $G(\breve{\mathbb{Q}}_p)$. We have an isomorphism $J_b(\mathbb{Q}_p) \simeq J_{b_0}(\mathbb{Q}_p)$ (induced by a conjugation in the ambient group $G(\breve{\mathbb{Q}}_p)$), canonical up to $J_b(\mathbb{Q}_p)$-conjugacy.

(14) We make a right action of $J_b(\mathbb{Q}_p)$ on $\mathfrak{Ig}_{b,K^p}$ so that it becomes a left action on the cohomology. In [CS17, §4.3], their arrow j is reverse to ours, from $A[p^\infty]$ to $\Sigma_b \times_{\overline{\mathbb{F}}_p} R$. The two conventions are identified via taking the inverse of j (with the understanding that the authors of loc. cit. are also using the right action of $J_b(\mathbb{Q}_p)$, though this does not appear there explicitly).

COROLLARY 6.1.3. — *There exists a $G(\mathbb{A}^{\infty,p})$-equivariant isomorphism*

$$\mathfrak{Ig}_b \xrightarrow{\sim} \mathfrak{Ig}_{b_0},$$

which is also equivariant for the actions of $J_b(\mathbb{Q}_p)$ and $J_{b_0}(\mathbb{Q}_p)$ through a suitable isomorphism $J_b(\mathbb{Q}_p) \simeq J_{b_0}(\mathbb{Q}_p)$ in its canonical $J_b(\mathbb{Q}_p)$-conjugacy class.

Proof. — Since $[b_x] = [b_{x_0}]$, there exists a quasi-isogeny $f : \Sigma_{b_0} \to \Sigma_b$ compatible with G-structures. Using the description of Lemma 6.1.2, we can give an isomorphism $\mathfrak{Ig}_b \xrightarrow{\sim} \mathfrak{Ig}_{b_0}$ on R-points by $(A, j) \mapsto (A, j \circ f)$. The equivariance property is straightforward. □

The $J_b(\mathbb{Q}_p)$-action on $\mathfrak{Ig}_{b,K^p}$ commutes with the Hecke action of $G(\mathbb{A}^{\infty,p})$ (as K^p varies) as it is clear on the moduli description. Now we would like to understand the $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-representation

$$H^i(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell) := \varinjlim_{K^p} H^i(\mathfrak{Ig}_{b,K^p}, \overline{\mathbb{Q}}_\ell), \qquad i \geqslant 0,$$

where the limit is over sufficiently small open compact subgroups of $G(\mathbb{A}^{\infty,p})$.

From Section 2.3 we obtain the following commutative diagram. Indeed, all maps and the commutativity are obvious except possibly the map $J_b(\mathbb{Q}_p)^{\mathrm{ab}} \twoheadrightarrow M_b(\mathbb{Q}_p)^{\mathrm{ab}}$, which comes from the proof of Corollary 2.3.3. (The latter also tells us that $M_b(\mathbb{Q}_p)^{\mathrm{ab}} = M_b(\mathbb{Q}_p)^\flat$ and $G(\mathbb{Q}_p)^{\mathrm{ab}} = G(\mathbb{Q}_p)^\flat$.)

(6.1.3)
$$\begin{array}{ccccc} J_b(\mathbb{Q}_p) & & M_b(\mathbb{Q}_p) & \longrightarrow & G(\mathbb{Q}_p) \\ \downarrow & & \downarrow & & \downarrow \\ J_b(\mathbb{Q}_p)^{\mathrm{ab}} & \twoheadrightarrow & M_b(\mathbb{Q}_p)^{\mathrm{ab}} & \longrightarrow & G(\mathbb{Q}_p)^{\mathrm{ab}}. \end{array}$$

The diagram yields the composite maps

(6.1.4)
$$\zeta_b : J_b(\mathbb{Q}_p) \longrightarrow G(\mathbb{Q}_p)^{\mathrm{ab}} \quad \text{and} \quad \zeta_b^* : M_b(\mathbb{Q}_p) \longrightarrow G(\mathbb{Q}_p)^{\mathrm{ab}}.$$

Thus every one-dimensional smooth representation π_p of $G(\mathbb{Q}_p)$ (necessarily factoring through $G(\mathbb{Q}_p)^{\mathrm{ab}}$) can be pulled back to one-dimensional representations of $J_b(\mathbb{Q}_p)$ and $M_b(\mathbb{Q}_p)$, to be denoted $\pi_p \circ \zeta_b$ and $\pi_p \circ \zeta_b^*$.

As a reminder from Section 5.3, every nonempty $\mathbb{Q}$-non-basic central leaf can be written as C_b, where b is a $\mathbb{Q}$-non-basic element contained in $G(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}}$ for a sufficiently divisible $r \in \mathbb{Z}_{\geqslant 1}$. (Conversely such a b gives rise to a nonempty leaf.) We are ready to state the main theorem of this paper.

THEOREM 6.1.4 (Main Theorem). — *Let $(G, X, p, \mathcal{G})$ be a Shimura datum of Hodge type. For every (nonempty) $\mathbb{Q}$-non-basic central leaf C_b, there exists a $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-module isomorphism*

$$\iota H^0(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\pi \in \mathcal{A}_{\mathbf{1}}(G)} \pi^{\infty,p} \otimes (\pi_p \circ \zeta_b),$$

where $\mathcal{A}_{\mathbf{1}}(G)$ stands for the set of one-dimensional automorphic representations π of $G(\mathbb{A})$ such that π_∞ is trivial on $G(\mathbb{R})_+$.

Proof. — After reduction to the completely slope divisible case by Lemma 6.2.2, the theorem will be established in Section 7 below. □

REMARK 6.1.5. — Since $\dim \pi_p = 1$, we have $(\pi_p \circ \zeta_b) \otimes \delta_{P_b} = J_{P_b^{\mathrm{op}}}(\pi_p) \otimes \delta_{P_b}^{1/2}$ as $J_b(\mathbb{Q}_p)$-representations. (The point is that the unipotent radical N_b^{op} acts trivially on π_p.) This is closely related to Lemma 3.1.2. It is also worth comparing with [HT01, Th. V.5.4] and [Shi12, Th. 6.7], where a similar expression appears in the description of cohomology of Igusa varieties.

6.2. Finite-level Igusa varieties in the completely slope divisible case

We recall the definition of finite-level Igusa varieties following [Man05, CS17, Ham19]. From Section 5.3 we have $r \in \mathbb{Z}_{\geqslant 1}$ such that $b \in G(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}}$. In this subsection, we further assume that b is completely slope divisible in the sense of [Kim19, Def. 2.4.1]. In particular, the decency equation holds:

$$b\sigma(b)\cdots\sigma^{r-1}(b) = r\nu_b(p). \tag{6.2.1}$$

A priori, (6.2.1) holds for *some* $r \in \mathbb{Z}_{\geqslant 1}$ but then it is still true for all multiples of r. So we may and will assume that (6.2.1) holds for the same r as in Section 5.3 by making r more divisible.

We start from the Siegel case. Write $\mathrm{Ig}_{b',m,K'^{,p}} \to C_{b',K'^{,p}}$ for Igusa varieties of level $m \in \mathbb{Z}_{\geqslant 1}$ as in [Man05, §4] or [Ham19, §3.1] (the definition works over $\mathbb{F}_{p^r}$ not just over $\overline{\mathbb{F}}_p$), defined to parametrize liftable isomorphisms on the p^m-torsion subgroup of each slope component. As shown in loc. cit. $\mathrm{Ig}_{b',m,K'^{,p}} \to C_{b',K'^{,p}}$ is a finite étale morphism, forming a projective system over varying m via the obvious projection maps. Write $\mathrm{Ig}_{b',K'^{,p}}$ for the projective limit of $\mathrm{Ig}_{b',m,K'^{,p}}$ over m. There are maps $\mathfrak{Ig}_{b',K'^{,p}} \to \mathrm{Ig}_{b',m,K'^{,p},\overline{\mathbb{F}}_p}$ for $m \geqslant 1$ compatible with each other, since the isomorphism (6.1.1) induces isomorphisms on isoclinic components. This induces an isomorphism $\mathfrak{Ig}_{b',K'^{,p}} \to \mathrm{Ig}^{\mathrm{perf}}_{b',K'^{,p},\overline{\mathbb{F}}_p}$. See [CS17, Prop. 4.3.8] and the preceding paragraph for details.

Following [Ham19, §4.1] (but working over $\mathbb{F}_{p^r}$ rather than $\overline{\mathbb{F}}_p$), define the $\mathbb{F}_{p^r}$-subscheme

$$\widetilde{\mathrm{Ig}}_{b,K^p} \subset \left(\mathrm{Ig}_{b',K'^{,p}} \times_{C_{b',K'^{,p}}} C_{b,K^p}\right)^{\mathrm{perf}}$$

to be the locus given by the same condition as in (6.1.2). Define Ig_{b,m,K^p} as the image of the composite map

$$\widetilde{\mathrm{Ig}}_{b,K^p} \longrightarrow \mathrm{Ig}_{b',K'^{,p}} \times_{C_{b',K'^{,p}}} C_{b,K^p} \longrightarrow \mathrm{Ig}_{b',m,K'^{,p}} \times_{C_{b',K'^{,p}}} C_{b,K^p}.$$

The projection onto the second component gives an $\mathbb{F}_{p^r}$-morphism $\mathrm{Ig}_{b,m,K^p} \to C_{b,K^p}$, which is finite étale by [Ham19, Prop. 4.1]. Via the canonical projection $\mathrm{Ig}_{b,m+1,K^p} \to \mathrm{Ig}_{b,m,K^p}$ commuting with the maps to C_{b,K^p}, we take the projective limit and denote it by Ig_{b,K^p}.

Besides the Hecke action of $G(\mathbb{A}^{\infty,p})$ on the tower of Ig_{b,K^p}, we also have an action on $\mathrm{Ig}_{b,K^p,\overline{\mathbb{F}}_p}$ by a submonoid $S_b \subset J_b(\mathbb{Q}_p)$ defined in [Man05, p. 586]. (The latter action is defined only over $\overline{\mathbb{F}}_p$ in general since self quasi-isogenies of Σ_b are not always defined

over finite fields.) The precise definition is unimportant, but it suffices to know two facts. Firstly, S_b generates $J_b(\mathbb{Q}_p)$ as a group. Secondly, S_b contains p^{-1} (the inverse of the multiplication by p map on Σ_b) and[15]

$$\mathrm{fr}^{-r} := r\nu_b(p) \in J_b(\mathbb{Q}_p).$$

By Lemma 5.3.1, $\mathrm{fr}^{-r} \in A_{J_b}(\mathbb{Q}_p)$. Let Fr denote the absolute Frobenius morphism on an $\mathbb{F}_p$-scheme.

LEMMA 6.2.1. — *The following hold true.*

(1) $\mathrm{fr}^{-r} \in A^{--}_{P_b^{\mathrm{op}}} \subset A_{M_b}(\mathbb{Q}_p) = A_{J_b}(\mathbb{Q}_p)$. *As an element of* $M_b(\mathbb{Q}_p)$, *we have* $\mathrm{fr}^{-r} \in A^{--}_{P_b^{\mathrm{op}}}$. *(Recall that* $A^{--}_{P_b^{\mathrm{op}}}$ *was defined in Section* 2.1. *For* $A_{M_b}(\mathbb{Q}_p) = A_{J_b}(\mathbb{Q}_p)$, *see Section* 5.3.*)*

(2) *The action of* $\mathrm{Fr}^r \times 1$ *on* $\mathrm{Ig}_{b,K^p} \times_{\mathbb{F}_{p^r}} \overline{\mathbb{F}}_p$ *induces the same action on* $\mathfrak{Ig}_{b,K^p}$ *as the action of* $\mathrm{fr}^{-r} \in J_b(\mathbb{Q}_p)$.

(3) *There is a canonical isomorphism* $\mathfrak{Ig}_{b,K^p} \simeq \mathrm{Ig}^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$ *over* $C_{b,K^p,\overline{\mathbb{F}}_p}$, *compatible with the* $G(\mathbb{A}^{\infty,p}) \times S_b$*-actions as* K^p *varies.*

Proof. — (1) We already know $\mathrm{fr}^{-r} \in A_{J_b}(\mathbb{Q}_p) = A_{M_b}(\mathbb{Q}_p)$. Since $r\nu_b$ is B-dominant (Section 5.3), we have $r\nu_b(p) \in A^{-}_{P_b^{\mathrm{op}}}$. Moreover $r\nu_b(p) \in A^{--}_{P_b^{\mathrm{op}}}$ as the centralizer of $r\nu_b(p)$ in G is exactly M_b.

(2) Write Fr_Σ for the absolute Frobenius action on $\Sigma_b/\mathbb{F}_{p^r}$. In view of (6.2.1), $\mathrm{fr}^{-r} = r\nu_b(p)$ acts on $\Sigma_b/\mathbb{F}_{p^r}$ as $(\mathrm{Fr}_\Sigma)^r$. Thus fr^{-r} sends (x, j) to $(x, j \circ \mathrm{Fr}_\Sigma^r)$ in the description of R-points in Lemma 6.1.2. On the other hand, $\mathrm{Fr}^r \times 1$ on $\mathfrak{Ig}_{b,K^p}$ sends (x, j) to $(x^{(r)}, j^{(r)})$, where $x^{(r)}$ corresponds to the p^r-th power Frobenius twist of x (so that $\mathcal{A}_{x^{(r)}} = (\mathcal{A}_x)^{(r)}$), and $j^{(r)}$ is the p^r-th power twist of j. Finally we observe that $(x^{(r)}, j^{(r)})$ is equivalent to $(x, j \circ \mathrm{Fr}_\Sigma^r)$ via the p^r-power relative Frobenius $\mathcal{A}_x \to \mathcal{A}_{x^{(r)}}$.

(3) We have the map $\mathfrak{Ig}_{b,K^p} \to \mathrm{Ig}_{b,K^p,\overline{\mathbb{F}}_p}$ over $C_{b,K^p,\overline{\mathbb{F}}_p}$ from the definition, which factors through $\mathfrak{Ig}_{b,K^p} \to \mathrm{Ig}^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$ since $\mathfrak{Ig}_{b,K^p}$ is perfect. This is shown to be an isomorphism exactly as in the proof of [CS17, Prop. 4.3.8], the point being a canonical splitting of the slope decomposition over the perfect scheme $\mathrm{Ig}^{\mathrm{perf}}_{b,K^p,\overline{\mathbb{F}}_p}$. □

LEMMA 6.2.2. — *In the setting of Theorem* 6.1.4, *if the theorem is true for every* b *which satisfies* (6.2.1) *for some* r, *then the theorem is true in general.*

Proof. — Let b_0 be arbitrary in the setting of Theorem 6.1.4. By [Kim19, Prop. 2.4.5] (alternatively by the argument of [Zha15, Lem. 4.2.8]), there exists $b \in G(\breve{\mathbb{Q}}_p)$ which is σ-conjugate to b_0 such that

– $b \in G(\breve{\mathbb{Z}}_p)\sigma\mu_p(p)^{-1}G(\breve{\mathbb{Z}}_p)$,
– b is completely slope divisible and satisfies (6.2.1) for some $r \in \mathbb{Z}_{\geqslant 1}$.

