Dynamics of Gauge Fields at High Temperature

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5 Counterterms for linear divergences

5.1 Introduction

In the previous chapter we have demonstrated that diagrams in a classical field theory contain linear divergences at one loop and logarithmic divergences at two loops. This indicates that certain quantities will not be calculable in a classical theory and the breakdown of the classical approximation. Examples we have encountered are the tadpole mass (2.12) and the plasmon frequency (4.5). Unfortunately, the divergences are not strictly confined to these quantities, but they also affect (superficially) finite quantities.

An illuminating and important example is given by the typical time scale for the non-perturbative modes with momenta \( p \sim g^2 T \). One may recall from (3.86) that this time scale for a quantum theory is

\[
t \sim \frac{\omega_{pl}^2}{p^3}.
\]  

(5.1)

The derivation in section 3.8 made use of the transverse propagator with the HTL self-energy inserted. Here we shall estimate this time scale for a classical theory (without HTL’s) with cut-off \( \Lambda \). Remember that the dominant (linearly divergent) contributions correspond exactly to the quantum HTL’s except for the value of the plasmon frequency, see section 4.2.1. This implies that the estimate (5.1) can be used, except that we have to insert the classical plasmon frequency \( \omega_{pl,cl}^2 \sim g^2 T \Lambda \) (4.5). This yields [12]

\[
t \sim \frac{\omega_{pl,cl}^2}{p^3} \sim g^{-4} \frac{\Lambda}{T^2}.
\]  

(5.2)

Because this time scale diverges, the Chern-Simons diffusion rate is proportional to \( \Lambda^{-1} \) (following the same reasoning as from (3.86) to (3.87)).

The cut-off dependence arises because the hard modes affect the soft modes in an essential way. Diagrammatically, this corresponds to superficially finite diagrams that acquire a cut-off dependence through divergent
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subdiagrams. The aim is now to improve the classical theory such that linear divergences are absent. In terms of the example above, the goal is to obtain an effective classical theory that yields the correct (quantum) time scale. This implies that the HTL’s have to be included in the classical theory; this was the subject of chapter 3. But in addition it requires the inclusion of counterterms for the classical divergences.

Counterterms for classical divergences are different from counterterms at zero temperature, since classical divergences are non-local and extend to diagrams with any number of external legs. For real-time classical gauge theories, counterterms were first studied by Bödeker, McLerran, and Smilga [25]. They derived an effective theory by integrating out modes with momenta $k > \Lambda_{\text{int}}$, where $\Lambda_{\text{int}}$ is an intermediate cut-off: $gT < \Lambda_{\text{int}} < T$. In the HTL approximation this yielded the usual HTL’s with subtractions linear in the cut-off $\Lambda_{\text{int}}$. These subtractions were interpreted as counterterms. Their treatment was not gauge invariant and, therefore, gauge invariance of the effective theory was broken by the counterterms. Later it was argued that in a gauge invariant approach the subtraction should be confined to the one-loop plasmon frequency [3,56]. We will confirm this conjecture here. Furthermore, on the basis of the results of the previous chapter we may conclude that no linear divergences will appear beyond one loop. A useful result that we will use in the reasoning is the fact that classical linear divergences are the classical analogues of HTL’s. This allows us to use the known facts on HTL’s, see chapter 3.

For practical calculations the implementation of counterterms for classical lattice theories is of some interest. After an introduction, the main part of this chapter will be devoted to this topic. We will find that exact lattice counterterms prevent a matching of the quantum HTL’s to the continuum. Approximate counterterms may be given by a lattice generalization of the model proposed by Iancu [56]. These approximate counterterms go beyond the counterterm that was used by Bödeker, Moore and Rummukainen [29].

5.2 Cut-off dependence

To start, we will consider a general formulation of the problem of classical divergences in a $SU(N)$ theory given by the Yang-Mills equations of motion

$$D_{\mu}^{ab} F^{\mu\nu} = j^{\nu}_{\text{HTL}},$$

(5.3)

with a cut-off $\Lambda$ to make the theory finite. The notation is the same as in chapter 3, except that the non-Abelian HTL source (3.65) generated by
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hard thermal loops is now denoted by $j_{\text{HTL}}$. The inclusion of the HTL-corrections in the classical theory was motivated by the need to resum diagrams that are dominated by soft momenta in the integration over internal momenta; see chapter 3. However, in an effective classical theory, we also have to deal with the Rayleigh-Jeans divergences, which are not removed by a HTL resummation. In the previous chapter we have determined the general structure of these divergences in perturbation theory. Here we will study the ensuing cut-off dependence in the equation of motion. In particular we will concerned with the question whether classical divergences can be removed by counterterms.

Let us first simplify to a purely classical theory without any source.

$$D^{ab}_\mu F^{\mu\nu b} = 0 . \quad (5.4)$$

To study the cut-off dependence of this theory it is useful to reduce the theory with cut-off $\Lambda$ to an effective theory with (smaller) cut-off $\Lambda'$ by integrating out the modes with momenta $k$: $\Lambda' < |k| < \Lambda$. This generates extra interactions in the equations of motion which we collect in a source

$$D^{ab}_\mu F^{\mu\nu b} = \delta j^{\nu a} . \quad (5.5)$$

From the preceding chapter we know that the leading behavior of these interactions is

$$\delta j^{\nu a} = g^2 T(\Lambda - \Lambda')j^{\nu a}_{\text{lin}} + (g^2 T)^2 \log \left( \frac{\Lambda}{\Lambda'} \right) j^{\nu a}_{\text{log}} + O(1/\Lambda) . \quad (5.6)$$

Here we have used the result that linear divergences occur at one loop and log divergences at two loops. The current $j^{\nu a}_{\text{lin}}$ generates the linear divergences in a similar manner as the induced source in the Vlasov equations generates the HTL's (3.69). Also contributions from linear subdivergences occur. These are however suppressed when $(\Lambda - \Lambda')/\Lambda << 1$. For instance consider $n$ HTL self-energy insertions into a certain loop; this gives an extra factor $[g^2 T(\Lambda - \Lambda')]^n/\Lambda^{2n}$ compared to the loop without HTL insertions.

