Dynamics of Gauge Fields at High Temperature

Nauta, B.J.

Citation for published version (APA):
Nauta, B. J. (2000). Dynamics of Gauge Fields at High Temperature
6 Baryon-number generation in the broken phase

6.1 Introduction

One of the important cosmological observations that may provide information about physics beyond the standard model is given by the matter anti-matter asymmetry in the universe [19, 73]. Quantitatively the asymmetry may be expressed by the ratio between the baryon-number density and the photon density [101]

\[
\frac{n_B}{n_\gamma} = (1.55 - 4.45) \times 10^{-10}.
\]  

(6.1)

The problem is to explain the observation (6.1) without assuming a special initial state for the universe\(^1\).

In the introduction of this thesis we have already described the standard scenario for electroweak baryogenesis. This scenario requires the electroweak phase-transition to be strongly first-order. For experimentally allowed Higgs masses this requirement is not satisfied by the standard model [64, 107]. Hence the standard scenario does not provide an explanation of the observed baryon asymmetry within the standard model. Extensions of the standard model, such as the minimal supersymmetric standard model, may allow for a phase-transition strong enough to generate (sufficient) baryons. However with the increasing experimental lower bound on the Higgs mass, for such models the parameter space consistent with the observed baryon asymmetry becomes quite small [37, 39]. This has triggered the search for alternative scenario's for baryogenesis at the electroweak scale. For instance, recently scenario's have been studied where baryon production occurs at the end of inflation during or after preheating [47, 51, 74].

\(^1\) To state the problem completely we have to specify the value of some conserved charges [69]. We do not assume special initial conditions and take for the difference between baryon and lepton number \(B - L = 0\), hypercharge \(Y = 0\) and isospin \(T_3 = 0\).
We study a different scenario. It requires the electroweak phase-transition to be weakly first-order. Therefore, this scenario will not provide an explanation of the observed asymmetry (6.1) within the standard model also. But, for extended models the allowed Higgs mass lies below that for the standard scenario.

The scenario discussed here is based on the following. At high temperatures, there is an effective potential for the baryon number given by the free energy at given baryon number, \( F(B) \). Before the phase transition in the symmetric phase this potential is symmetric and quadratic for small baryon-number densities, we write \( F(B) = \alpha B^2 \). In equilibrium the expectation value of the baryon number vanishes: \( \langle B \rangle = 0 \). During the phase transition, when the particles acquire a mass by the Higgs mechanism, the potential will change to \( F(B) = \alpha' B^2 \), with \( \alpha' < \alpha \). Also in the broken phase the baryon-number expectation value vanishes in equilibrium. But, the point is that when C and CP-violation is present the baryon-number expectation value may acquire a non-zero value after the transition before it relaxes to its equilibrium value zero. If the Higgs expectation value has grown enough to effectively stop baryon-number violating processes before this relaxation, the baryon-number will remain at its non-zero value and baryons will have been created.

To handle baryon-number violation in a non-equilibrium situation is extremely complicated (at least when linear response theory does not apply). Therefore most of this chapter deals with a simpler situation, namely with a system in equilibrium without potential \( F(B) = 0 \). In this case, the expectation value \( \langle B \rangle \) is constant. We assume that initially it is zero, then for all times the expectation value vanishes. However, the distribution function of the baryon number may develop an asymmetry. We will show this happens indeed. In particular, we will argue that the position of the maximum of the distribution will not remain at \( B = 0 \), but move as \( B \sim \delta t \), with \( \delta \) the strength of the CP-violation. (The analysis is based on an expansion in \( \delta \), it may well be that the \( O(\delta^2) \) contribution will change this behavior. But for \( t \sim \delta^{-1} \) we expect this linear increase or decrease in time.) To keep the expectation value equal to zero this means that the tail of the distribution function is much larger in the direction opposite of which the peak moves.

Let us now sketch how such an asymmetric developing distribution may lead to the a temporary non-zero baryon-number expectation value when the effective potential is included. We consider an initial distribution that is peaked more sharply around \( B = 0 \) than the equilibrium distribution (the situation after the first-order phase transition in the above described
6.2. Sakharov requirements

In 1967 Sakharov was the first to address the problem of baryon number generation \[108\]. He noted that there are three requirements to be met:

1. baryon-number non-conservation,
2. C- and CP-violation,
3. departure from equilibrium.

Since we are interested in the possibility of baryogenesis at the electroweak scale, we consider if and how these requirements may be satisfied in the standard model.

1. baryon-number non-conservation

As was discovered by 't Hooft \[54\] baryon-number is not conserved. This is due to the anomaly equation

\[ \partial_\mu j^\mu_B = \frac{3g^2}{32\pi^2} F^{a}_{\mu\nu} \tilde{F}^{\mu\nu a}, \]  

(6.2)

with baryon current \( j^\mu_B \), the SU(2) field strength \( F^{a}_{\mu\nu} \), its dual \( \tilde{F}^{\mu\nu a} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F^{\rho\sigma} \), and gauge coupling \( g \). Together with the vacuum structure of the SU(2)-Higgs sector of the standard model. (The contribution of the \( U(1) \)-fields to the anomaly equation is not given in (6.2). Since, due to the trivial vacuum structure of the \( U(1) \)-fields, such a contribution cannot lead to a permanent change in baryon-number.) A transition from one (classical) vacuum to the...
next in the positive (negative) Chern-Simons direction yields a change in baryon-number of $+3(-3)$. In the broken phase these vacua are separated by energy barriers. At zero temperature, transitions from one vacuum to another occur through instanton processes, and the rate of baryon-number non-conservation is very much suppressed. At high temperatures however, the system can go over the barrier due to thermal fluctuations. Then the transition rate is proportional to the Boltzmann factor $\exp -\beta E_{\text{sph}}$, where $E_{\text{sph}}$ is the energy of the minimal energy configuration at the barrier, called the sphaleron. This sphaleron energy is $E_{\text{sph}} = \text{"number" } \times 4\pi v/g$, with $v$ is the expectation value of the Higgs field. The "number" depends on the Higgs mass, for typical values of about $100 - 300$ GeV the "number" is approximately 2. At zero temperature the Higgs vacuum expectation value $v \approx 250$ GeV determines the sphaleron energy $E_{\text{sph}} \approx 10$ TeV. In the next section we will discuss sphaleron transitions in more detail.

2. C- and CP-violation
In the standard model C symmetry is violated. In our scenario (as in most scenario's for electroweak baryogenesis) C-violation is included through the relation (6.2).

