Games, walks and grammars: Problems I've worked on

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Chapter 6

Random Walks

In this chapter we will introduce the basic random walk. The stereotypical example of a random walk is that of the Drunkard's Walk, where a drunken vagrant, starting from Times Square, wanders the streets of Manhattan aimlessly and totally at random. The behavior of this famous alcoholic has been studied in many papers [25, 32]. One of the more interesting aspects of this stochastic process is that it has the property of *recurrence*. It can be shown that, given enough time, the drunkard is certain to return to Times Square eventually. Indeed, even if we extend Manhattan to some hypothetical infinite city (preserving the characteristic street pattern, of course), not only is the drunkard certain to return to Times Square, but eventually he will visit each and every corner of the city infinitely often. Or to be exact, the probability of his doing so is equal to 1.

We will start this chapter by reviewing notational conventions, defining the non-reinforced random walk and giving several characterizations of recurrence for this walk. Next we will introduce such basic concepts as martingales, stopping times and harmonic functions, and show how martingales naturally arise from random walks. Finally, we will characterize recurrence of non-reinforced random walks on graphs in terms of the existence of certain superharmonic functions on the vertices of these graphs, and give several examples of the application of these theorems to specific graphs. This chapter presumes some basic knowledge of graph theory and probability theory, but an effort has been made to make it as self-contained as possible.

6.1 The Non-Reinforced Random Walk

6.1.1. REMARK. In this dissertation, random walks are always considered to be walks on the edges of weighted graphs with finitely or countably infinitely many vertices. We will assume that any given graph is connected, that there are no 'degenerate' edges of weight 0, and that each vertex has only finitely many neighbors. To avoid needless notational complications we will also assume that any given graph is countably infinite, and simple (i.e. without loops or parallel edges) unless explicitly stated otherwise.

The reader is invited to verify for him- or herself that all definitions, proofs and results in these chapters can easily be extended to non-simple graphs. Indeed, the generalization to non-simple graphs of Lemma 8.3.4 will be used in the proof of Theorem 8.3.9. However, since this extension does not add anything conceptually, and since it is convenient to be able to denote edges and arcs by their endpoints, we will concern ourselves with simple graphs, and postulate generalizations to non-simple graphs when necessary.

NOTATION. We denote a weighted graph G as $G = (V, E, w)$, where V and E are the sets of vertices and edges of G, and $w : E \to I\!R_{>0}$ is its weight function. Edges are denoted by their endpoints, as in 'the edge uv'. Note that uv and vu denote the same edge. Whenever the order of the vertices is important (for instance, when we want to indicate the direction in which an edge has been traversed), we use *arcs* (oriented edges), denoted as in 'the arc $\overrightarrow{uv}$', instead of edges. u and v are called the *tail* and *head* of $\overrightarrow{uv}$, respectively.
Some other notational conventions:

- v and u are used for vertices.
- $N_G(v)$ denotes the *neighbor set* of a vertex v in a graph $G = (V, E, w)$, i.e. the set of vertices u such that $uv \in E$.
- $w_G(v)$ denotes the total weight $\sum_{u \in N(v)} w(vu)$ of the edges adjacent to v.
- $\rho_G(v)$ denotes the *degree* of v in G, i.e. the number of adjacent edges.
- $d_G(v, u)$ denotes the *distance* in G between the vertex v and the vertex u (i.e. the number of edges contained in the shortest $v - u$ path in G).
- $d_G(v, F)$ denotes the distance in G between a vertex v and a vertex-set F.

The index G is omitted when no confusion is possible.

A random walk is a stochastic process of traversing the edges of a graph, where, each time a vertex is reached, the random walk continues over a randomly selected adjacent edge. Specifically, the *non-reinforced random walk* on a graph $G = (V, E, w)$ starting at a vertex $v_0 \in V$, is the following stochastic process:

- We start with the vertex v_0.
- Next, we randomly pick an edge $v_0v_1 \in E$ that connects v_0 with some other vertex $v_1 \in V$. All candidate edges have a probability of being picked proportional to their weight. The random walk is said to *traverse* the edge v_0v_1, and to *visit* the vertex v_1 at *time* 1.

