Games, walks and grammars: Problems I've worked on

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Chapter 5

Non-Borel Blackwell Games

Up to this point, we have implicitly assumed that any payoff function f is Borel measurable. Indeed, properly speaking Blackwell games are (as yet) only defined for Borel measurable payoff functions. In this chapter we will look at non-Borel Blackwell games.

5.1 Definitions

If f is a bounded non-Borel function, then we can play the corresponding Blackwell game in exactly the same manner as before. However, when we try to define the value of a strategy, we run into a problem: f is not necessarily integrable under arbitrary measures $\mu_{\sigma,\tau}$. So instead of the value of the integral itself, we have to use approximations from above and below.

5.1.1. Definition. Let f be a bounded but not necessarily Borel measurable function. The *Blackwell game* $\Gamma(f)$ with *payoff function* f is played exactly as in Definition 3.1.1. For any strategies σ and τ for players I and II, we define the probability measure $\mu_{\sigma,\tau}$ as in Definition 3.1.5. However, instead of the expected income of player I (if she plays according to σ and player II plays according to τ) we now define the *lower* and *upper expected income* of player I:

$$E^{-}(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) = \sup_{g \leq f, g \text{ Borel measurable}} \int g(w) d\mu_{\sigma,\tau}(w) \tag{5.1}$$

$$E^{+}(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) = \inf_{g \geq f, g \text{ Borel measurable}} \int g(w) d\mu_{\sigma,\tau}(w) \tag{5.2}$$

The definition of the value of a strategy σ or τ in $\Gamma(f)$ is slightly different as well:

$$\text{val}(\sigma \text{ in } \Gamma(f)) = \inf_{\tau} E^{-}(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{5.3}$$

$$\text{val}(\tau \text{ in } \Gamma(f)) = \sup_{\sigma} E^{+}(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{5.4}$$

The definitions of lower and upper values and of determinacy are the same as in Definition 3.1.7.

Note that when f is measurable, these definitions are all equivalent to the old definitions.
It is easy to see that Lemmas 4.5.4 and 4.5.5 also hold for non-Borel Blackwell games. This means that determinacy of a particular class of perfect information games implies determinacy of the corresponding class of Blackwell games. The determinacy of many classes of perfect information games can be deduced from so-called large cardinal axioms, and hence we have corresponding results for Blackwell games[15]. Martin gives the example that, for $n \geq 0$, determinacy of Σ^1_{n+1} Blackwell games follows from determinacy of Σ^1_{n+1} perfect information games, which in turn follows from the existence of n Woodin cardinals with a measurable cardinal above them.

5.2 The Axiom of Blackwell Determinacy

A well-known axiom in set theory is the Axiom of Determinacy[17].

5.2.1. DEFINITION. The *Axiom of Determinacy* (AD) is the assertion that

> all game of Perfect Information and finite or countable choice of moves are determined.

Using the Axiom of Choice, a non-measurable payoff function f can be constructed such that $\Gamma(f)$ is not determined [6], hence AD contradicts AC, hence AD contradicts the Axiom of Choice, and has many interesting consequences, such as the existence of an ultrafilter on $\aleph_1$ and of a complete measure on $\mathbb{R}$. It is commonly used in large cardinal theory as an alternative to AC [8, 17]. We can formulate an analogue of this axiom for Blackwell games.

5.2.2. DEFINITION. The *Axiom of Determinacy for Blackwell Games* (AD-Bl) is the assertion that

> all Blackwell games are determined

In this case as well, we can use Lemmas 4.5.4 and 4.5.5 to obtain

5.2.3. THEOREM. *Working in ZF without the Axiom of Choice, AD implies AD-Bl.*

It is unknown whether the converse also holds, i.e. whether AD-Bl also implies AD. For a given game of Perfect Information, we can easily construct a Blackwell game that is 'equivalent', and assuming AD-Bl we can find an ϵ-optimal *mixed* strategy for that equivalent Blackwell-game. But to derive AD from AD-Bl, we

need to have a *pure* strategy, and even though we can interpret any mixed strategy as a probability distribution on pure strategies, there is no guarantee that any of these pure strategies by itself will do as well as the mixed strategy.
However, a number of consequences of AD *can* be derived from AD-Bl. In fact, one of them is almost trivial:

5.2.4. THEOREM. *Assuming AD-Bl, it follows that all sets of reals are Lebesgue measurable.*

Proof
It suffices to show that the Lebesgue measure on $[0, 1]$ is complete. Set $X = Y = \{0, 1\}$, and define $\phi : W \to [0, 1]$ by

$$\phi((x_1, y_1, x_2, y_2, \ldots)) = \sum_{i=1}^{\infty} 2^{-i}(x_i \oplus y_i) \tag{5.5}$$

where $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$. Now let σ, τ be strategies, and suppose that one of those strategies is the strategy that assigns the $\frac{1}{2}$-$\frac{1}{2}$ probability distribution on X or Y, respectively. Then for any $i \in I\!N$, $x_i \oplus y_i$ has equal chances of being 0 or 1, and the distribution of $\phi(w)$ on $[0, 1]$ is the uniform distribution on $[0, 1]$ under the Lebesgue measure on $[0, 1]$. It follows that for any subset $S \subset [0, 1]$,

$$\begin{aligned} \mu^{\text{inner}}_{\sigma,\tau}(\phi^{-1}[S]) &= \mu^{\text{inner}}_{\lambda}(S) && (5.6)\\ \mu^{\text{outer}}_{\sigma,\tau}(\phi^{-1}[S]) &= \mu^{\text{outer}}_{\lambda}(S) && (5.7) \end{aligned}$$

where $\mu^{\text{inner}}(A) = \sup_{B \subseteq A \text{ measurable}} \mu(B)$, $\mu^{\text{outer}}(A) = \inf_{B \supseteq A \text{ measurable}} \mu(B)$.
Let $S \subset [0, 1]$. No strategy for player I in the game $\Gamma(\phi^{-1}[S])$ can have value greater than $\mu^{\text{inner}}_{\lambda}(S)$, since this is the lower expected income of any strategy for player I against the $\frac{1}{2}$-$\frac{1}{2}$ strategy. Similarly, no strategy for player II in the game $\Gamma(\phi^{-1}[S])$ can have value less than $\mu^{\text{outer}}_{\lambda}(S)$. Therefore

$$\text{val}^{\downarrow}(\Gamma(\phi^{-1}[S])) \leq \mu^{\text{inner}}_{\lambda}(S) \leq \mu^{\text{outer}}_{\lambda}(S) \leq \text{val}^{\uparrow}(\Gamma(\phi^{-1}[S])). \tag{5.8}$$

From the determinacy of the game $\Gamma(\phi^{-1}[S])$, it now follows that

$$\mu^{\text{inner}}_{\lambda}(S) = \mu^{\text{outer}}_{\lambda}(S) \tag{5.9}$$

Since this holds for arbitrary sets $S \subset [0, 1]$, all subsets of $[0, 1]$ are Lebesgue measurable.

□

5.2.5. COROLLARY. *AD-Bl is not consistent with AC, and the consistency of ZF + AD-Bl cannot be proven in ZFC.*

5.3 Mixed Strategies in Games of Perfect Information

There are a number of other consequences of AD that are also consequences of AD-Bl [11]. In some cases, it is possible to adapt the proofs for AD to use mixed strategies instead of pure strategies (an example will be given in the next section). In general, a requirement for such an adaption is that we may assume the mixed strategy is of value 0 or 1, rather than somewhere in between. The theorems in this section show that for games of perfect information, this is a valid assumption.

5.3.1. REMARK. The games of perfect information that fall under AD, and are used in this section and the next, are usually defined on a *game tree*. Aside from the fact that only one player moves at a time, games defined on game trees differ from the games as defined in definition 3.1.1, in that the sets from which the players select their moves are dependent on the current position, and may even be countably infinite. However, all such games are equivalent to games with binary choice-of-moves. For instance, a selection of a natural number can be emulated by letting the player repeatedly choose between 0 and 1 until the player chooses 1, and interpreting the number of times the player selected 0 as the natural number selected. In turn, a binary perfect information game can be simulated by a Blackwell game with $X = Y = \{0, 1\}$, simply by constructing a payoff function which only depends on the moves made in the Blackwell game by the player whose turn it was to move in the perfect information game (at that position).
It follows that all the definitions and statements relating to Blackwell games (including AD-Bl) also apply to mixed strategies in countable perfect information games. Therefore we will simply treat these games as Blackwell games, ignoring the formal differences.