Also CP-violation is present in the standard model, namely in the CKM-matrix. However an order of magnitude estimate of CP-violation in the CKM-matrix indicates that it is too small to account for the observed matter-antimatter asymmetry [106,111]. In extensions of the standard model, such as the minimal supersymmetric standard model and the two Higgs doublet model, the amount of CP-violation may be sufficient. We will use the effective action approach and include CP-violation through the following nonrenormalizable dimension-eight operators

$$S_{\text{CP}} = \int d^4x \frac{1}{M^4} \left[ \delta_{\text{CP}}^1 (D_\rho \phi)^\dagger (D^\rho \phi) - \delta_{\text{CP}}^2 \frac{1}{4} F_{\rho \sigma}^a F^{\rho \sigma a} \right] \frac{3g^2}{32\pi^2} F_{\mu \nu}^b \tilde{F}^{\mu \nu b}, \ (6.3)$$

where the mass $M$ and the coefficients $\delta_{\text{CP}}^1, \delta_{\text{CP}}^2$ can (in principle) be expressed in the parameters of a fundamental theory. The action (6.3) contains the lowest-dimensional operators in the SU(2)-Higgs effective action that contribute to the asymmetry (6.1). We will see that the dimension-six operator $\phi^\dagger \phi F_{\mu \nu}^a \tilde{F}^{\mu \nu a}$ does not contribute to the baryon asymmetry in the scenario that we study here.

3. departure from equilibrium.
6.3. Sphaleron transitions

We first recall the reasoning for the necessity for a departure from equilibrium. There are basically two arguments for this. One states that if we start with an initial state with zero baryon-number and end up in a state with non-zero baryon number, somewhere between the final and initial state the system must have been out of equilibrium (see e.g. [106]). Another, stronger statement is that in equilibrium the baryon-number expectation value is zero (see e.g. [105]). The argument runs as follows. The equilibrium value of the baryon number $B$ is given by

$$\langle B \rangle_{eq} = \text{Tr} \ e^{-\beta H} B,$$

where we have assumed that conserved charges (such as the difference between lepton- and baryon-number) are zero. Note that we did not include a chemical potential for the baryon-number, since it is not a conserved quantity. Using the fact that the Hamiltonian $H$ is CPT-even and that $B$ is CPT-odd, the manipulations

$$\text{Tr} \ e^{-\beta H} B = \text{Tr} \ e^{-\beta H} [\text{CPT}] [\text{CPT}]^{-1} B$$

$$= \text{Tr} \ [\text{CPT}] e^{-\beta H} [\text{CPT}]^{-1} B$$

$$= \text{Tr} \ e^{-\beta H} [\text{CPT}]^{-1} B [\text{CPT}]$$

$$= -\text{Tr} \ e^{-\beta H} B$$

show that

$$\langle B \rangle_{eq} = 0.$$

The standard scenario for electroweak baryogenesis assumes that the electroweak phase-transition was strongly first-order. This provides then the necessary departure from equilibrium.

In the scenario that we study here, the departure from equilibrium is introduced by the change in the effective potential for the baryon-number during the weakly first-order phase-transition. The required strength of the phase transition for our scenario to work is determined by the time-scale that the baryon-number expectation value differs from zero after the phase transition, because the stronger the transition the faster sphaleron transition are effectively stopped and baryon-number is conserved.

6.3 Sphaleron transitions

Since sphaleron transitions form the crucial physical process in scenarios for electroweak baryogenesis, we will review these first.
The anomaly equation relates the baryon number $B$ to the Chern-Simons number $N_{CS}$

$$B(t) - B(0) = 3 \left[ N_{CS}(t) - N_{CS}(0) \right] = \frac{3g^2}{32\pi^2} \int_0^t dt \int d^3x F_{\mu\nu}^a F^{\mu\nu a}. \quad (6.7)$$

This equation relates the change in baryon number to the time evolution of the gauge fields. The practical implication is that $B$-changing processes can be studied by focusing on the gauge field dynamics. And we will do so in the following.

Here, we will take the baryon density equal to zero, in the next section we will review the effect of a non-zero baryon density.

The potential of the Chern-Simons number along the minimal-energy path is sketched in fig. 6.1. The different (classical) vacua are separated by an energy barrier. As already mentioned in the introduction, at zero-temperature the transitions from one vacuum to another occur through tunneling and are very much suppressed. At high temperatures however the system can go over the barrier. The transition rate is $[9, 69, 75]$

$$\Gamma_{\text{sph}} \sim \exp - \beta E_{\text{sph}}, \quad (6.8)$$

in the broken phase.

The physical picture is that once in a while the mode along the Chern-Simons direction gets thermally activated and can cross the energy barrier. After the transition to the neighboring vacuum at the right or left, this mode gets damped and looses its energy to the other modes. Subsequently, another transition will take place after some time. When the temperature is small compared to the energy barrier between different vacua ($T \ll E_{\text{sph}}$), it is

2. The sphaleron has zero modes [9], for example those related to simple translations and rotations. Therefore there is not a unique minimal energy path.
expected that subsequent transitions are uncorrelated. This implies that the system follows a random walk and one expects that

$$\langle [N_{CS}(t) - N_{CS}(0)]^2 \rangle = V \Gamma_{sph} t, \quad \tag{6.9}$$

with $V$ the volume of the system. The sphaleron rate equals the Chern-Simons diffusion rate.

In section 6.4 we will show that in the presence of CP-odd operators, the probabilities for a transition to the right or left differ, with the effect that the most probable value of $N_{CS}$ grows linearly in time.

### 6.3.1 Including a baryon density

The non-conservation of baryon-number, although required for baryogenesis, poses also a serious problem. Namely, a once created baryon asymmetry may be washed out by sphaleron transitions. To discuss this issue, we review the sphaleron rate in the presence of a baryon density [9,69].

A useful starting point is the free energy at a given baryon number [69]

$$F(B) = \frac{237}{216} \frac{B^2}{V T^2}, \quad \tag{6.10}$$

where we assumed that the difference between baryon and lepton number $B - L = 0$ and that the baryon density is small: $B/V \ll T^3$. Equation (6.10) holds when the temperature is much larger than the masses of the fermions. The coefficient $\alpha$ used in the introduction as coefficient of the quadratic part of the potential $F(B)$ in the symmetric phase may be read of from (6.10). The other coefficient $\alpha'$ differs by a mass correction, as will be discussed later on.

