Games, walks and grammars: Problems I've worked on

Vervoort, M.R.

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Chapter 3

Definitions, Lemmas and Terminology

In this chapter, we define the basic concepts of Blackwell games, introduce some terminology, and derive some lemmas that we'll be using in the chapters to come.

3.1 Games, Strategies and Values

The definitions in this section are fairly standard, and merely formalize the intuitive concepts from the introduction. The lemmas are all basic properties of game-values.

First we will define the concept of Blackwell games itself, and the basic concepts of plays and positions. Fix two finite, nonempty sets X and Y, and put $Z = X \times Y$, $W = Z^{\mathbb{N}}$.

3.1.1. DEFINITION. Let $f : W \to \mathbb{R}$ be a bounded Borel-measurable[6] Borel-measurable function. The *Blackwell game* $\Gamma(f)$ with *payoff function* f is the two-person zero-sum infinite game of imperfect information played as follows:

> Player I selects an element $x_1 \in X$ (we say that she *makes the move* x_1) and simultaneously player II selects an element $y_1 \in Y$. Then both players are told $z_1 = (x_1, y_1)$. Next, player I selects $x_2 \in X$, and simultaneously player II selects $y_2 \in Y$. Then both players are told $z_2 = (x_2, y_2)$. Next, both players simultaneously selects $x_3 \in X$ and $y_3 \in Y$, etc. In this manner they produce an infinite sequence $w = (z_1, z_2, \ldots)$. Finally, player II pays player I the amount $f(w)$, ending the game.

[6]These conditions on the payoff function ensure that expectations and values of strategies can be easily defined. In the next chapters, if a function is introduced as the payoff function of a Blackwell game, it may be implicitly assumed to be bounded and Borel-measurable unless explicitly stated otherwise.

If the payoff function is the indicator function I_S of a set $S \subseteq W$, then we will often write $\Gamma(S)$ for $\Gamma(I_S)$.

3.1.2. DEFINITION. Let $\Gamma(f)$ be a Blackwell game. A countably infinite sequence w of pairs $(x, y) \in Z$ is called an *(infinite) play* of $\Gamma(f)$. A finite sequence p of length k of pairs $(x, y) \in Z$ is called a *finite play* or *position* of $\Gamma(f)$, of length k. Note that a finite play of length k contains k moves of each of the players, not k moves total.

If an infinite play w starts with a finite play p of length k, we say that *w hits p on round k*, or that *w passes through p on round k*. Analogously for a finite play p' instead of w. If p' passes through p and $p' \neq p$, we also say that *p' follows p*, or that *p precedes p'*.

NOTATION. We use the following notational conventions:

- w and p are used to denote infinite plays and positions, respectively.
- W denotes the set of infinite plays, as previously defined.
- P denotes the set of positions $Z^{<\omega}$.
- For all $n \in I\!N$, W_n denotes the set Z^n of finite plays of length n.
- $\text{len}(p)$ denotes the length of a finite play p.
- $w_{|k}$ $[p_{|k}]$ denote the position w $[p]$ passes through on round k.
- $p \subset w$ $[p \subseteq p']$ denote that w $[p']$ passes through p on some round.
- $p \subset p'$ denotes that p precedes p'.
- ϵ denotes the position of length 0, i.e. the empty sequence.
- $p^\frown p'$ $[p^\frown w]$ denotes the sequence consisting of the finite sequence p followed by the finite sequence p' [the infinite sequence w].
- $[p]$ denotes the set $\{w \in W \mid w \supset p\}$ of all plays hitting the position p.
- $[H]$ denotes the set $\{w \in W \mid \exists p \in H : w \supset p\}$ of all plays hitting any position in a set of positions H.
- We sometimes write $(x_1, y_1, x_2, y_2, \ldots)$ instead of $((x_1, y_1), (x_2, y_2), \ldots)$.

