Games, walks and grammars: Problems I've worked on

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Publication date
2000

Citation for published version (APA):
Vervoort, M. R. (2000). *Games, walks and grammars: Problems I've worked on*. ILLC.

Chapter 2

An Introduction to Game Theory

2.1 Games of Perfect Information

Imagine two players playing a game of blind chess. The only board they have is in their minds, and they make their moves merely by announcing them. Someone who doesn't know the rules would find a game like this difficult to follow. If that someone was of a literal bend, he might describe it like this:

> "There were two players, playing against each other. The first player said something, and I was told it was her move, and that she had made the move by saying it. The other player thought for a while, and then announced his own move. Then the first player made a move again, then the second player, and so forth. The moves always sounded similar, something like 'pawn from ee-four to ee-five'. So I think they couldn't just say anything, but had to select their moves from only a few possible options. And suddenly they stopped, and shook hands, and I was told that the first player had won, apparently because of the moves she and her opponent had played."

If no one gave the poor fellow a copy of the rules of chess, the way a sequence of moves determines which player wins would probably seem quite arbitrary. And our hypothetical observer might be quite impressed that apparently chess-players are able to memorize this long list of what the result is of each possible sequence of moves.

Of course, the game of chess is not really that arbitrary, and those of us who play chess only need to know a few simple rules to figure out which player has won. But we can use this concept of a game to construct a quite general mathematical game $\Gamma_{p.i.}(f)$.

> Let there be given two finite sets X and Y, an integer n, and a function f assigning to each sequence w of length n of pairs $(x_i, y_i) \in X \times Y$,

> a *payoff* $f(w) \in \mathbb{R}$. Two players are playing against each other. First, player I makes a move by selecting an element $x_1 \in X$, and announcing his or her selection. Then player II makes a move by selecting an element $y_1 \in Y$ and announcing his or her selection. Then they each in turn make a second move in this fashion, and a third move, and continue making moves until n rounds have been played. This generates a sequence w of length n of pairs $(x_i, y_i) \in X \times Y$. Then they stop, and player II pays player I the amount $f(w)$.

With the right choices for X, Y, n and f, the game $\Gamma_{p.i.}(f)$ can 'emulate' the game of chess. For if we let X and Y be the set of all possible chess moves, and $n = 6350$[1] then a sequence w corresponds to a finished game of chess. We now set $f(w) = 1$, $f(w) = 0$, or $f(w) = \frac{1}{2}$, depending on whether the corresponding game is a win for White, a win for Black, or a draw.[3] And voilá, we have our chess emulator.

But chess is not the only game that can be 'emulated' in this manner. The same can be done with Noughts-and-Crosses, Connect-Four, Go and Checkers. In general, the games $\Gamma_{p.i.}(f)$ can emulate any game G that has the following properties:

- There are two players.
- There is no element of chance
- Moves are essentially made by selecting them and announcing them.
- There is no hidden information: a player knows all the moves made so far when making his or her current move, and there is nothing going on simultaneously either (*Perfect Information*).
- If one player loses (a certain amount) the other player wins (that same amount) (*Zero-Sum*).
- The game can last no more than a certain number of rounds (*Finite Duration*).

[1] There exists a rule in chess (the fifty-move rule) stating that if no piece has been captured and no pawn has been moved for fifty turns, the game is a draw. Since there are only 30 pieces that can be captured, and each of the 16 pawns can make at most 6 moves, it is easy to show that no game can last longer than $(30 + 16 \cdot 6) \cdot 50 + 50 = 6350$ moves[2].

[2] With a little more effort, this upper bound can be improved to 5950 by observing that pawns in the same column cannot pass one another without one of them making a capture. A game of length 5950 can be constructed, so this is also the maximum length of a game. Optionally the arbiter may allow for greater intervals between captures and/or pawn moves in certain endgames which are known to require this: the maximum length of games under this optional rule may be different.