As before, it is assumed that, in the broken phase, the sphaleron transitions are slow, so that after a transition the system is thermalized (in that baryon sector) before the next transition. Then one may use (6.10) as an effective potential that generates a force towards $B = 0$. Hence, at non-zero baryon-density the rate towards positive $N_{CS}$, $\Gamma^{\uparrow}$, will differ from the rate towards negative $N_{CS}$, $\Gamma^{\downarrow}$. The rate equation reads

$$\dot{B} = 3V \left( \Gamma^{\uparrow} - \Gamma^{\downarrow} \right), \quad \tag{6.11}$$

with

$$\Gamma^{\uparrow(\downarrow)} = \Gamma_{sph} \left[ 1 - (+) \frac{3}{2} \beta \partial_B F(B) \right], \quad \tag{6.12}$$
where we defined the discretized derivative \( \partial_B F(B) = [F(B + 3) - F(B)] / 3 \).

The number 3 in (6.11) and the discretized derivative comes from the fact that the baryon number is changed by that amount in one transition. From (6.10), (6.11), and (6.12), one obtains the result [69]

\[
\dot{B} = -\frac{237}{24} \Gamma_{\text{sph}} \frac{B}{T^3}.
\]

(6.13)

It follows that an initial baryon number will (exponentially) decrease in time.

As already mentioned in the introduction, in the standard scenario for electroweak baryogenesis the baryon asymmetry is generated at the electroweak phase-transition. To avoid the wash out of baryon number, it is required that \( \Gamma_{\text{sph}} \) is sufficiently small after the phase transition. Since the sphaleron energy is proportional to the Higgs expectation value \( v \), it is required that \( v \) is sufficiently large directly after the phase transition. This can be translated in a model-dependent upper-bound on the Higgs mass. For the standard model the requirement of a first-order phase transition allows for a Higgs mass \( m_H \lesssim 72 \text{ GeV} \) [64,107]. Also requiring that a generated asymmetry is not washed out, one can bring down the upper bound to 45 GeV [24]. Since, the experimental lower bound on the Higgs mass is now 106 GeV, this scenario will not work within the Standard model. For the minimal supersymmetric standard model (MSSM) the upper bound on the Higgs mass reads 116 GeV (this upper bound depends on the allowed values for the mass of the heavy stop). If one restricts this mass to values below 1 TeV one can bring down the upper bound on the Higgs mass to 107 GeV) [37,39]. A thorough discussion of the allowed parameter space in the MSSM may be found in references [37,39].

### 6.4 Effect of CP-violation on the rate

The Chern-Simons number \( N_{CS} \) is a CP-odd operator. Therefore the inclusion of the CP-odd operators in (6.3) may break the symmetry between sphaleron transitions towards positive and negative Chern-Simons number. We study the effect of the CP-violating operators in (6.3) on the motion along a particular path, that goes from a vacuum to the sphaleron. We use the path of Manton [83] and parameterize it by the time-dependent coordinate \( \Theta \) (in [83] this coordinate is called \( \mu \)). This path is not the minimal energy path which was constructed in [5]. But we expect that the precise path will not be important for the following rather general arguments and
that the final result is sufficient as an order of magnitude estimate. We use the following parameterization for the fields

\[
\begin{align*}
\text{gauge} & \quad A^a_\mu \sigma^a = \frac{-2i}{g} f(r) [\partial_\mu U(\Theta)] U^{-1}(\Theta), \\
\text{Higgs} & \quad \phi = \frac{1}{2} \sqrt{2} v h(r) U(\Theta) \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{align*}
\]

(6.14) \quad (6.15)

with the $\Theta$-dependent SU(2)-matrix

\[
U(\Theta) = \frac{1}{r} \begin{pmatrix} z & x + iy \\ -x + iy & z \end{pmatrix} \sin \Theta + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cos \Theta.
\]

(6.16)

The functions $f$ and $h$ satisfy the boundary conditions

\[
\begin{align*}
f & \to 0 \quad h \to 0 \quad r \to 0, \\
f & \to 1 \quad h \to 1 \quad r \to \infty.
\end{align*}
\]

(6.17)

The parameterization (6.14), (6.15) is a non-static generalization of the fields considered in [71,83]. It is convenient, since the field strength vanishes for $r \to \infty$. As is verified in appendix 6.A, this parametrization yields the correct Chern-Simons number for the sphaleron configuration.

We use Ansatz b of Klinkhamer and Manton [71] for the functions $f$ and $h$

\[
\begin{align*}
f(\rho) &= \begin{cases} 
\frac{\rho^2}{A(A+4)} & \rho \leq A \\
1 - \frac{4}{A+4} \exp\left[\frac{1}{2}(A - \rho)\right] & \rho \geq A
\end{cases}, \\
h(\rho) &= \begin{cases} 
\frac{\sigma B+1}{B(\sigma B+2)} \rho & \rho \leq B \\
1 - \frac{B}{\sigma B+2} \frac{1}{\rho} \exp\left[\frac{1}{2}(A - \rho)\right] & \rho \geq B
\end{cases},
\end{align*}
\]

(6.18) \quad (6.19)

with $\rho = gvr$ and $\sigma = (\lambda/2g^2)^{\frac{1}{2}}$. The parameters $A, B$ are determined by minimizing the energy for the static field configuration at $\Theta = \frac{1}{2}\pi$. Then the static fields provide a very good approximation for the sphaleron configuration at $\Theta = \frac{1}{2}\pi$ [71]. In this way the parameters depend only on the Higgs mass at zero temperature. We take $M_H = 230$ GeV for which $A = 1.15$ and $B = 1.25$ [71] (the parameters depend only slightly on the Higgs mass;
also for a Higgs mass of about 100 GeV the following calculations are expected
to provide a reasonable estimate).