3.1.3. REMARK. We give W the usual topology by letting the basic open sets be the sets of the form $[H]$ for some some set $H \subseteq W_n$ of positions of fixed length n. Then the open subsets of W are exactly those of the form $[H]$ for some set H of positions. The G_δ subsets of W are exactly those of the form $\{w \in W \mid \#\{p \in H \mid w \text{ hits } p\} = \infty\}$ for some set H of positions. Note that under this topology, W is a compact space.

Next we will give the definitions of strategies and values, and some basic properties thereof. Proofs are omitted for reasons of conciseness.

3.1.4. DEFINITION. A *strategy* for player I in a Blackwell game $\Gamma(f)$ is a function σ assigning to each position p a probability distribution on X. More formally, σ is a function $P \to [0,1]^X$ satisfying $\forall p \in P : \sum_{x \in X} \sigma(p)(x) = 1$.
Analogously, a *strategy* for player II is a function τ assigning to each position p a probability distribution on Y.

3.1.5. DEFINITION. Let σ and τ be strategies for players I and II in a Blackwell game $\Gamma(f)$. σ and τ determine a probability measure $\mu_{\sigma,\tau}$ on W, induced by

$$\mu_{\sigma,\tau}[p] = P\{w \mid w \text{ hits } p\} = \prod_{i=1}^{n} \left(\sigma(p_{|(i-1)})(x_i) \bullet \tau(p_{|(i-1)})(y_i)\right) \tag{3.1}$$

for any position $p = (x_1, y_1, \ldots, x_n, y_n) \in P$.
The *expected income* of player I in $\Gamma(f)$, if she plays according to σ and player II plays according to τ, is the expectation of $f(w)$ under this probability measure:

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) = \int f(w) d\mu_{\sigma,\tau}(w) \tag{3.2}$$

3.1.6. DEFINITION. Let $\Gamma(f)$ be a Blackwell game. The *value* of a strategy σ for player I in $\Gamma(f)$ is the expected income player I can guarantee if she plays according to σ. Similarly, the *value* of a strategy τ for player II in $\Gamma(f)$ is the amount to which player II can restrict player I's income if he plays according to τ. I.e.

$$\text{val}(\sigma \text{ in } \Gamma(f)) = \inf_{\tau} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{3.3}$$

$$\text{val}(\tau \text{ in } \Gamma(f)) = \sup_{\sigma} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{3.4}$$

3.1.7. DEFINITION. Let $\Gamma(f)$ be a Blackwell game. The *lower value* of $\Gamma(f)$ is the smallest upper bound on the income that player I can guarantee. Similarly, the *upper value* of $\Gamma(f)$ is the largest lower bound on the restrictions player II can put on player I's income. I.e.

$$\text{val}^{\downarrow}(\Gamma(f)) = \sup_{\sigma} \text{val}(\sigma \text{ in } \Gamma(f)) = \sup_{\sigma} \inf_{\tau} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{3.5}$$

$$\text{val}^{\uparrow}(\Gamma(f)) = \inf_{\tau} \text{val}(\tau \text{ in } \Gamma(f)) = \inf_{\tau} \sup_{\sigma} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{3.6}$$

Clearly, for all games $\Gamma(f)$, $\text{val}^{\downarrow}(\Gamma(f)) \leq \text{val}^{\uparrow}(\Gamma(f))$. If $\text{val}^{\uparrow}(\Gamma(f)) = \text{val}^{\downarrow}(\Gamma(f))$, then $\Gamma(f)$ is called *determined*, and we write $\text{val}(\Gamma(f)) = \text{val}^{\uparrow}(\Gamma(f)) = \text{val}^{\downarrow}(\Gamma(f))$.