We first need an auxiliary lemma:

5.3.2. LEMMA. *Let $\Gamma_n(f)$ be a finite perfect information game, and let σ, τ be mixed strategies for players I and II of values v and v'. Then*

$$\forall x > v' : \mu_{\sigma,\tau}\{w \in W_n \mid f(w) \geq x\} \leq \frac{v' - v}{x - v} \tag{5.10}$$

and

$$\forall x < v : \mu_{\sigma,\tau}\{w \in W_n \mid f(w) \leq x\} \leq \frac{v' - v}{v' - x} \tag{5.11}$$

Proof
We will prove this by induction on the length n of the game $\Gamma_n(f)$. For $n = 0$, it is trivial. For $n + 1$, we may assume without loss of generality that in the first round, player I is to move. Let $p_1, p_2, \ldots$ denote the positions of Γ_n that can be

reached from the starting position e in a single move. For $i \in I\!N$, let z_i denote the probability (according to player I's strategy σ) that player I will move to p_i, and let v_i and v_i' denote the values of the strategies σ and τ at that position. Then for all $i \in I\!N$, $v_i \leq v_i' \leq v'$, and $v = \sum_{i \in I\!N} z_i v_i$.
From the induction hypothesis it follows that for $x > v'$ and $i \in I\!N$,

$$\mu_{\sigma,\tau}(\{w \in [p_i] \mid f(w) \geq x\}) \leq z_i \frac{v_i' - v_i}{x - v_i} \leq z_i \frac{v' - v_i}{x - v_i} = z_i - z_i \frac{x - v'}{x - v_i} \tag{5.12}$$

and hence for $x > v'$

$$\mu_{\sigma,\tau}(\{w \in W_n \mid f(w) \geq x\}) = \sum_{i \in I\!N} \mu_{\sigma,\tau}(\{w \in [p_i] \mid f(w) \geq x\}) \tag{5.13}$$

$$\leq \sum_{i \in I\!N} \left(z_i - z_i \frac{x - v'}{x - v_i} \right) \leq 1 - \frac{x - v'}{x - v} = \frac{v' - v}{x - v} \tag{5.14}$$

Similarly it follows from the induction hypothesis that for $x < v$ and $i \in I\!N$,

$$\mu_{\sigma,\tau}(\{w \in [p_i] \mid f(w) \leq x\}) \leq z_i \cdot \begin{cases} \frac{v_i' - v_i}{v_i' - x} & \text{if } x < v_i \\ 1 & \text{if } v_i \leq x \end{cases} \leq z_i \frac{v' - v_i}{v' - x} \tag{5.15}$$

and hence for $x < v$

$$\mu_{\sigma,\tau}(\{w \in W_n \mid f(w) \leq x\}) = \sum_{i \in I\!N} \mu_{\sigma,\tau}(\{w \in [p_i] \mid f(w) \leq x\}) \tag{5.16}$$

$$\leq \sum_{i \in I\!N} z_i \frac{v' - v_i}{v' - x} = \frac{v' - v}{v' - x} \tag{5.17}$$

□

5.3.3. THEOREM (0-1 LAW FOR MIXED STRATEGIES). *Let $\Gamma(S)$ be an countably infinite perfect information game, whose payoff function is the characteristic function of a set S. If $\Gamma(S)$ is determined (in the sense of mixed strategies), then its value is either* 0 *or* 1.

Proof

The following proof is based on communications with D.A. Martin. Let v be the value of $\Gamma(S)$, and suppose that $0 < v < 1$. Then we can select $\epsilon > 0$ such that $2\epsilon < v$, $2\epsilon < 1 - v$ and $v - 2\epsilon > 2\epsilon/(1 - v - 2\epsilon)$. Let σ and τ be ϵ-optimal mixed strategies for players I and II in $\Gamma(S)$, and assume that players I and II play according to σ and τ. Let H be the set of positions

$$H := \{p \in P \mid E(\sigma \text{ vs } \tau \text{ in } \Gamma(S, p)) \geq 1 - \epsilon\} \tag{5.18}$$