Now that the dynamics has been restricted to the path described by
(6.19) and (6.18) we may rewrite the SU(2)-Higgs action $S$ and the CP-
violating action (6.3) in terms of the coordinate $\Theta$

$$S = \frac{4\pi v^2}{g} \int dt \left[ (a_1 + a_2 \sin^2 \Theta) \frac{\Theta^2}{(g v)^2} - (a_3 \sin^2 \Theta + a_4 \sin^4 \Theta) \right], \quad (6.20)$$

$$S_{CP} = \frac{4\pi v^2}{M^4} \int dt \left( b_1 \delta^1_{CP} + b_2 \delta^2_{CP} + b_3 \delta^3_{CP} \sin^2 \Theta \right) \dot{\Theta}^3 \sin^2 \Theta, \quad (6.21)$$

where we have neglected total time-derivatives. Had we included the dimension-
six operator $\phi^\dagger \phi F F$, it would only have given a total time-derivative. The
coefficients $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ are given by the integrals

$$a_1 = \int_0^{\infty} d\rho \rho^2 \left[ (\partial_\rho f)^2 + \frac{1}{2} h^2 (1 - f)^2 \right] = 2.51, \quad (6.22)$$

$$a_2 = 8 \int_0^{\infty} d\rho f^2 (1 - f)^2 = 1.35, \quad (6.23)$$

$$a_3 = \int_0^{\infty} d\rho \left[ 4 (\partial_\rho f)^2 + \frac{1}{2} (\partial_\rho h)^2 + h^2 (1 - f)^2 \right] - 2fh(1 - f)(1 - h) + f^2(1 - h)^2 = 1.58, \quad (6.24)$$

$$a_4 = \int_0^{\infty} d\rho \left[ 2f^2(1 - f)^2 + 2fh(1 - f)(1 - h) - f^2(1 - h)^2 + \frac{1}{4} (1 - h^2)^2 \right] = 0.53, \quad (6.25)$$

$$b_1 = \frac{9}{2} \int_0^{\infty} d\rho h^2 (\partial_\rho f) f (1 - f)^3 = 0.14, \quad (6.26)$$

$$b_2 = 9 \int_0^{\infty} d\rho (\partial_\rho f)^3 f (1 - f) = 0.096, \quad (6.27)$$

$$b_3 = 72 \int_0^{\infty} d\rho \frac{1}{\rho^2} (\partial_\rho f)^3 f (1 - f)^3 = 0.23. \quad (6.28)$$

In terms of the coordinate $\Theta$ the CP-transformation is simply

$$\Theta \rightarrow -\Theta. \quad (6.29)$$

The action $S_{CP}$ (6.21) is CP-odd and CPT-even. The effect of these CP-odd
operators on sphaleron transitions is studied in the next section.
6.4. Effect of CP-violation on the rate

**Figure 6.2:** The force $F_{CP}$ for a motion to the right (left figure) and a motion to the left (right figure).

### 6.4.1 Asymmetry

The CP-odd action (6.21) introduces a velocity-dependent force in the equations of motion. For the moment we ignore the $\sin \Theta$-dependence in the action (6.21), and denote this force as

$$F_{CP} = -\delta \dot{\Theta} \hat{\Theta},$$

(6.30)

with $\delta$ a positive coefficient that can be expressed in the parameters in the CP-odd action (6.21). This force points in the direction of motion when the system moves from the vacuum towards the sphaleron at $N_{CS} = +\frac{1}{2}$, whereas the force is opposite to the direction of motion when the motion is towards the sphaleron at $N_{CS} = -\frac{1}{2}$, see fig 6.2. As a consequence, the system will find it easier to cross the barrier to the right than to the left. Therefore, the probability of crossing the barrier to the right, $P^\uparrow$, is larger than the probability of crossing the barrier to the left, $P^\downarrow$. This difference in probabilities implies that the diffusion of the Chern-Simons number will evolve in an asymmetric manner. This, however, does not imply that the Chern-Simons number develops a non-zero expectation value. Indeed, from the fact that the average velocity vanishes in equilibrium, it follows that it will not. The asymmetry will manifest itself in the distribution function of the Chern-Simons number and expectation values such as $\langle N_{CS}^3 \rangle$. In a non-equilibrium situation, the asymmetric evolution may result in a non-zero expectation value of $N_{CS}$. In the following, we will estimate the asymmetry in the probabilities.

To obtain a quantitative estimate for the effect of the CP-odd terms on the motion over the barrier, we consider the shift in the energy caused by the extra CP-violating terms (6.21)

$$E_{CP}(\Theta, \dot{\Theta}) = \frac{8\pi v^2}{M^4} (b_1 \delta_{CP}^1 + b_2 \delta_{CP}^2 + b_3 \delta_{CP}^2 \sin^2 \Theta) \dot{\Theta}^3 \sin^2 \Theta.$$  

(6.31)
Especially the typical energy shift at the sphaleron configuration is important. To calculate this energy shift, we need the typical velocity $\dot{\Theta}$. To zeroth-order in $\delta_{CP}^1$ and $\delta_{CP}^2$ the velocity has a Gaussian distribution at the sphaleron and we find

$$\langle \dot{\Theta}^2 \delta(\Theta - \frac{1}{2}\pi) \rangle = \frac{(gv)^3T}{4\pi v^2(a_1 + a_2)},$$

where the $\delta$-function enforces that the average over the velocity is taken at the sphaleron configuration. With this estimate for the velocity we find for the typical energy shift

$$\delta E_{\text{sph}} = \frac{1}{\sqrt{\pi vM_4}} (b_1\delta_{CP}^1 + b_2\delta_{CP}^2 + b_3\delta_{CP}^3) \left[ \frac{(gv)^3T}{(a_1 + a_2)} \right]^{\frac{3}{2}},$$

which provides a quantitative measure for the amount of CP-violation.

As an estimate for $P^\uparrow$ we may take the probability that a configuration at the barrier moves in the positive Chern-Simons direction

$$P^\uparrow = \langle \delta(\Theta - \frac{1}{2}\pi)H(\dot{\Theta}) \rangle / \langle \delta(\Theta - \frac{1}{2}\pi) \rangle,$$

where $H(\dot{\Theta})$ is the Heaviside function. In a similar manner $P^\downarrow$ can be calculated. We get

$$P^\uparrow(\downarrow) = \frac{1}{2} + (-0.80)0.80 \beta \delta E_{\text{sph}}.$$

In the estimate for these probabilities in the presence of CP-violating interactions (6.3) an uncertainty arises from the path that we have chosen, because the fields (6.14) and (6.15) do not satisfy the (SU(2)-Higgs) equations of motion. However, for $\Theta = \frac{1}{2}\pi, \dot{\Theta} = 0$ these fields do provide a very good approximation to the solution of the (static) field equations [71]. Hence, we expect that close to the sphaleron and for small velocities $\dot{\Theta} << gv$, the estimates (6.33) and (6.41) provide a reasonable approximation. The parametric dependence on $g$, $v$, $M$, and $T$ is expected to be correct.