[3] If w does not correspond to a *legal* chess game, we count it as a win for White if the first illegal move is made by Black, and vice versa.

- There is a maximum number of alternatives each player can select from (*Finite Choice-of-Moves*).

These games are called *games of perfect information*, and any results for the games $\Gamma_{p.i.}(f)$ apply to all the games with these properties.

2.2 Strategies and Values

D. Blackwell described the concept of a *strategy* as [4]:

> Imagine that you are to play the white pieces in a single game of chess, and that you discover you are unable to be present for the occasion. There is available a deputy, who will represent you on the occasion, and who will carry out your instructions exactly, but who is absolutely unable to make any decisions of his own volition. Thus, in order to guarantee that your deputy will be able to conduct the white pieces throughout the game, your instructions to him must envisage every possible circumstance in which he may be required to move, and must specify, for each such circumstance, what his choice is to be. Any such complete set of instructions constitutes what we shall call a *strategy*.

Thus, a strategy for a given player in a given game consists of a specification, for each position in which he or she is required to make a move, of the particular move to make in that position. In turn, a position can be specified by the moves made to get to that position. If we apply this to the game $\Gamma_{p.i.}(f)$, a strategy becomes a function from the set of sequences of length $< n$ of pairs $(x_i, y_i) \in X \times Y$ (followed by single elements $x_j \in X$ in case of a strategy for player II), to the set of possible selections X or Y.
Given strategies for each of the players, the outcome of the game is fixed: each move follows from the current position and the strategy of the player whose turn it is to move, and determines the next position. So, the totality of all the decisions to be made can be described by a single decision - the choice of a strategy. This is the *normal form* of a game: the two players independently make a single move, which consists of selecting a strategy, and then payoff is calculated and made.
Of course, there are good strategies and bad strategies. The *value* of a strategy for a given player is the result of that strategy against the best counter-strategy. The *value* of a game for a given player is the best result that that player can guarantee, i.e. the value of that player's best strategy. A game is called *determined* if its value is the same for both players. That value is the result that will occur if both players are playing perfectly.[4]

[4]In more general cases, we allow ϵ-approximation, i.e. a game is *determined* if and only if there exists a value v such that for any $\epsilon > 0$, the two players have strategies guaranteeing them a payoff of at least $v - \epsilon$ or at most $v + \epsilon$, respectively.

V. Allis [1] recently demonstrated that in a game of Connect-Four, the first player can win, i.e. has a strategy that wins against any counter-strategy. And countless persons throughout the ages have independently discovered that in the game of Noughts-and-Crosses, both players can force a draw. These are both examples of determinacy. It can be shown (using induction) that any game $\Gamma(f)$ as defined above is determined, and hence any game with all of the properties mentioned above is determined. In the case of Go, chess, and checkers, this means that either one of the players has a winning strategy, or both players have a drawing strategy.

2.3 Games of Imperfect Information

Now consider the game of Scissors-Paper-Stone. In this game, the two players simultaneously 'throw' one of three symbols: 'stone' (hand balled in a fist), 'paper' (hand flat with the palm down) or 'scissors' (middle and forefinger spread, pointing forwards). If both players throw the same symbols, the result is a draw; otherwise, 'paper' beats 'stone', 'stone' beats 'scissors', and 'scissors' beats 'paper' (the mnemonic being "paper wraps stone, stone blunts scissors, and scissors cut paper"). In this game, the players do not make moves in turn, but simultaneously. In other words, both players make moves, and neither player knows what move the other is making. This is an example of a game of *imperfect information*. In general, games of imperfect information have all the properties that games of perfect information have, except that players make moves at the same time instead of one after the other.