3.1.8. Definition. Let $\Gamma(f)$ be a Blackwell game, and let $\epsilon > 0$. A strategy σ for player I in $\Gamma(f)$ is *optimal* if $\mathrm{val}(\sigma \text{ in } \Gamma(f)) = \mathrm{val}^{\downarrow}(\Gamma(f))$. A strategy σ for player I in $\Gamma(f)$ is *ϵ-optimal* if $\mathrm{val}(\sigma \text{ in } \Gamma(f)) > \mathrm{val}^{\downarrow}(\Gamma(f)) - \epsilon$. Similarly, a strategy τ for player II in $\Gamma(f)$ is *optimal* if $\mathrm{val}(\tau \text{ in } \Gamma(f)) = \mathrm{val}^{\uparrow}(\Gamma(f))$, and *$\epsilon$-optimal* if $\mathrm{val}(\tau \text{ in } \Gamma(f)) < \mathrm{val}^{\uparrow}(\Gamma(f)) + \epsilon$.

3.1.9. Lemma. *Let f, g be two payoff functions such that for all $w \in W$, $f(w) \leq g(w)$. Then $\mathrm{val}^{\downarrow}(\Gamma(f)) \leq \mathrm{val}^{\downarrow}(\Gamma(g))$ and $\mathrm{val}^{\uparrow}(\Gamma(f)) \leq \mathrm{val}^{\uparrow}(\Gamma(g))$.*

3.1.10. Lemma. *Let f be a payoff function, and let $a, c \in \mathbb{R}, a \geq 0$. Then $\mathrm{val}^{\downarrow}(\Gamma(af+c)) = a\,\mathrm{val}^{\downarrow}(\Gamma(f)) + c$ and $\mathrm{val}^{\uparrow}(\Gamma(af+c)) = a\,\mathrm{val}^{\uparrow}(\Gamma(f)) + c$.*

3.1.11. Lemma. *Let f be a payoff function, and let $f_{sw} : (Y \times X)^{\mathbb{N}} \to \mathbb{R}$ be defined by*

$$f_{sw}((y_1, x_1), (y_2, x_2), \ldots) = f((x_1, y_1), (x_2, y_2), \ldots) \tag{3.7}$$

Then

$$\mathrm{val}^{\downarrow}(\Gamma(-f)) = -\mathrm{val}^{\uparrow}(\Gamma_{sw}(f_{sw})) \tag{3.8}$$

$$\mathrm{val}^{\uparrow}(\Gamma(-f)) = -\mathrm{val}^{\downarrow}(\Gamma_{sw}(f_{sw})) \tag{3.9}$$

where $\Gamma_{sw}(f_{sw})$ is the Blackwell game with payoff function f_{sw} in which player I selects moves from Y and player II selects moves from X.

3.1.12. Lemma. *Let $(f_i)_i$ be a sequence of payoff functions $f_i : W \to [a, b]$ such that $(f_i)_i$ converges pointwise to a function $f : W \to [a, b]$. Then for any two strategies σ, τ, $\lim_{i\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f_i)) = E(\sigma \text{ vs } \tau \text{ in } \Gamma(f))$*

3.1.13. Lemma. *Let $(f_i)_i$ be a sequence of payoff functions $f_i : W \to [a, b]$ such that $(f_i)_i$ converges uniformly to a function $f : W \to [a, b]$. Then $\mathrm{val}^{\downarrow}(\Gamma(f)) = \lim_{i\to\infty} \mathrm{val}^{\downarrow}(\Gamma(f_i))$ and $\mathrm{val}^{\uparrow}(\Gamma(f)) = \lim_{i\to\infty} \mathrm{val}^{\uparrow}(\Gamma(f_i))$.*

3.2 Starting and Stopping

On occasion, we would like to 'fix' a finite sequence of initial moves, and to consider the game starting from that position rather than the game starting from the empty position ϵ. Similarly, we sometimes like to consider a game with positions such that if the game ever hits such a position, the outcome is known and the players may stop playing. This section will lay the basics for using starting and stopping positions. The next section will give some tools for using starting and stopping positions to 'combine' games.

3.2.1. DEFINITION. Let $f : W \to \mathbb{R}$ be a bounded Borel function, and $p = ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n))$ a position. The *subgame* $\Gamma(f, p)$ *starting from position* p is the game played like $\Gamma(f)$, except that the players start at round $n+1$, and the first n moves are supposed to have been $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$. The game $\Gamma(f, p)$ is played exactly the same as the game $\Gamma(g)$, where g is the payoff function defined by $g(w) = f(p^\frown w)$.

