Games, walks and grammars: Problems I've worked on

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Chapter 7

Why The Gambler Came Home Late

Consider the following stochastic experiment. We have n players, each possessing $a_1, a_2, \ldots, a_n$ coins. They play a gambling game, where each round, one of the players loses a coin to one of the other players (both selected randomly). The game continues until one of the players has no more coins. Let us denote the expected duration of the game (in rounds) by $T_n(a_1, a_2, \ldots, a_n)$.

7.0.9. REMARK. This process of gambling can be viewed as a random walk on a finite graph. Consider the graph $G_n^s = (V_n^s, E_n^s)$ with vertices and edges

$$V_n^s = \{\vec{a} \in I\!N^n : a_1 + \ldots + a_n = s\} \tag{7.1}$$

$$E_n^s = \{\vec{a}\vec{b} \mid \vec{a}, \vec{b} \in V_n^s, \exists i \exists j : a_i = b_i + 1 \wedge a_j = b_j - 1 \wedge \forall k \neq i, j : a_k = b_k\} \tag{7.2}$$

Then the gambling process above corresponds to a random walk in G_n^s, where we take s to be the total number of coins the players possess. In particular, the expected duration of the gambling process corresponds to the expected time until the random walk reaches a vertex $\vec{a}$ with $\exists i : a_i = 0$. Since the graph on which the random walk takes place is finite, the expectation of this time is finite. Hence $T_n(a_1, \ldots, a_n)$ exists for all $n \geq 1$ and all $a_1, \ldots, a_n \geq 0$.

Question: What can be said about T_n as a function on $I\!N^n$?

7.1 Basic properties of T_n

To get an idea of the properties of the function T_n, let us consider first the case where $n = 2$. Here we have two players, and each round, one of the players loses one coin to the other player, until one of the players is broke. If the players currently possess a_1 and a_2 coins, respectively, then after one round, with probability $1/2$ they will possess $a_1 - 1$ and $a_2 + 1$ coins, and otherwise they will possess $a_1 + 1$

and $a_2 - 1$ coins, respectively. It is easily seen that the function T_2 satisfies the equation

$$T_2(a_1, a_2) = \begin{cases} 0 & \text{if } a_1 = 0 \text{ or } a_2 = 0 \\ 1 + \frac{T_2(a_1-1, a_2+1) + T_2(a_1+1, a_2-1)}{2} & \text{otherwise} \end{cases} \tag{7.3}$$

Furthermore, for constant $s = a_1 + a_2$, this is a finite linear set of equations in the variables $T(0, s)$, $T(1, s-1)$, ..., $T(s, 0)$. It can be shown that this system of equations has a unique solution. Hence the function T_2 is uniquely determined by equation 7.3 Since the function $a_1 a_2$ satisfies this condition,

$$T_2(a_1 a_2) = a_1 a_2 \tag{7.4}$$

For $n = 3$, we can do something similar. In this case, each round there are 6 possibilities (one of three players loses a coin, and one of the remaining two players gains one), and we obtain the equation

$$T_3(a_1, a_2, a_3) = \begin{cases} 0 & \text{if } \exists i : a_i = 0 \\ 1 + \frac{T_3(a_1-1, a_2+1, a_3) + \ldots + T_3(a_1, a_2+1, a_3-1)}{6} & \text{otherwise} \end{cases} \tag{7.5}$$

Again, T_3 is uniquely defined by this equation (for a proof, see the general proof in the next lemma), and with some searching we can find the formula

$$T_3(a_1, a_2, a_3) = \frac{3a_1 a_2 a_3}{a_1 + a_2 + a_3} \tag{7.6}$$

In general

7.1.1. LEMMA. *If we define*

$$P_n = \{\vec{d} \in \mathbb{Z}^n \mid \exists i \exists j : d_i = 1 \wedge d_j = -1 \wedge \forall k \neq i, j : d_k = 0\} \tag{7.7}$$

then we have, for general n, that T_n satisfies and is uniquely defined by the equation

$$T_n(\vec{a}) = \begin{cases} 0 & \text{if } \exists i : a_i = 0 \\ 1 + \frac{1}{n(n-1)} \sum_{\vec{d} \in P_n} T_n(\vec{a} + \vec{d}) & \text{otherwise} \end{cases} \tag{7.8}$$