[15] Here is a note on the sign. On the slope $0 \leqslant \lambda \leqslant 1$ component, the action of fr^r is $p^{\lambda r}$, but ν_b records slope $-\lambda$ since we use the covariant Dieudonné theory.

It follows that we have (br1) for b, namely $b \in G(\mathbb{Z}_{p^r})\sigma\mu_p(p)^{-1}G(\mathbb{Z}_{p^r})$ for some r. By making r more divisible (note that (6.2.1) still holds for the new r), we can ensure (br2) and (br3) for b. Thus $b \in G(\mathbb{Q}_p^{\mathrm{ur}})_{r\text{-good}}$ and $C_b \neq \varnothing$. By hypothesis, Theorem 6.1.4 is true for this b. On the other hand, we see from Corollary 6.1.3 that, fixing an isomorphism $J_b(\mathbb{Q}_p) \simeq J_{b_0}(\mathbb{Q}_p)$ as in there,

$$H^0(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell) \simeq H^0(\mathfrak{Ig}_{b_0}, \overline{\mathbb{Q}}_\ell) \quad \text{as } G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)\text{-modules.}$$

Therefore, Theorem 6.1.4 for b implies that the same theorem holds for b_0. (Note that the transfer of one-dimensional representations via $J_b(\mathbb{Q}_p) \simeq J_{b_0}(\mathbb{Q}_p)$ is canonical.) □

7. Cohomology of Igusa varieties

The main purpose of this section is to prove Theorem 6.1.4. Throughout we are in the setting of Section 6.2, namely we are assuming (6.2.1) on b and r in addition to (br1)–(br3) of Section 5.3, since this is sufficient in light of Lemma 6.2.2. We will switch to compactly supported cohomology via Poincaré duality and apply Mack-Crane's Langlands–Kottwitz style formula to bring in techniques from the trace formula and harmonic analysis. All ingredients will be combined together in Section 7.6 to identify the leading term in the Lang–Weil estimate.

7.1. Compactly supported cohomology in top degree. — In Section 6.2 we constructed Ig_{b,K^p} over $\mathbb{F}_{p^r}$ such that $\mathfrak{Ig}_{b,K^p}$ is isomorphic to the perfection of $\mathrm{Ig}_{b,K^p,\overline{\mathbb{F}}_p}$ (compatibly with the transition maps as K^p varies). Recall that $\dim \mathrm{Ig}_b = \langle 2\rho, \nu_b \rangle$. Define for $i \in \mathbb{Z}_{\geqslant 0}$,

$$H_c^i(\mathrm{Ig}_{b,m,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) := \varinjlim_{K^p} H_c^i(\mathrm{Ig}_{b,m,K^p,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell), \quad H_c^i(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) := \varinjlim_{m \geqslant 0} H_c^i(\mathrm{Ig}_{b,m,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell).$$

As for $H_c^i(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$, we have a $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-module structure on $H_c^i(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$. This is an admissible $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-module as the cohomology is finite-dimensional at each finite level. It is convenient to prove the following dual version of Theorem 6.1.4.

Theorem 7.1.1. — *Assume that b is $\mathbb{Q}$-non-basic, and that Σ_b is completely slope divisible. Then there is a $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-module isomorphism*

$$\iota H_c^{\langle 4\rho,\nu_b\rangle}(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\pi \in \mathcal{A}_1(G)} \pi^{\infty,p} \otimes ((\pi_p \circ \zeta_b) \otimes \delta_{P_b}).$$

Proof. — The proof will be carried out in Section 7.6 after recalling a stabilized trace formula (Theorem 7.5.1), by employing the estimates in Section 4. □

Theorem 7.1.1 *implies Theorem* 6.1.4. — We may put ourselves in the completely slope divisible case by Lemma 6.2.2. Write $d := \langle 2\rho, \nu_b \rangle$. Applying Poincaré duality to finite-level Igusa varieties Ig_{b,m,K^p} and taking direct limit over m and K^p, we obtain a pairing

$$H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \times H_c^{2d}(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(d)) \longrightarrow \overline{\mathbb{Q}}_\ell,$$

where $\overline{\mathbb{Q}}_\ell(d)$ denotes the d-th power Tate twist. The construction of duality (Exp. XVIII, §3 in [SGA4-3]) goes through a family of canonical isomorphisms $Rf^!_{m,K^p}\overline{\mathbb{Q}}_\ell \simeq \overline{\mathbb{Q}}_\ell(d)[-2d]$ (concentrated in degree $2d$) over m and K^p, where $f_{m,K^p} : \mathrm{Ig}_{b,m,K^p} \to \operatorname{Spec}\mathbb{F}_{p^r}$ denotes the structure map. Thus the action of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ on $\mathrm{Ig}_b = \{\mathrm{Ig}_{b,m,K^p}\}$ induces an action on $\overline{\mathbb{Q}}_\ell(d)[-2d]$, through a character $\varsigma : G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_\ell^\times$. (As in Section 6.2.1, the action of $J_b(\mathbb{Q}_p)$ is defined a priori on a submonoid S_b and then extended to $J_b(\mathbb{Q}_p)$. Alternatively, this action can be defined directly after perfectifying Ig_b.) Together with the $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-action on Ig_b, this yields an action of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ on $H^{2d}_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(d))$ and $H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$, respectively. It follows from the functoriality of Poincaré duality that the above pairing is $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-equivariant. Thus $H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$ is isomorphic to the (smooth) contragredient of $H^{2d}_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(d))$, which is isomorphic to $H^{2d}_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \otimes \varsigma$. Therefore, Theorem 7.1.1 implies that

$$\iota H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\pi \in \mathcal{A}_{\mathbf{1}}(G)} ((\pi^{\infty,p}) \otimes ((\pi_p \circ \zeta_b) \otimes \delta_{P_b}))^\vee \otimes \varsigma^{-1}. \tag{7.1.1}$$

On the other hand, $H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$ is the space of smooth $\overline{\mathbb{Q}}_\ell$-valued functions on $\pi_0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p})$, on which $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$ acts through right translation. (Here smoothness means invariance under an open compact subgroup of $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$.) In particular, the trivial representation appears in $H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$ as the subspace of constant functions on $\pi_0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p})$. Hence $\varsigma^{-1} = (\pi_0^{\infty,p}) \otimes ((\pi_{0,p} \circ \zeta_b) \otimes \delta_{P_b})$ for some $\pi_0 \in \mathcal{A}_{\mathbf{1}}(G)$. Plugging this formula into (7.1.1) and using the fact that $\mathcal{A}_{\mathbf{1}}(G)$ is invariant under taking dual and twisting by π_0, we can rewrite (7.1.1) as

$$\iota H^0(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\pi \in \mathcal{A}_{\mathbf{1}}(G)} \pi^{\infty,p} \otimes (\pi_p \circ \zeta_b).$$

Finally, the same holds with $\mathfrak{Ig}_b$ in place of $\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}$ thanks to Lemma 6.2.1(3). □

REMARK 7.1.2. — It may be possible to compute the character ς in the proof, but we have got around it. As we know the Frobenius action on $\overline{\mathbb{Q}}_\ell(d)[-2d]$, Lemma 6.2.1(2) tells us that $\mathrm{fr}^{-r} \in J_b(\mathbb{Q}_p)$ acts by p^{rd}. We guess that ς is trivial on $G(\mathbb{A}^{\infty,p})$ and equal to $\delta_{P_b}^{-1}$ on $J_b(\mathbb{Q}_p)$.

7.2. PREPARATIONS IN HARMONIC ANALYSIS. — Let $\phi^{\infty,p} = \otimes'_{v \neq \infty,p} \phi_v \in \mathcal{H}(G(\mathbb{A}^{\infty,p}))$ and $\phi_p \in \mathcal{H}(J_b(\mathbb{Q}_p))$. With a view towards Theorem 7.1.1, we want to compute

$$\operatorname{Tr}\left(\phi^{\infty,p}\phi_p^{(j)} \,\middle|\, \iota H_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)\right) := \sum_{i \geqslant 0} (-1)^i \operatorname{Tr}\left(\phi^{\infty,p}\phi_p^{(j)} \,\middle|\, \iota H^i_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)\right).$$

We keep T, B, $b \in G(\breve{\mathbb{Q}}_p)$, and $r \in \mathbb{Z}_{\geqslant 1}$ as before, so that $r\nu_b \in X_*(T)_+$. Recall that $r\nu_b(p) \in A_{J_b}$. Given $\phi_p \in \mathcal{H}(J_b(\mathbb{Q}_p))$, define

$$\phi_p^{(j)} \in \mathcal{H}(J_b(\mathbb{Q}_p)) \quad \text{by} \quad \phi_p^{(j)}(\delta) := \phi_p(j\nu_b(p)^{-1}\delta), \qquad j \in r\mathbb{Z}_{\geqslant 1}.$$

This coincides with the analogous definition of $\phi_p^{(k)}$ in Section 3.1, namely $\phi_p^{(j)} = \phi_p^{(k)}$ via $k = j/r$ and $\nu = r\nu_b$. (The difference is that ν is a cocharacter but ν_b is only a fractional cocharacter.)

An element $\delta \in J_b(\overline{\mathbb{Q}}_p)$ is *acceptable* if its image in $M_b(\overline{\mathbb{Q}}_p)$ is acceptable (Definition 3.1.1) under the isomorphism $J_b(\overline{\mathbb{Q}}_p) \simeq M_b(\overline{\mathbb{Q}}_p)$ induced by some (thus any) inner twist at the end of Section 5.3. As in Section 3.1, let $\mathcal{H}_{\mathrm{acc}}(J_b(\mathbb{Q}_p)) \subset \mathcal{H}(J_b(\mathbb{Q}_p))$ denote the subspace of functions supported on acceptable elements. Choose $j_0 \in \mathbb{Z}_{\geqslant 0}$ such that

$$\phi_p^{(j)} \in \mathcal{H}_{\mathrm{acc}}(J_b(\mathbb{Q}_p)), \qquad j \in r\mathbb{Z},\ j \geqslant j_0.$$

Such a j_0 exists by the argument of Lemma 3.1.7. By Lemma 6.2.1 and the definition of $\phi_p^{(j)}$,

$$\mathrm{Tr}\Big(\phi^{\infty,p}\phi_p^{(j)} \;\Big|\; \iota H_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)\Big) = \mathrm{Tr}\Big(\phi^{\infty,p}\phi_p \times (\mathrm{Fr}^j \times 1) \;\Big|\; \iota H_c(\mathrm{Ig}_{b,\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)\Big), \tag{7.2.1}$$

where Fr^j is the j/r-th power of the relative Frobenius of Ig_b over $\mathbb{F}_{p^r}$. Since the action of Fr^j is the same as the action of a central element of $J_b(\mathbb{Q}_p)$, it commutes with the action of $\phi^{\infty,p}\phi_p$. Thus (7.2.1) and the Lang–Weil bound tell us that the top degree compactly supported cohomology in Theorem 7.1.1 is captured by the leading term as $j \to \infty$. This will be the basic idea underlying the proof of the theorem in Section 7.6 below.

We fix the global central character datum $(\mathfrak{X}, \chi_0) = (A_{G,\infty}, \mathbf{1})$ for G, which can also be viewed as a central character datum for G^* via $Z(G) = Z(G^*)$. (Since we compute the cohomology with constant coefficients, we do not need to consider nontrivial χ_0.)

We also fix a z-extension $1 \to Z_1 \to G_1 \to G \to 1$ over $\mathbb{Q}$ once and for all, which is unramified over $\mathbb{Q}_p$. As explained in [KSZ, §7.3.3], we can promote G_1 to a Shimura datum (G_1, X_1) lifting (G, X) together with the conjugacy class $\{\mu_{X_1}\}$ of cocharacters of $G_{1,\mathbb{C}}$ lifting $\{\mu_X\}$. Therefore, in $\iota_p\{\mu_{X_1}\}$, we can find a cocharacter

$$\mu_{p,1} : \mathbb{G}_m \longrightarrow G_1$$

which lifts μ_p and is defined over an unramified extension of $\mathbb{Q}_p$ (since the reflex field of (G_1, X_1) is unramified at p). Making r more divisible, we arrange that $\mu_{p,1}$ is defined over $\mathbb{Q}_{p^r}$. We apply Lemma 5.3.8 to find a lift $b_1 \in G_1(\mathbb{Q}_{p^r})$ of b; we also ensure that $r\nu_{b_1}$ is a cocharacter as in the last assertion of that lemma. We mention that $\mu_{p,1}$ and b_1 are going to enter the construction of test functions at ∞ and p, respectively.

For each $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$, fix a $\mathbb{Q}$-rational minimal parabolic subgroup of $G^{\mathfrak{e}}$ and its Levi component as at the start of Section 4.2 (with $G^{\mathfrak{e}}$ in place of G there). Call them P_0^* and M_0^* in the case $\mathfrak{e} = \mathfrak{e}^*$. On the other hand, we have $G_{\mathbb{Q}_p} \supset B_{\mathbb{Q}_p} \supset T_{\mathbb{Q}_p}$ from Section 5.3. Since $G_{\mathbb{Q}_p}$ is quasi-split, there is a canonical $G^{\mathrm{ad}}(\mathbb{Q}_p)$-conjugacy class of isomorphisms $G_{\mathbb{Q}_p} \simeq G^*_{\mathbb{Q}_p}$. We fix one such isomorphism such that $B_{\mathbb{Q}_p}$ (resp. $T_{\mathbb{Q}_p}$) is carried into $P^*_{0,\mathbb{Q}_p}$ (resp. $M^*_{0,\mathbb{Q}_p}$). The images of $B_{\mathbb{Q}_p}$ and $T_{\mathbb{Q}_p}$ in $G^*_{\mathbb{Q}_p}$ will play the roles of B and T in Section 4.2.