The asymmetry in the probabilities of moving left or right at the sphaleron configuration implies that there is a difference in the average velocity of configurations that move left or right. To consider the average velocity along the $\Theta$-trajectory at the sphaleron for configurations that move to the right is

$$v^\uparrow = \langle |\dot{\Theta}|H(\dot{\Theta})\delta(\Theta - \frac{1}{2}\pi) \rangle / \langle H(\dot{\Theta})\delta(\Theta - \frac{1}{2}\pi) \rangle$$

(6.36)
6.4. Effect of CP-violation on the rate

For configurations moving in the opposite direction the average velocity is

\[ v^\dagger = \langle \dot{\Theta} | H(-\dot{\Theta})\delta(\Theta - \frac{1}{2}\pi) \rangle / \langle H(-\dot{\Theta})\delta(\Theta - \frac{1}{2}\pi) \rangle \]  \hspace{1cm} (6.37)

From the observation that the flux vanishes (this is discussed in section 6.4.4)

\[ \langle \dot{\Theta}\delta(\Theta - \frac{1}{2}\pi) \rangle = \langle |\dot{\Theta}| H(\dot{\Theta})\delta(\Theta - \frac{1}{2}\pi) \rangle = \langle \dot{\Theta}| H(-\dot{\Theta})\delta(\Theta - \frac{1}{2}\pi) \rangle = 0, \]  \hspace{1cm} (6.38)

it follows that the asymmetry in the probabilities (6.35) results in a difference in the average velocities (6.36) and (6.37). In particular, when \( \delta E_{\text{sph}} > 0 \) we have

\[ v^\dagger > v^\dagger. \]  \hspace{1cm} (6.39)

Further, we note that the asymmetry in the velocities and probabilities vanishes at \( \Theta = 0 \), but is everywhere else of the same sign. This means that if we consider the time evolution of the probability distribution it will not only spread due to diffusion, but also develop an asymmetry. Namely, since the average velocity towards negative Chern-Simons numbers is larger, the tail of the distribution in the negative Chern-Simons direction will be longer than the tail in the positive direction.

As mentioned in the introduction of this chapter, we will argue that the peak of the distribution will increase linearly in time. We should remark here, that the above derived asymmetry in the the probabilities is not sufficient to conclude that this will happen. This may be illustrated by the following simple model. Consider a particle on a one-dimensional lattice, that has a probability of \( 2/3 \) of moving one step to the right and a probability \( 1/3 \) of moving two steps to the left. In this way the average flux is zero, as it should in equilibrium, see subsection 6.4.4. It is easy to derive that for this simple system the peak of the probability distribution function remains located at the initial position of the particle.

The notion that the peak of the distribution moves, is based on the argument presented in the next section. Here we conjecture that the velocity of the peak is proportional to the asymmetry in the probabilities. We write

\[ \langle N_{CS}(t) - N_{CS}(t_{in}) \rangle_{mp} = V \left( \Gamma^\dagger - \Gamma^\dagger \right) (t - t_{in}), \]  \hspace{1cm} (6.40)

with \( V \) the volume. The brackets \( \langle . . \rangle_{mp} \) denote the peak of the distribution function. The difference in rates towards negative or positive Chern-Simons number is expected to be proportional to the difference in the probabilities (6.35)

\[ \Gamma^\dagger (1) = \Gamma_{\text{sph}} \left[ 1 + (-) c \beta \delta E_{\text{sph}} \right], \]  \hspace{1cm} (6.41)
where \( c \) is a coefficient of order one.

### 6.4.2 Alternative derivation

Here we present the argument that the asymmetry in the distribution function will manifest itself also through a linear increase of the most probable Chern-Simons number.

We define an effective force for the random walk by averaging the force (6.30) over one transition

\[
\bar{F}_{CP} = \frac{1}{\pi} \int_{0}^{\pi} d\Theta \bar{F}_{CP}.
\]

Inserting (6.30) we get

\[
\bar{F}_{CP} = \frac{-\delta}{\pi} \int_{0}^{\pi} d\Theta \dot{\Theta} \ddot{\Theta} = \frac{-\delta}{\pi} \int_{t_b}^{t_e} dt \dot{\Theta}^{2} \ddot{\Theta},
\]

where \( t_b \) is time that the system starts its barrier-crossing motion, and \( t_e \) is the time it ends in the other vacuum. We find

\[
\bar{F}_{CP} = \frac{-\delta}{3\pi} (v^3_e - v^3_b),
\]

where \( v_b \) is the velocity at the beginning of the motion and \( v_e \) is the velocity at the end. When the temperature is much smaller than the sphaleron energy, \( T << E_{\text{ sph}} \), the velocity \( v_b \) is much larger than the average velocity. Due to damping from the coupling to other degrees of freedom \( v_e \) will be closer to the average velocity, and (in most transitions) smaller than \( v_b \). Hence,

\[
\bar{F}_{CP} > 0.
\]

Note that in this derivation the damping by the modes plays an essential role. It may be remarked that the difference between the velocities \( v_b \) and \( v_e \) is only present sufficiently deep in the broken phase, where \( T << E_{\text{ sph}} \). Here the velocity \( v_b \) has to be exceptionally large to cross the barrier. Especially in the symmetric phase it is to expected that on average the begin velocity and end velocity on average are equal (in the symmetric phase the begin- and endpoint of a crossing is not even well-defined). Therefore, we expect that in the symmetric phase the rates \( \Gamma^\uparrow \) and \( \Gamma^\downarrow \) are equal.
6.4. Effect of CP-violation on the rate

6.4.3 Numerical check

In this section we present a numerical analysis to verify the linear growth of the Chern-Simons number (6.40). We consider the model system

\[
L = \frac{1}{2} \left[ \dot{x}^2 + \frac{1}{6} \dot{x}^3 + \frac{1}{36} \dot{x}^4 \right] - 2 \sin(x)^2 + \sum_{i=1}^{2} L_{od}(\dot{x}_i, x_i) - V_{int}(x, x_i). \tag{6.46}
\]

Here the coordinate \( x \) plays the role of Chern-Simons number. The \( \dot{x}^3 \)-term is the analog of the CP-violating operator. We have also included a \( \dot{x}^4 \)-term so that the energy is bounded for large velocities. The coefficient of this term is sufficiently large that the Lagrangian is convex and a Hamiltonian analysis is possible. The other degrees of freedom \( x_i, i = 1, 2 \), introduce the necessary damping. The Lagrangian of the these degrees of freedom and the interaction potential reads

\[
L_{od}(\dot{x}_i, x_i) = \frac{1}{2} \dot{x}_i^2 - \frac{1}{2} x_i^2, \tag{6.47}
\]

\[
V_{int}(x, x_i) = \frac{1}{30} \left[ \sin(x)^2 + \sum_{i=1}^{2} x_i^2 \right]^2. \tag{6.48}
\]