Proof

By Remark 7.0.9, T_n exists, and that T_n satisfies this equation follows directly from the definition of the gambling process, analogously to the cases with two and three players. To show that it is uniquely defined, fix $s \in \mathbb{N}$, and view the linear system of equations given by equation 7.8 for all $\vec{a} \in \mathbb{N}^n$ with $a_1 + a_2 + \ldots + a_n = s$, i.e. for all $\vec{a} \in V_n^s$. This system has exactly as many equations as it has variables, so to show that it has a unique solution, it suffices to show that the related system of equations

$$T'_n(\vec{a}) = \begin{cases} 0 & \text{if } \exists i : a_i = 0 \\ \frac{1}{n(n-1)} \sum_{\vec{d} \in P_n} T'_n(\vec{a} + \vec{d}) & \text{otherwise} \end{cases} \tag{7.9}$$

for all $\vec{a} \in V_n^s$, only has the solution with $T_n'(\vec{a}) = 0$ for all $\vec{a} \in V_n^s$. Now, if T_n' satisfies this system of equations, and if for some $a \in V_n^s$, $T_n'(\vec{a})$ is nonzero and either maximal or minimal in V_n^s, then we have that $T_n'(\vec{a} + \vec{d}) = T_n'(\vec{a})$ for all $d \in P_n$. Since G_s^n is a connected and finite graph, this would imply that $T_n'(\vec{a})$ is constant and non-zero on V_s^n, a contradiction.

□

Unfortunately, for $n > 3$ there is no formula known to satisfy this equation. Simply generalizing of the formulas for $n = 2$ and $n = 3$ to

$$T_n(\vec{a}) = \frac{C a_1 a_2 \ldots a_n}{(a_1 + \ldots + a_n)^m}, \text{ for some } C > 0,\ m \in I\!N \tag{7.10}$$

does not work: the differential

$$T_n(\vec{a}) - \frac{1}{n(n-1)} \sum_{\vec{d} \in P_n} T_n(\vec{a} + \vec{d}) \tag{7.11}$$

should be constant in order to satisfy equation (7.8), but if we write it out we get

$$\frac{C}{(a_1 + \ldots + a_n)^m} \sum_{1 \le i < j \le n} \prod_{k \ne i,j} a_k \tag{7.12}$$

For $n > 3$, this expression is not constant for any $C > 0$ and $m \in I\!N$, and hence T_n cannot be expressed in this particular form. In Theorem 7.3.1, we will show that if such a formula exists, it must be considerably more complicated than the formulas for $n = 2$ and $n = 3$.

7.2 T_n, H_n and T_n^*

For each s, T_n can be thought of as a function on the graph G_n^s having constant curvature. There are known functions with constant curvature, such as $1/2|\vec{a}|^2$, which unfortunately do not have the right boundary values. However, the difference between such functions and T_n would be a function of *zero* curvature, i.e. a harmonic function.

7.2.1. LEMMA. *T_n can be written as*

$$T_n(\vec{a}) = \frac{1}{2}(H_n(\vec{a}) - |\vec{a}|^2) \tag{7.13}$$

where H_n is the unique function satisfying

$$H_n(\vec{a}) = |\vec{a}|^2 \quad \textit{if } \exists i : a_i = 0 \tag{7.14}$$

$$\sum_{\vec{d} \in P_n} (H_n(\vec{a} + \vec{d}) - H_n(\vec{a})) = 0 \quad \textit{otherwise} \tag{7.15}$$

Proof
It is straightforward to see that $H_n = 2T_n + |\vec{a}|^2$ satisfies the given equations. Uniqueness can be proven exactly as in the previous lemma.

□

7.2.2. REMARK. For any s, H_n is harmonic on G_n^s everywhere except on vertices $\vec{a}$ with $\exists i : a_i = 0$. As such, for $\vec{a} \in V_n^s$ $H_n(\vec{a})$ is equal to the *expected value* of $|\vec{b}|^2$, where $\vec{b}$ is the first vertex of G_n^s with $\exists i : b_i = 0$ which is visited by the random walk on G_n^s starting in $\vec{a}$.