Equations (5.5) and (5.6) show that no matter what the momentum scale of interest is, the (dynamics of the) gauge fields will be sensitive to the cut-off. The time-scale (5.2) is an example of this sensitivity. Let us remark here that for static quantities the cut-off dependence is less severe. Especially the dimensionally reduced theory (3.80) valid for the non-perturbative length scale $1/g^2 T$ is cut-off independent [63].
5.3 Effective theory with counterterms

The simple nature of the cut-off dependence (5.6) suggests that the divergences can be removed by a subtraction of linear and logarithmic terms. In the Vlasov equation (5.3) including HTL corrections, we propose to do this as follows

\[ D_\mu F^{\mu\nu} = j_{\text{HTL}} - j_{\text{ct}} \]

with a counterterm for the linear divergences of the form \( j_{\text{ct}}^\nu = g^2 T\Lambda j_\text{lin}^\nu \).

Let us discuss the diagrams that this theory generates in perturbation theory. Firstly, there are classical diagrams, that is, diagrams that are constructed from classical propagators and tree-level vertices. The Feynman rules for the case of a scalar theory have been given in section 2.7 and one-loop and two-loop divergent diagrams relevant to \( SU(N) \) gauge theories have been studied in the previous chapter. Secondly, there are diagrams that contain HTL self-energy insertions and/or HTL vertices. The HTL vertices are of the general form (3.70), and the HTL self-energy has been worked out in detail in (3.10), (3.14), and (3.15). Finally, also diagrams with self-energy insertions and vertices from the counterterm current \( j_{\text{ct}}^\nu \) occur. The counterterm current is chosen to be equal to the linearly divergent current generated in the classical theory. It subtracts all linearly divergent one-loop vertex functions (without HTL resummation). These vertex functions have been discussed in section 4.2.1, where it was shown that they equal HTL vertex functions, except that the plasmon frequency is the classical one (4.5).

Since HTL's and classical counterterms are non-local and may contain any number of fields, see sections 3.2, 3.7, and 4.2.1, one might expect that these terms themselves give rise to new (and perhaps even worse) divergences. However, we shall now argue that such new terms are at most superficially logarithmically divergent. Consider a diagram in the effective theory (5.7) with some interactions from \( j_\text{lin}^\nu \). Such vertices come from linearly divergent diagrams (with more loops) in the purely classical theory. For such diagrams the power counting of the previous chapter applies. Hence, we know that its superficial degree of divergence is at most logarithmic, with a linear sub-divergence. The logarithmic divergence must be subtracted by a logarithmic counterterm. However, an explicit form cannot be given without actually doing the calculations. In the remainder of this chapter we shall confine ourselves to linear counterterms leaving the logarithmic problem to further research.

We find by this reasoning that diagrams with vertices from \( j_\text{lin}^\nu \) do not generate new linear divergences. The same may be argued for diagrams that
contain vertices from the HTL source $j^{\nu a}_{\text{HTL}}$, since these vertices may be seen as classical diagrams with a cut-off of the order of the temperature.

The subtraction of linear diverges as in (5.7) may be compared to the standard method of (HTL) resummations in the quantum theory. Then one uses the action

$$S = S_{\text{cl}} + \Gamma_{\text{HTL}} - \Gamma_{\text{ct}} = S_{\text{resum}} - \Gamma_{\text{ct}},$$

(5.8)

where the counterterm action $\Gamma_{\text{ct}}$ is in fact equal to the HTL action $\Gamma_{\text{HTL}}$, but is treated as a counterterm to the resummed action represented by the first two terms. In the simplest case, this amounts to the introduction of the HTL self-energy into the (resummed) propagator. The counterterm action corrects for overcounting in the resummed theory, because otherwise the resummed action would generate HTL’s which are already included in $\Gamma_{\text{HTL}}$.

Now consider the classical case. The classical diagrams that are generated by the resummed action in (5.8) give rise to linearly and logarithmically divergent terms controlled by a cut-off $\Lambda$, and finite terms. The linear divergences are the classical analogous of HTL’s and should as such be incorporated in the effective theory. However, in the resummed action they have already been taken into account explicitly. Therefore, any linear divergence that appears should be subtracted by the counterterm action in (5.8). The classical equations of motion (5.7) are then just the variational equations of the action (5.8). In a manner of speaking, we could say that (5.7) constitutes a classical resummation of HTL’s.

We should mention here that these arguments do not ensure that (5.7) provides a consistent theory. This requires that current and energy conservation as well as stability of the system need to be checked separately. We will find that these requirements (especially the stability of the system) limit the applicability of counterterms.

### 5.4 Continuum

Let us consider what this means for a classical theory on the continuum. We have noticed already that the HTL’s and classical linear divergences are the same except for a proportionality factor, see section 4.2. Therefore the two sources on the right hand side of (5.7) can be combined. This yields a HTL source whose strength is $\Lambda$-dependent

$$j^{\nu a}_{\text{HTL,ct}} = j^{\nu a}_{\text{HTL}} - j^{\nu a}_{\text{ct}} = \frac{3 \omega_{p1}^2}{4 \pi} V^\nu W^a(x, \nu).$$

(5.9)
The $W^a$-fields satisfy the standard equation (3.66). The $\Lambda$-dependent plasmon frequency is given by the difference between the quantum (4.4) and classical plasmon frequency (4.5)

$$\omega_{pl}^2(\Lambda) = -\frac{1}{3\pi^2}g^2N\int_0^\infty dk\ k^2\ [n'(k) - n'_{cl,\Lambda}(k)]$$

$$= \frac{1}{9}g^2NT\left( T - \frac{6}{\pi^2}\Lambda \right), \quad (5.10)$$

where the cut-off is introduced according to

$$n_{cl,\Lambda}(k) = \frac{T}{k}\Theta(\Lambda - k). \quad (5.11)$$

Thus we find that a subtraction in the plasmon frequency suffices to renormalize the classical linear divergences, confirming the proposal of [3,56].