For games of perfect information, we need to redefine the concept of strategy. If we keep to the existing definition, then the only possible strategies in (for example) the game of Scissors-Paper-Stone, are strategies of the type 'throw *this*'. However, any such strategy, for either player, is a losing strategy: for instance, the strategy 'throw stone' loses against the counter-strategy 'throw paper'. So in terms of the concept of strategy described above, this game is not determined. On the other hand, consider the 'strategy' 'throw scissors 1/3 of the time, throw paper 1/3 of the time, and throw stone the remaining 1/3 of the time'. Against any other strategy, this strategy loses, draws and wins 1/3 of the time each, for an 'average result' of a draw. This strategy does not fit in the concept of strategy given above, but it is clearly worth considering.

Strategies of this new type are called *mixed* strategies, as opposed to the old type of strategies, the *pure* strategies. A mixed strategy for a given player in a given game consists of a specification, for each position in which he or she is required to make a move, of the *probability distribution* to be used to determine what move to make in that position.[5] Given mixed strategies for each of the players,

[5]Standard game theory defines a mixed strategy as a probability distribution on pure strategies, but the above definition can be shown to be equivalent to that one.

the outcome of the game is not determined, but we can calculate the *probability* of each outcome. If we assign values to winning and losing ('the loser pays the winner one dollar'), then we can calculate the average profit/loss one player can expect to make from the other, playing those strategies.
The *value* of a mixed strategy is therefore the *expected average result* against the best counter-strategy. And a game is called determined if, for some value v, one of the players has a strategy with which she can always expect to make (on average) at least v, no matter what the other plays, while the other player has a strategy with which he can always expect to lose (on average) at most v, no matter what the other plays. As before, it can be shown (using induction and a theorem of von Neumann) that all finite two-person zero-sum games with Imperfect Information (i.e. the games with the properties mentioned above, except that players make moves at the same time instead of one after the other) are determined.

2.4 Infinite Games

All the games mentioned so far are of *finite* duration. Let, as before, X and Y be two finite sets, and let f be a function assigning to each countably infinite sequence w of pairs $(x_i, y_i) \in X \times Y$, a payoff $f(w) \in \mathbb{R}$. We first consider games of infinite duration and *perfect* information:

> Two players are playing against each other. Each player, in turn, makes a move by selecting an element $x_1 \in X$ or $y_1 \in Y$, respectively, and announcing his or her selection. Then they each in turn make a second move, and a third move, and continue making moves for a *countably infinite* number of rounds. This generates an infinite sequence w of pairs $(x_i, y_i) \in X \times Y$. 'Then' they stop, and player II pays player I the amount $f(w)$.

The problem with infinite games, of course, is that the outcome is only known after an infinite number of moves, and thus it is impractical to play the game as it is. But our concept of a strategy as a specification of which move to make in each position, is still valid in the case of games of infinite duration. And given strategies for both players we can construct the infinite sequence of moves that will be played (or the probability distribution thereof), and apply the payoff function to obtain our (expectation of the) outcome. Hence we can still play the game in a fashion, by using its *normal form*.
The concepts of values and determinacy carry over as well. But it is no longer provable that all such games are determined. For some payoff functions f, such as bounded Borel-measurable functions f, it has been proven that the infinite game of perfect information $\Gamma_{p.i.}(f)$ is determined. But using the Axiom of Choice, a non-measurable payoff function f can be constructed such that $\Gamma(f)$ is not

determined [6]. The *Axiom of Determinacy*, the axiom that all games $\Gamma(f)$ are determined, is a commonly used alternative to AC [8, 17].

A game of infinite duration and *imperfect* information is similar, except that both players make their n^{th} move at the same time. These games are called *Blackwell* games, named after D. Blackwell, the first person to describe and study these games [2]. For Blackwell games, it was quickly proven that the game $\Gamma(f)$ is determined if f is the indicator function of an open or G_δ set, [2, 3], and after a considerable period determinacy was also shown for the case where f is the indicator function of a $G_{\delta\sigma}$ set[20]. In 1996, D.A. Martin finally proved determinacy of $\Gamma(f)$ for the case that f is Borel[16].