Interestingly, if we look at a variation on the first game where, once one gambler is broke, the game continues with the remaining gamblers until one gambler has won all the money, we get an expected duration function T_n^* which *does* have a simple form:

7.2.3. THEOREM. *Let $T_n^*(\vec{a})$ denote the expected duration of the variation of the game where play continues until all but one gambler is broke. Then*

$$T_n^*(\vec{a}) = \frac{1}{2}((\sum_i a_i)^2 - |\vec{a}|^2) \tag{7.16}$$

Proof
We will prove this by induction on n. For $n = 1$, both sides of equation (7.16) are equal to 0, and the equation holds. For $n > 1$, using the methods of Lemma 7.1.1, it follows from the Induction Hypothesis that that T_n^* is the unique function satisfying

$$T_n(\vec{a}) = \begin{cases} \frac{1}{2}((\sum_i a_i)^2 - |\vec{a}|^2) & \text{if } \exists i : a_i = 0 \\ 1 + \frac{1}{n(n-1)} \sum_{\vec{d} \in P_n} T_n(\vec{a} + \vec{d}) & \text{otherwise} \end{cases} \tag{7.17}$$

and it is straightforward to check that this is satisfied by the formula of equation (7.16).

□

7.2.4. REMARK. Some calculation shows that

$$T_n^*(\vec{a}) = T_n(\vec{a}) - \frac{1}{2}H_n(\vec{a}) + \frac{1}{2}(\sum_i a_i)^2 \tag{7.18}$$

In other words, with respect to this other game, T_n is the expectation of the time *until* the first gambler goes broke, and $\frac{1}{2}((\sum_i a_i)^2 - H_n(\vec{a}))$ is the expectation of the time the game will last *after* the first gambler goes broke.

7.2.5. COROLLARY. *For all $\vec{a} \in \mathbb{N}^n$,*

$$0 \leq T_n(\vec{a}) \leq \frac{1}{2}((\sum_i a_i)^2 - |\vec{a}|^2) \tag{7.19}$$

7.2.6. COROLLARY. *For all $\vec{a} \in \mathbb{N}^n$,*

$$|\vec{a}|^2 \leq H_n(\vec{a}) \leq (a_1 + \ldots + a_n)^2 \tag{7.20}$$

7.3 Simple Rational Polynomials

Now, the formulas we have for T_2 and T_3 are rational polynomials. It would be nice to show that for $n \geq 4$, T_n *can't* be expressed as a rational polynomial. Although Lemma 7.4.1 is an effort in that direction, the proposition is still unproven. However, we can prove that T_n can't be expressed as a rational polynomial such that the denominator only depends on $a_1 + \ldots + a_n$:

7.3.1. THEOREM. *For all $n \geq 4$, T_n is not of the form*

$$T_n(\vec{a}) = \frac{P(a_1, \ldots, a_n)}{Q(a_1 + \ldots + a_n)} \tag{7.21}$$

for any two polynomials $P(a_1, \ldots, a_n)$ and $Q(s)$.

This theorem follows directly from the following two lemmas

7.3.2. LEMMA. *Suppose that, for some $n \geq 3$, T_n is as in equation (7.21), for some polynomials $P(a_1, \ldots, a_n)$ and $Q(s)$. Then there exists a nonzero polynomial $R(a_1, \ldots, a_{n-1})$ of degree $\geq n$, that is divisible by $a_1 \ldots a_{n-1}$ and $a_1 + \ldots + a_{n-1}$ and satisfies*

$$\sum_{\vec{d} \in P_n} D_{\vec{d}_{|n-1}} D_{\vec{d}_{|n-1}} R(\vec{a}) = 0 \tag{7.22}$$

where $D_{\vec{u}} f$ denotes the partial derivative of f in the direction $\vec{u}$, and $\vec{d}_{|n-1}$ denotes the vector obtained from $\vec{d}$ by discarding the final coordinate.

Proof
Without loss of generality we may assume that $P(a_1, \ldots, a_n)$ and $Q(s)$ have no common factors. Now, P and Q are polynomials, and $Q(a_1 + \ldots + a_n)$ only depends on the *total* amount of money, which does not change during the gambling. Combining these facts with the properties of T_n, we find that $P(a_1, \ldots, a_n)$ is divisible by $a_1 \ldots a_n$ and that for all $\vec{a} \in \mathbb{R}^n$

$$2n(n-1)Q(a_1 + \ldots + a_n) = \sum_{\vec{d} \in P_n} \left(P(\vec{a} + \vec{d}) + P(\vec{a} - \vec{d}) - 2P(\vec{a}) \right) \tag{7.23}$$