We like also to mention that the subtraction in the plasmon frequency can be found from first principles [97]. Then one starts from the quantum theory and integrates out all modes except the classical ones with momentum $k < \Lambda$ 1, in the HTL approximation. This yields precisely the HTL's with a subtraction in the plasmon frequency. From consistency it then follows that this should provide the correct counterterm. (This is only straightforward for one-loop diagrams. At two loops it may be that non-local (non-divergent) vertices need to be included into the effective classical theory for the subtractions to match the divergences [59].) This mechanism for generation of the one-loop counterterm we have already encountered in the simple case of scalar $\lambda\phi^4$-theory in section 2.3. Namely, the counterterm for the linear divergence in the classical (zero-mode) contribution to the tadpole (2.12) was generated by the non-zero mode contribution (2.13).

Since the subtraction only enters the plasmon frequency, the system has the same properties as the HTL equations. The current is conserved. Also there is a conserved energy (the non-Abelian generalization of (3.29))

$$E = \frac{1}{2}\int d^3x\left[ (E^a)^2 + (B^a)^2 + 3\omega_{pl}^2(\Lambda)\int \frac{dQ}{4\pi}W^a(x,v)W^a(x,v) \right]. \quad (5.12)$$

We note that for $\omega_{pl}^2(\Lambda) < 0$ the energy is not bounded. This implies that the system is unstable. For $\omega_{pl}^2(\Lambda) > 0$, the cut-off has to satisfy $\Lambda < \pi^2T/6$.

1. This analysis is performed in perturbation theory in a fixed gauge. The difficulty is to divide the modes in soft (classical) and hard modes, while preserving BRS invariance. But this can be done [97].
Hence the we cannot interpret $\Lambda$ as a true UV-cut-off, but only as an intermediate cut-off. Therefore contributions that are proportional to inverse powers of $\Lambda$ cannot be made to vanish by sending the cut-off to infinity. But for $\Lambda \sim T$ they are suppressed by powers of the coupling.

As an example we estimate the part of classical one-loop self-energy proportional to $\Lambda^{-1}$, denoted as $\Pi_{1,cl}$, for soft momenta $p_0 \sim p \sim gT$. Since we consider a one-loop contribution, the expression will contain one classical distribution function. Hence, the one-loop self-energy is proportional to $T$. Combined with a dimensional analysis, we obtain $\Pi_{1,cl} \sim g^2 T p^2 \Lambda^{-1} \sim g^4 T^2$, for $\Lambda \sim T$. The part suppressed by inverse powers of the cut-off is of order $g^2$ compared to the HTL contribution. When we consider a classical diagram that is not divergent, we should compare the suppressed part (proportional to an inverse power of $\Lambda$) to the unsuppressed classical contribution. Then we find that it is of order $g$. Hence, even though the cut-off cannot be send to infinity, to leading order in the coupling $g$, suppressed contributions may be neglected.

5.5 Perturbative renormalization on a lattice

5.5.1 Static

Before turning to the HTL equations of motion, we shortly review the static classical theory on a lattice, as far as the linear divergences are concerned. The appropriate classical theory is the dimensionally reduced theory that we did already encounter in (3.79)

$$L_{DR} = \int d^3 x \left[ \frac{1}{4} F_i^a F_i^{ja} + \frac{1}{2} (D_i^{ab} A_0^b)^2 + \frac{1}{2} \mu_0^2 (A_0^a)^2 + \frac{1}{4} \lambda_0 (A_0^a A_0^b)^2 \right].$$

We now consider this theory on a lattice. Ideally, one would like to mimic the continuum theory as best as possible. This means that the thermal corrections that one has to include, should be calculated in the continuum, whereas the counterterms for the divergences need to be calculated on the lattice.

Consider for instance the Debye mass. It contains the only linear divergence in the static 3d theory. A counterterm for this divergence may be introduced in the mass of the temporal gauge field [63,87]

$$\mu_0^2 = m_D^2 - m_{cl,lat}^2,$$

with the continuum HTL contribution

$$m_D^2 = -2g^2 N \int \frac{d^3 k}{(2\pi)^3} n'(k) = \frac{1}{3} g^2 N T^2,$$
and the classical mass (for a simple cubic lattice with lattice spacing $a$)

$$m_{\text{cl,lat}}^2 = -2g^2 N \int \frac{d^3p}{(2\pi)^3} n'(\Omega_p) \approx 0.51 g^2 N T a^{-1}. \quad (5.16)$$

The momentum $p$ is restricted to the first Brillouin zone $|p_i| \leq \pi/a$ and the energy $\Omega_p$ is

$$\Omega_p^2 = \frac{4}{a^2} \left[ \sin^2 \left( \frac{p_x a}{2} \right) + \sin^2 \left( \frac{p_y a}{2} \right) + \sin^2 \left( \frac{p_z a}{2} \right) \right]. \quad (5.17)$$

The mass (5.16) is the linearly divergent contribution to the Debye mass on the lattice. Its subtraction in (5.14) ensures that no linear divergences are present in the static theory with the mass counterterm included. The continuum HTL contribution (5.15) to the mass (5.14) provides the finite renormalization. It ensures that the leading-order Debye screening in this effective lattice model is the same as in the continuum.