Now, we can find open sets $O \supseteq S$ with $\mu_{\sigma,\tau}(O)$ arbitrarily close to $\mu_{\sigma,\tau}(S)$, and for any such O, $(1-\epsilon)\mu_{\sigma,\tau}(O) + \epsilon\mu_{\sigma,\tau}([H]) \geq \mu_{\sigma,\tau}(S)$. It follows that

$$\mu_{\sigma,\tau}([H]) \geq \mu_{\sigma,\tau}(S) \geq \mathrm{val}(\sigma \text{ in } \Gamma(S)) \geq v-\epsilon \tag{5.19}$$

i.e. with probability at least $v-\epsilon$, the game will eventually visit a position in H. Let $n \in I\!\!N$ be such that the probability of visiting H before time n is at least $v-2\epsilon$, and let $\Gamma_{H'}(f)$ be the truncated subgame which stops at time n or when H is visited (whichever happens first), with payoff $f(p) = E(\sigma \text{ vs } \tau \text{ in } \Gamma(S,p))$. Now, since for all $p \in H'$, $f(p) \geq \mathrm{val}(\sigma \text{ in } \Gamma(S,p))$, we have

$$\mathrm{val}(\tau \text{ in } \Gamma(f)) \geq \mathrm{val}(\sigma \text{ in } \Gamma(f)) \geq \mathrm{val}(\sigma \text{ in } \Gamma(S)) > v-\epsilon \tag{5.20}$$

and similarly,

$$\mathrm{val}(\sigma \text{ in } \Gamma(f)) \leq \mathrm{val}(\tau \text{ in } \Gamma(f)) \leq \mathrm{val}(\tau \text{ in } \Gamma(S)) < v+\epsilon \tag{5.21}$$

so by applying the previous Lemma we obtain,

$$\mu_{\sigma,\tau}\{w \in W_n \mid f(w) \geq 1-\epsilon\} \leq \frac{2\epsilon}{1-v-2\epsilon} < v-2\epsilon \tag{5.22}$$

contradicting our choice of n.

□

5.3.4. THEOREM. *Assuming AD-Bl, in every countably infinite perfect information game $\Gamma(S)$ whose payoff function is the characteristic function of a set S, either player I has a strategy of value 1 or player II has a strategy of value 0*[10].

Proof

By AD-Bl and the 0-1 Law for Mixed Strategies, for any position $p \in P$ the value of the game $\Gamma(S,p)$ is either 0 or 1. Without loss of generality we may assume that $\Gamma(S)$ itself is of value 1. If we eliminate positions of value 0 (or equivalently, constrain player I to avoid those positions), the value of the game will not change, so we may also assume without loss of generality that for any position $p \in P$, $\mathrm{val}(\Gamma(S,p)) = 1$.

Now, for any $p \in P$, let σ_p be a strategy of value $> 2/3$ in the game $\Gamma(S,p)$[11]. Set $H_0 = \{e\}$. We can inductively define nested stopping sets H_i such that, for all $i \geq 0$, H_{i+1} consists of all the first positions p' following a position p in H_i,

[10] To avoid confusion, remember: for player II, lower values are better. A strategy for player II of value 0 is a strategy such that the expected payoff is at most 0, i.e. such that player II wins almost surely.

[11] This does not require the Axiom of Choice. Consider an auxiliary game where player II first selects a position $p \in P$, and the players then play $\Gamma(S,p)$. Obviously player II cannot have a strategy of value < 1, so by AD-Bl, player I has a strategy of value $> 2/3$. This strategy contains all the strategies h_p.

such that val(σ_p in $\Gamma(S, p')) < 1/3$. Let σ be the strategy where player I starts out by playing according to σ_e, and whenever a position $p \in H_0 \cup H_1 \cup \ldots$ is hit, player I switches to playing according to σ_p. Then for any $i \geq 0$, and any $p \in P$, val(σ in $\Gamma_{H_i}(S \cup [H_i], p)) \geq 1/3$.
If player I uses this strategy, and player II uses some strategy τ, then for any $i \geq 0$, the probability (conditional on hitting H_i) of hitting H_{i+1} is at most $1/2$. Hence $\mu_{\sigma,\tau}(\bigcap_{i\geq 0} H_i) = 0$. It follows that for all $p \in P$,

$$\begin{aligned} E(\sigma \text{ vs } \tau \text{ in } \Gamma(S, p)) &= E(\sigma \text{ vs } \tau \text{ in } \Gamma(S \cup \bigcap_{i\geq 0}[H_i], p)) && (5.23)\\ &= \lim_{i\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma(S \cup H_i, p)) && (5.24)\\ &\geq 1/3 && (5.25)\end{aligned}$$