We want to replace $Q(a_1 + \ldots + a_n)$ by 0, and $P(\vec{a} + \vec{d}) + P(\vec{a} - \vec{d}) - 2P(\vec{a})$ by the partial derivative $D_{\vec{d}} D_{\vec{d}} P(\vec{a})$. Both can be accomplished by taking the limit of an appropriate scaling, *provided* we can take $Q(a_1 + \ldots + a_n)$ to be constant. The easiest way to do this is to set $a_n = -(a_1 + \ldots + a_{n-1})$. So consider the polynomial

$$P^*(a_1, \ldots, a_{n-1}) = P(a_1, \ldots, a_{n-1}, -(a_1 + \ldots + a_{n-1})) \tag{7.24}$$

This polynomial is divisible by $a_1 \ldots a_{n-1}$ and $a_1 + \ldots + a_{n-1}$, and satisfies for $\vec{a} \in \mathbb{R}^{n-1}$

$$2n(n-1)Q(0) = \sum_{\vec{d}\in P_n} \left(P^*(\vec{a} + \vec{d}_{|n-1}) + P^*(\vec{a} - \vec{d}_{|n-1}) - 2P^*(\vec{a})\right) \qquad (7.25)$$

Now, if $P^*(a_1, \ldots, a_{n-1})$ were the zero polynomial, then we would have $Q(0) = 0$, and then $P(a_1, \ldots, a_n)$ and $Q(a_1 + \ldots + a_n)$ would have a common factor $a_1 + \ldots + a_n$, contradicting one of our starting assumptions. So $P^*(a_1, \ldots, a_{n-1})$ is nonzero, and of total degree $d \geq n$. Let R be the uniformization of P, i.e. set

$$R(a_1, \ldots, a_{n-1}) = \lim_{\delta\to 0} \delta^d P^*(\frac{a_1}{\delta}, \ldots, \frac{a_{n-1}}{\delta}) \qquad (7.26)$$

Then $R(a_1, \ldots, a_{n-1})$ is a uniform polynomial[15] of total degree d, is divisible by $a_1 \ldots a_{n-1}$ and $a_1 + \ldots + a_{n-1}$, and satisfies

$$\begin{aligned}
&\sum_{\vec{d}\in P_n} D_{\vec{d}_{|n-1}} D_{\vec{d}_{|n-1}} R(\vec{a}) && (7.27)\\
&= \sum_{\vec{d}\in P_n} \lim_{\epsilon\to 0} \frac{1}{\epsilon^2} \left(R(\vec{a} + \epsilon\vec{d}_{|n-1}) + R(\vec{a} - \epsilon\vec{d}_{n-1}) - 2S(\vec{a})\right) && (7.28)\\
&= \sum_{\vec{d}\in P_n} \lim_{\epsilon\to 0}\lim_{\delta\to 0} \frac{\delta^d}{\epsilon^2} \left(P^*(\frac{\vec{a}}{\delta} + \frac{\epsilon}{\delta}\vec{d}_{|n-1}) + P^*(\frac{\vec{a}}{\delta} - \frac{\epsilon}{\delta}\vec{d}_{|n-1}) - 2P^*(\frac{\vec{a}}{\delta})\right) && (7.29)\\
&= \lim_{\delta\to 0} \delta^{d-2} \sum_{\vec{d}\in P_n} \left(P^*(\frac{\vec{a}}{\delta} + \vec{d}_{|n-1}) + P^*(\frac{\vec{a}}{\delta} - \vec{d}_{|n-1}) - 2P^*(\frac{\vec{a}}{\delta})\right) && (7.30)\\
&= \lim_{\delta\to 0} \delta^{d-2} 2n(n-1)Q(0) && (7.31)\\
&= 0 && (7.32)
\end{aligned}$$

Note that in the third equality, we are allowed to set $\epsilon = \delta$ and swap limits and sums, because the expression after the limit can be written as a polynomial in $a_1, \ldots, a_{n-1}$, δ and ϵ.