In fig. 6.3 a numerical solution to the equations of motion is shown for an energy equal to six. We see the expected behavior: there are transitions from one vacuum to another, and in between transitions the system oscillates around the local minimum of the potential. Of interest is the long-time
behavior of the system. From (6.40) we expect a linear growth of the angle $x$. In fig. 6.4 the long-time behavior of two solutions is shown both with an energy equal to six. The initial conditions of the two solutions are chosen such, that they interchange under the transformation $x \rightarrow -x$. Therefore without the $x^3$-term in the Lagrangian the solution curves should interchange under $x \rightarrow -x$ (which indeed they do). We see that with the (equivalent of a CP-odd) $x^3$-term in the Lagrangian, the angle $x$ grows linearly in time, even though the fluctuations in $x$ are quite large. This is in qualitative agreement with (6.40).

We have also performed a more quantitative analysis of the model system (6.46). We solved numerically the equations of motion for 20 different initial conditions with the energy fixed, and the same initial conditions for $x, p$: $x_{\text{in}} = 1.3$, $p_{\text{in}} = 0$. We let the system evolve for $t = 200,000$. For each initial condition the final value of $x$ is positive, and the average (over initial conditions) differs from 0 by $10\sigma$. Hence, for these initial conditions $x$ grows in time. When we would have taken a thermal average over initial conditions however, the average of $x$ should remain equal to 0. That we find such a clear increase of the average of $x$ implies that the system is not ergodic or, at least, that the equilibration time is much larger than 200,000. This conclusion is further supported by simulations where we started with a large part of the energy in the $x$ and $p$ coordinates. Then we found that the system moves over the barriers, without slowing down, and without energy redistribution. Also, when we start out with a small amount of energy for the coordinates $x, p$, the system stays in one vacuum for extremely long times.
6.4. Effect of CP-violation on the rate

The main point is the following. When the system is not trapped in one vacuum or moves without slowing down over the, that it is makes transitions from one vacuum to another in a random fashion, the coordinate $x$ increases in time. This implies that for thermal initial conditions the peak of the distribution will move towards larger values, in agreement with the arguments in sections 6.4.1 and 6.4.2.

6.4.4 Some remarks on an asymmetric distribution function

Let us first present the argument why in equilibrium the expectation value $\langle N_{CS}(t) - N_{CS}(0) \rangle$ remains zero, or at least will not grow linear in time. We use again the coordinate $\Theta$ along the Chern-Simons direction, with conjugate momentum $p_\Theta$, and Hamiltonian $H$. The Hamiltonian is periodic in $\Theta$ with period $\pi$. To argue that the Chern-Simons number expectation value is zero we should show that $\langle \Theta \rangle = 0$.

An argument similar as used in section 6.2 to show that the baryon-number expectation value vanishes (6.6), cannot be given. Since it would rely on the phase-space average

$$\langle \Theta \rangle = Z^{-1} \int dp_\Theta \int_{-\infty}^{\infty} d\Theta \, \Theta \, e^{-\beta H}. \quad (6.49)$$

However this quantity is not well defined, since the equilibrium distribution function, $\exp -\beta H$, is not normalizable on the full real axis $\Theta \in ] - \infty, \infty [$. Hence we should restrict the equilibrium distribution to a finite interval, for instance $\Theta \in ] - \frac{1}{2} \pi, \frac{1}{2} \pi [$. Then we cannot calculate the expectation value of the winding number. But we can calculate the average of its velocity

$$\langle \dot{\Theta} \rangle = \int dp_\Theta \int_{-\frac{1}{2} \pi}^{\frac{1}{2} \pi} d\Theta \, \dot{\Theta} \, e^{-\beta H}$$

$$= \int dp_\Theta \int_{-\frac{1}{2} \pi}^{\frac{1}{2} \pi} d\Theta \frac{\partial H}{\partial p_\Theta} \, e^{-\beta H} = 0. \quad (6.50)$$

The phase space average of the velocity vanishes, also when the kinetic energy is complicated. For an ergodic system this implies that

$$\lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} dt' \dot{\Theta} = 0. \quad (6.51)$$

Hence, $\Theta$ can not grow linearly in time.
Other asymmetric functions like $\Theta^3$ may be non-static in equilibrium. The thermal average of the time-derivative $3\Theta^2\dot{\Theta}$ seems to vanish in a similar manner as for (6.50). However the thermal average is not well defined when $\Theta$ is not bounded. For the same reason the quantity $\Theta^2$ may be non-static, as expected for systems where diffusion plays a role. From the argument below (6.38), we infer that $\langle \Theta^3 \rangle$ becomes negative and grows in time.

6.4.5 Fokker-Planck equation

Here we will derive the Fokker-Planck equation for systems with a complicated kinetic energy, as we have encountered in section 6.4. This allows us to verify that the time-evolution of the distribution function may result in an asymmetry.

It may be useful to specify a typical Hamiltonian for the coordinate $x$ along the Chern-Simons direction and its conjugate momentum $p$

$$H = \frac{1}{2} \left( p^2 - \delta p^3 + \delta^2 p^4 \right) + V(x).$$

(6.52)

In terms of the velocity the kinetic energy reads

$$\frac{1}{2} \left( \dot{x}^2 + \delta \dot{x}^3 + \delta^2 \dot{x}^4 \right) + O(\delta^3).$$

(6.53)

The contribution $+\delta \dot{x}^3$ corresponds to the CP-violating term.

The degree of freedom, that represents the Chern-Simons number, interacts with other modes of the plasma with coordinates $x_i$. We assume interactions of the form $V_{\text{int}}(x, x_i)$ (no interactions involving the momentum $p$). We include this interaction with the other modes by a damping term and a stochastic force in the equations of motion

$$\dot{p} = -\partial_x V(x) - \sigma \dot{x} + \xi,$$

(6.54)

$$\dot{x} = v(p) = p - \frac{3}{2} \delta p^2 + 2\delta^2 p^3,$$

(6.55)

with

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t)\xi(t') \rangle = T \sigma \delta(t - t').$$

(6.56)

The subtlety in the introduction of the damping and the stochastic force lies in the use of the velocity instead of the momentum in the damping term.
This is motivated as follows. For interactions including only the position $x$ and not the momentum $p$, integrating out the other modes generates a memory kernel involving $x$ at earlier times. For a slowly varying $x$ this may be approximated by a damping term. The point is now that this memory kernel involves only the position, and is independent of the kinetic part of the Hamiltonian. Therefore, in a local approximation of the memory kernel time-derivatives of $x$ naturally occur. This motivates the use of the velocity instead of the momentum in the damping term in (6.54).