As before, strategies σ and τ determine a probability measure $\mu_{\sigma,\tau \text{ in } \Gamma(f,p)}$ on W. This measure is equal to the conditional probability measure obtained from $\mu_{\sigma,\tau}$ given $[p]$, i.e.

$$\mu_{\sigma,\tau \text{ in } \Gamma(f,p)}(S) = \frac{\mu_{\sigma,\tau}(S \cap [p])}{\mu_{\sigma,\tau}[p]} \quad \text{if } \mu_{\sigma,\tau}[p] = 0 \tag{3.10}$$

The expected income of player I, the value of a strategy σ, etc. are defined for the games $\Gamma(f, p)$ in the same manner as for the games $\Gamma(f)$.

3.2.2. DEFINITION. A *stopping position* in a Blackwell game $\Gamma(f)$ is a position p, such that for all plays $w, w' \in [p]$, $f(w) = f(w')$. We will denote this value by $f(p)$. A *stopset* in a Blackwell game $\Gamma(f)$ is a set H of stopping positions, such that no stopping position $p \in H$ precedes another stopping position $p' \in H$.

We will often define a payoff function f using the following format:

$$\begin{aligned} f(p) &= \text{some formula} && \text{for } p \in H \\ f(w) &= \text{some other formula} && \text{for } w \notin [H] \end{aligned}$$

where H is a set of positions such that no position $p \in H$ precedes another position $p' \in H$. In the game $\Gamma(f)$ constructed in this fashion, H is a stopset.

3.2.3. REMARK. If p is a stopping position, any moves made at or after p will not affect the outcome of the game. It is often convenient to assume that both players will stop playing if a stopping position is hit. If $\Gamma(f)$ is a Blackwell game, and H is a stopset, we write $\Gamma_H(f)$ to explicitly denote that players stop playing at the positions in H. In this case, we only require strategies to be defined on non-stopping positions. Similarly, with respect to a subgame $\Gamma(f, p)$, we only require strategies to be defined on positions that are following or equal to p.

Using stopsets, a finite game can be treated as a special type of infinite game.

3.2.4. DEFINITION. Let $\Gamma(f)$ be a Blackwell game. If, for some n, all positions in W_n are stopping positions, then $\Gamma(f)$ is called *finite (of length n)*. If $\Gamma(f)$ is finite, we can stop after playing n rounds, and we will denote this by writing $\Gamma_n(f)$.

3.3 Equivalent Truncated Subgames

In games like chess, Go, or even Risk or Monopoly, a player is usually allowed to give up if he has no hope of winning. He or she doesn't have to play it out in the hope that the other player will make a mistake. Two players can agree beforehand to stop in certain positions, and pay out the value of the game at that position rather than continue playing. Provided their assessment of that value is accurate, this does not change the value of the total game. We will call a game resulting from such an alteration a *truncated subgame*.

3.3.1. DEFINITION. Let f, g be two payoff functions, and H a stopset in $\Gamma(g)$. $\Gamma_H(g)$ is an *equivalent truncated subgame* of $\Gamma(f)$ (*truncated at* H), if for any play $w \notin [H]$, $f(w) = g(w)$, and for any $p \in H$, $g(p) = \text{val}(\Gamma(f,p))$. $\Gamma_H(g)$ is a *truncated subgame, equivalent for player I [for player II]*, if for any play $w \notin H$, $f(w) = g(w)$, and for any $p \in H$, $g(p) = \text{val}^{\downarrow}(\Gamma(f,p))$ $[g(p) = \text{val}^{\uparrow}(\Gamma(f,p))]$. In all three cases, $\Gamma(f)$ is called an *extension* of $\Gamma_H(g)$.

Note that $\Gamma_H(g)$ is an equivalent truncated subgame of $\Gamma(f)$ if and only if it is a truncated subgame equivalent for both player I and player II.