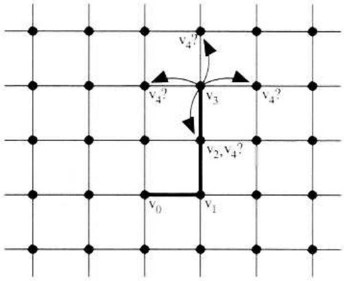

Figure 6.1: The first few steps of a random walk on the square lattice on $\mathbb{Z}^2$.

- Next, we randomly pick an edge $v_1v_2 \in E$ that connects v_1 with some other vertex $v_2 \in V$, in the same manner as in the previous step.
- Continuing in this manner, we obtain a path $v_0v_1v_2v_3\ldots$.

More formally,

6.1.2. DEFINITION. A *non-reinforced random walk* on a weighted graph $G = (V, E, w)$ is a series of stochastic variables $v_0, v_1, \ldots \in V$ such that for any *time* $t \in \mathbb{N}$,

$$P(v_{t+1} = u \mid \mathcal{F}_t) = \begin{cases} \frac{w(v_t u)}{w(v_t)} & \text{if } u \in N(v_t) \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

where $\mathcal{F}_t$ denotes the σ-algebra of the history up to time t. Note that by our assumptions on graphs, $N(v) > 0$ for all $v \in V$.

NOTATION. v_t always denotes the location of the random walk at time t. Sometimes we write $v_0v_1v_2\ldots$ for the random walk itself. Throughout these chapters s and t are used for (integer) times, and the use of t as a subscript indicates a (stochastic) variable whose contents changes over time (such as v_t).

6.1.3. DEFINITION. A realization of a random walk is said to be *recurrent* if every vertex is visited infinitely often, and *transient* if every vertex is visited only finitely many times.

The question we are mainly concerned with in these chapters , is under what conditions a random walk is recurrent almost surely (i.e. with probability 1). For non-reinforced random walks we have the following observations:

6.1.4. LEMMA. *Let $G = (V, E, w)$ be a weighted graph, and consider the non-reinforced random walk on G starting in a vertex v_0. Then, depending on G, the random walk is either almost surely recurrent or almost surely transient.*

Proof

Whenever the random walk is at v_0, there is a probability p that v_0 is revisited at some later time. The probability that the random walk will revisit v_0 at least n times is exactly p^n. Hence, if $p = 1$ then almost surely v_0 will be visited infinitely often, and if $p < 1$ then almost surely v_0 will be visited only finitely often.
If two vertices $u, u' \in V$ are neighbors in G, then whenever u is visited, there is a probability $p' > 0$ that the next vertex visited will be u'. It follows that

$$P(u \text{ is visited infinitely often and } u' \text{ only finitely often}) = 0 \tag{6.2}$$

By induction on $d_G(u, u')$ we can show that the same holds for any two vertices $u, u' \in V$. The result follows.

□

6.1.5. LEMMA. *Let $G = (V, E, w)$ be a weighted graph, $F \subset V$ a finite set of vertices of G, and $v \in F$. Then the following are equivalent:*

(i) Any non-reinforced random walk on G is almost surely recurrent.

(ii) The non-reinforced random walk on G starting in v is almost surely recurrent.

(iii) The non-reinforced random walk on G starting in v returns to v almost surely.

(iv) Any non-reinforced random walk on G visits F almost surely.

Proof

$(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are obvious.
If we assume (iv) does not hold, then for some $u \in V$, the non-reinforced random walk starting in u will not almost surely visit F. Since there is a path between v and u, the non-reinforced random walk starting in v will visit u with positive probability, and hence will not almost surely return to v, and (iii) fails.
Finally assume (iv) holds. Then starting at any vertex $u \in V$, F is visited almost surely. After this first visit F is almost surely visited again, and repeating this process we find that F is almost surely visited infinitely often. Therefore the non-reinforced random walk starting in u is almost surely not transient, and hence by Lemma 6.1.4 almost surely recurrent, proving (i).