### 5.5.2 Real-time

The above approach may be extended to a real-time classical theory. We consider again the equation of motion for the gauge fields

$$D^b_{\mu} F^{\mu \nu c} = j^{\nu b}_{\text{HTL,ct}}, \quad (5.18)$$

but now space is a simple cubic lattice with lattice spacing $a$. A simple subtraction in the plasmon frequency will not suffice to remove the linear divergences, as it did for the continuum. Therefore we start anew from (5.7). Similar to the static mass (5.14), that consists of the continuum HTL Debye mass with the classical lattice mass subtracted, we construct a source to contain a continuum HTL contribution with a classical lattice contribution subtracted

$$j^{\nu b}_{\text{HTL,ct}} = j^{\nu b}_{\text{HTL}} - j^{\nu b}_{\text{ct}}. \quad (5.19)$$

To this end, we introduce two particle distribution functions $\delta N(x, k)$ and $\delta N_{\text{ct}}(x, p)$ for particles with energies $E_k = |k|$ and $\Omega_p$ respectively. The idea is that the particle distribution function $\delta N$ generates the continuum HTL source $j^{\nu b}_{\text{HTL}}$, and $\delta N_{\text{ct}}$ generates the counterterms for the linear lattice divergences in the current $j^{\nu b}_{\text{ct}}$. The particle distribution function $\delta N$ satisfies the equation (the non-Abelian generalization of (3.20))

$$V^\mu D^b_{\mu} \delta N^c(x, k) = g_v \cdot E^b(x) n'(k), \quad (5.20)$$
with $V^\mu = (1, k/k)$. It contributes to the HTL current as

$$j_{\text{HTL}}^b(x) = 2gN \int \frac{d^3k}{(2\pi)^3} V^\nu \delta N^b(x, k). \tag{5.21}$$

To obtain an effective lattice theory free of linear divergences, the current $j_{\text{ct}}^b$ should subtract the linear classical lattice divergences. To achieve this, the current $j_{\text{ct}}^b$ is chosen equal to the induced source of the classical lattice Vlasov theory. The latter generates classical lattice HTL's which are exactly the linear divergences that need to be subtracted (remember that classical HTL's correspond to linear divergences). In the classical lattice Vlasov theory the distribution function satisfies the equation [12]

$$V_{\text{lat}}^\mu D^{bc}_{\mu} \delta N^c_{\text{ct}}(x, p) = g v_{\text{lat}} \cdot E^b(x) n'_{\text{cl}}(\Omega_p), \tag{5.22}$$

with the four-velocity on the lattice $V_{\text{lat}} = (1, v_{\text{lat}})$ with

$$v_{\text{lat}}^i = \partial_p, \Omega_p = \frac{1}{a\Omega_p} \sin(a p_i), \tag{5.23}$$

and $|v_{\text{lat}}| \neq 1$ in general. The counterterm current is then given by

$$j_{\text{ct}}^{\mu b}(x) = 2gN \int \frac{d^3p}{(2\pi)^3} V^\nu_{\text{lat}} \delta N^b_{\text{ct}}(x, p). \tag{5.24}$$

Here the integration over $p$ is restricted to the first Brillouin zone $|p_i| < \pi/a$. As in the continuum, it is useful to define a field $W^b(x, v)$ that satisfies

$$\partial_t W^b(x, v) + v \cdot D^{bc} W^c(x, v) = v \cdot E^b, \tag{5.25}$$

in the $A^b_0 = 0$ gauge. Since the lattice velocity (5.23) is not restricted to the speed of light, we have to allow for general velocities $v$ in (5.25). Hence, the $W^b$-field lives on a 6+1 dimensional space instead of the 5+1 dimensional space that is sufficient in the continuum case. The current (5.19) reads

$$j_{\text{HTL,ct}}^{\mu b}(x) = 3\omega_{\text{pl}}^2 \int \frac{d\Omega}{4\pi} V^\nu W^b(x, v) - 2g^2 NTA^{-1} \int \frac{d^3\hat{p}}{(2\pi)^3} \hat{\Omega}_p^{-2} V^\nu_{\text{lat}} W^b(x, v_{\text{lat}}). \tag{5.26}$$

with the dimensionless quantities $\hat{p}_i = a p_i$, $\hat{\Omega}_p = a \Omega_p$ and the integration restricted to $|\hat{p}_i| < \pi$. It may be verified that the induced current (5.26) is covariantly conserved: $D^{\mu c}_{\nu} j_{\text{HTL,ct}}^{\mu c} = 0$, because.
The first term on the right hand side of (5.26) is the continuum contribution for which the $k$-integration decouples and has been performed. In the second term on the right hand side of (5.26), the integration cannot be simplified since the velocity not only depends on the direction of the momentum $\mathbf{p}$, but also on its magnitude. The lattice contribution requires fields that depend also on the magnitude of the velocity $|v_{\text{lat}}| < 1$. This in contrast to the calculation of the continuum contribution to the induced current a field $W(x, \mathbf{v})$ depending on the direction of $\mathbf{v}$ only is sufficient, and a subtraction in the plasmon frequency renders the effective classical theory free of linear divergences. In section 5.6.2 we will study the question whether, for the calculation of the Chern-Simons diffusion rate, we may approximate the induced current with fields that only depend on the direction of the velocity.