3.3.2. LEMMA. *Let $\Gamma(f)$ be a Blackwell game, and let $\Gamma_H(g)$ be a truncated subgame of $\Gamma(f)$, truncated at a set of positions H, equivalent for player I [for player II]. Then* $\text{val}^{\downarrow}(\Gamma(f)) = \text{val}^{\downarrow}(\Gamma_H(g))$ $[\text{val}^{\uparrow}(\Gamma(f)) = \text{val}^{\uparrow}(\Gamma_H(g))]$. *Furthermore, for any $\epsilon > 0$, any ϵ-optimal strategy for player I [player II] in $\Gamma_H(g)$ (if it is undefined on all positions at or after positions in H) can be extended to an ϵ-optimal strategy for player I [player II] in $\Gamma(f)$, and conversely, any ϵ-optimal strategy for player I [player II] in $\Gamma(f)$ is also an ϵ-optimal strategy for that player in $\Gamma_H(g)$.*

Proof

Let σ_0 be an ϵ-optimal strategy for player I in $\Gamma_H(g)$. Let $\delta = \text{val}^{\downarrow}(\Gamma_H(g)) - \text{val}(\sigma \text{ in } \Gamma_H(g))$, and for each stopping position $p' \in H$, let $\sigma_{p'}$ be an $(\epsilon - \delta)$-optimal strategy for player I in $\Gamma(f,p')$. We can combine these strategies in a single strategy σ for player I in $\Gamma(f)$, by setting

$$\sigma(p) = \begin{cases} \sigma_{p'}(p) & \text{if for some } p' \in H,\ p' \subseteq p \\ \sigma_0(p) & \text{otherwise} \end{cases} \tag{3.11}$$

It is easy to verify that this is an ϵ-optimal strategy for player I in $\Gamma(f)$. The converse holds trivially.

□

3.3.3. COROLLARY. *Let $\Gamma(f)$ be a Blackwell game, and let $\Gamma_H(g)$ be an equivalent truncated subgame of $\Gamma(f)$ (truncated at H). If $\Gamma_H(g)$ is determined, then $\Gamma(f)$ is determined, and* $\text{val}(\Gamma(f)) = \text{val}(\Gamma_H(g))$. *Furthermore, for any $\epsilon > 0$, any ϵ-optimal strategy for player I or player II in $\Gamma_H(g)$ can be extended to an ϵ-optimal strategy for player I or player II in $\Gamma(f)$.*

3.3.4. COROLLARY. *Let $\Gamma(f), \Gamma_H(g)$ be Blackwell games. If for any $p \in H$, $g(p) \leq \mathrm{val}^{\downarrow}(\Gamma(f, p))$, and for any $w \notin [H]$, $g(w) \leq f(w)$, then $\mathrm{val}^{\downarrow}(\Gamma_H(g)) \leq \mathrm{val}^{\downarrow}(\Gamma(f))$. Similarly for the value and the upper value, and for $\geq$ instead of $\leq$.*

Truncated subgames may be nested. If we have a nested series of truncated subgames, then we may extend a strategy for the smallest subgame to a strategy for *all* subgames. This allows us to approximate complicated games with a series of simpler, truncated subgames, obtain a strategy that is (ϵ-)optimal in all the subgames. The final lemma in this section allows us to prove results for that strategy in the original game.

3.3.5. DEFINITION. Let, for $n \in I\!N$, f_n be a payoff function, and H_n a set of stopping positions in $\Gamma(f_n)$. If for all $n \in I\!N$, $\Gamma_{H_n}(f_n)$ is a truncated subgame of $\Gamma_{H_{n+1}}(f_{n+1})$, and equivalent to $\Gamma_{H_{n+1}}(f_{n+1})$ [for player I, II], then the series of games $(\Gamma_{H_n}(f_n))_{n\in I\!N}$ is called a *nested series of equivalent truncated subgames* [equivalent for player I, II].