Now, for any strategy τ for player II, if $E(\sigma \text{ vs } \tau \text{ in } \Gamma(S)) < 1$, then for any $\epsilon > 0$ there would be positions $p \in P$ such that

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma(S, p)) < \epsilon \qquad (5.26)$$

We conclude that such a strategy τ does not exist, and that therefore σ is of value 1 in $\Gamma(S)$.

□

5.4 Constructing a Free and σ-Complete Ultrafilter on ω_1

A *σ-complete ultrafilter* on a set X is a collection U of subsets of X, closed under countable intersection and taking supersets, such that for each set $V \subseteq X$, exactly one of $V, X - V$ is in the ultrafilter. For any cardinal α, U is called α-complete if it is closed under the intersection of less than α sets. An ultrafilter is called *free* if it is not of the form $\{V \subseteq X \mid x \in V\}$ for some $x \in X$. If there exists a free, α-complete ultrafilter on a set of cardinality α, we say that α is *measurable*.
Under AC, $\aleph_1$ (the first uncountable cardinal) is not measurable, i.e. there exists no free, σ-complete ultrafilter on a set of cardinality $\aleph_1$. It is a well-known theorem of large cardinal theory that under AD, $\aleph_1$ *is* measurable[8]. In this section we take a construction of a free σ-complete ultrafilter on the set $\omega_1 = \{\alpha \in ORD \mid \alpha \text{ is finite or countable}\}$ (a set of cardinality $\aleph_1$) which uses the Axiom of Determinacy [19], and modify it to use AD-Bl instead.
Let $V \subseteq \omega_1$. We define auxiliary perfect information games $\Gamma^{\omega_1}(V)$ in which players I and II independently construct countably many countable ordinals, represented as subsets of $\mathbb{Q}$. The two players maintain separate (countable) collections of (initially empty) subsets of $\mathbb{Q}$, and each round adds finitely many elements to finitely many subsets. Player I wins at the 'end' of the game if the supremum of

the ordinals represented by the constructed subsets is in V, otherwise player II wins. Formally:

5.4.1. DEFINITION. Let $\mathcal{A}$ and $\mathcal{B}$ be two countably infinite, disjoint sets. For any subset $V \subseteq \omega_1$, the game $\Gamma^{\omega_1}_{\mathcal{A},\mathcal{B}}(V)$ is defined as follows:

- In any round $i \geq 1$, first player I selects a finite set a_i of pairs $(a, q) \in \mathcal{A} \times \mathbb{Q}$, and then player II selects a finite set b_i of pairs $(b, r) \in \mathcal{B} \times \mathbb{Q}$.
- Set $\mathcal{I} := \mathcal{A} \cup \mathcal{B}$, and define the *result* z by $z := a_1 \cup b_1 \cup a_2 \cup b_2 \cup \ldots \subseteq \mathcal{I} \times Q$.
- Let the function $\pi : \mathcal{P}(\mathbb{Q}) \to \omega_1$ be defined by

$$\pi(R) := \begin{cases} \text{the order type of } (R, <) & \text{if } (R, <) \text{ is well-ordered} \\ 0 & \text{otherwise} \end{cases} \tag{5.27}$$

  Set $\Pi_{\mathcal{A},\mathcal{B}}(z) := \sup_{i \in \mathcal{I}} \pi(\{q \in \mathbb{Q} \mid (i, q) \in z\})$.
  Player I wins if $\Pi_{\mathcal{A},\mathcal{B}}(z) \in V$, otherwise player II wins.

If we interpret $\Pi_{\mathcal{A},\mathcal{B}}$ as a function from $\mathcal{P}(\mathcal{I} \times \mathbb{Q})$ to ω_1, then we can set

$$\underline{V}_{\mathcal{A},\mathcal{B}} := \Pi^{-1}_{\mathcal{A},\mathcal{B}}[V] = \{z \subseteq \mathcal{I} \times \mathbb{Q} \mid \Pi_{\mathcal{A},\mathcal{B}}(z) \in V\} \tag{5.28}$$

and the winning condition of the game can be reformulated as "Player I wins if $z \in \underline{V}_{\mathcal{A},\mathcal{B}}$, otherwise player II wins".