□

7.3.3. LEMMA. *For $n \geq 4$, there is no polynomial $R(a_1, \ldots, a_{n-1})$, other than the zero polynomial, that is divisible by $a_1 \ldots a_{n-1}$ and satisfies*

$$\sum_{\vec{d}\in P_n} D_{\vec{d}_{|n-1}} D_{\vec{d}_{|n-1}} R(\vec{a}) \text{ is constant} \qquad (7.33)$$

Proof

Let $n \geq 4$ and let $R(a_1, \ldots, a_{n-1})$ be a polynomial. Set

$$R^{(2)} = \frac{1}{2} \sum_{\vec{d}\in P_n} D_{\vec{d}_{|n-1}} D_{\vec{d}_{|n-1}} R(\vec{a}) \qquad (7.34)$$

[15] R may be obtained from P^* by omitting all terms of total degree less than d

A bit of calculating shows that

$$R^{(2)} = (n-1)\sum_{1\le i\le n-1}\frac{\partial^2 R}{\partial a_i^2} \quad - \sum_{1\le i,j\le n-1, i\neq j}\frac{\partial^2 s}{\partial a_i \partial a_j} \tag{7.35}$$

Now suppose that R is divisible by $a_1, \ldots, a_{n-1}$, and that $R^{(2)}$ is constant. Since R contains no terms of total degree 2 or less, we immediately conclude that $R^{(2)} = 0$ everywhere. Now we shall prove by induction on m, that for all $m \in I\!N$ the following holds

> For all $i < j \le n-1$, R contains no terms such that the exponents of a_i and a_j in the term sum to m or less

Since R is divisible by $a_1, \ldots, a_{n-1}$, this is trivial for $m = 0, 1$. So let $m \ge 2$, let $1 \le i < j \le n-1$ and let P be any product of variables other than a_i and a_j. Set C to be the collection of terms of R such that the exponents of a_i and a_j sum to m and the remaining variables form P, i.e. C can be written as

$$\{c_k a_i^k a_j^{m-k} P \mid 0 \le k \le m\} \tag{7.36}$$

Each term of R contributes terms to $R^{(2)}$, as given by equation (7.35). Amongst the terms contributed by a term $c_k a_i^k a_j^{m-k} P$ of C are $(n-1)k(k-1)c_k a_i^{k-2} a_j^{m-k} P$, $-2k(m-k)c_k a_i^{k-1} a_j^{m-k-1} P$ and $(n-1)(m-k)(m-k-1)c_k a_i^k a_j^{m-k-2} P$. In these three terms of $R^{(2)}$, the exponents of a_i and a_j sum to $m-2$ and the remaining variables form P. It is easily seen that such terms are not contributed by any terms of R outside of C, since by the Induction Hypothesis, R does not contain any terms such that the exponents of a_i and a_j sum to $m-1$ or less. Since $R^{(2)} = 0$, this implies that for $k = 1, 2, \ldots, m-1$,

$$(n-1)(m-k+1)(m-k)c_{k-1} \;-\; 2k(m-k)c_k \;+\; (n-1)k(k+1)c_{k+1} \;=\; 0 \tag{7.37}$$

Multiplying by $(k-1)!(m-k-1)!$ and writing c_k^* for $(m-k)!k!c_k$ yields, for $k = 1, 2, \ldots, m-1$,

$$(n-1)c_{k-1}^* - 2c_k^* + (n-1)c_{k+1}^* = 0 \tag{7.38}$$

Furthermore, since a_i and a_j divide R, $c_0 = c_m = c_0^* = c_m^* = 0$. This system of linear equations has a nontrivial solution if and only if $\det B_{m-1}^{n-1} = 0$, where B_{m-1}^{n-1} is the $(m-1) \times (m-1)$ matrix

$$\begin{pmatrix} -2 & n-1 & 0 & 0 & \ldots & 0 & 0 \\ n-1 & -2 & n-1 & 0 & \ldots & 0 & 0 \\ 0 & n-1 & -2 & n-1 & \ldots & 0 & 0 \\ 0 & 0 & n-1 & -2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -2 & n-1 \\ 0 & 0 & 0 & 0 & \ldots & n-1 & -2 \end{pmatrix} \tag{7.39}$$

Defining the matrices B_i^{n-1} analogously, we get the recursive equation

$$\det B_i^{n-1} = -2 \det B_{i-1}^{n-1} - (n-1)^2 \det B_{i-2}^{n-1} \tag{7.40}$$

with $\det B_0^{n-1} = 1$, $\det B_1^{n-1} = -2$. Now, if $n > 3$, then either $4|n-1$, or $p|n-1$ for some prime number $p > 2$. In the first case, it is easy to show by induction on i that for all i, $\det B_i^{n-1}$ is a multiple of 2^i but not of 2^{i+1}. In the second case, it is easy to show by induction on i that for all i, $\det B_i^{n-1}$ is not a multiple of p. In both cases, $\det B_i^{n-1} \neq 0$ for all $i \geq 0$. In particular, $\det B_{m-1}^{n-1} \neq 0$. [16]
It follows that all coefficients of terms of R in C are 0. We conclude that R contains no terms such that for some i and j, the exponents of a_i and a_j in the term sum to m or less. Since this holds for all $m \geq N$, R is the zero polynomial.