For each $\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$, we have a central extension $1 \to Z_1 \to G_1^{\mathfrak{e}} \to G^{\mathfrak{e}} \to 1$ over $\mathbb{Q}$, determining an endoscopic datum $\mathfrak{e}_1$ for G_1 and a central character datum $(\mathfrak{X}_1^{\mathfrak{e}}, \chi_1^{\mathfrak{e}})$ for $G_1^{\mathfrak{e}}$ as in Section 2.7.

7.3. THE TEST FUNCTIONS AWAY FROM p. — For each $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$, Let us introduce the test functions to enter the statement of Theorem 7.5.1 below. Here we consider the places away from p. The place p will be treated in the next subsection.

The first case is away from p and ∞. When $\mathfrak{e} = \mathfrak{e}^*$, we have $(f^{\mathrm{Ig},*})^{\infty,p} = \otimes'_{v\neq\infty,p} f_v^*$, where $f_v^* \in \mathcal{H}(G^*(\mathbb{Q}_v))$ is a transfer of ϕ_v as in Section 2.5. In case $\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$, at each $v \neq \infty, p$, the function ϕ_v admits a transfer $f_{1,v}^{\mathfrak{e}} \in \mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{Q}_v), (\chi_{1,v}^{\mathfrak{e}})^{-1})$. Then we take

$$(f^{\mathrm{Ig},\mathfrak{e}})_1^{\infty,p} = \otimes'_{v\neq\infty,p} f_{1,v}^{\mathfrak{e}} \in \mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{A}^{\infty,p}), (\chi_1^{\mathfrak{e},\infty,p})^{-1}).$$

The next case is the real place. We construct the test function $f_{1,\infty}^{\mathrm{Ig},\mathfrak{e}} \in \mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{R}), (\chi_{1,\infty}^{\mathfrak{e}})^{-1})$ by adapting [Kot90, §7] to the case with central characters. In the easier case of $\mathfrak{e} = \mathfrak{e}^* = (G^*, {}^LG^*, 1, \mathrm{id})$, we take $f_\infty^{\mathrm{Ig},*} := e(G_\infty) f_{\mathbf{1}}$ in the notation of Section 2.4. Now let $\mathfrak{e} \in \mathcal{E}^{<}_{\mathrm{ell}}(G)$. In the notation of Section 2.4, both ξ and ζ are trivial in the current setup (since we are focusing on the constant coefficient case). Write ξ_1 and ζ_1 for the pullbacks of ξ and ζ from G to G_1; they are again trivial. We obtain a discrete L-packet $\Pi(\xi_1, \zeta_1)$ for $G_1(\mathbb{R})$ along with an L-parameter $\phi_{\xi_1,\zeta_1} : W_{\mathbb{R}} \to {}^LG_1$ as in Section 2.4. Let $\Phi_2(G_{1,\mathbb{R}}^{\mathfrak{e}}, \phi_{\xi_1,\zeta_1})$ denote the set of discrete L-parameters $\phi' \in \Phi(G_1^{\mathfrak{e}}(\mathbb{R}))$ such that $\eta_1^{\mathfrak{e}}\phi' \simeq \phi_{\xi_1,\zeta_1}$. Then define (cf. [Kot90, p. 186])

$$f_{1,\infty}^{\mathrm{Ig},\mathfrak{e}} := (-1)^{q(G_1)} \langle \mu_1, s_1^{\mathfrak{e}} \rangle \sum_{\phi'} \det(\omega_*(\phi')) f_{\phi'},$$

where $f_{\phi'}$ is the averaged Lefschetz function for the L-packet of ϕ' defined in Section 2.4, and the sum runs over $\phi' \in \Phi(G_{1,\mathbb{R}}^{\mathfrak{e}}, \phi_{\xi_1,\zeta_1})$. As in [KSZ, §8.2.5] we check that $f_{1,\infty}^{\mathrm{Ig},\mathfrak{e}}$ is $(\chi_{1,\infty}^{\mathfrak{e}})^{-1}$-equivariant and compactly supported modulo $\mathfrak{X}_{1,\infty}^{\mathfrak{e}}$.

7.4. THE TEST FUNCTIONS AT p. — We apply the contents of Section 3 to the cocharacter $\nu := r\nu_b$ over $F = \mathbb{Q}_p$ with uniformizer $\varpi = p$. In particular, we have $P_b := P_\nu$ whose Levi factor is $M_b = M_\nu$.

Consider the case $\mathfrak{e} = \mathfrak{e}^*$. Each function $\phi_p \in \mathcal{H}(J_b(\mathbb{Q}_p))$ admits a transfer $\phi_p^* \in \mathcal{H}(M_b(\mathbb{Q}_p))$ as explained in Section 2.3. When $\phi_p \in \mathcal{H}_{\mathrm{acc}}(J_b(\mathbb{Q}_p))$, we can arrange that $\phi_p^* \in \mathcal{H}_{\mathrm{acc}}(M_b(\mathbb{Q}_p))$ after multiplying by the indicator function on the set of acceptable elements in $M_b(\mathbb{Q}_p)$. (This is possible as the subset of acceptable elements is nonempty, open, and stable under $J_b(\mathbb{Q}_p)$-conjugacy.) The image of ϕ_p^* in $\mathcal{S}(M_b)$ depends only on ϕ_p (as an element of $\mathcal{S}(J_b)$). In the notation of Section 3.1, define

$$f_p^{\mathrm{Ig},*,(j)} := \mathscr{J}_\nu\big(\delta_{P_\nu}^{1/2} \cdot \phi_p^{*,(j)}\big) \in \mathcal{S}(G), \qquad j \in \mathbb{Z}_{\geqslant 0},$$

As before, we still write $f_p^{\mathrm{Ig},*,(j)}$ for a representative in $\mathcal{H}(G(\mathbb{Q}_p))$. Lemma 3.1.2 implies that

$$\mathrm{Tr}\big(f_p^{\mathrm{Ig},*,(j)} | \pi_p\big) = \mathrm{Tr}\big(\phi_p^{*,(j)} | J_{P_\nu^{\mathrm{op}}}(\pi_p) \otimes \delta_{P_\nu}^{1/2}\big), \qquad \forall \pi_p \in \mathrm{Irr}(G(\mathbb{Q}_p)).$$

REMARK 7.4.1. — In the definition of $f_p^{\mathrm{Ig},*,(j)}$, we have not multiplied the constant c_{M_H} appearing in [Shi10, §6] (with H, M_H there corresponding to G^*, M_b here). In the sign convention of Remark 6.4 therein, the transfer factor between J_b and M_b equals $e(J_b)$,

resulting in $c_{M_H} = e(J_b)$. In contrast, we have taken the transfer factor between inner forms to be 1 (cf. Remark 2.3.6), so $c_{M_H} = 1$ in our convention.

Now let $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}^{<}(G)$. Recall from (7.2) that $b_1 \in G_1(\mathbb{Q}_{p^r})$ was chosen. Take $\nu_1 := r\nu_{b_1}$. By pulling back the z-extension $1 \to Z_1 \to G_1 \to G \to 1$ via $M_b \hookrightarrow G$, and using the definition of J_b and J_{b_1}, we obtain z-extensions over $\mathbb{Q}_p$ as follows:

$$1 \longrightarrow Z_1 \longrightarrow M_{b_1} \longrightarrow M_b \longrightarrow 1, \qquad 1 \longrightarrow Z_1 \longrightarrow J_{b_1} \longrightarrow J_b \longrightarrow 1.$$

(For J_b, the point is that the σ-stabilizer subgroup of $\mathrm{Res}_{\breve{\mathbb{Q}}_p/\mathbb{Q}_p}\mathbb{G}_m$ is simply $\mathbb{G}_m$.) We pull back $\phi_p^{(j)} \in \mathcal{H}(J_b(\mathbb{Q}_p))$ to obtain $\phi_{1,p}^{(j)} \in \mathcal{H}(J_{b_1}(\mathbb{Q}_p), \chi_{1,p})$. (Recall that $\chi_1 = \prod_v \chi_{1,v}$ is the trivial character on $\mathfrak{X}_1 = Z_1(\mathbb{A})$.) Write $\phi_p^* \in \mathcal{H}(M_b(\mathbb{Q}_p))$ for a transfer of ϕ_p, and $\phi_{1,p}^* \in \mathcal{H}(M_{b_1}(\mathbb{Q}_p), \chi_{1,p})$ for the pullback of ϕ_p^*. Then $\phi_p^{*,(j)}$ (defined in Section 3.1) is a transfer of $\phi_p^{(j)}$ (namely $\phi^{*,(j)} = (\phi_p^{(j)})^*$ in $\mathcal{S}(J_b)$), and $\phi_{1,p}^{*,(j)}$ is a transfer of $\phi_{1,p}^{(j)}$, for all $j \in \mathbb{Z}$.

The desired test function $f_{1,p}^{\mathrm{Ig},\mathfrak{e}}$ is described by the process in [Shi10, §6] (which is applicable since G_1 has simply connected derived subgroup), with $J_{b_1}, G_1^{\mathfrak{e}}, G_1$ in place of J_b, H, G therein, followed by averaging over $\mathfrak{X}_1 = \mathfrak{X}_1^{\mathfrak{e}}$. We summarize the construction as follows:

$$f_{1,p}^{\mathrm{Ig},\mathfrak{e},(j)} := \sum_{\omega \in \Omega_{\mathfrak{e}_1,\nu_1}} f_{1,p,\omega}^{\mathrm{Ig},\mathfrak{e},(j)}, \qquad \text{where} \tag{7.4.1}$$

$$f_{1,p,\omega}^{\mathrm{Ig},\mathfrak{e},(j)} := c_\omega \cdot \mathscr{J}_{\nu_{1,\omega}}\big(\mathrm{LS}^{\mathfrak{e}_1,\omega}(\delta_{P_{\nu_1}}^{1/2} \cdot \phi_{1,p}^{*,(j)})\big) \in \mathcal{H}(G_1^{\mathfrak{e}}(\mathbb{Q}_p), \chi_{1,p}^{\mathfrak{e},-1}),$$

Here $c_\omega \in \mathbb{C}$ are constants (possibly zero) independent of ϕ_p. Note that $\mathscr{J}_{\nu_{1,\omega}}$ and $\mathrm{LS}^{\mathfrak{e}_1,\omega}$ denote the maps in the setup with fixed central character as in Section 3.5. We observe the following about the right hand side of (7.4.1).

$$\delta_{P_{\nu_1}}^{1/2} \cdot \phi_{1,p}^{*,(j)} = \delta_{P_{\nu_1}}^{1/2}(\nu_1(p))\big(\delta_{P_{\nu_1}}^{1/2} \cdot \phi_{1,p}^*\big)^{(j)} = p^{\langle \rho, \nu \rangle}\big(\delta_{P_{\nu_1}}^{1/2} \cdot \phi_{1,p}^*\big)^{(j)}. \tag{7.4.2}$$

7.5. THE STABLE TRACE FORMULA FOR IGUSA VARIETIES. — Continuing from the preceding subsections, we freely use the notation from Section 2.9 . The following stabilized formula for Igusa varieties is of key importance to us.

THEOREM 7.5.1. — *Given $\phi^{\infty,p} \in \mathcal{H}(G(\mathbb{A}^{\infty,p}))$ and $\phi_p \in \mathcal{H}(J_b(\mathbb{Q}_p))$, there exists $j_0 = j_0(\phi^{\infty,p}, \phi_p) \in \mathbb{Z}_{\geqslant 1}$ such that $\phi_p^{(j)} \in \mathcal{H}_{\mathrm{acc}}(J_b(\mathbb{Q}_p))$ and*

$$\mathrm{Tr}\big(\phi^{\infty,p}\phi_p^{(j)} \mid \iota H_c(\mathrm{Ig}_b, \overline{\mathbb{Q}}_\ell)\big) = \mathrm{ST}_{\mathrm{ell},\chi_0}^{G^*}(f^{\mathrm{Ig},*,(j)}) + \sum_{\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}^{<}(G)} \iota(G, G^{\mathfrak{e}}) \mathrm{ST}_{\mathrm{ell},\chi_1^{\mathfrak{e}}}^{G_1^{\mathfrak{e}}}\big(f_1^{\mathrm{Ig},\mathfrak{e},(j)}\big). \tag{7.5.1}$$

for every integer $j \geqslant j_0$ divisible by r.

Proof. — The point is to stabilize the main result of [MC22]. This is carried out in [BMS22]; see Theorems 3.3.9 and 4.4.2 therein. □

REMARK 7.5.2. — The coefficients c_ω in (7.4.1) for $\mathfrak{e} \neq \mathfrak{e}^*$ can be made precise with a normalization of transfer factors as in [BMS22], but we do not need the information in this paper.

7.6. COMPLETION OF THE PROOF OF THEOREM 7.1.1. — The main term in the right hand side of Theorem 7.5.1 will turn out to be the following. Recall $\mathcal{A}_{\mathbf{1}}(G)$ from Theorem 6.1.4.