□

6.2 Random Walks and Martingales

Our major tools for showing recurrence of random walks will be the concept of *martingales* and the Optional Stopping Theorem.

6.2.1. DEFINITION. A series of stochastic variables $(M_t)_{t\in \mathbb{N}}$ is called a *martingale* if for all $t \in \mathbb{N}$,

$$M_t = E(M_{t+1} \mid \mathcal{F}_t) \tag{6.3}$$

where $E(M_{t+1} \mid \mathcal{F}_t)$ denotes the expectation, at time t, of the value of M_{t+1}.

6.2.2. DEFINITION. A series of stochastic variables $(M_t)_{t\in \mathbb{N}}$ is called a *supermartingale* [*submartingale*] if for all $t \in \mathbb{N}$,

$$M_t \geq [\leq]\ E(M_{t+1} \mid \mathcal{F}_t) \tag{6.4}$$

It is easy to see that if M is a martingale, then $E(M_t) = M_0$ for any time $t \in \mathbb{N}$. The Optional Stopping Theorem for Martingales basically states that the same holds for the expectation of the value of the martingale at times which are defined in terms of states or conditions, such as the first time at which the value of the martingale is < 0 or > 100. To state the theorem, we need the concept of *stopping times.*

6.2.3. DEFINITION. A *stopping time* is a stochastic variable τ, taking values in $\mathbb{N} \cup \{\infty\}$, such that for all $t \in \mathbb{N}$, $\{\tau = t\} \in \mathcal{F}_t$.

Stopping times are usually defined in the manner of 'let τ be the first time at which some condition holds'. Often we are only interested in the course of a random walk up to a certain event, such as its first visit to some given vertex. In that case we write 'the random walk which stops at time τ', or sometimes simply 'the random walk which stops as soon as some condition holds'.

6.2.4. THEOREM (OPTIONAL STOPPING THEOREM FOR MARTINGALES). *Let M_t be a martingale [supermartingale, submartingale] and τ a stopping time such that $\tau < \infty$ almost surely. If M_t is bounded [bounded from below, bounded from above] for $t < \tau$, then*

$$M_0 = [\geq, \leq] E(M_\tau) \tag{6.5}$$

and more generally

$$M_{t_0} = [\geq, \leq] E(M_\tau \mid \mathcal{F}_{t_0}) \tag{6.6}$$

if $t_0 \leq \tau$.

Kakutani[26] found that random walks give rise to martingales naturally, if we can find a function on the vertex-set of the graph with the property of *harmonicity*:

6.2.5. DEFINITION. Let $G = (V, E, w)$ be a weighted graph, and let $h : V \to \mathbb{R}$ be a function. We say that h is *harmonic* [*superharmonic, subharmonic*] on a vertex-set $V' \subset V$ if for all $v \in V'$,

$$h(v) = [\geq, \leq] \sum_{u\in N(v)} h(u)\frac{w(vu)}{w(v)} \tag{6.7}$$

or, equivalently,

$$\sum_{u \in N(v)} w(vu)\Delta_h(\overrightarrow{vu}) = [\leq, \geq]\ 0 \tag{6.8}$$

where $\Delta_h(\overrightarrow{vu})$ denotes $h(u) - h(v)$.