Just as the usual HTL equations, the equations (5.25) and (5.26) (or equivalently (5.20), (5.22) and (5.19)), together with the equation for the gauge fields, define a perturbation theory. Taking retarded initial conditions the retarded propagator (and higher-order retarded vertex functions) can be obtained, as in [22]. The classical KMS condition then fixes the entire propagator, including its thermal part. Using perturbation theory we may verify that also the time-dependent counterterms are correct. We calculate the retarded propagator to one-loop order. In a general linear gauge it takes the form

$$D^{\mu\nu}_{\text{cl}}(Q) = \left[ g^{\mu\nu} Q^2 - Q^\mu Q^\nu + F^\mu F^\nu + \Pi^{\mu\nu}_{\text{cl}}(Q) + \Pi^{\mu\nu}_{\text{HTL,ct}}(Q) \right]^{-1}, \quad (5.27)$$

with $F^\mu$ the gauge fixing vector and $\Pi^{\mu\nu}_{\text{cl}}$ the classical self-energy and $\Pi^{\mu\nu}_{\text{HTL,ct}}$ the counterterm self-energy introduced in the induced source (5.26). The classical self-energy to one-loop order reads [25, 26]

$$\Pi^{\mu\nu}_{\text{cl}}(Q) = 2 g^2 N a^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} n^\prime_{\text{cl}}(\Omega_p) \left[ -\delta^{\mu0}\delta^{\nu0} + \frac{V^\mu_{\text{lat}} V^\nu_{\text{lat}} q_0}{q_0 + i\epsilon - v_{\text{lat}} \cdot q} \right]. \quad (5.28)$$

At this order the classical self-energy contains no contribution from the induced source. The linearized induced source

$$j^{\mu}_{\text{HTL,ct}}(x) = \int d^4 x' \Pi^{\mu\nu}_{\text{HTL,ct}}(x, x') A_\nu(x') \quad (5.29)$$

defines the retarded self energy

$$\Pi^{\mu\nu}_{\text{HTL,ct}}(Q) = \Pi^{\mu\nu}_{\text{HTL}}(Q) - \Pi^{\mu\nu}_{\text{cl}}(Q), \quad (5.30)$$
with the continuum HTL self-energy

$$\Pi_{\text{HTL}}^{\mu\nu}(Q) = 3\omega_{pl}^2 \left[ -\delta^{\mu0}\delta^{\nu0} + \int \frac{d\Omega}{4\pi} \frac{V^{\mu}V^{\nu}}{q_0 + i\epsilon - \mathbf{v} \cdot \mathbf{q}} \right] .$$

(5.31)

Inserting the self-energy (5.30) in the propagator (5.27), the linear divergent classical self-energy in (5.27) is compensated by the subtraction in (5.30). The resulting self-energy in the propagator (5.27) is the correct (continuum) HTL self-energy (5.31). Furthermore one may note that in the static limit the self-energy (5.30) reduces to the counterterm mass (5.14), as it should.

Unfortunately the system defined by (5.25), (5.26) is unsuitable for numerical implementation [90]. This follows from the conserved energy of the system

$$E = \int d^3x \frac{1}{2} \left[ (\mathbf{E}^b)^2 + (\mathbf{B}^b)^2 + 3\omega_{pl}^2 \int \frac{d\Omega}{4\pi} \mathcal{W}^b(x,\mathbf{v})\mathcal{W}^b(x,\mathbf{v}) 
- 2g^2NTa^{-1} \int \frac{d^3p}{(2\pi)^3} \hat{\Omega}^{-2}\mathcal{W}^b(x,\mathbf{v}_{\text{lat}})\mathcal{W}^b(x,\mathbf{v}_{\text{lat}}) \right] ,$$

(5.32)

with \( \mathbf{B} \) the chromo-magnetic field and \( b \) the adjoint index. The energy is unbounded from below and this means that the system is unstable. Perturbatively there is no problem, the effect of the counterterm particle distribution function is precisely neutralized by the hard modes of the classical gauge fields. However in a non-perturbative lattice simulation the evolution of the particle density and the hard modes will differ, which means that after some time the (wrong) effect of the counterterm particle distribution function is no longer compensated by the hard modes, and the fields will (exponentially) blow up.

5.6 Two stable lattice models

5.6.1 Model with lattice dispersion relation

The goal is now to obtain a model that is defined on the lattice, that is stable and can be used to calculate IR-sensitive real-time properties of a non-Abelian plasma without linear divergences. Such a model should meet the following three requirements:
1. In the small lattice spacing limit the continuum HTL equations of motion should be obtained.
2. Counterterms for the linear divergences (on the lattice) should be included.
The energy must be bounded from below.

As a reminder, the model considered in the previous section failed to have bounded energy. To obtain a model with a bounded energy one may consider a model where the modes inducing the finite renormalization have the same dispersion relation as the counterterm modes. In this section we focus on a model where both the counterterm modes and the modes generating the finite renormalization satisfy a lattice dispersion relation. Perhaps we should warn the practical-minded readers that the model considered below will not allow for a useful continuum limit. Those readers may be more interested in the next section, where the other possibility of enforcing the continuum dispersion relation on the counterterm modes is explored.

To obtain HTL equations where the both types of modes satisfy a lattice dispersion relation, we do not match to a continuum quantum theory as in the previous section, but to a quantum theory on the lattice, with a (small) lattice spacing $a_S$. The trick is that we can then combine the required generation of quantum HTL's and classical counterterms into one distribution function $\delta \tilde{N}(x, \hat{p})$, where $\hat{p}$ is the dimensionless lattice momentum. With this distribution function the Vlasov equations (in the $A_0^b = 0$ gauge) become

$$D^b_c F^\mu_{uc}(x) = j^\nu(x) = 2gN \int \frac{d^3 \hat{p}}{(2\pi)^3} V^\nu c \delta \tilde{N}^b(x, \hat{p}),$$

$$\partial_t \delta \tilde{N}^b(x, \hat{p}) - v_{\text{lat}} \cdot D^b_c \delta \tilde{N}^c(x, \hat{p}) = -g v_{\text{lat}} \cdot E^b(x) \partial_{\hat{p}_c} \tilde{N}(\hat{p}),$$

with $x = (t, x)$, where the time $t$ is continuous and the position $x$ is an element of a cubic lattice with (large) lattice spacing $a_L$. The dimensionless momentum $\hat{p}$ is restricted to the first Brillouin zone $|\hat{p}_i| < \pi$, the dimensionless energy is $\hat{E} = 2\sqrt{\sum_i \sin(\hat{p}_i/2)^2}$ and the velocity is $v_i = \partial_{\hat{p}_i} \hat{E}$.