PROPOSITION 7.6.1. — *Fix $\phi^{\infty,p}\phi_p \in \mathcal{H}(G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p))$, from which $f^{*,(j)} \in \mathcal{H}(G^*(\mathbb{A}))$ is given as in Section 7.4 for every $j \in \mathbb{Z}_{\geqslant 1}$ such that $j \geqslant j_0 = j_0(\phi^{\infty,p}\phi_p)$. As $j \geqslant j_0$ varies over positive integers divisible by r, we have the estimate*

$$T^{G^*}_{\mathrm{disc},\chi_0}\big(f^{\mathrm{Ig},*,(j)}\big) = \sum_{\pi \in \mathcal{A}_{\mathbf{1}}(G)} \mathrm{Tr}\,(\phi^{\infty,p}|\pi^{\infty,p}) \cdot \mathrm{Tr}\big(\phi_p^{(j)} \mid (\pi_p \circ \zeta_b) \otimes \delta_{P_b}\big) + o\big(p^{j\langle 2\rho,\nu_b\rangle}\big).$$

Proof. — We have

$$T^{G^*}_{\mathrm{disc},\chi_0}\big(f^{\mathrm{Ig},*,(j)}\big) = \sum_{\pi^*} m(\pi^*)\,\mathrm{Tr}\big(f^{\mathrm{Ig},*,p} \mid \pi^{*,p}\big)\,\mathrm{Tr}\big(f_p^{\mathrm{Ig},*,(j)}|\pi_p^*\big). \tag{7.6.1}$$

Let $JH(J_{P_b^{\mathrm{op}}}(\pi_p^*))$ denote the multi-set of irreducible subquotients of $J_{P_b^{\mathrm{op}}}(\pi_p^*)$ (up to isomorphism). The central character of $\tau \in JH(J_{P_b^{\mathrm{op}}}(\pi_p^*))$ is denoted ω_τ. We see from Lemma 3.1.2 (ii) that

$$\begin{aligned} \mathrm{Tr}\big(f_p^{\mathrm{Ig},*,(j)} \mid \pi_p^*\big) &= \mathrm{Tr}\big(\delta_{P_b}^{1/2}\phi_p^{*,(j)} \mid J_{P_b^{\mathrm{op}}}(\pi_p^*)\big) = \mathrm{Tr}\big(\phi_p^{*,(j)} \mid J_{P_b^{\mathrm{op}}}(\pi_p^*) \otimes \delta_{P_b}^{1/2}\big) \\ &= \sum_{\tau \in JH(J_{P_b^{\mathrm{op}}}(\pi_p^*))} \omega_\tau(j\nu_b(p))\delta_{P_b}^{1/2}(j\nu_b(p))\ \mathrm{Tr}(\phi_p^*|\tau). \end{aligned} \tag{7.6.2}$$

We have $j\nu_b(p) \in A^{--}_{P_b^{\mathrm{op}}}$, cf. Section 3.1. Since our running assumption that b is $\mathbb{Q}$-non-basic implies ($\mathbb{Q}$-nb(P_b^{op})) by Lemma 5.3.7, it follows from Corollary 2.5.2 that the largest growth of $\omega_\tau(j\nu_b(p))$ as a function in j is achieved exactly when $\dim \pi^* = 1$. In that case, we have $m(\pi^*) = 1$ and π_p^* is a unitary character. Via Lemma 2.5.3, π^* corresponds to a unique one-dimensional automorphic representation π of $G(\mathbb{A})$. We have $\pi_p^* \simeq \pi_p$ via $G^*(\mathbb{Q}_p) \simeq G(\mathbb{Q}_p)$. Thus

$$\begin{aligned} \mathrm{Tr}(\phi_p^{*,(j)}|J_{P_b^{\mathrm{op}}}(\pi_p^*) \otimes \delta_{P_b}^{1/2}) &\overset{\text{Rem. 6.1.5}}{=\!=} \mathrm{Tr}(\phi_p^{*,(j)}|(\pi_p^* \circ \zeta_b^*) \otimes \delta_{P_b}) \\ &= \mathrm{Tr}(\phi_p^{(j)}|(\pi_p \circ \zeta_b) \otimes \delta_{P_b}) = \delta_{P_b}(j\nu_b(p))\,\mathrm{Tr}(\phi_p|(\pi_p \circ \zeta_b) \otimes \delta_{P_b}) \\ &= p^{j\langle 2\rho,\nu_b\rangle}\,\mathrm{Tr}(\phi_p|(\pi_p \circ \zeta_b) \otimes \delta_{P_b}). \end{aligned} \tag{7.6.3}$$

We used Lemma 2.3.7 for the second equality above. Indeed, $(\pi_p^* \circ \zeta_b^*) \otimes \delta_{P_b}$ as a character of $M_b(\mathbb{Q}_p)$ and $(\pi_p \circ \zeta_b) \otimes \delta_{P_b}$ as a character of $J_b(\mathbb{Q}_p)$ correspond to each other via the diagram (6.1.3).

Let $f_{\mathbf{1}}$ denote the averaged Lefschetz function on $G(\mathbb{R})$ as in Section 2.4 with $\xi = \mathbf{1}$ and $\zeta = \mathbf{1}$. Write $e(G^{\infty,p}) := \prod_{v \neq \infty,p} e(G_v)$ for the product of Kottwitz signs.

We rewrite (7.6.1) as

$$\begin{aligned} &T^{G^*}_{\mathrm{disc},\chi_0}\big(f^{\mathrm{Ig},*,(j)}\big) \\ &= \sum_{\substack{\pi^* \\ \dim \pi^*=1}} \mathrm{Tr}\left(f^{\mathrm{Ig},*}_\infty|\pi^*_\infty\right) \mathrm{Tr}\left(f^{\mathrm{Ig},*,\infty,p}|\pi^{*,\infty,p}\right) \mathrm{Tr}\left(f^{\mathrm{Ig},*,(j)}_p|\pi^*_p\right) + o\big(p^{j\langle 2\rho,\nu_b\rangle}\big) \\ &= \sum_{\substack{\pi \\ \dim \pi=1}} \mathrm{Tr}(f_{\mathbf{1}}|\pi_\infty)\,\mathrm{Tr}(\phi^{\infty,p}|\pi^{\infty,p})\,\mathrm{Tr}\left(\phi^{(j)}_p|(\pi_p\circ\zeta_b)\otimes\delta_{P_b}\right) + o\big(p^{j\langle 2\rho,\nu_b\rangle}\big), \end{aligned}$$

where the last equality was obtained from (7.6.2) at p, Lemma 2.4.2 at ∞, and Lemma 2.3.7 at the places away from p. To conclude, we invoke Lemma 2.4.3 to see that $\mathrm{Tr}(f_{\mathbf{1}}|\pi_\infty) = 1$ if $\pi_\infty|_{G(\mathbb{R})_+} = \mathbf{1}$ and $\mathrm{Tr}(f_{\mathbf{1}}|\pi_\infty) = 0$ otherwise. □

Finally we complete the proof of Theorem 7.1.1 employing the main estimates of Section 4.

COROLLARY 7.6.2. — *Theorem* 7.1.1 *is true.*

Proof. — Let $q \neq p$ be an auxiliary prime such that $G_{\mathbb{Q}_q}$ is split. Fix $\phi^{\infty,p,q}\phi_p \in \mathcal{H}(G(\mathbb{A}^{\infty,p,q}) \times J_b(\mathbb{Q}_p))$. Let $\mathfrak{e} \in \mathcal{E}^<_{\mathrm{ell}}(G)$. There exists a constant $C_{\mathfrak{e}} > 0$, depending on $\phi^{\infty,p,q}, \phi_p$, such that for each $\phi_q \in \mathcal{H}(G(\mathbb{Q}_q))_{C_{\mathfrak{e}}\text{-reg}}$, we have the following bound on endoscopic terms in the stabilization of Theorem 7.5.1 by applying the last bound in Corollary 4.2.3 to $k = j/r$, $\nu = r\nu_{1,\omega}$, $\phi^{(k)}_p = c_\omega(\delta^{1/2}_{P_{\nu_1}}\phi^*_{1,p})^{(k)}$, and $\chi = \chi_0$ for each $\omega \in \Omega_{\mathfrak{e}_1,\nu_1}$. Notice that $f^{\mathrm{Ig},\mathfrak{e},(j)}_{1,p,\omega}$ is $p^{\langle\rho^{\mathfrak{e}},\nu_{1,\omega}\rangle}$ times $f^{(k)}_p$ of that corollary, in light of (7.4.1) and (7.4.2). We have

$$\begin{aligned} \mathrm{ST}^{G^{\mathfrak{e}}_1}_{\mathrm{ell},\chi^{\mathfrak{e}}_1}\big((f^{\mathrm{Ig},\mathfrak{e},p}_1)f^{\mathrm{Ig},\mathfrak{e},(j)}_{1,p,\omega}\big) &= O\big(p^{k(\langle 2\rho^{\mathfrak{e}},r\nu_{1,\omega}\rangle - \langle\chi^{\mathfrak{e}}_1,r\overline{\nu}_{1,\omega}\rangle)}\big) \\ &= O\big(p^{j(\langle 2\rho^{\mathfrak{e}},\nu_{1,\omega}\rangle - \langle\chi^{\mathfrak{e}}_1,\overline{\nu}_{1,\omega}\rangle)}\big). \end{aligned}$$

To turn this into a more manageable bound, we use (a) and (b) from the proof of Corollary 4.2.3 and the fact that $\langle\chi_{0,\infty},\overline{\nu}\rangle = 0$ since χ_0 (which plays the role of χ there) is trivial. Thereby we see that the right hand side is $o\left(p^{j\langle 2\rho,\nu_b\rangle}\right)$. Taking the sum over $\omega \in \Omega_{\mathfrak{e}_1,\nu_1}$, we obtain

$$\mathrm{ST}^{G^{\mathfrak{e}}_1}_{\mathrm{ell},\chi^{\mathfrak{e}}_1}\big(f^{\mathrm{Ig},\mathfrak{e},(j)}_1\big) = o\big(p^{j\langle 2\rho,\nu_b\rangle}\big), \qquad \mathfrak{e} \in \mathcal{E}^<_{\mathrm{ell}}(G). \tag{7.6.4}$$

By Lemma 2.9.2, there are only finitely many $\mathfrak{e}$ contributing to the sum in Theorem 7.5.1 for a fixed choice of $\phi^{\infty,p,q}\phi_p$. Thus the coefficients $\iota(G,G^{\mathfrak{e}})$ are bounded by a uniform constant (depending on $\phi^{\infty,p,q}\phi_p$). We deduce the following by applying Theorem 7.5.1, (7.6.4), Corollary 4.2.3 (the first estimate therein), and Proposition 7.6.1 in the order: there exists a constant $C = C(\phi^{\infty,p,q},\phi_p) > 0$ (e.g., the maximum of $C_{\mathfrak{e}}$ over the set of finitely many $\mathfrak{e}$ which contribute) such that for every $\phi_q \in \mathcal{H}(G(\mathbb{Q}_q))_{C\text{-reg}}$, we have

$$\begin{aligned} &\mathrm{Tr}\big(\phi^{\infty,p}\phi^{(j)}_p \;\big|\; \iota H_c(\mathrm{Ig}_b,\overline{\mathbb{Q}}_\ell)\big) \\ &\qquad = \mathrm{ST}^{G^*}_{\mathrm{ell},\chi}\big(f^{\mathrm{Ig},*,(j)}\big) + o\big(p^{j\langle 2\rho,\nu_b\rangle}\big) = T^{G^*}_{\mathrm{disc},\chi}\big(f^{\mathrm{Ig},*,(j)}\big) + o\big(p^{j\langle 2\rho,\nu_b\rangle}\big) \\ &\qquad = \sum_{\pi\in\mathcal{A}_{\mathbf{1}}(G)} \mathrm{Tr}\left(\phi^{\infty,p}|\pi^{\infty,p}\right)\cdot \mathrm{Tr}\big(\phi^{*,(j)}_p \;\big|\; (\pi_p\circ\zeta_b)\otimes\delta_{P_b}\big) + o\big(p^{j\langle 2\rho,\nu_b\rangle}\big). \end{aligned}$$

We have seen in (7.6.3) that $\mathrm{Tr}\big(\phi_p^{(j)} \mid (\pi_p \circ \zeta_b) \otimes \delta_{P_b}\big)$ is either 0 or a nonzero multiple of $p^{j\langle 2\rho,\nu_b\rangle}$ as j varies over multiples of r. Since $\dim \mathrm{Ig}_b = \langle 2\rho, \nu_b\rangle$, it is implied by (7.2.1) and the Lang–Weil bound that the leading term should be of order $p^{j\langle 2\rho,\nu_b\rangle}$.[16] Therefore

$$\mathrm{Tr}\big(\phi^{\infty,p}\phi_p^{(j)} \mid \iota H_c^{\langle 4\rho,\nu_b\rangle}(\mathrm{Ig}_b, \overline{\mathbb{Q}}_\ell)\big) = \sum_{\pi \in \mathcal{A}_{\mathbf{1}}(G)} \mathrm{Tr}\,(\phi^{\infty,p}|\pi^{\infty,p})\,\mathrm{Tr}\big(\phi_p^{(j)}|(\pi_p \circ \zeta_b) \otimes \delta_{P_b}\big).$$

Let B_q be a Borel subgroup of $G_{\mathbb{Q}_q}$ over $\mathbb{Q}_q$ with a Levi component T_q. According to Lemma 3.4.9, we have an isomorphism of $G(\mathbb{A}^{\infty,p,q}) \times J_b(\mathbb{Q}_p) \times T_q(\mathbb{Q}_q)$-representations

$$J_{B_q}\big(H_c^{\langle 4\rho,\nu_b\rangle}(\mathrm{Ig}_b, \overline{\mathbb{Q}}_\ell)\big) \simeq \sum_{\pi \in \mathcal{A}_{\mathbf{1}}(G)} \pi^{\infty,p,q} \otimes ((\pi_p \circ \zeta_b) \otimes \delta_{P_b}) \otimes J_{B_q}(\pi_q).$$

(A priori the isomorphism exists up to semi-simplification, but distinct one-dimensional representations have no extensions with each other.) Repeating the same argument for any other prime $q' \notin \{p, q\}$ such that $G(\mathbb{Q}_{q'})$ is split, the above isomorphism exists with q' in place of q. Comparing the two consequences, we deduce that as $G(\mathbb{A}^{\infty,p}) \times J_b(\mathbb{Q}_p)$-modules,

$$\iota H_c^{\langle 4\rho,\nu_b\rangle}(\mathrm{Ig}_b, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\pi \in \mathcal{A}_{\mathbf{1}}(G)} \pi^{\infty,p} \otimes ((\pi_p \circ \zeta_b) \otimes \delta_{P_b})$$

by multiplicity one for $\mathcal{A}_{\mathbf{1}}(G)$ and weak approximation for G. □

8. Applications to geometry

This section is devoted to working out geometric consequences of Theorem 6.1.4, continuing in the setting of Hodge-type Shimura varieties with hyperspecial level at p.

8.1. Irreducibility of Igusa varieties.

In Section 1.2, we reviewed earlier results on irreducibility of Igusa towers over the μ-ordinary Newton strata of certain PEL-type Shimura varieties. Now we explain that our main theorem implies a generalization thereof to Hodge-type Shimura varieties and to non-μ-ordinary strata.

Let $(G, X, p, \mathcal{G}) \in \mathcal{SD}^{\mathrm{ur}}_{\mathrm{Hodge}}$ and $b \in G(\breve{\mathbb{Q}}_p)$ be as in Section 6.1. Assume that b is $\mathbb{Q}$-non-basic. Define $J(\mathbb{Q}_p)' := \ker(\zeta_b : J_b(\mathbb{Q}_p) \to G(\mathbb{Q}_p)^{\mathrm{ab}})$, cf. (6.1.4). Recall that $\mathrm{pr} : \mathfrak{Ig}_b \to C^{\mathrm{perf}}_{b,\overline{\mathbb{F}}_p}$ is a pro-étale J_b^{int}-torsor. From Theorem 6.1.4, we deduce that Igusa varieties are "as irreducible as possible".