6.2.6. LEMMA. *Let $G = (V, E, w)$ be a weighted graph, and let $h : V \to I\!R$ be a harmonic [superharmonic, subharmonic] function on a subset $V' \subset V$. Consider a non-reinforced random walk on G, and define*

$$M_t = \sum_{t'=0}^{t} \begin{cases} \Delta_h(v_{t'}v_{t'+1}) & \text{if } v_{t'} \in V' \\ 0 & \text{otherwise} \end{cases} \tag{6.9}$$

for $t \in I\!N$. Then M is a martingale [supermartingale, submartingale]. Furthermore, as long as $V - V'$ has not yet been visited,

$$M_t = h(v_t) - h(v_0) \tag{6.10}$$

Proof
If $v_t \in V - V'$ then $M_{t-1} = M_t$, otherwise

$$\begin{aligned} M_t &= [\geq, \leq] \quad M_t + \frac{1}{w(v_t)} \cdot \sum_{u \in N(v_t)} \Delta_h(v_t u) w(v_t u) && (6.11) \\ &= \quad M_t + \sum_{u \in N(v_t)} P(v_{t+1} = u \mid \mathcal{F}_t)\Delta_h(v_t u) && (6.12) \\ &= \quad E(M_{t+1} \mid \mathcal{F}_t) && (6.13) \end{aligned}$$

The proof of the final statement is trivial.

□

More about martingales may be found in [24].

6.3 Recurrence and Superharmonic Functions

Now, if h is a superharmonic function on (a subset of) the vertex-set V of a graph G then the Optional Stopping Theorem for Martingales places bounds on the expected values of $h(v_t)$. We can use this to characterize recurrence of random walks in terms of the existence of superharmonic functions with certain properties.

6.3.1. DEFINITION. Let $G = (V, E, w)$ be a weighted graph, and let $h : V \to I\!R$ be a function. We say that $h(v)$ *goes to infinity if v goes to infinity* if

$$\forall r \in I\!R\ \exists n \in I\!N\ \forall v \in V\ (d_G(v_0, v) > n \ \Rightarrow\ h(v) > r) \tag{6.14}$$

For the graphs we are concerned about, in which no vertex has infinitely many neighbors, this is equivalent to the condition that

$$\text{for each } r \in I\!R,\ \{v \in V \mid h(v) < r\} \text{ is finite} \tag{6.15}$$

6.3.2. THEOREM. *Let $G = (V, E, w)$ be a weighted graph. Then non-reinforced random walks on G are almost surely recurrent if there exists a function $h : V \to \mathbb{R}$ satisfying*

1. *h is superharmonic everywhere except on some finite set F.*

2. *$h(v)$ goes to infinity if v goes to infinity.*

Conversely, if non-reinforced random walks on G are almost surely recurrent, then a function h as above exists, and F may be chosen to be an arbitrary non-empty finite set.

Proof

First assume such a function h exists. Then h is bounded from below. We may assume without loss of generality that $h \geq 0$. So consider the random walk, starting at an arbitrary point $v_0 \in V$. By Lemma 6.1.5, it suffices to show that F will be visited almost surely. Let M_t be the martingale from Lemma 6.2.6, and let for $r > 0$ the stopping time τ_r be the first time at which the random walk leaves the finite set of vertices $\{v \in V \mid v \notin F \wedge h(v) < r\}$.

By Lemma 6.1.4, almost surely the random walk is either transient or recurrent, and in both cases the random walk visits infinitely many vertices. It follows that $\tau_r < \infty$ almost surely. Furthermore, $M_t = h(v_t) \geq 0$ for $t \leq \tau$. Hence we can use the Optional Stopping Time Theorem to obtain

$$M_0 \geq E(M_{\tau_r}) \geq (1 - P(v_{\tau_r} \in F))r \qquad (6.16)$$

and hence $P(v_{\tau_r} \in F) \geq 1 - M_0/r$ for all $r > 0$. We conclude that $P(\exists t : v_t \in F) \geq 1 - \epsilon$ for arbitrarily small $\epsilon > 0$, and hence the random walk is almost surely recurrent.