The lattice spacing has been scaled out of the above equations and enters only in the equilibrium distribution function $\tilde{N}$. The distribution function $\tilde{N}$ should contain a contribution that generates, after solving (5.34), the quantum HTL source and a contribution that generates the counterterms for the classical divergences. The important step is now to allow for different lattice spacings $a_L, a_S$ in the the different parts of the equilibrium distribution function

$$\tilde{N}(\hat{p}) = a_S^{-2} n^S(\hat{p}) - a_L^{-2} n^L(\hat{p}),$$

with

$$n^S(\hat{p}) = \frac{1}{e^{\hat{E} / (a_S T)} - 1},$$
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\[ n_{\text{cl}}^{L}(\hat{\Omega}_{\hat{p}}) = \frac{T a_{L}}{\hat{\Omega}_{\hat{p}}}, \quad (5.36) \]

and \( T \) the temperature of the system.

To see that the model (5.33) and (5.34) contains the counterterms for the linear divergences it is useful to introduce the field

\[ \tilde{W}^{b}(x, \hat{p}) = \delta \tilde{N}^{b}(x, \hat{p}) \left( -g \tilde{N}'(\hat{\Omega}_{\hat{p}}) \right), \quad (5.37) \]

where \( \tilde{N}'(\hat{\Omega}_{\hat{p}}) = \partial_{\hat{\Omega}_{\hat{p}}} \tilde{N}(\hat{\Omega}_{\hat{p}}) \). It satisfies the equation

\[ \partial_{t} \tilde{W}^{b}(x, \hat{p}) - v_{\text{lat}} \cdot D^{ab}\tilde{W}^{c}(x, \hat{p}) = v_{\text{lat}} \cdot B^{b}(x). \quad (5.38) \]

The source can be split into a part generating the finite quantum HTL source and a part subtracting the linear divergent classical source

\[ j_{\text{ind}}^{\nu b} = j_{\text{fin}}^{\nu b} - j_{\text{ct}}^{\nu b}. \quad (5.39) \]

In terms of the field \( \tilde{W} \) these sources read

\[ j_{\text{fin}}^{\nu b} = 2 g^{2} N \int \frac{d^{3}p_{S}}{(2\pi)^{3}} V_{\text{lat}}^{\nu}(\Omega_{S}) \tilde{W}^{b}(x, p_{S} a_{S}), \quad (5.40) \]
\[ j_{\text{ct}}^{\nu b} = 2 g^{2} N \int \frac{d^{3}p_{L}}{(2\pi)^{3}} V_{\text{lat}}^{\nu} n_{\text{cl}}^{L}(\Omega_{L}) \tilde{W}^{b}(x, p_{L} a_{L}), \quad (5.41) \]

with \( p_{S} = a_{S}^{-1} \hat{p}, \Omega_{S} = a_{S}^{-1} \hat{\Omega}_{\hat{p}} \) and similar for \( p_{L}, \Omega_{L} \). Both sources (5.40) and (5.41) are covariantly conserved.

Written in dimensionful quantities we recognize the source \( j_{\text{ct}} \) (5.41) as the classical HTL source on a lattice with lattice spacing \( a_{L} \). The difference with the perturbative model of the previous section is the choice of the finite renormalization. The source \( j_{\text{fin}} \) (5.40) is the quantum HTL source on a lattice with lattice spacing \( a_{S} \). To extract continuum results from this model we should require \( a_{S}^{-1} \gg T \). Also \( a_{L} \) cannot be too large, since the relevant field configurations for the sphaleron rate have size \( (g^{2}T)^{-1} \). Therefore we should at least require \( a_{L}^{-1} \gg g^{2}T \). However, as Bödeker [27] has shown, modes of spatial size \( (gT)^{-1} \) give corrections of \( O(1) \); to take these corrections into account requires a smaller lattice spacing \( a_{L}^{-1} \gg gT \).

To ensure the stability of the model (5.33) and (5.34) we demand that the energy,

\[ E = \int d^{3}x \, \frac{1}{2} \left[ (E^{b})^{2} + (B^{b})^{2} + 2N \int \frac{d^{3}p}{(2\pi)^{3}} \delta \tilde{N}^{b}(x, \hat{p}) \delta \tilde{N}^{b}(x, \hat{p})/\tilde{N}'(\hat{\Omega}_{\hat{p}}) \right], \quad (5.42) \]
Table 5.1: The maximum value of $a_S^{-1}/T$ given the ratio $a_L/a_S$. This follows from the requirement that the energy is bounded from below.

<table>
<thead>
<tr>
<th>$a_L/a_S$</th>
<th>1.1</th>
<th>1.5</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>25</th>
<th>50</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max(a_S^{-1})/T$</td>
<td>0.31</td>
<td>0.64</td>
<td>0.86</td>
<td>1.36</td>
<td>1.68</td>
<td>1.97</td>
<td>2.06</td>
<td>2.33</td>
<td>2.59</td>
<td>3.42</td>
</tr>
</tbody>
</table>

is bounded from below. This leads to the requirement

$$-\tilde{N}'(\hat{\Omega}_p) > 0.$$  (5.43)

For $\hat{p} = 0$, this requirement implies $a_S < a_L$, which is in accordance with the general idea that the classical theory is matched to a quantum theory with a smaller lattice spacing.