Theorem 8.1.1 (Irreducibility of Igusa varieties). — *In the setting of Section* 6.1*, the stabilizer of each connected component of* $\mathfrak{Ig}_b$ *under the* $J_b(\mathbb{Q}_p)$*-action is equal to* $J_b(\mathbb{Q}_p)'$*.*

Proof. — Fix a component $I \subset \mathfrak{Ig}_b$ and write $\mathrm{Stab}(I)$ for the stabilizer of I in $J_b(\mathbb{Q}_p)$. Since the $J_b(\mathbb{Q}_p)$-action on every π_p in Theorem 6.1.4 factors through $J_b(\mathbb{Q}_p)/J_b(\mathbb{Q}_p)'$, we see that $\mathrm{Stab}(I) \supset J_b(\mathbb{Q}_p)'$. To prove the reverse inclusion, we show that every $\delta \in J_b(\mathbb{Q}_p) \backslash J_b(\mathbb{Q}_p)'$ acts nontrivially on $H^0(\mathfrak{Ig}_b, \overline{\mathbb{Q}}_\ell)$. Write $\delta^{\mathrm{ab}} := \zeta_b(\delta) \in G(\mathbb{Q}_p)^{\mathrm{ab}}$.

[16] In fact, the Lang–Weil bound proves that $\dim \mathrm{Ig}_b = \langle 2\rho, \nu_b\rangle$ even if we did not know it a priori. This gives an alternative proof of the dimension formula in Proposition 5.3.4.

Then $\delta^{\mathrm{ab}} \neq 1$ by assumption. It suffices to show that some π in the summation of Theorem 6.1.4 has the property that $\pi_p(\delta^{\mathrm{ab}}) \neq 1$. This follows from Lemma 2.5.4. □

COROLLARY 8.1.2. — *Let S be a connected component of $C^{\mathrm{perf}}_{b,\overline{\mathbb{F}}_p}$. Then the set $\pi_0(\mathrm{pr}^{-1}(S)) \subset \pi_0(\mathfrak{Ig}_b)$ is a torsor under the group $J_b^{\mathrm{int}}/(J_b^{\mathrm{int}} \cap J_b(\mathbb{Q}_p)')$. Every component of $\mathrm{pr}^{-1}(S)$ is a pro-étale torsor under $J_b^{\mathrm{int}} \cap J_b(\mathbb{Q}_p)'$, and conversely, if $I \subset \mathfrak{Ig}_b$ is an open subscheme such that $I \to S$ is a pro-étale $J_b^{\mathrm{int}} \cap J_b(\mathbb{Q}_p)'$-torsor via* pr*, then I is irreducible.*

Proof. — As pr is a J_b^{int}-torsor, J_b^{int} acts transitively on $\pi_0(\mathrm{pr}^{-1}(S))$. Theorem 8.1.1 tells us that the action factors through a simply transitive action of $J_b^{\mathrm{int}}/J_b^{\mathrm{int}} \cap J_b(\mathbb{Q}_p)'$, implying the first assertion. The second assertion also follows from the same theorem and the fact that pr is a J_b^{int}-torsor. □

For the remainder of this subsection, we compare with similar irreducibility results in the μ-ordinary case. Thus we specialize to the case when $[b] \in B(G, \mu_p^{-1})$ is *μ-ordinary*, meaning either of the following equivalent conditions [Wor13, Rem. 5.7(2)]:

– $[b] = [\mu_p^{-1}(p)]$ in $B(G)$ (which implies $[b] \in B(G, \mu_p^{-1})$).
– $[b]$ is the unique minimal element in $B(G, \mu_p^{-1})$ for the partial order $\preceq$ therein.

In this case, we may and will take $b = b^\circ = \mu_p^{-1}(p)$. Indeed, we can change b within its σ-conjugacy class thanks to Corollary 6.1.3. Put $r := [k(\mathfrak{p}) : \mathbb{F}_p]$. By the convention of Section 5.3, μ_p is defined over $\mathbb{Q}_{p^r}$. Then we have $\nu_b = \frac{1}{r}\sum_{i=0}^{r-1} \sigma^i \mu_p^{-1}$ (this follows from (4.3.1)–(4.3.3) of [Kot85] with $n = r$ and $c = 1$), which is defined over $\mathbb{Q}_p$, and conditions (br2) and (br3) are satisfied.

We define the μ-ordinary Newton stratum N_{b,K^p} as in [Wor13], that is, by changing the definition of C_{b,K^p} (Section 5.3) to require the existence of an isomorphism only after inverting p. Then $C_{b,K^p} \subset N_{b,K^p}$ is closed by [Ham17, §2.3, Prop. 2]. It is worth verifying that $C_{b,K^p} = N_{b,K^p}$, so that $\mathfrak{Ig}_b$ is a pro-étale torsor over $N^{\mathrm{perf}}_{b,K^p}$ (not just $C^{\mathrm{perf}}_{b,K^p}$).

LEMMA 8.1.3. — *In the μ-ordinary setup above, $C_{b,K^p} = N_{b,K^p}$.*

Proof. — This is a consequence of two facts: that the μ-ordinary Newton stratum is an Ekedahl-Oort stratum [Wor13, Th. 6.10], and that every Newton stratum contains an Ekedahl-Oort stratum that is a central leaf [SZ22, Th. D]. (We thank Pol van Hoften for communicating this proof to us.) □

We explain that Corollary 8.1.2 gives another proof for the irreducibility of Igusa towers in the μ-ordinary case, for unitary similitude PEL-type Shimura varieties as in [CEF+16, EM21], cf. [Hid11, §§2, 3]. Analogous arguments can be made in the elliptic/Hilbert/Siegel modular cases.

Write $\left(\mathrm{Ig}^{\mu\text{-ord}}_{m,K^p}\right)_{m\geqslant 1}$ for the Igusa tower $(\mathrm{Ig}_\mu)_{m,1}$, $m \geqslant 1$, over the μ-ordinary stratum N_{b,K^p} in [EM21, §3.2] (relative to the same K^p) with finite étale transition maps. The scheme $\mathrm{Ig}^{\mu\text{-ord}}_{K^p} = \varprojlim_m \mathrm{Ig}^{\mu\text{-ord}}_{m,K^p}$ is a pro-étale J_b^{int}-torsor over N_{b,K^p}. Then

$\mathfrak{Ig}_{b,K^p} \simeq (\mathrm{Ig}_{K^p}^{\mu\text{-ord}})^{\mathrm{perf}}$ compatibly with the actions of $G(\mathbb{A}^{\infty,p}) \times S_b$ (see Prop. 4.3.8 and the paragraph above Cor. 4.3.9 in [CS17]; see also [CS19, Rem. 2.3.7]), and we have a $J_b(\mathbb{Q}_p)$-equivariant bijection

$$\pi_0(\mathfrak{Ig}_{b,K^p}) \simeq \pi_0((\mathrm{Ig}_{K^p}^{\mu\text{-ord}})^{\mathrm{perf}}) \simeq \pi_0(\mathrm{Ig}_{K^p}^{\mu\text{-ord}}).$$

Therefore, each connected component of $\mathrm{Ig}_{K^p}^{\mu\text{-ord}}$ has stabilizer $J_b(\mathbb{Q}_p)'$ in $J_b(\mathbb{Q}_p)$.

The $\mathbb{Z}_p$-group J_μ in [EM21, Rem. 2.9.3] has the property that $J_b^{\mathrm{int}} = J_\mu(\mathbb{Z}_p)$. Let $I \subset \mathrm{Ig}_{K^p}^{\mu\text{-ord}}$ denote the open subscheme Ig_μ^{SU} over a fixed component S of N_{b,K^p} as defined in [EM21, §3.3] (more precisely, we mean the special fiber of Ig_μ^{SU} over $\overline{\mathbb{F}}_p$). Then I is a pro-étale $J_b^{\mathrm{int}} \cap J(\mathbb{Q}_p)'$-torsor over S by construction. (The determinant map of [EM21, §3.3] goes from a J_b^{int}-torsor to a torsor under $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$, so the fiber is a torsor under $\ker(J_b^{\mathrm{int}} \to \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p))$.)[17] Hence I is irreducible by the preceding paragraph, cf. the proof of Corollary 8.1.2.

If $[b]$ is moreover *ordinary*, namely if $[b]$ is μ-ordinary and ν_b is conjugate to μ_p^{-1}, then μ_p is defined over $\mathbb{Q}_p$ (since the conjugacy class of ν_b is always defined over $\mathbb{Q}_p$). Also $r = [E_{\mathfrak{p}} : \mathbb{Q}_p] = 1$. By our choice $b = \mu_p(p)^{-1}$, we have $\nu_b = \mu_p^{-1}$ (not just conjugate) in this case. The following lemma is handy when comparing with results in the ordinary case such as [Hid09, Hid11]. Note that trivially $\varrho(G_{\mathrm{sc}}(\mathbb{Q}_p)) = G_{\mathrm{der}}(\mathbb{Q}_p)$ when $G_{\mathrm{der}} = G_{\mathrm{sc}}$.

Lemma 8.1.4. — *If $[b]$ is ordinary, then $J_b = M_b$, $J_b^{\mathrm{int}} = M_b(\mathbb{Z}_p)$, and $J_b(\mathbb{Q}_p)' = M_b(\mathbb{Q}_p) \cap \varrho(G_{\mathrm{sc}}(\mathbb{Q}_p))$.*

Proof. — By definition, M_b is the centralizer of $\nu_b = \mu_p^{-1}$ in G. From the definition (5.3.1) with $b = \mu_p^{-1}(p)$, we see that M_b is a closed $\mathbb{Q}_p$-subgroup of J_b. On the other hand, M_b is an inner form of J_b, so we conclude $M_b = J_b$. Then $J_b^{\mathrm{int}} = J_b(\mathbb{Q}_p) \cap \mathcal{G}(\breve{\mathbb{Z}}_p) = M_b(\mathbb{Q}_p) \cap \mathcal{G}(\breve{\mathbb{Z}}_p) = M_b(\mathbb{Z}_p)$. The description of $J_b(\mathbb{Q}_p)'$ is obvious from $J_b = M_b$. □

8.2. The discrete Hecke orbit conjecture. — We state the Hecke orbit conjecture for Shimura varieties of Hodge type with hyperspecial level at p. We prove the discrete part of the Hecke orbit conjecture, and find purely local criteria for the irreducibility of central leaves.

Fix $(G, X, p, \mathcal{G}) \in \mathcal{SD}^{\mathrm{ur}}_{\mathrm{Hodge}}$. Let $x \in \mathscr{S}_{K^pK_p,k(\mathfrak{p})}(\overline{\mathbb{F}}_p)$. Denote by $\widetilde{x} \subset |\mathscr{S}_{K_p,k(\mathfrak{p})}|$ the preimage of x in the topological space $|\mathscr{S}_{K_p,k(\mathfrak{p})}|$ via the projection map $\mathscr{S}_{K_p,k(\mathfrak{p})} \to \mathscr{S}_{K^pK_p,k(\mathfrak{p})}$. Define the prime-to-$p$ *Hecke orbit* as a set:

$$H(x) := \widetilde{x} \cdot G(\mathbb{A}^{\infty,p}) \ \subset \ |\mathscr{S}_{K_p,k(\mathfrak{p})}|.$$

Write $H_{K^p}(x)$ for the image of $H(x)$ in $|\mathscr{S}_{K^pK_p,k(\mathfrak{p})}|$. By $C_{K^p}(x)$ we mean the central leaf through x, namely C_{b_x,K^p}. Since the action of $G(\mathbb{A}^{\infty,p})$ does not change the $(G(\breve{\mathbb{Z}}_p), \sigma)$-conjugacy class $[[b_x]]$, we see that

$$H_{K^p}(x) \subset |C_{K^p}(x)|.$$

[17] In fact we have not understood the definition of the determinant map in [EM21, §3.3] unless B is a field, so we should restrict our comparison with loc. cit. to this setup.

Following Chai and Oort, cf. [Cha05, Cha06], we formulate the Hecke Orbit Problem as follows.

QUESTION 8.2.1 (Hecke Orbit Problem). — *Let $x \in \mathscr{S}_{K^pK_p,k(\mathfrak{p})}(\overline{\mathbb{F}}_p)$ such that $[b_x]$ is $\mathbb{Q}$-non-basic. Does the subset $H_{K^p}(x)$ have the following properties?*

(HO) *$H_{K^p}(x)$ is Zariski dense in the central leaf $C_{K^p}(x)$.*

($\mathrm{HO}_{\mathrm{cont}}$) *The Zariski closure of $H_{K^p}(x)$ in $C_{K^p}(x)$ is a union of irreducible components of $C_{K^p}(x)$.*

($\mathrm{HO}_{\mathrm{disc}}$) *$H_{K^p}(x)$ meets every irreducible component of $C_{K^p}(x)$.*

($\mathrm{HO}^+_{\mathrm{disc}}$) *For every $x \in \mathscr{S}_{K^pK_p}(\overline{k(\mathfrak{p})})$ such that $[b_x]$ is $\mathbb{Q}$-non-basic, the immersion $C_{K^p}(x) \hookrightarrow \mathscr{S}_{K^pK_p,k(\mathfrak{p})}$ induces a bijection $\pi_0(C_{K^p}(x)) \xrightarrow{\sim} \pi_0(\mathscr{S}_{K_pK^p,\overline{k(\mathfrak{p})}})$.*

REMARK 8.2.2. — The hypothesis on $[b_x]$ cannot be weakened to only requiring that $[b_x]$ be non-basic. For example, for Shimura varieties arising from $(G \times \cdots \times G, X \times \cdots \times X)$, with (G, X) a Shimura datum, we see the necessity to assume $[b_x]$ to be basic in every copy of G (which is a $\mathbb{Q}$-factor).

Note that (HO) is the analogue of the Hecke orbit conjecture for Hodge-type Shimura varieties, which is divided into discrete and continuous parts in the sense that ($\mathrm{HO}_{\mathrm{disc}}$) and ($\mathrm{HO}_{\mathrm{cont}}$) combined is obviously equivalent to (HO). We usually refer to ($\mathrm{HO}^+_{\mathrm{disc}}$) as "irreducibility of central leaves", as it states that the central leaf through x is irreducible in each connected component of the ambient Shimura variety. Regarding ($\mathrm{HO}_{\mathrm{disc}}$) and ($\mathrm{HO}^+_{\mathrm{disc}}$), we have the following relationship and representation-theoretic interpretations.

LEMMA 8.2.3. — *Let b be $\mathbb{Q}$-non-basic such that C_b is nonempty. Between the following statements, there are logical implications* (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4).