For the converse implication, let $F \subset V$ be a non-empty finite set of vertices, and assume random walks on G almost surely visit F. Let, for any vertex $v \in V$, τ_v be the time that the random walk starting from v first visits F. Then for any vertex $v \in V$, $\tau_v < \infty$ almost surely. Now set $h(v) = E(f(\tau_v))$, where $f : \mathbb{N} \to \mathbb{R}$ is such that such that f monotonely diverges to infinity and $E(f(\tau_v))$ is finite for all $v \in V$.[13] Then h is well-defined, and by the monotonicity of f we have that for $v \in V - F$

$$\begin{aligned} h(v) &\geq E(f(\tau_v - 1)) && (6.17)\\ &= \sum_{u \in N(v)} E(f(\tau_v - 1) \mid v_1 = u)\frac{w(vu)}{w(v)} && (6.18)\\ &= \sum_{u \in N(v)} E(f(\tau_u))\frac{w(vu)}{w(v)} && (6.19) \end{aligned}$$

[13]For example, $f(n) = \min_{v \in V} \left(d(v, F) + (P(\tau_v \geq n))^{-1/2}\right)$ can be shown to have these properties. Unfortunately $E(\tau_v)$ is not generally finite, or we would not *need* f.

$$= \sum_{u \in N(v)} h(u) \frac{w(vu)}{w(v)} \tag{6.20}$$

So h is superharmonic on $V - F$. Furthermore, since starting from a vertex v F cannot be reached before time $t = d(v, F)$,

$$h(v) = E(f(\tau_v)) \geq f(d(v, F)) \to \infty \text{ if } v \to \infty \tag{6.21}$$

□

6.3.3. EXAMPLE. The random walk on the square lattice graph on $\mathbb{Z}^2$ with unit weights is almost surely recurrent.

Proof

Let $h : \mathbb{Z}^2 \to \mathbb{R}$ be defined by

$$h(x, y) = \begin{cases} \log(1/12) & \text{if } (x, y) = (0, 0) \\ \log(1/4) & \text{if } (x, y) = (0, \pm 1) \text{ or } (x, y) = (\pm 1, 0) \\ \log(x^2 + y^2 - 1) & \text{otherwise} \end{cases} \tag{6.22}$$

Then h satisfies the conditions of theorem 6.3.2, with $F = \{(0, 0)\}$.

□

6.3.4. EXAMPLE. For any $n \in \mathbb{N}_{>0}$, the random walks on the square lattice graphs on $\mathbb{Z} \times \{1, \ldots, n\}$ and $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$ with unit weights are almost surely recurrent.

Proof

Let $h : (\mathbb{Z} \times \{1 \ldots n\}) \to \mathbb{R}$ be defined by

$$h(x, y) = |x| \tag{6.23}$$

Then h satisfies the conditions of theorem 6.3.2, with $F = \{(0, y) \mid 1 \leq y \leq n\}$. The proof for the cylinder lattice $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$ is completely analogous.

□

Interestingly enough, the non-recurrence of random walks on a graph can *also* be characterized in terms of the existence of certain superharmonic functions.

6.3.5. THEOREM. *Let $G = (V, E, w)$ be a weighted graph. Then non-reinforced random walks on G are not almost surely recurrent if and only if there exists a bounded non-constant function $h : V \to \mathbb{R}$ that is superharmonic on V.*

Proof

First assume that such a function h exists. Let $v_0, u \in V$ be vertices with $h(v_0) < h(u)$. By Lemma 6.1.5, it suffices to prove that the random walk starting at the vertex v_0 does not almost surely visit u. So consider the random walk, starting at the vertex v_0, which halts on visiting the vertex u, and assume that it does so almost surely. Then the stopping time $\tau = \min\{t \geq 1 \mid v_t = u\}$ is finite almost surely. By Lemma 6.2.6, the stochastic process $M_t = h(v_t)$ is a supermartingale, and by our initial assumption it is bounded. Hence we can use the Optional Stopping Times Theorem to obtain

$$h(v_0) = M_0 \geq E(M_\tau) = h(u) \tag{6.24}$$

contradicting our choice of v and u.