The function $-\tilde{N}'(\hat{\Omega}_p)$, with $a_S < a_L$, decreases from plus infinity at $\hat{\Omega}_p = 0$, to its minimum below zero, after which it increases and asymptotically reaches zero. The maximum value of the dimensionless energy is $\hat{\Omega}_p = 2\sqrt{3}$. Demanding that

$$-\tilde{N}'(2\sqrt{3}) > 0,$$  (5.44)

together with $a_S < a_L$ is sufficient for (5.43) to hold for any $\hat{p}$. In this way, we obtain a maximum value for $a_S^{-1}$ given the ratio $a_L/a_S$. In table 5.1 the smallest possible lattice spacings $a_S$ are listed for some values of the ratio $a_L/a_S$.

The conclusion is that it is possible to match a real-time classical lattice theory, with lattice spacing $a_L$, to a real-time quantum lattice theory at smaller lattice spacing $a_S$. But that this is restricted by the constraint that the energy must be bounded from below. Given the lattice spacing of the classical theory this restricts the lattice spacing of the quantum theory to which can be matched.

We see from table 5.1 that in order to obtain continuum-like HTL contributions, the ratio $a_L/a_S$ should be very (exponentially) large. Since we want $a_L^{-1} >> gT$, the coupling coupling $g$ should be chosen extremely small. For instance, if we fix $a_S^{-1} = 2.59T$, then stability requires $a_L/a_S \geq 100$, so $a_L^{-1} \leq 2.59 \times 10^{-2}T$ and $g << 2.59 \times 10^{-2}$.

The very small coupling that is required to reach the continuum limit makes this model useless for practical purposes. It is interesting to
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note, however, that matching to a quantum lattice model is not a problem. Rather, the problem is to match lattice with the continuum than classical with quantum. In the next section we will discuss an approximate matching of lattice classical to continuum quantum that may be useful for numerical calculations.

5.6.2 Model with a continuum dispersion relation

The other approach that we want to investigate is a model where we enforce the continuum dispersion relation on the counterterm modes. Such a model has the advantage that instead of a 6+1d field \( \delta N \), a 5+1d auxiliary field \( W^b(x, \tilde{v}_{\text{lat}}) \), that depends only on the direction of the velocity \( \tilde{v}_{\text{lat}} = v_{\text{lat}}/|v_{\text{lat}}| \), can be used. The counterterms that we obtain in this model are not exact, but for the calculation of the Chern-Simons diffusion rate they provide a reasonable approximation.

The model that we consider is given by the replacement of the induced source (5.26) by the expression

\[
j_{\text{app}}^{\nu b}(x) = 3\omega_{\text{pl}}^2 \int \frac{d\Omega}{4\pi} V^{\nu} W^b(x, v) - 2g^2 NTa^{-1} \int \frac{d^3 p}{(2\pi)^3} \hat{\Omega}_{p}^{-2} v_{\text{lat}} |\tilde{v}_{\text{lat}} W^b(x, \tilde{v}_{\text{lat}}), \tag{5.45}\]

with \( \tilde{v}_{\text{lat}} = (1, \tilde{v}_{\text{lat}}) \). We use this construct since it reproduces the induced vector current for a field configuration with \( W^b(x, \tilde{v}_{\text{lat}}) = W^b(x, v_{\text{lat}}) \). And the vector current is essential in the dynamics of the soft fields. The density is then determined by requiring current conservation \( D^{bc} j_{\text{app}}^{\mu c} = 0 \). As a consequence the induced density \( j_{\text{HTL,ct}}^{0b} \) in (5.26) is not correctly reproduced by the density \( j_{\text{app}}^{0b} \). This can be understood as follows, changing the velocity of the particles and requiring current conservation either the vector current or the density can remain unaltered. The expression (5.45) is the lattice equivalent of the approximation for the induced source used by Iancu in [56].

We may also write (5.45) as

\[
j_{\text{app}}^{\nu b}(x) = \int \frac{d\Omega}{4\pi} m^2(v) V^{\nu} W^b(x, v), \tag{5.46}\]

with the velocity dependent mass

\[
m^2(v) = 3\omega_{\text{pl}}^2 - 2g^2 NTa^{-1} \int \frac{d^3 p}{(2\pi)^3} \hat{\Omega}_{p}^{-2} v_{\text{lat}} |\tilde{v}_{\text{lat}} \delta^S(v - \tilde{v}_{\text{lat}}). \tag{5.47}\]
Chapter 5. Counterterms for linear divergences

The second term contains a linear divergence in the direction \( \mathbf{v} = (1,1,1)/\sqrt{3} \) [90] and logarithmic divergences in directions \( \mathbf{v} = (1,1,s)/\sqrt{2 + s^2} \) with \(-1 < s < 1\) (and directions related by symmetry). Therefore, the mass and the energy are not strictly positive. To obtain a bounded energy some averaging over the direction of the velocity \( \mathbf{v} \) should be performed. This can be achieved by expanding the field \( W^b(x, \mathbf{v}) \) in spherical harmonics

\[
W^b(x, \mathbf{v}) = \sum_{lm} W^b_{lm}(x) Y_{lm}(\mathbf{v}), \tag{5.48}
\]

and keeping a finite number terms. The induced source can then be written as

\[
j^\nu_{\text{app}}(x) = \sum_{lm} a^\nu_{lm} W_{lm}(x), \tag{5.49}
\]

with coefficients

\[
a^\nu_{lm} = \int \frac{d\Omega}{4\pi} m^2(\mathbf{v}) V^\nu Y_{lm}(\mathbf{v}). \tag{5.50}
\]

Given the lattice spacing \( a \), the requirement that the energy is bounded from below, puts an upper bound \( l_{\text{max}} \) on allowed values of \( l \). It was found in [29] that the Chern-Simons diffusion rate is insensitive to \( l_{\text{max}} \) for even \( l_{\text{max}} \). In the following we will therefore focus on the approximation made in (5.45).