(1) ($\mathrm{HO}^+_{\mathrm{disc}}$) *holds true for all neat $K^p \subset G(\mathbb{A}^{\infty,p})$ and all $x \in C_{b,K^p}(\overline{\mathbb{F}}_p)$.*

(2) *$H^0(C_b, \overline{\mathbb{Q}}_\ell) \simeq H^0(\mathrm{Sh}_{K_p}, \overline{\mathbb{Q}}_\ell)$ as $G(\mathbb{A}^{\infty,p})$-modules. (This asserts the existence of an isomorphism, which need not be induced by the natural map $C_b \to \mathrm{Sh}_{K_p}$.)*

(3) ($\mathrm{HO}_{\mathrm{disc}}$) *holds true for all neat $K^p \subset G(\mathbb{A}^{\infty,p})$ and all $x \in C_{b,K^p}(\overline{\mathbb{F}}_p)$.*

(4) $\dim_{G(\mathbb{A}^{\infty,p})} \mathrm{Hom}(\mathbf{1}, H^0(C_b, \overline{\mathbb{Q}}_\ell)) = 1$.

REMARK 8.2.4. — The non-emptiness condition in the lemma is essentially equivalent to the condition that $b \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\sigma\mu_p(p)^{-1}\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$. Indeed, by Proposition 5.3.5, the latter implies that C_b (as well as C_{b,K^p} for all neat K^p) is nonempty; conversely, if C_b is nonempty then we can re-choose b without changing C_b such that $b \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\sigma\mu_p(p)^{-1}\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$.

Proof. — As K^p varies, the immersion $C_{b,K^p} \hookrightarrow \mathscr{S}_{K^pK_p,k(\mathfrak{p})}$ induces a $G(\mathbb{A}^{\infty,p})$-equivariant map

$$\pi_0(C_b) \longrightarrow \pi_0(\mathscr{S}_{K_p,\overline{k(\mathfrak{p})}}), \tag{8.2.1}$$

which is surjective since $G(\mathbb{A}^{\infty,p})$ acts transitively on $\pi_0(\mathscr{S}_{K_p,\overline{k(\mathfrak{p})}})$ by Lemma 5.2.2. Condition (1) is equivalent to the condition that the above map is an isomorphism,

and (3) is equivalent to the condition that $G(\mathbb{A}^{\infty,p})$ acts transitively on $\pi_0(C_b)$. From this, it is clear that (1) $\Rightarrow$ (2) and that (1) $\Rightarrow$ (3) $\Leftrightarrow$ (4).

Now suppose that (2) holds. Then the $G(\mathbb{A}^{\infty,p})$-equivariant injection $H^0(\mathrm{Sh}_{K_p}, \overline{\mathbb{Q}}_\ell) \hookrightarrow H^0(C_b, \overline{\mathbb{Q}}_\ell)$ *induced by* (8.2.1) must be an isomorphism by (2), since $\iota H^0(\mathrm{Sh}_{K_p}, \overline{\mathbb{Q}}_\ell)$ is a semisimple module in which each $\pi^{\infty,p}$ appears with finite multiplicity (in fact multiplicity one) by Lemma 5.1.1. Hence (8.2.1) is a bijection, and (1) follows. □

Now we allow $b \in G(\breve{\mathbb{Q}}_p)$ which need not be $\mathbb{Q}$-non-basic for the moment. The map $\zeta_b : J_b(\mathbb{Q}_p) \to G(\mathbb{Q}_p)^{\mathrm{ab}}$ from (6.1.4) is an open map as it is the composite of open maps.

Write $\mathcal{G}^{\mathrm{ab}}$ for the abelianization of $\mathcal{G}$ as an algebraic group over $\mathbb{Z}_p$. Then $\mathcal{G}^{\mathrm{ab}}$ is a torus over $\mathbb{Z}_p$, and $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ is a unique maximal subgroup of $G^{\mathrm{ab}}(\mathbb{Q}_p)$. On the level of points, denote by $\mathcal{G}(\mathbb{Z}_p)^{\mathrm{ab}}$ the image of $\mathcal{G}(\mathbb{Z}_p)$ under the projection $G(\mathbb{Q}_p) \to G(\mathbb{Q}_p)^{\mathrm{ab}}$. When $G_{\mathrm{der}} = G_{\mathrm{sc}}$, then $\mathcal{G}(\mathbb{Z}_p)^{\mathrm{ab}} = \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ as subgroups of $G(\mathbb{Q}_p)^{\mathrm{ab}} = G^{\mathrm{ab}}(\mathbb{Q}_p)$.

Given a cocharacter $\upsilon : \mathbb{G}_m \to G$ over $\breve{\mathbb{Q}}_p$, define the affine Deligne–Lusztig set

$$X_\upsilon(b) := \{g \in G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p) : g^{-1} b \sigma(g) \in \mathcal{G}(\breve{\mathbb{Z}}_p) \upsilon(p) \mathcal{G}(\breve{\mathbb{Z}}_p)\},$$

equipped with the left multiplication action by $J_b(\mathbb{Q}_p)$. In fact $X_\upsilon(b)$ is the set of closed points of a perfect variety over $\overline{\mathbb{F}}_p$ [Zhu17, BS17].

Lemma 8.2.5. — *The subgroup $\zeta_b(J_b^{\mathrm{int}}) \subset G(\mathbb{Q}_p)^{\mathrm{ab}}$ is open, compact, and contained in $\mathcal{G}(\mathbb{Z}_p)^{\mathrm{ab}}$. Furthermore, there exists $b_0 \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \sigma\mu_p(p)^{-1} \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ which is σ-conjugate to b in $G(\breve{\mathbb{Q}}_p)$ such that $\zeta_{b_0}(J_{b_0}^{\mathrm{int}}) = \mathcal{G}(\mathbb{Z}_p)^{\mathrm{ab}}$.*

Proof. — Since ζ_b is an open map, it carries the open subgroup J_b^{int} of $J_b(\mathbb{Q}_p)$ onto an open subgroup of $G(\mathbb{Q}_p)^{\mathrm{ab}}$. Since $\zeta_b(J_b^{\mathrm{int}})$ is contained in both $G(\mathbb{Q}_p)^{\mathrm{ab}}$ and the image of $\mathcal{G}(\breve{\mathbb{Z}}_p)$ under the abelianization map, it is contained in $\mathcal{G}(\mathbb{Z}_p)^{\mathrm{ab}}$. This proves the first assertion.

As for the second assertion, we start by claiming that there exists an element b_0 in the double coset $\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}}) \sigma\mu_p(p)^{-1} \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$ which is σ-conjugate to b such that $J_{b_0}^{\mathrm{int}}$ contains an Iwahori subgroup of $J_{b_0}(\mathbb{Q}_p)$. This follows from the proof of [ZZ20, Prop. 3.1.4], which is based on results of He [He14]. (The claim amounts to the existence of a point on $X_{\sigma\mu_p^{-1}}(b)$ whose stabilizer in $J_b(\mathbb{Q}_p)$ contains an Iwahori subgroup. It is enough to check this on the level of Iwahori affine Deligne–Lusztig varieties. Moreover, the assertion is invariant under $J_b(\mathbb{Q}_p)$-equivariant bijections between Iwahori affine Deligne–Lusztig varieties. With this in mind, take w and x as in the first two paragraphs of the proof of [ZZ20, Prop. 3.1.4]. Then the claim follows from the fact that $J_{\dot{x}}(\mathbb{Q}_p) \cap \mathcal{P}(\breve{\mathbb{Z}}_p)$ is a parahoric subgroup of $J_{\dot{x}}(\mathbb{Q}_p)$, cf. p. 168, line 14 in loc. cit. where $\mathcal{P}$ is a parahoric subgroup of $G_{\breve{\mathbb{Q}}_p}$ defined therein.)

By the last claim, it suffices to show that

$$\zeta_{b_0}(\mathrm{Iw}) = \mathcal{G}(\mathbb{Z}_p)^{\mathrm{ab}}$$

for just one Iwahori subgroup Iw of $J_{b_0}(\mathbb{Q}_p)$ since all Iwahori subgroups are $J_{b_0}(\mathbb{Q}_p)$-conjugate. As the statement is now only about b_0, we drop the subscript 0 to simplify notation.

By using an unramified z-extension $1 \to Z_1 \to G_1 \to G \to 1$ over $\mathbb{Q}_p$ (which gives rise to a smooth map of reductive models $\mathcal{G}_1 \to \mathcal{G}$ with connected kernel over $\mathbb{Z}_p$; the induced map $\mathcal{G}_1(\mathbb{Z}_p) \to \mathcal{G}(\mathbb{Z}_p)$ is thus surjective) and choosing $b_1 \in G_1(\breve{\mathbb{Q}}_p)$ as in Lemma 5.3.8, we reduce to the case when $G_{\mathrm{der}} = G_{\mathrm{sc}}$. So M_b and J_b also have simply connected derived subgroups. In particular, $G(\mathbb{Q}_p)^{\mathrm{ab}} = G^{\mathrm{ab}}(\mathbb{Q}_p)$ and likewise for M_b and J_b.

Since M_b splits over $\mathbb{Q}_p^{\mathrm{ur}}$, we see from (5.3.3) that J_b also splits over $\mathbb{Q}_p^{\mathrm{ur}}$. By [DeB06, §2.4], J_b contains an unramified elliptic maximal torus T_b over $\mathbb{Q}_p$. Write $\mathcal{T}_b$ for the torus over $\mathbb{Z}_p$ extending T_b. Then $\mathcal{T}_b$ is contained in some Iwahori subgroup of $J_b(\mathbb{Q}_p)$ (associated with the chamber whose closure contains the facet F in [DeB06, §2.4]). In view of (6.1.3), we can think of ζ_b as the map on the set of $\mathbb{Q}_p$-points arising from the composite $\mathbb{Q}_p$-morphism

$$J_b \longrightarrow J_b^{\mathrm{ab}} \simeq M_b^{\mathrm{ab}} \longrightarrow G^{\mathrm{ab}}.$$

Composing with $T_b \hookrightarrow J_b$, we obtain a $\mathbb{Q}_p$-morphism $T_b \to G^{\mathrm{ab}}$ of unramified tori. This uniquely extends to a $\mathbb{Z}_p$-morphism $\mathcal{T}_b \to \mathcal{G}^{\mathrm{ab}}$, inducing the map

$$\mathcal{T}_b(\mathbb{Z}_p) \longrightarrow \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p).$$

We will be done if this map is surjective. By smoothness and Lang's theorem over finite fields, it is enough to check that $\ker(\mathcal{T}_b \to \mathcal{G}^{\mathrm{ab}})$ is connected. To see this, observe that $T_b' := \ker(T_b \to G^{\mathrm{ab}})$ is connected, since it becomes a maximal torus of G_{der} after base change from $\mathbb{Q}_p$ to $\overline{\mathbb{Q}}_p$. Thus we have a short exact sequence $1 \to T_b' \to T_b \to G^{\mathrm{ab}} \to 1$ of unramified tori over $\mathbb{Q}_p$. It follows that $\ker(\mathcal{T}_b \to \mathcal{G}^{\mathrm{ab}})$ is the torus over $\mathbb{Z}_p$ extending T_b', hence connected. □

REMARK 8.2.6. — In an earlier version of this paper, we incorrectly asserted that $\zeta_b(J_b^{\mathrm{int}})$ equals $\zeta_{b_0}(J_{b_0}^{\mathrm{int}})$. This led us to mistakenly claim that $(\mathrm{HO}_{\mathrm{disc}}^{+})$ was true for non-$\mathbb{Q}$-basic b. As illustrated by Example 8.2.12 below, $(\mathrm{HO}_{\mathrm{disc}}^{+})$ is false in general.

Write $U_p(b)$ for the preimage of $\zeta_b(J_b^{\mathrm{int}})$ under the projection $G(\mathbb{Q}_p) \to G^{\mathrm{ab}}(\mathbb{Q}_p)$. Then $U_p(b)$ is an open subgroup of $G(\mathbb{Q}_p)$ by Lemma 8.2.5. Now we can describe H^0 of every non-$\mathbb{Q}$-basic central leaf and deduce the discrete part of the Hecke orbit conjecture.

THEOREM 8.2.7. — *Let b be as in Lemma 8.2.3. As a $G(\mathbb{A}^{\infty,p})$-module,*

$$\iota H^0(C_b, \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\pi \in \mathcal{A}_{\mathbf{1}}(G)} \dim \pi_p^{U_p(b)} \cdot \pi^{\infty,p}.$$

Moreover, $(\mathrm{HO}_{\mathrm{disc}})$ holds true for all neat $K^p \subset G(\mathbb{A}^{\infty,p})$ and all $x \in C_{b,K^p}(\overline{\mathbb{F}}_p)$.

Proof. — According to Remark 8.2.4 we may assume that $b \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\sigma\mu_p(p)^{-1}\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$, hence that (br1), (br2), and (br3) are satisfied by a sufficiently divisible $r \in \mathbb{Z}_{\geqslant 1}$. The first assertion is a consequence of Theorem 6.1.4 and Lemma 6.1.1(1), noting that perfection does not change cohomology. For the second assertion, let $\pi \in \mathcal{A}_{\mathbf{1}}(G)$ with $\pi^{\infty,p} = \mathbf{1}$. In light of Lemma 8.2.3 it suffices to check that if $\pi_p|_{U_p(b)} = \mathbf{1}$, then π is trivial. We have $\pi_\infty|_{G(\mathbb{R})_+} = \mathbf{1}$ from $\pi \in \mathcal{A}_{\mathbf{1}}(G)$. Thus π as a continuous character

$G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}^\times$ is trivial on $G(\mathbb{A}^{\infty,p})U_p(b)G(\mathbb{R})_+$. Since $G(\mathbb{Q}) \hookrightarrow G(\mathbb{Q}_p) \times G(\mathbb{R})$ has dense image (cf. proof of Lemma 5.2.2), π is trivial. □

REMARK 8.2.8. — The only fact about $U_p(b)$ used in the above proof is that $U_p(b) \subset G(\mathbb{Q}_p)$ is an open subgroup. Thus the same argument shows the obvious analogue of $(\mathrm{HO}_{\mathrm{disc}})$ for finite-level Igusa varieties Ig_{b,m,K^p} since $\mathfrak{Ig}_{b,K^p}$ is a pro-étale J-torsor over Ig_{b,m,K^p} after perfection for an open subgroup J of J_b^{int}. Indeed, we only need to replace C_b with the projective limit of Ig_{b,m,K^p} over K^p, and $U_p(b)$ with the preimage of $\zeta_b(J)$ in $G(\mathbb{Q}_p)$. (This remark was suggested by the referee, whom we thank.)

In light of (1) ⇔ (2) in Lemma 8.2.3, the following theorem gives criteria for $(\mathrm{HO}^+_{\mathrm{disc}})$. Observe that (c2) and (c3) are purely local conditions at p, depending only on the data pertaining to $G_{\mathbb{Q}_p}$.