From the stochastic equation (6.54) with velocity (6.55), we can derive the Fokker Planck equation using standard methods, see for instance paragraph 3.2 of [120]. The result is

$$
\dot{P} = \partial_p \{ T \sigma \partial_p P + [\partial_x V + \sigma v(p)] P \} - v(p) \partial_x P,
$$

(6.57)

with the probability distribution $P = P(x,p,t)$.

We note that the static solution of the Fokker-Planck equation (6.57) is the equilibrium distribution $\exp -\beta H$, as expected.

The equilibrium distribution is invariant under the transformations

$$
x \rightarrow -x
$$

(6.58)

and, to order $\delta$,

$$
p \rightarrow -p + \delta p^2 + \mathcal{O}(\delta^2).
$$

(6.59)

But these transformations are not a symmetry of the Fokker-Planck equation (even when the transformations are applied together). This implies that an initial equilibrium distribution restricted to the interval $x \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ may develop an asymmetry. Which according to the arguments in section 6.4.1 will happen indeed.

### 6.5 Baryon-number generation

In this section we study the evolution of the baryon-number expectation value after the phase transition. The basic idea is, as discussed in the introduction, that during the phase-transition the effective potential changes: from $F(B) \sim a B^2$ to $F(B) \sim a'B^2$ with $a' < a$. The asymmetric evolution of the distribution function of the baryon-number/Chern-Simons number may lead, in this non-equilibrium situation, to a (temporary) non-zero baryon-number expectation value, for a simple example see appendix 6.B. The argument is as follows. When the distribution function spreads, to adjust
itself to the new potential, the tail in the negative baryon-number direction will become larger than the tail in the positive baryon-number direction, as discussed below (6.38). Therefore, the distribution function "feels" the potential in the negative baryon-number direction first. Then the tail in the negative direction can no longer compensate for the positive part of the distribution function. This results in a positive value of the baryon-number expectation value. To generate a lasting asymmetry, it is necessary that the baryon-number is frozen out, before it relaxes back to its equilibrium value, which is zero. Here, we calculate the maximum (largest deviation from equilibrium) baryon-number expectation value possible in this scenario. Then we include a suppression factor to account for the smallness of the deviation from equilibrium. Finally, we estimate the time that the expectation value is non-zero. Which gives a bound on the strength of the phase transition.

To estimate the maximum baryon-number the peak of the distribution function can reach, we superimpose the effect of a non-zero baryon density on the asymmetry of the sphaleron rates, see (6.12) and (6.41). Combining these effects, we find for the rates

$$
\Gamma^{\uparrow(\downarrow)}(n_B) = \Gamma_{\text{sph}} \left[ 1 - (+) 0.80 \frac{n_B}{n_\gamma} + (-) c \frac{\delta E_{\text{sph}}}{T} \right], \quad (6.60)
$$

where $n_\gamma = 0.24 T^3$ is the photon density. Similar as in (6.11) the rate equation reads

$$
\frac{dn_B}{dt} = 3 \left[ \Gamma^{\uparrow}(n_B) - \Gamma^{\downarrow}(n_B) \right]. \quad (6.61)
$$

This equation is not CPT-invariant. As explained in section 6.4.2 this is due to the effect of damping. From the rate equation with (6.60) we find the stationary (and stable) solution

$$
\left. \frac{n_B}{n_\gamma} \right|_{\text{max}} = 1.25 c \frac{\delta E_{\text{sph}}}{T}. \quad (6.62)
$$

This provides a maximum that the baryon number can grow in an effective potential $F(B)$ (6.10).

This maximum value will only be reached for a maximal deviation from equilibrium initially. That is when the initial distribution is much sharper peaked than the equilibrium distribution. In our scenario, after the weakly first-order phase transition, the distribution is not that sharply peaked and we expect a suppression. When we parameterize, as in the introduction, the initial distribution as $\exp -\alpha B^2$ and the equilibrium distribution as $\exp -\alpha' B^2$, 

...
the suppression factor \((1 - \alpha'/\alpha)\) may be expected (we find this suppression factor in simple examples, see for instance the one in appendix 6.B). The difference between \(\alpha\) and \(\alpha'\) arises from the mass that the baryons acquire in the broken phase due to the Higgs mechanism. The largest contribution to the difference comes from the top quark mass \(m_t(v_{pt})\) after the phase transition, (we have indicated the dependence on the Higgs expectation value after the phase transition, \(v_{pt}\)). The suppression factor may be estimated as \((1 - \alpha'/\alpha) \sim m_t(v_{pt})^2/T^2\). The precise relation may be calculated by the methods employed in [76]. For the phase-transition temperature \(T \sim 100\) GeV and a typical Higgs expectation value after a weakly first-order phase-transition \(v_{pt} \sim 70\) GeV, we may estimate \(m_t(v_{pt})^2/T^2 \sim 0.1\).

Let us estimate the baryon asymmetry that may be generated in this scenario. We evaluate the maximal baryon-asymmetry (6.62) at the temperature \(T^* \approx v(T^*) \approx 100\) GeV, at which the baryon-number is frozen out [106]. From (6.22)-(6.28), (6.33), and (6.62), we obtain for the resulting baryon-number at \(T = T^*\)

\[
\frac{n_B}{n_\gamma}_{T=T^*} = (2 \delta_{CP}^1 + 4 \delta_{CP}^2) \times 10^{-4} \left(\frac{100\ \text{GeV}}{M}\right)^4,
\]  

where we have used \(c = 1\) and included the suppression factor \((1 - \alpha'/\alpha) \sim 0.1\). The baryon-photon ratio is not constant under expansion of the universe. The relation between the ratio at \(T = T^*\) and now \((T = T_{\text{now}})\) is given by

\[
\frac{n_B}{n_\gamma}_{\text{now}} = \frac{g_*S(T_{\text{now}})}{g_*S(T^*)} \left. \frac{n_B}{n_\gamma} \right|_{T=T^*},
\]  

with \(g_*S\) the (effective) number of particle species contributing to the entropy at a given temperature [73]. We get from \(g_*S(T_{\text{now}})/g_*S(T^*) = 0.037\) the final result

\[
\left. \frac{n_B}{n_\gamma} \right|_{\text{now}} = (7 \delta_{CP}^1 + 16 \delta_{CP}^2) \times 10^{-6} \left(\frac{100\ \text{GeV}}{M}\right)^4.
\]  

In the standard model the magnitude of \(\delta_{CP}^1, \delta_{CP}^2\) is too small (about \(10^{-20}\)) to explain the observed matter anti-matter asymmetry (6.1). However, for extensions of the standard model \(\delta_{CP}^1, \delta_{CP}^2\) can be as large as \(10^{-3}\) and we see that (6.65) may explain the observed baryon-number excess (6.1), at least when the new mass-scale \(M\) is not too large.