As was already mentioned, the approximation (5.45) changes the charge density. For instance, for the coefficient \( a^0_{00} \) we have

\[
a^0_{00} = m_D^2 - 2g^2 N Ta^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} \Omega^{-2} |\mathbf{v}_{\text{lat}}|. \tag{5.51}
\]

Comparing (5.51) with (5.16), we see that the expression (5.45) does not correctly reproduce the counterterm for the Debye mass. This implies that the current is not suitable to describe the behavior of fields at length scale \((gT)^{-1}\).

To see whether the approximation (5.45) is valid for fields at the length scale \((g^2T)^{-1}\), we consider the spatial components of the counterterm self-energy generated by the source (5.45) (for \( l_{\text{max}} \to \infty \))

\[
\Pi_{\text{app}}^{ij}(q_0, \mathbf{q}) = \Pi_{\text{HTL}}^{ij}(q_0, \mathbf{q}) - \Pi_{\text{app,ct}}^{ij}(q_0, \mathbf{q}), \tag{5.52}
\]

with

\[
\Pi_{\text{app,ct}}^{ij}(q_0, \mathbf{q}) = 2g^2 N Ta^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} \Omega^{-2} |\mathbf{v}_{\text{lat}}| \frac{\mathbf{v}_{\text{lat}} \cdot \mathbf{q}_0}{q_0 + i\epsilon - \mathbf{v}_{\text{lat}} \cdot \mathbf{q}}, \tag{5.53}
\]
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which should be compared with the classical self-energy (5.28). It is important
to realize that the relevant fields for the Chern-Simons diffusion rate we
are interested in, have typical momenta of order \( q_0 \sim g^4 T, q \sim g^2 T \) [12],
see section 3.9. For the gauge fields that are relevant for the Chern-Simons
diffusion rate \( q_0 \ll |q| \) and we may neglect \( q_0 \) in the denominator of the
counterterm (5.53) and the classical self-energy (5.28). We then note that
these two expressions are equal and that they cancel. For these fields the
effective theory is finite and reproduces the HTL contributions.

However, as was realized by Bödeker [27], interactions between semi-
hard and soft fields give corrections to the dynamics of the soft fields which
are not suppressed by powers of \( g \). On the contrary, even \( \log(1/g) \) enhanced
contributions arise, resulting in the Chern-Simons diffusion rate [27]

\[
\Gamma_{CS} = \left[ \kappa_1 \log \frac{1}{g} + \kappa_2 \right] g^{10} T^4. \tag{5.54}
\]

The counterterms in the approximated source (5.45) and the classical HTL’s
do not cancel for the semi-hard modes (with momenta \( q_0, q \sim g T \)). Therefore
the semi-hard modes are sensitive to the cut-off \( a^{-1} \).

The leading log contribution arises from the IR-sensitive part of the
contribution of the semi-hard modes with momenta \( k_0 \ll k \sim \mu, \) with
\( \mu \sim g^2 T \) an IR cut-off. For these momenta the approximation is correct to
leading order. Therefore a calculation of the Chern-Simons diffusion rate with
approximation (5.45) produces the correct leading-log contribution, that is
the coefficient \( \kappa_1 \) in (5.54) is independent of the lattice spacing.

The \( \mathcal{O}(1) \) correction from the semi-hard modes does depend on the cut-off.
An estimate of the cut-off dependence can be obtained from a comparison
of the classical HTL self-energy (5.28) with the counterterm (5.53). To be
explicit, we compare the diagonal components at zero spatial momentum

\[
\Pi^{ii}_{cl}(q_0, q = 0) = 2g^2 NT a^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} \hat{\Omega}_{-2}^- |v_{\text{lat}}|^2 = 0.26 g^2 NT a^{-1}, \tag{5.55}
\]

\[
\Pi^{ii}_{\text{app}}(q_0, q = 0) = 2g^2 NT a^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} \hat{\Omega}_{-2}^- |v_{\text{lat}}| = 0.34 g^2 NT a^{-1}. \tag{5.56}
\]

Comparing the difference between (45) and (46) with the HTL self-energy
at zero spatial momentum \( \Pi^{ii}_{\text{HTL}}(q_0, q = 0) = 3 \omega_{pl}^2 \sim g^2 T^2/5, \) we obtain
an estimate for the maximal error of about 25% for \( a^{-1} = T/\hbar. \) However,
the semi-hard modes that give the \( \mathcal{O}(1) \) correction have space-like momenta
\( q_0 < |q| \) [28]. For these modes we expect (5.53) to be a better approximation
to the classical self-energy (5.28).
Besides the mismatch between classical HTL's and the counterterms from (5.45), the lattice spacing dependence of \( \kappa_2 \) depends on the magnitude of the \( \mathcal{O}(1) \) correction from the semi-hard modes. Especially when the soft modes dominate the contribution to \( \kappa_2 \) this model is suitable for a calculation of the Chern-Simons diffusion rate.

5.7 Conclusion

In this chapter, we studied the linear divergences in classical SU(\( N \)) gauge theories at finite temperature. Counterterms for these divergences can be incorporated in an (induced) source. Although the divergences are non-local the equations of motion including these counterterms can be given in a local form by introducing auxiliary fields. In the continuum a subtraction in the plasmon frequency is sufficient to render the classical theory free of linear divergences. For a lattice theory this is not the case.

We have presented two lattice models that are stable. The first matches the classical lattice model to a real-time quantum lattice theory with a small lattice spacing \( a_S \). The requirement that the energy is bounded, presents a lower bound on \( a_S \), given the lattice spacing \( a_L \) of the classical model. To obtain the continuum limit \( a_L \) has to be extremely large, which requires an unrealistically small coupling \( g \) to keep the interesting excitations on the lattice. In the second model we argued that the restriction to auxiliary fields depending on the direction of the velocity allows for a reasonable approximation (5.45) for the calculation of quantities dominated by fields with momenta \( (q_0, q) \sim (g^4T, g^2T) \), such as the Chern-Simons diffusion rate.