THEOREM 8.2.9. — *Let $b \in \mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})\sigma\mu_p(p)^{-1}\mathcal{G}(\mathbb{Z}_p^{\mathrm{ur}})$. Assume that b is $\mathbb{Q}$-non-basic. The following are equivalent.*

(c0) *$(\mathrm{HO}^+_{\mathrm{disc}})$ holds true for all neat $K^p \subset G(\mathbb{A}^{\infty,p})$ and all $x \in C_{b,K^p}(\overline{\mathbb{F}}_p)$.*
(c1) *$H^0(C_b, \overline{\mathbb{Q}}_\ell) \simeq H^0(\mathrm{Sh}_{K_p}, \overline{\mathbb{Q}}_\ell)$ as $G(\mathbb{A}^{\infty,p})$-modules.*
(c2) *$\zeta_b(J_b^{\mathrm{int}}) = \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$.*
(c3) *The stabilizer in $J_b(\mathbb{Q}_p)$ of $1 \in X_{\sigma\mu_p^{-1}}(b)$ maps onto $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ under ζ_b.*

Proof. — (c0) ⇔ (c1). Already shown in Lemma 8.2.3.

(c2) ⇒ (c1). For each $\pi \in \mathcal{A}_{\mathbf{1}}(G)$, it is enough to check the claim that $\pi_p^{U_p(b)} \neq 0$ if and only if $\pi_p^{\mathcal{G}(\mathbb{Z}_p)} \neq 0$. As a character π_p factors through $G(\mathbb{Q}_p) \to G^{\mathrm{ab}}(\mathbb{Q}_p)$, condition (c2) ensures that the images of $U_p(b)$ and $\mathcal{G}(\mathbb{Z}_p)$ in $G^{\mathrm{ab}}(\mathbb{Q}_p)$ are both equal to $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$. The claim follows.

(c1) ⇒ (c2). Assuming $\zeta_b(J_b^{\mathrm{int}}) \subsetneq \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$, it is enough to find $\pi \in \mathcal{A}(G)_{\mathbf{1}}$ such that $\pi_p|_{\zeta_b(J_b^{\mathrm{int}})} = \mathbf{1}$ but $\pi_p|_{\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)} \neq \mathbf{1}$, where π_p is viewed as a character of $G^{\mathrm{ab}}(\mathbb{Q}_p)$. This is proved in the same way as Lemma 2.5.4. (When reducing to the torus case, use a z-extension which is unramified at p.)

(c2) ⇔ (c3). Clear since the stabilizer of in $J_b(\mathbb{Q}_p)$ of $1 \in X_{\sigma\mu_p^{-1}}(b)$ is nothing but J_b^{int}. □

COROLLARY 8.2.10. — *$(\mathrm{HO}^+_{\mathrm{disc}})$ is true on the μ-ordinary Newton stratum.*

Proof. — Let $b = b_x \in G(\breve{\mathbb{Q}}_p)$ for x in the μ-ordinary stratum. Choose b_0 as in Lemma 8.2.5 so that $\zeta_{b_0}(J_{b_0}^{\mathrm{int}}) = \mathcal{G}(\mathbb{Z}_p)^{\mathrm{ab}}$. Then $(\mathrm{HO}^+_{\mathrm{disc}})$ holds true for C_{b_0,K^p} for every K^p, through criterion (c2) of Theorem 8.2.9. Since C_{b_0,K^p} is the entire μ-ordinary Newton stratum by Lemma 8.1.3, the proof is finished. □

We also have a partial analogue of Theorem 8.2.9 that is isogeny-invariant, i.e., depending on b only through $[b]$.

COROLLARY 8.2.11. — *Let $b \in G(\breve{\mathbb{Q}}_p)$. Assume that $[b] \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$ and that b is $\mathbb{Q}$-non-basic. The following are equivalent.*

(C0) ($\mathrm{HO}^+_{\mathrm{disc}}$) *is true on the Newton stratum N_{b,K^p}, for all neat open compact $K^p \subset G(\mathbb{A}^{\infty,p})$.*

(C3) *The stabilizer in $J_b(\mathbb{Q}_p)$ of each closed point of $X_{\sigma\mu_p^{-1}}(b)$ maps onto $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ under ζ_b.*

Proof. — (C3) ⇒ (C0). Let $x \in N_{b,K^p}(\overline{\mathbb{F}}_p)$. Then $b_x = g^{-1}b\sigma(g)$ for some $g \in G(\breve{\mathbb{Q}}_p)$. The map $x \mapsto xg$ induces an isomorphism $X_{\sigma\mu_p^{-1}}(b_x) \simeq X_{\sigma\mu_p^{-1}}(b)$ equivariantly with respect to the actions by the $\mathbb{Q}_p$-groups $J_{b_x}(\mathbb{Q}_p) \simeq J_b(\mathbb{Q}_p)$. The latter isomorphism comes from the conjugation by g on $G(\breve{\mathbb{Q}}_p)$, and it commutes with the maps ζ_{b_x} and ζ_b to $G(\mathbb{Q}_p)^{\mathrm{ab}}$. Hence (c3) of Theorem 8.2.9 for b_x is implied by (C3) of this corollary. We deduce (C0) from the same theorem.

(C0) ⇒ (C3). Fix a neat subgroup $K^p \subset G(\mathbb{A}^{\infty,p})$. Let $x \in N_{b,K^p}(\overline{\mathbb{F}}_p)$. Since $[b_x] = [b]$, we may assume $b = b_x$. Now, for each $b' \in X_{\sigma\mu_p^{-1}}(b)$, there exists $y \in N_{b,K^p}(\overline{\mathbb{F}}_p)$ such that $b' = b_y$ by [Kis10, Prop. 1.4.4]. As in the proof of (c3) ⇒ (c1), the stabilizer of b' in $J_b(\mathbb{Q}_p)$ maps onto $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ under ζ_b if and only if (c3) of Theorem 8.2.9 holds for b_y (in place of b). The latter condition holds as we are assuming (C0), again via the same theorem. Hence (C3) holds. □

EXAMPLE 8.2.12. — Condition (C3) of the last corollary makes it convenient to generate a counterexample to ($\mathrm{HO}^+_{\mathrm{disc}}$) by utilizing facts about affine Deligne–Lusztig varieties. We learned such an example from Rong Zhou in an email correspondence together with Pol van Hoften, via a Shimura datum (G, X) of PEL type A such that G^{ad} is $\mathbb{Q}$-simple but (C3) is violated by some non-basic element $b \in B(G_{\mathbb{Q}_p}, \mu_p^{-1})$. It comes down to an explicit affine Deligne–Lusztig variety associated with GL_2 which is a union of irreducible components isomorphic to projective lines. In this case, it can be shown that some component contains a closed point whose stabilizer is too small to satisfy (C3). See [vHX21, §6.3] for details.

REFERENCES

[ACC+23] P. B. ALLEN, F. CALEGARI, A. CARAIANI, T. GEE, D. HELM, B. V. LE HUNG, J. NEWTON, P. SCHOLZE, R. TAYLOR & J. A. THORNE – "Potential automorphy over CM fields", *Ann. of Math. (2)* **197** (2023), no. 3, p. 897–1113.

[AIP16] F. ANDREATTA, A. IOVITA & V. PILLONI – "The adic, cuspidal, Hilbert eigenvarieties", *Res. Math. Sci.* **3** (2016), article no. 34 (36 pages).

[AIP18] ———, "Le halo spectral", *Ann. Sci. École Norm. Sup. (4)* **51** (2018), no. 3, p. 603–655.

[Art89] J. ARTHUR – "The L^2-Lefschetz numbers of Hecke operators", *Invent. Math.* **97** (1989), no. 2, p. 257–290.

[Art96] ———, "On local character relations", *Selecta Math. (N.S.)* **2** (1996), no. 4, p. 501–579.

[BC17] N. BERGERON & L. CLOZEL – "Sur la cohomologie des variétés hyperboliques de dimension 7 trialitaires", *Israel J. Math.* **222** (2017), no. 1, p. 333–400.

[BZ77] I. N. BERNSTEIN & A. V. ZELEVINSKY – "Induced representations of reductive $\mathfrak{p}$-adic groups. I", *Ann. Sci. École Norm. Sup. (4)* **10** (1977), no. 4, p. 441–472.

[BDK86] J. BERNSTEIN, P. DELIGNE & D. KAZHDAN – "Trace Paley-Wiener theorem for reductive p-adic groups", *J. Analyse Math.* **47** (1986), p. 180–192.

[BMS22] A. BERTOLONI MELI & S. W. SHIN – "A stable trace formula for Igusa varieties, II", 2022, `arXiv:2205.05462`.

[BS15] B. BHATT & P. SCHOLZE – *The pro-étale topology for schemes*, Astérisque, vol. 369, Société Mathématique de France, Paris, 2015.

[BS17] ———, "Projectivity of the Witt vector affine Grassmannian", *Invent. Math.* **209** (2017), no. 2, p. 329–423.

[Bor79] A. BOREL – "Automorphic L-functions", in *Automorphic forms, representations and L-functions (Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., vol. XXXIII, American Mathematical Society, Providence, R.I., 1979, p. 27–61.

[Bor91] A. BOREL – *Linear algebraic groups*, 2nd ed., Graduate Texts in Math., vol. 126, Springer-Verlag, New York, 1991.

[Boy07] P. BOYER – "On the irreductibility of some Igusa varieties", 2007, `arXiv:0702329`.

[CEF+16] A. CARAIANI, E. EISCHEN, J. FINTZEN, E. MANTOVAN & I. VARMA – "p-adic q-expansion principles on unitary Shimura varieties", in *Directions in number theory*, Assoc. Women Math. Ser., vol. 3, Springer, Cham, 2016, p. 197–243.

[CS17] A. CARAIANI & P. SCHOLZE – "On the generic part of the cohomology of compact unitary Shimura varieties", *Ann. of Math. (2)* **186** (2017), no. 3, p. 649–766.

[CS19] ———, "On the generic part of the cohomology of non-compact unitary Shimura varieties", 2019, `arXiv:1909.01898`.

[Cas95] W. CASSELMAN – "Introduction to the theory of admissible representations of p-adic reductive groups", 1995, `https://www.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf`.

[Cha05] C.-L. CHAI – "Hecke orbits on Siegel modular varieties", in *Geometric methods in algebra and number theory*, Progress in Math., vol. 235, Birkhäuser Boston, Boston, MA, 2005, p. 71–107.

[Cha06] ———, "Hecke orbits as Shimura varieties in positive characteristic", in *International Congress of Mathematicians. Vol. II*, European Mathematical Society, Zürich, 2006, p. 295–312.

[Cha08] ———, "Methods for p-adic monodromy", *J. Inst. Math. Jussieu* **7** (2008), no. 2, p. 247–268.

[CO11] C.-L. CHAI & F. OORT – "Monodromy and irreducibility of leaves", *Ann. of Math. (2)* **173** (2011), no. 3, p. 1359–1396.

[CO19] ———, "The Hecke orbit conjecture: a survey and outlook", in *Open problems in arithmetic algebraic geometry*, Adv. Lect. Math. (ALM), vol. 46, International Press, Somerville, MA, 2019, p. 235–262.

[CD85] L. CLOZEL & P. DELORME – "Pseudo-coefficients et cohomologie des groupes de Lie réductifs réels", *C. R. Acad. Sci. Paris Sér. I Math.* **300** (1985), no. 12, p. 385–387.

[COU01] L. CLOZEL, H. OH & E. ULLMO – "Hecke operators and equidistribution of Hecke points", *Invent. Math.* **144** (2001), no. 2, p. 327–351.

[CL10] R. CLUCKERS & F. LOESER – "Constructible exponential functions, motivic Fourier transform and transfer principle", *Ann. of Math. (2)* **171** (2010), no. 2, p. 1011–1065.

[Dal22] R. DALAL – "Sato-Tate equidistribution for families of automorphic representations through the stable trace formula", *Algebra Number Theory* **16** (2022), no. 1, p. 59–137.

[DeB06] S. DEBACKER – "Parameterizing conjugacy classes of maximal unramified tori via Bruhat-Tits theory", *Michigan Math. J.* **54** (2006), no. 1, p. 157–178.

[Del79] P. DELIGNE – "Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques", in *Automorphic forms, representations and L-functions (Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., vol. XXXIII, American Mathematical Society, Providence, R.I., 1979, p. 247–289.

[Del71] P. DELIGNE – "Travaux de Shimura", in *Séminaire Bourbaki, (1970/71)*, Lect. Notes in Math., vol. 244, Springer, Berlin, 1971, Exp. No. 389, p. 123–165.

[vD72] G. VAN DIJK – "Computation of certain induced characters of p-adic groups", *Math. Ann.* **199** (1972), p. 229–240.

[EMO01] S. J. EDIXHOVEN, B. J. J. MOONEN & F. OORT – "Open problems in algebraic geometry", *Bull. Sci. Math.* **125** (2001), no. 1, p. 1–22.

[EM21] E. EISCHEN & E. MANTOVAN – "p-adic families of automorphic forms in the μ-ordinary setting", *Amer. J. Math.* **143** (2021), no. 1, p. 1–52.

[EO06] A. ESKIN & H. OH – "Ergodic theoretic proof of equidistribution of Hecke points", *Ergodic Theory Dynam. Systems* **26** (2006), no. 1, p. 163–167.

[Far04] L. FARGUES – "Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales", in *Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales*, Astérisque, vol. 291, Société Mathématique de France, Paris, 2004, p. 1–199.

[FK88] Y. Z. FLICKER & D. A. KAZHDAN – "A simple trace formula", *J. Analyse Math.* **50** (1988), p. 189–200.

[Fra98] J. FRANKE – "Harmonic analysis in weighted L_2-spaces", *Ann. Sci. École Norm. Sup. (4)* **31** (1998), no. 2, p. 181–279.

[FLN10] E. FRENKEL, R. LANGLANDS & B. C. NGÔ – "Formule des traces et fonctorialité: le début d'un programme", *Ann. Sci. Math. Québec* **34** (2010), no. 2, p. 199–243.

[GK21] E. Z. GOREN & P. L. KASSAEI – "p-adic dynamics of Hecke operators on modular curves", *J. Théor. Nombres Bordeaux* **33** (2021), no. 2, p. 387–431.

[GKM97] M. GORESKY, R. KOTTWITZ & R. MACPHERSON – "Discrete series characters and the Lefschetz formula for Hecke operators", *Duke Math. J.* **89** (1997), no. 3, p. 477–554.