3.3.6. LEMMA. *Let $(\Gamma_{H_i}(f_i))_{i\in I\!N}$ be a nested series of truncated games equivalent for player I [player II]. Then all the games have the same lower value [upper value]. Furthermore, we can find a strategy for player I [player II] that is ϵ-optimal in all the games $\Gamma_{H_i}(f_i)$.*

Proof

We may assume without loss of generality that $H_0 = \{e\}$ and that for any $i \in I\!N$, any stopping position in $\Gamma_{H_i}(f_i)$ is either equal to or preceded by a stopping position from the set H_i. Note that for $j > i$, all elements of H_j are stopping positions in $\Gamma_{H_i}(f_i)$.

For $i > 0$ and $p' \in H_{i-1}$, let $\sigma_{i,p'}$ be a $2^{-i}\epsilon$-optimal strategy for player I in $\Gamma_{H_i}(f_i, p')$. We can combine these strategies in a single strategy σ for player I in $\Gamma(f)$, by setting

$$\sigma(p) = \begin{cases} \sigma_{i,p'}(p) & \text{if } p' \in H_i,\ p' \subseteq p, \text{ and } \neg\exists p'' \in H_{i+1} : p'' \subseteq p \\ \text{arbitrary} & \text{if } \forall i \exists p' \in H_i : p' \subseteq p \end{cases} \tag{3.12}$$

It is easy to verify that for all i, this is an $(1-2^{-i})\epsilon$-optimal strategy for player I in the game $\Gamma_{H_i}(f_i)$. Conversely, any ϵ-optimal strategy for player I in a game $\Gamma_{H_i}(f_i)$ is also an ϵ-optimal strategy for player I in the games $\Gamma_{H_j}(f_j)$, for all $j < i$.

□

3.3.7. LEMMA. *Let $(\Gamma_{H_i}(f_i))_{i\in I\!N}$ be a nested series of equivalent truncated subgames equivalent for player I [player II]. Let $f : W \to I\!R$ be a bounded Borel function such that*

$$\forall w \in W : \liminf_{i\to\infty} f_i(w) \leq f(w) \leq \limsup_{i\to\infty} f_i(w) \tag{3.13}$$

Then the game $\Gamma(f)$ has the same lower value [upper value] as the games $\Gamma_{H_i}(f_i)$.

Proof

The proof of this lemma is based on work of D.A. Martin[16, proof of Lemma 1.1]. We may assume without loss of generality that $H_0 = \{e\}$ and that for any $i \in \mathbb{N}$, any stopping position in $\Gamma_{H_i}(f_i)$ is either equal to or preceded by a stopping position from the set H_i. Note that for $j > i$, all elements of H_j are stopping positions in $\Gamma_{H_i}(f_i)$.

Let v be the value of the games $\Gamma_{H_i}(f_i)$. First we will show that the lower value of the game $\Gamma(f)$ is at least v. Let $\epsilon > 0$, and consider the strategy σ defined in Lemma 3.3.6. The value of this strategy in $\Gamma(f)$ is at least $v - 3\epsilon$. For let τ be an arbitrary counter-strategy for player II, and suppose that $E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) < v - 3\epsilon$. We can find a continuous bounded function $g : W \to \mathbb{R}$ such that $f \le g$ and[7]

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma(g)) = \int g(w) d\mu_{\sigma,\tau}(w) \le \int f(w) d\mu_{\sigma,\tau}(w) + \epsilon < v - 2\epsilon \qquad (3.14)$$

Define the functions $g_i : W \to \mathbb{R}$, for $i \in \mathbb{N}$, by setting

$$\begin{array}{rclr} g_i(p) & = & E(\sigma \text{ vs } \tau \text{ in } \Gamma(g, p)) & \text{for } p \in H_i \\ g_i(w) & = & g(w) & \text{for } w \notin [H_i] \end{array} \qquad (3.15)$$

Then we have for any $i \in \mathbb{N}$ and any position $p \in H_i$,

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_{i+1}}(g_{i+1}, p)) = g_i(p) \qquad (3.16)$$

By the construction of σ, we know that for any $i \in \mathbb{N}$ and any position $p \in H_i$,