A remaining question is, how long after the phase transition the baryon asymmetry will remain at a substantial fraction of the maximum (6.62). This is important since this time scale determines the window of Higgs expectation
values \( v_{\text{pt}} \) for which the scenario may explain the observed baryon asymmetry (6.1). For instance, if this time scale is short \( v_{\text{pt}} \) should be very close to \( v(T^*) \) to prevent the relaxation back to the equilibrium value \( \langle B \rangle = 0 \).

The typical time scale for a sphaleron transition is \( t_{\text{sph}} \sim (\Gamma_{\text{sph}})^{-1} T^3 \). The time scale to develop the asymmetry is longer, since it is inversely proportional to the asymmetry in the rates \( t_{\text{as}} \sim (\Gamma^+ - \Gamma^-)^{-1} T^3 \). We do not expect that there is another time scale for the relaxation back to equilibrium, since \( t_{\text{as}} \) determines the time after which the asymmetry starts to "feel" the quadratic potential. The required Higgs expectation value may be obtained by comparing the time scale \( t_{\text{as}} \) to the time scale for sphaleron transitions at the point baryon number freezes out: \( t_{\text{as}} = t_{\text{sph}}(T^*) \). At the freeze out temperature the sphaleron rate is \( \Gamma_{\text{sph}}(T^*) \sim \exp(-45) \) [111]. We use

\[
\Gamma^+ - \Gamma^- \sim 10^{-6} \Gamma_{\text{sph}}. \tag{6.66}
\]

The required Higgs expectation value after the phase transition is determined by

\[
10^{-6} \Gamma_{\text{sph}}|_{v=v_{\text{pt}}} > \exp(-45), \tag{6.67}
\]

which leads to

\[
v_{\text{pt}} > 70 \text{ GeV.} \tag{6.68}
\]

This is somewhat lower than the required \( v_{\text{pt}} \) for the standard scenario (see the introduction of this thesis) where the baryons must be frozen out immediately after the transition, this demands \( v_{\text{pt}} > 100 \text{ GeV.} \)

### 6.6 Conclusion

We have shown that in the absence of a baryon density, the dimension-eight CP-odd operators in (6.3) introduce an asymmetry in the diffusion of the Chern-Simons number. This implies that the distribution function of the Chern-Simons number will become asymmetric. We have argued that this effect may lead to baryon-number generation after a weakly first-order electroweak transition. The estimated baryon asymmetry may, depending on the strength of the CP-violation, be sufficient to agree with the observed asymmetry (6.1).
6.A Chern-Simons number of the sphaleron

In this appendix we calculate the difference between the Chern-Simons number of the vacuum and sphaleron configuration. The difference in Chern-Simons number is

$$\Delta N_{CS} = \frac{1}{32\pi^2} \int_0^{t_f} dt \int d^3 x F_{\mu\nu}^{\alpha} \tilde{F}^{\mu\nu\alpha},$$

(6.69)

where at the initial time $t = 0$ the system starts at a classical vacuum, it ends at $t = t_e$ at a sphaleron.

We calculate the r.h.s. of (6.69) for a (general) motion from a vacuum to a sphaleron along the $\Theta$-path. Since we do not evaluate (6.69) for general paths the calculation presented here is not so much a (re-)derivation of the Chern-Simons number of the sphaleron [71], but rather a check on the field parameterization (6.14), (6.15). Using this parameterization, we may rewrite the r.h.s. of (6.69) and we get

$$\Delta N_{CS} = \frac{12}{\pi} \int_0^{t_f} dt \dot{\Theta} \sin^2 \Theta \int_0^\infty dr (\partial_r f)(1 - f).$$

(6.70)

We note that the time and spatial integration are factorized. For the spatial integration it is sufficient to know the boundary values of the function $f$ (6.17). The result is

$$\int_0^\infty dr (\partial_r f)(1 - f) = \frac{1}{6}.$$  

(6.71)

The time integral can also be easily performed

$$\int_0^{t_f} dt \dot{\Theta} \sin^2 \Theta = \frac{1}{2} \pi d\Theta \sin^2 \Theta = \frac{1}{4} \pi.$$  

(6.72)

where we have chosen a path from a vacuum to the nearest sphaleron in the positive $N_{CS}$ (the value of the integral for paths to other sphalerons differs by an integer times $\pi/2$). The resulting Chern-Simons number is

$$\Delta N_{CS} = \frac{1}{2},$$  

(6.73)

in agreement with [71].
6.B Simple example of asymmetry generation

We consider a particle in a harmonic potential with Hamiltonian

\[ H = \frac{1}{2}(p^2 - \delta p^4 + \delta^2 p^4) + \alpha' x^2, \]  

(6.74)

which at the initial time \( t = 0 \) its position and momentum are distributed according to

\[ \rho_{\text{in}} = N \exp -\beta \left[ \frac{1}{2}(p^2 - \delta p^4 + \delta^2 p^4) + \alpha x^2 \right], \]

(6.75)

with \( N \) a normalization factor. In the case \( \alpha = \alpha' \), the system is in equilibrium, and the expectation value of \( x \) vanishes at all later times. When \( \alpha \neq \alpha' \), we find to first order in \( \delta \)

\[ \langle x(t) \rangle = \frac{1}{2} \frac{\delta}{\pi} \left( 1 - \frac{\alpha'}{\alpha} \right) \sin^3 \sqrt{\alpha'} t + O(\delta^2). \]

(6.76)

As expected, the expectation value becomes non-zero, even though the equilibrium expectation value of the system (6.74) vanishes. Since we did not include damping the expectation value remains oscillating.