5.4.2. REMARK. For technical reasons, it is necessary at some places in the proof to be able to react to one's own moves as if they had been made by the opponent. This is done by temporarily considering some of one's own subsets-under-construction to belong to the other player, for the purpose of reacting to the moves made in them. The 'index-structure' $(\mathcal{A}, \mathcal{B})$ used in the definition above is used to facilitate this.
For any two index-structures $(\mathcal{A}, \mathcal{B})$, $(\mathcal{A}', \mathcal{B}')$, there exist bijective mappings $\mathcal{A} \leftrightarrow \mathcal{A}'$ and $\mathcal{B} \leftrightarrow \mathcal{B}'$, which in turn induce bijective mappings between the moves, games and strategies of $\Gamma^{\omega_1}_{\mathcal{A},\mathcal{B}}(V)$ and those of $\Gamma^{\omega_1}_{\mathcal{A}',\mathcal{B}'}(V)$. Therefore, when the distinction is not important, we write $\Gamma^{\omega_1}(V)$, $\Pi(z)$ and $\underline{V}$ for $\Gamma^{\omega_1}_{\mathcal{A},\mathcal{B}}(V)$, $\Pi_{\mathcal{A},\mathcal{B}}(z)$ and $\underline{V}_{\mathcal{A},\mathcal{B}}$. Note that $\Pi_{\mathcal{A},\mathcal{B}}$ and $\underline{V}_{\mathcal{A},\mathcal{B}}$ depend on $\mathcal{A} \cup \mathcal{B}$ only.

An ultrafilter U can be thought of as a partitioning of the subsets of X into 'large' subsets (those in U) and 'small' subsets (those not in U). The property 'player I can almost surely force the supremum to be in V' intuitively seems likely to be a 'largeness'-type property. And indeed, we will show that

5.4.3. THEOREM. *Under AD-Bl, the set*

$$U := \{V \subseteq \omega_1 \mid \textit{player I has a strategy of value } 1 \textit{ in } \Gamma^{\omega_1}(V)\} \tag{5.29}$$

is a free and σ-complete ultrafilter on ω_1.

To prove this, first we, for several different constructions of V from other sets, lemmas constructing strategies in $\Gamma^{\omega_1}(V)$.

5.4.4. LEMMA. *If player I has a strategy of value* 1 *in the game* $\Gamma^{\omega_1}(V)$, *and* $V \subseteq W$, *then player I has a strategy of value* 1 *in* $\Gamma^{\omega_1}(W)$.

Proof

Any strategy of value 1 for player I in $\Gamma^{\omega_1}(V)$ is also of value 1 in $\Gamma^{\omega_1}(W)$.

□

5.4.5. LEMMA. *If V is a singleton, then player II has a strategy of value* 1 *in the game* $\Gamma^{\omega_1}(V)$.

Proof

If $V = \{\beta\}$, then player II can win by constructing $\beta + 1$.

□

5.4.6. LEMMA. *If player I has a strategy of value* 1 *in the game* $\Gamma^{\omega_1}(V)$, *then player II has a strategy of value* 0 *in* $\Gamma^{\omega_1}(\omega_1 - V)$, *and vica versa.*

Proof

Suppose that player I has a strategy f of value 1 in the game $G_{\mathcal{A},\mathcal{B}}(V)$. Then this is also a strategy for player II of value 0 in the game $G_{\mathcal{B},\mathcal{A}}(\omega_1 - V)$, except that since II does not have the first move, player II's response to any move is always 'delayed' by one round. Formally, we construct a strategy g for player II by setting

$$g(\langle b_1, a_1, b_2, a_2, \ldots, a_{k-1}, b_k\rangle) := f(\langle a_1, b_1, a_2, b_2, \ldots, a_{k-1}, b_{k-1}\rangle) \qquad (5.30)$$