Now assume that random walks on G are not almost surely recurrent. Then there are vertices $v_0, u \in V$ such that starting at v_0, the random walk will not almost surely visit u. Define the function h by

$$h(v) = P(\text{the random walk starting at } v \text{ will reach } u) \tag{6.25}$$

Then $h : V \to [0, 1]$ is bounded, $h(v_0) < h(u) = 1$, h is harmonic on $V - \{u\}$ and h is superharmonic on $\{u\}$.

□

6.3.6. EXAMPLE. The random walk on the cubic lattice graph on $\mathbb{Z}^3$ is not almost surely recurrent.

Proof

Let $h : \mathbb{Z}^3 \to [0, 6^{-1/2}]$ be defined by

$$h(x, y, z) = \frac{1}{(x^2 + y^2 + z^2 + 6)^{1/2}} \tag{6.26}$$

Using a truncated Taylor series expansion of h, we can show that for all $x, y, z \in \mathbb{Z}$,

$$h(x+1, y, z) + h(x-1, y, z) \leq 2h(x, y, z) + \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2 + 6)^{5/2}} \tag{6.27}$$

Analogous inequalities hold for $h(x, y+1, z) + h(x, y-1, z)$ and $h(x, y, z+1) + h(x, y, z-1)$. Taking the sum of these inequalities yields the superharmonicity inequality.

□

6.3.7. EXAMPLE. Let $G = (V, E, w)$ be a weighted graph with $V = \{v^n \mid n \in \mathbb{Z}\}$, and $E = \{v^n v^{n+1} \mid n \in \mathbb{Z}\}$.[14] Then random walks on G are almost surely recurrent if and only if $\sum_{n=-\infty}^{-1}(1/w(v^n v^{n+1}))$ and $\sum_{n=0}^{\infty}(1/w(v^n v^{n+1}))$ both diverge.

[14] The superscript index v^n is used here to avoid confusion with the temporal index v_t.

Proof

If $\sum_{n=0}^{\infty}(1/w(v^n v^{n+1}))$ converges to $c \in \mathbb{R}$, then define $h : V \to [0, c]$ by setting $h(v^n) = \max(c, \sum_{k=n}^{\infty}(1/w(v^k v^{k+1})))$ for $n \in \mathbb{Z}$. It is easily verified that h is non-constant, harmonic on $V - \{v^0\}$ and superharmonic on $\{v^0\}$, fulfilling the conditions of Theorem 6.3.5. Likewise for the case that $\sum_{n=-\infty}^{-1}(1/w(v^n v^{n+1}))$ converges.

Now suppose $\sum_{n=-\infty}^{-1}(1/w(v^n v^{n+1}))$ and $\sum_{n=0}^{\infty}(1/w(v^n v^{n+1}))$ both diverge to ∞. Then define $h : V \to \mathbb{R}_{\geq 0}$ by setting $h(v^n) = \sum_{k=0}^{n-1}(1/w(v^k v^{k+1}))$ for $n \geq 0$ and $h(v^n) = \sum_{k=-n}^{-1}(1/w(v^k v^{k+1}))$ for $n < 0$. Again it is easily verified that h is non-constant, $h(v^n) \to \infty$ if $n \to \infty$ or $n \to -\infty$, and h is harmonic on $V - \{v^0\}$, fulfilling the conditions of Theorem 6.3.2

□

The next Theorem is included because it will be used in a later chapter. The proof, unfortunately, is beyond the scope of these pages. A beautiful proof was given by Doyle and Snell in [25].

6.3.8. THEOREM. *Let $G = (V, E, w)$ be a weighted graph such that non-reinforced random walks on G are almost surely recurrent. If $G' = (V', E', w')$ is a connected subgraph of G, possibly with lesser weights (i.e. $V' \subset V$, $E' \subset E$ and for all $e \in E'$, $w'(e) \leq w(e)$), then random walks on G' are almost surely recurrent.*