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_{i+1}}(f_{i+1}, p)) > f_i(p) - 2^{-i-1}\epsilon \qquad (3.17)$$

Using induction, we can now construct a sequence of positions p_i such that for all i, $p_i \in H_i$, $p_i \subseteq p_{i+1}$, and

$$g_i(p_i) < f_i(p_i) - (1 + 2^{-i})\epsilon \qquad (3.18)$$

For $i = 0$, we can take $p_0 = e$, since by our assumption that $H_0 = \{e\}$ we can write

$$g_0(e) = E(\sigma \text{ vs } \tau \text{ in } \Gamma(g)) < v - 2\epsilon = f_0(e) - 2\epsilon \qquad (3.19)$$

For $i + 1$, we use the induction hypothesis and equations (3.16) and (3.17) to derive

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_{i+1}}(g_{i+1})) < E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_{i+1}}(f_{i+1})) - (1 + 2^{-i-1})\epsilon \qquad (3.20)$$

which, together with the observation that for all $w \in W - [H_i]$, $g_i(w) = g(w) \ge f(w) = f_i(w)$, implies that there must be a position $p_{i+1} \in H_{i+1}$ extending p_i and satisfying (3.18).

[7] The existence of g is guaranteed by the observation that f is bounded, and that we can contain the discontinuities of f in an open set of arbitrary small $\mu_{\sigma,\tau}$-measure[9, Theorem 17.12].

If for some i, $p_j = p_i$ for all $j > i$, then we are done, for then p_i is a stopping position of f and $E(\sigma \text{ vs } \tau \text{ in } \Gamma(g, p_i)) < f(p) - \epsilon$, contradicting $g \geq f$. So assume that the sequence $(p_i)_{i \in \mathbb{N}}$ is not eventually constant. For all $i \in \mathbb{N}$ we can find $w_i \in [p_i]$ such that

$$g(w_i) \leq E(\sigma \text{ vs } \tau \text{ in } \Gamma(g, p_i)) = g_i(p_i) \tag{3.21}$$

Since $(p_i)_{i \in \mathbb{N}}$ is not eventually constant, the sequence $(w_i)_{i \in \mathbb{N}}$ converges to some $w \in W$. Since g is continuous, we now have

$$g(w) = \lim_{i \to \infty} g(w_i) \leq \liminf_{i \to \infty} g_i(p_i) \leq \liminf_{i \to \infty} f_i(p_i) - \epsilon \leq f(w) - \epsilon \tag{3.22}$$

again contradicting $g \geq f$. It follows that there exists no counter-strategy τ for player II with $E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) < v - 3\epsilon$, as required.
To show that the lower value of the game $\Gamma(f)$ is not greater than v, let σ be a strategy for player I in $\Gamma(f)$, and suppose that σ is of value $> v + 3\epsilon$. In the manner of the construction of Lemma 3.3.6, we can construct a counter-strategy τ for player II such that for any $i \in \mathbb{N}$ and any position $p \in H_i$,

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_{i+1}}(f_{i+1}, p)) < f_i(p) + 2^{-i-1}\epsilon \tag{3.23}$$

From here on, we can continue in the manner of the first part of this proof, to again derive a contradiction.

□

3.3.8. COROLLARY. *Let $(\Gamma_{H_i}(f_i))_{i \in \mathbb{N}}$ be a nested series of equivalent truncated subgames. If $\Gamma_{H_0}(f_0)$ is determined, then all the games $\Gamma_{H_i}(f_i)$ are determined, as well as the game $\Gamma(f)$, where f is as in Lemma 3.3.7. Furthermore, all the games have the same value, and we can find strategies for player I and player II that are ϵ-optimal in all the games $\Gamma_{H_i}(f_i)$ and the game $\Gamma(f)$.*

3.3.9. REMARK. If the component games involved all have optimal strategies, then we can extend optimal strategies with optimal strategies to optimal strategies, i.e. drop the ϵ in the above lemmas and corollaries.