For any moves for player I in $G_{\mathcal{B},\mathcal{A}}(\omega_1 - V)$, if player II plays according to g, then the resulting sequence of moves $\langle b_1, a_1, b_2, a_2 \ldots\rangle$ corresponds to a sequence of moves $\langle a_1, b_1, a_2, b_2, \ldots\rangle$ in the game $G_{\mathcal{A},\mathcal{B}}(V)$, such that the probability distribution of player I's moves is according to the strategy f. Hence we have

$$z = b_1 \cup a_1 \cup b_2 \cup a_2 \cup \ldots = a_1 \cup b_1 \cup a_2 \cup b_2 \cup \ldots \in \underline{V} \text{ almost surely} \qquad (5.31)$$

So g is a strategy for player II of value 0 in $G_{\mathcal{B},\mathcal{A}}(V)$.

Now suppose that player II has a strategy g of value 0 in the game $G_{\mathcal{B},\mathcal{A}}(\omega_1 - V)$. Then this is also a strategy for player I of value 1 in the game $G_{\mathcal{A},\mathcal{B}}(V)$, except that player I has a first move in which she does nothing. Formally, we construct a strategy f for player I by setting

$$f(\langle\rangle) := \text{'play } \emptyset\text{'}, f(\langle a_1, b_1, a_2, b_2, \ldots, a_{k-1}, b_{k-1}\rangle) := g(\langle b_1, a_2, b_2, \ldots, a_{k-1}, b_{k-1}\rangle) \qquad (5.32)$$

For any moves for player II in $G_{\mathcal{A},\mathcal{B}}(V)$, if player I plays according to f, then the resulting sequence of moves $\langle \emptyset, b_1, a_2, b_2, a_3, \ldots \rangle$ corresponds to a sequence of moves $\langle b_1, a_2, b_2, a_3, \ldots \rangle$ in the game $G_{\mathcal{B},\mathcal{A}}(\omega_1 - V)$, such that the probability distribution of player II's moves is according to the strategy g. Hence we have

$$z = \emptyset \cup b_1 \cup a_2 \cup b_2 \cup a_3 \cup \ldots = b_1 \cup a_2 \cup b_2 \cup a_3 \cup \ldots \in \underline{V} \text{ almost surely} \quad (5.33)$$

So f is a strategy for player I of value 1 in $G_{\mathcal{A},\mathcal{B}}(V)$.

□

5.4.7. LEMMA. *Let $(V^i)_{i \geq 0}$ be a countable sequence of subsets of ω_1. If for all $i \geq 0$, player II has a strategy of value 0 in $\Gamma^{\omega_1}(V^i)$, then player II has a strategy of value 0 in $\Gamma^{\omega_1}(\bigcup_{i \geq 0} V^i)$.*

Proof

Let $(V^i)_{i \geq 0}$ be a countable sequence of subsets of ω_1, and suppose that for all $i \geq 0$, player II has a strategy of value 0 in $\Gamma^{\omega_1}(V^i)$. Let $(\mathcal{A}, \mathcal{B})$ be an index-structure for the game. We will construct a strategy g of value 0 for player II in $\Gamma^{\omega_1}_{\mathcal{A},\mathcal{B}}(\bigcup_{i \geq 0} V^i)$.

Let $(\mathcal{B}^i)_{i \geq 0}$ be a partitioning of $\mathcal{B}$ into a countably infinite number of disjoint countably infinite sets. Define $\mathcal{A}^i = (\mathcal{A} \cup \mathcal{B}) - \mathcal{B}^i$ for $i \geq 1$. Then for all $i \geq 1$, $(\mathcal{A}^i, \mathcal{B}^i)$ is an index-structure. By assumption, we can find[12] strategies g^i of value 0 for player II in each of the games $\Gamma^{\omega_1}_{\mathcal{A}^i,\mathcal{B}^i}(V^i)$.