[Ham17] P. HAMACHER – "The almost product structure of Newton strata in the deformation space of a Barsotti-Tate group with crystalline Tate tensors", *Math. Z.* **287** (2017), no. 3-4, p. 1255–1277.

[Ham19] ———, "The product structure of Newton strata in the good reduction of Shimura varieties of Hodge type", *J. Algebraic Geom.* **28** (2019), no. 4, p. 721–749.

[HK19] P. HAMACHER & W. KIM – "ℓ-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients", *Math. Ann.* **375** (2019), no. 3-4, p. 973–1044.

[HT01] M. HARRIS & R. TAYLOR – *The geometry and cohomology of some simple Shimura varieties*, Annals of Math. Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001.

[He14] X. HE – "Geometric and homological properties of affine Deligne-Lusztig varieties", *Ann. of Math. (2)* **179** (2014), no. 1, p. 367–404.

[HZZ21] X. HE, R. ZHOU & Y. ZHU – "Stabilizers of irreducible components of affine Deligne–Lusztig varieties", 2021, `arXiv:2109.02594`.

[HMRL20] S. HERRERO, R. MENARES & J. RIVERA-LETELIER – "p-adic distribution of CM points and Hecke orbits I: Convergence towards the Gauss point", *Algebra Number Theory* **14** (2020), no. 5, p. 1239–1290.

[Hid04] H. HIDA – *p-adic automorphic forms on Shimura varieties*, Springer Monogr. in Math., Springer-Verlag, New York, 2004.

[Hid09] ———, "Irreducibility of the Igusa tower", *Acta Math. Sinica (N.S.)* **25** (2009), no. 1, p. 1–20.

[Hid11] ———, "Irreducibility of the Igusa tower over unitary Shimura varieties", in *On certain L-functions*, Clay Math. Proc., vol. 13, American Mathematical Society, Providence, RI, 2011, p. 187–203.

[vH10] P. VAN HOFTEN – "Mod p points on Shimura varieties of parahoric level", 2010, `arXiv:2010.10496`.

[vHX21] P. VAN HOFTEN & L. X. XIAO – "Monodromy and irreducibility of Igusa varieties", 2021, `arXiv:2102.09870`.

[HM79] R. E. HOWE & C. C. MOORE – "Asymptotic properties of unitary representations", *J. Funct. Anal.* **32** (1979), no. 1, p. 72–96.

[How20] S. HOWE – "A unipotent circle action on p-adic modular forms", *Trans. Amer. Math. Soc. Ser. B* **7** (2020), p. 186–226.

[Igu68] J.-I. IGUSA – "On the algebraic theory of elliptic modular functions", *J. Math. Soc. Japan* **20** (1968), p. 96–106.

[KM85] N. M. KATZ & B. MAZUR – *Arithmetic moduli of elliptic curves*, Annals of Math. Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985.

[Kaz86] D. KAZHDAN – "Cuspidal geometry of p-adic groups", *J. Analyse Math.* **47** (1986), p. 1–36.

[KV16] D. KAZHDAN & Y. VARSHAVSKY – "Geometric approach to parabolic induction", *Selecta Math. (N.S.)* **22** (2016), no. 4, p. 2243–2269.

[KST20] J.-L. KIM, S. W. SHIN & N. TEMPLIER – "Asymptotic behavior of supercuspidal representations and Sato-Tate equidistribution for families", *Adv. Math.* **362** (2020), article no. 106955 (57 pages).

[Kim19] W. Kim – "On central leaves of Hodge-type Shimura varieties with parahoric level structure", *Math. Z.* **291** (2019), no. 1-2, p. 329–363.

[KMP16] W. Kim & K. Madapusi Pera – "2-adic integral canonical models", *Forum Math. Sigma* **4** (2016), article no. e28 (34 pages).

[Kis10] M. Kisin – "Integral models for Shimura varieties of abelian type", *J. Amer. Math. Soc.* **23** (2010), no. 4, p. 967–1012.

[Kis17] ———, "Mod p points on Shimura varieties of abelian type", *J. Amer. Math. Soc.* **30** (2017), no. 3, p. 819–914.

[KMPS22] M. Kisin, K. Madapusi Pera & S. W. Shin – "Honda-Tate theory for Shimura varieties", *Duke Math. J.* **171** (2022), no. 7, p. 1559–1614.

[KSZ] M. Kisin, S. W. Shin & Y. Zhu – "The stable trace formula for certain Shimura varieties of abelian type".

[Kot82] R. E. Kottwitz – "Rational conjugacy classes in reductive groups", *Duke Math. J.* **49** (1982), no. 4, p. 785–806.

[Kot84a] ———, "Shimura varieties and twisted orbital integrals", *Math. Ann.* **269** (1984), no. 3, p. 287–300.

[Kot84b] ———, "Stable trace formula: cuspidal tempered terms", *Duke Math. J.* **51** (1984), no. 3, p. 611–650.

[Kot85] ———, "Isocrystals with additional structure", *Compositio Math.* **56** (1985), no. 2, p. 201–220.

[Kot86] ———, "Stable trace formula: elliptic singular terms", *Math. Ann.* **275** (1986), no. 3, p. 365–399.

[Kot88] ———, "Tamagawa numbers", *Ann. of Math. (2)* **127** (1988), no. 3, p. 629–646.

[Kot90] ———, "Shimura varieties and λ-adic representations", in *Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988)*, Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, p. 161–209.

[Kot92a] ———, "On the λ-adic representations associated to some simple Shimura varieties", *Invent. Math.* **108** (1992), no. 3, p. 653–665.

[Kot92b] ———, "Points on some Shimura varieties over finite fields", *J. Amer. Math. Soc.* **5** (1992), no. 2, p. 373–444.

[Kot97] ———, "Isocrystals with additional structure. II", *Compositio Math.* **109** (1997), no. 3, p. 255–339.

[KS99] R. E. Kottwitz & D. Shelstad – *Foundations of twisted endoscopy*, Astérisque, vol. 255, Société Mathématique de France, Paris, 1999.

[Lab99] J.-P. Labesse – *Cohomologie, stabilisation et changement de base*, Astérisque, vol. 257, Société Mathématique de France, Paris, 1999.

[Lan13] K.-W. Lan – *Arithmetic compactifications of PEL-type Shimura varieties*, Lond. Math. Soc. Monogr. Ser., vol. 36, Princeton University Press, Princeton, NJ, 2013.

[Lan79] R. P. Langlands – "Stable conjugacy: definitions and lemmas", *Canad. Appl. Math. Quart.* **31** (1979), no. 4, p. 700–725.

[Lan83] ———, *Les débuts d'une formule des traces stable*, Publications Mathématiques de l'Université Paris VII, vol. 13, Université de Paris VII, U.E.R. de Mathématiques, Paris, 1983.

[Lan89] ———, "On the classification of irreducible representations of real algebraic groups", in *Representation theory and harmonic analysis on semisimple Lie groups*, Math. Surveys Monogr., vol. 31, American Mathematical Society, Providence, RI, 1989, p. 101–170.

[LR87] R. P. Langlands & M. Rapoport – "Shimuravarietäten und Gerben", *J. reine angew. Math.* **378** (1987), p. 113–220.

[LS87] R. P. Langlands & D. Shelstad – "On the definition of transfer factors", *Math. Ann.* **278** (1987), no. 1-4, p. 219–271.

[Lau97] G. Laumon – "Sur la cohomologie à supports compacts des variétés de Shimura pour $\mathrm{GSp}(4)_{\mathbf{Q}}$", *Compositio Math.* **105** (1997), no. 3, p. 267–359.

[Lau05] ———, "Fonctions zêtas des variétés de Siegel de dimension trois", in *Formes automorphes. II. Le cas du groupe* GSp(4), Astérisque, vol. 302, Société Mathématique de France, Paris, 2005, p. 1–66.

[Lov17] T. LOVERING – "Filtered F-crystals on Shimura varieties of abelian type", 2017, arXiv:1702.06611.

[MC22] S. MACK-CRANE – "Counting points on Igusa varieties of Hodge type", 2022, arXiv:2203.01448.

[MP19] K. MADAPUSI PERA – "Toroidal compactifications of integral models of Shimura varieties of Hodge type", *Ann. Sci. École Norm. Sup. (4)* **52** (2019), no. 2, p. 393–514.

[Man05] E. MANTOVAN – "On the cohomology of certain PEL-type Shimura varieties", *Duke Math. J.* **129** (2005), no. 3, p. 573–610.

[MST22] D. MAULIK, A. N. SHANKAR & Y. TANG – "Picard ranks of K3 surfaces over function fields and the Hecke orbit conjecture", *Invent. Math.* **228** (2022), no. 3, p. 1075–1143.

[Mil05] J. S. MILNE – "Introduction to Shimura varieties", in *Harmonic analysis, the trace formula, and Shimura varieties*, Clay Math. Proc., vol. 4, American Mathematical Society, Providence, RI, 2005, p. 265–378.

[Ngô10] B. C. NGÔ – "Le lemme fondamental pour les algèbres de Lie", *Publ. Math. Inst. Hautes Études Sci.* (2010), no. 111, p. 1–169.

[Oki23] Y. OKI – "On the connected components of Shimura varieties for CM unitary groups in odd variables", *J. Number Theory* **245** (2023), p. 1–64.

[Oor19] F. OORT – "Appendix 1: Some questions in algebraic geometry", in *Open problems in arithmetic algebraic geometry*, Adv. Lect. Math. (ALM), vol. 46, International Press, Somerville, MA, 2019, p. 263–283.

[PST21] J. PILA, A. SHANKAR & J. TSIMERMAN – "Canonical heights on Shimura varieties and the André-Oort conjecture", 2021, arXiv:2109.08788.

[PR94] V. PLATONOV & A. RAPINCHUK – *Algebraic groups and number theory*, Pure and Applied Math., vol. 139, Academic Press, Inc., Boston, MA, 1994.

[Pra82] G. PRASAD – "Elementary proof of a theorem of Bruhat-Tits-Rousseau and of a theorem of Tits", *Bull. Soc. math. France* **110** (1982), no. 2, p. 197–202.

[RR96] M. RAPOPORT & M. RICHARTZ – "On the classification and specialization of F-isocrystals with additional structure", *Compositio Math.* **103** (1996), no. 2, p. 153–181.

[RZ96] M. RAPOPORT & T. ZINK – *Period spaces for p-divisible groups*, Annals of Math. Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.

[Rie70] C. RIEHM – "The norm 1 group of a $\mathfrak{P}$-adic division algebra", *Amer. J. Math.* **92** (1970), p. 499–523.

[Rog83] J. D. ROGAWSKI – "Representations of GL(n) and division algebras over a p-adic field", *Duke Math. J.* **50** (1983), no. 1, p. 161–196.

[Ser96] J.-P. SERRE – "Two letters on quaternions and modular forms (mod p)", *Israel J. Math.* **95** (1996), p. 281–299, With introduction, appendix and references by R. Livné.

[SGA4-3] M. ARTIN, A. GROTHENDIECK & J.-L. VERDIER – *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lect. Notes in Math., vol. 305, Springer-Verlag, Berlin-New York, 1972–1973, Séminaire de Géométrie Algébrique du Bois–Marie 1963–64 (SGA 4).

[SZ22] X. SHEN & C. ZHANG – "Stratifications in good reductions of Shimura varieties of abelian type", *Asian J. Math.* **26** (2022), no. 2, p. 167–226.

[Shi11] S. W. SHIN – "Galois representations arising from some compact Shimura varieties", *Ann. of Math. (2)* **173** (2011), no. 3, p. 1645–1741.

[Shi09] S. W. SHIN – "Counting points on Igusa varieties", *Duke Math. J.* **146** (2009), no. 3, p. 509–568.

[Shi10] ———, "A stable trace formula for Igusa varieties", *J. Inst. Math. Jussieu* **9** (2010), no. 4, p. 847–895.

[Shi12] ———, "On the cohomology of Rapoport-Zink spaces of EL-type", *Amer. J. Math.* **134** (2012), no. 2, p. 407–452.

[Sta21] STACKS PROJECT AUTHORS – "The stacks project", https://stacks.math.columbia.edu, 2021.

[Ste65] R. STEINBERG – "Regular elements of semisimple algebraic groups", *Publ. Math. Inst. Hautes Études Sci.* (1965), no. 25, p. 49–80.

[Tsi18] J. TSIMERMAN – "The André-Oort conjecture for $\mathcal{A}_g$", *Ann. of Math. (2)* **187** (2018), no. 2, p. 379–390.

[Wal97] J.-L. Waldspurger – "Le lemme fondamental implique le transfert", *Compositio Math.* **105** (1997), no. 2, p. 153–236.

[Wal06] ———, "Endoscopie et changement de caractéristique", *J. Inst. Math. Jussieu* **5** (2006), no. 3, p. 423–525.

[Wal08] ———, *L'endoscopie tordue n'est pas si tordue*, Mem. Amer. Math. Soc., vol. 194, no. 908, American Mathematical Society, Providence, RI, 2008.

[Wor13] D. Wortmann – "The μ-ordinary locus for Shimura varieties of Hodge type", 2013, arXiv:1310.6444.

[Xia20] L. Xiao – "On the Hecke orbit conjecture for PEL type Shimura varieties", 2020, arXiv:2006.06859.

[Xu16] B. Xu – "On a lifting problem of L-packets", *Compositio Math.* **152** (2016), no. 9, p. 1800–1850.

[Xu17] ———, "On the cuspidal support of discrete series for p-adic quasisplit $Sp(N)$ and $SO(N)$", *Manuscripta Math.* **154** (2017), no. 3-4, p. 441–502.

[YCO20] C.-F. Yu, C.-L. Chai & F. Oort – "Stratifying Lie strata of Hilbert modular varieties", *Taiwanese J. Math.* **24** (2020), no. 6, p. 1307–1352.

[Zha15] C. Zhang – "Stratifications and foliations for good reductions of Shimura varieties of Hodge type", 2015, arXiv:1512.08102.

[Zho19] R. Zhou – "Motivic cohomology of quaternionic Shimura varieties and level raising", 2019, arXiv:1901.01954.

[Zho20] ———, "Coherent sheaves on the stack of Langlands parameters", 2020, arXiv:2008.02998.

[ZZ20] R. Zhou & Y. Zhu – "Twisted orbital integrals and irreducible components of affine Deligne-Lusztig varieties", *Camb. J. Math.* **8** (2020), no. 1, p. 149–241.

[Zhu17] X. Zhu – "Affine Grassmannians and the geometric Satake in mixed characteristic", *Ann. of Math. (2)* **185** (2017), no. 2, p. 403–492.

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