Now let $w = (a_1, b_1, a_2, b_2, \ldots)$ be a play of $\Gamma^{\omega_1}_{\mathcal{A},\mathcal{B}}(V)$ such that for all $i \geq 0$ and $k \geq 1$, $b_{2^i(2k-1)} \in \mathcal{B}^i$. If we define for $i \geq 0$ and $k \geq 1$,

$$\begin{aligned}
a^i_1 &= a_1 \cup b_1 \cup a_2 \cup \ldots \cup a_{2^i} \\
a^i_k &= a_{2^i(2k-3)+1} \cup b_{2^i(2k-3)+1} \cup a_{2^i(2k-3)+2} \cup \ldots \cup a_{2^i(2k-1)} \text{ for } k > 1 \\
b^i_k &= b_{2^i(2k-1)}
\end{aligned}$$

then for all $i \geq 0$ and $k \geq 1$, a^i_k and b^i_k are finite subsets of $\mathcal{A}^i \times \mathbb{Q}$ and $\mathcal{B}^i \times \mathbb{Q}$, so $w^i = (a^i_1, b^i_1, a^i_2, b^i_2 \ldots)$ is a valid play of the game $\Gamma^{\omega_1}_{\mathcal{A}^i,\mathcal{B}^i}(V^i)$. So construct the strategy g for player II in $\Gamma^{\omega_1}_{\mathcal{A},\mathcal{B}}(V)$ by setting, for $i \geq 0$ and $k \geq 1$,

$$g((a_1, b_1, \ldots, a_{2^i(2k-1)})) := g^i((a^i_1, b^i_1, \ldots, a^i_k)) \quad (5.34)$$

It can easily be shown (inductively) that for all $i \geq 0$ and $k \geq 1$, $b_{2^i(2k-1)} \in \mathcal{B}^i$ always, so g is well-defined. Moreover, for all $i \geq 0$ and $k \geq 1$, the probability distribution of b^i_k is given by $g^i((a^i_1, b^i_1, a^i_2, \ldots, a^i_k))$, so for all $i \geq 0$, the probability distribution of w^i in the game $\Gamma^{\omega_1}_{\mathcal{A}^i,\mathcal{B}^i}(V^i)$ is consistent with player II's strategy g^i. It follows that for all $i \geq 0$,

$$z = a_1 \cup b_1 \cup a_2 \cup b_2 \cup \ldots = a^i_1 \cup b^i_1 \cup a^i_2 \cup b^i_2 \cup \ldots \in \underline{\omega_1 - V^i} \text{ almost surely} \quad (5.35)$$

[12] Again, we do not need the Axiom of Choice for this. See footnote 11.

and hence

$$z = a_1 \cup b_1 \cup a_2 \cup b_2 \cup \ldots \in \bigcap_{i \geq 0} \underline{\omega_1 - V^i} = \underline{\omega_1 - V} \text{ almost surely} \tag{5.36}$$

So g is a strategy of value 0 for player II in the game $\Gamma^{\omega_1}_{\mathcal{A},\mathcal{B}}(V)$.

□

Proof of Theorem 5.4.3

In any game $\Gamma^{\omega_1}_{\mathcal{A},\mathcal{B}}(V)$, there are only countably many possible moves each turn, since $\mathcal{A}$, $\mathcal{B}$ and $\boldsymbol{Q}$ are all countable, and therefore there are only countably many different finite collections of pairs $(a, q) \in \mathcal{A} \times \boldsymbol{Q}$ or $(b, q) \in \mathcal{B} \times \boldsymbol{Q}$. Hence Theorem 5.3.4 applies, and we have that for all $V \subseteq \omega_1$:

$$\begin{aligned} V \in U \quad &\Leftrightarrow \quad \text{player I has a strategy of value 1 in } \Gamma^{\omega_1}(V) \\ V \notin U \quad &\Leftrightarrow \quad \text{player II has a strategy of value 0 in } \Gamma^{\omega_1}(V) \end{aligned}$$

By the above equivalences, the previous lemmas correspond to the following properties of U:

1. For any $V, W \subseteq \omega_1$, if $V \in U$ and $V \subseteq W$, then $W \in U$.
2. For any $V \subseteq \omega_1$, if V is a singleton, then $V \notin U$.
3. For any $V \subseteq \omega_1$, $V \in U$ if and only if $\omega_1 - V \notin U$.
4. For any sequence $V_i \subseteq \omega_1$, if $V_i \notin U$ for all $i \geq 0$, then $\bigcup_{i \geq 0} V_i \notin U$.

and from the third and fourth property we can derive

5. For any sequence $V_i \subseteq \omega_1$, if $V_i \in U$ for all $i \geq 0$, then $\bigcap_{i \geq 0} V_i \in U$.

So U is a free and σ-complete ultrafilter on ω_1.

□