□

7.3.4. REMARK. The proof of Lemma 7.3.3 also goes through, with only minor modifications, for the case where R can be written as a power series. However, scaling in the manner of Lemma 7.3.2 is not generally possible for analytic functions. When it is, i.e. when for some $d \geq 2$ and some $s \in \mathbb{R}$, the function

$$R(a_1, \ldots, a_{n-1}) := \lim_{\delta \to 0} \delta^d P(\frac{a_1}{\delta}, \ldots, \frac{a_{n-1}}{\delta}, \frac{-a_1 - \ldots - a_{n-1}}{\delta}) \tag{7.41}$$

can be written as a power series in $a_1, \ldots, a_{n-1}$ and equation (7.30) holds, then Lemma 7.3.3 can be applied. But aside from the polynomials there appear to be very few functions for which this is the case. For instance, if P is a rational polynomial, the denominator of R is 0 for $\vec{a} = \vec{0}$, and then R cannot be written as a power series in $a_1, \ldots, a_{n-1}$.

Unfortunately, when we try to adapt this proof to rational polynomials in general, we run into a number of problems. For instance, the formula for the second-order derivatives of the quotient of two functions involves taking products of the numerator, the denominator and their derivatives, and as a result the equations that the coefficients of the polynomials must satisfy are no longer *linear* equations.

7.4 General Rational Polynomials

Now, the following lemma *may* be useful in order to prove that T_n cannot be expressed as a rational polynomial, by giving a consequence of this premise which seems refutable. Unfortunately, there appear to be no general results in this area

[16] From the recursive equation we can derive an explicit formula, which turns out to be $B_i^{n-1} = (n-1)^i \sin(i+1)\alpha / \sin \alpha$, where $0 \leq \alpha \leq \pi$ and $\cos \alpha = -1/(n-1)$. Hence this result is distantly related to the fact that the only rational $c \in \mathbb{Q}$ such that $\cos \beta = c$ for some rational multiple β of π, are $-1, -0.5, 0, 0.5$ and 1.

that could be used to refute it. Please note that it is not a complete reduction of the problem: the implication is one-way. In particular, if a function h_n is found with the properties given below, it is not at all clear how to obtain T_n from h_n.

7.4.1. LEMMA. *Suppose that $T_n(a_1, \ldots, a_n)$ can be expressed as a rational polynomial (the quotient of two polynomials). Then there exists a rational polynomial function h_n on an n-simplex $S \subset \mathbb{R}^{n-1}$, such that*

1. *h_n is harmonic on S (i.e. the Laplacian Δh_n is 0 on S).*
2. *for some point $\vec{m}$, $h_n(\vec{x}) = d^2(\vec{x}, \vec{m})$ for all $x \in \partial S$.*

Proof

Suppose that T_n can be expressed as a rational polynomial. Then so can H_n. Furthermore, whenever the denominator of T_n is non-zero, H_n is defined and satisfies

$$\sum_{\vec{d} \in P_n} (H_n(\vec{a} + \vec{d}) + H_n(\vec{a} - \vec{d}) - 2H_n(\vec{a})) = 0 \tag{7.42}$$

and

$$H_n(\vec{a}) = |\vec{a}|^2 \text{ if } \exists i : a_i = 0 \tag{7.43}$$

As in the proof of Theorem 7.3.1, we will substitute partial derivatives for their discrete counterparts, by taking the limit of an appropriate scaling. From Lemma 7.2.6 it follows that the total degree of the numerator is exactly 2 more than the total degree of the denominator. Let R be the uniformization of H_n, i.e. set

$$R(\vec{a}) = \lim_{\delta \to 0} \delta^2 H_n(\vec{a}/\delta) \tag{7.44}$$

on $\mathbb{R} \times \mathbb{R}^n$. R is a uniform rational polynomial[17] in $a_1, \ldots, a_n$ satisfying

$$R(\vec{a}) = |\vec{a}|^2 \text{ if } \exists i : a_i = 0 \tag{7.45}$$

and

$$|\vec{a}|^2 \leq R(\vec{a}) \leq (a_1 + \ldots + a_n)^2 \text{ if } \vec{a} \in R^n_{\geq 0} \tag{7.46}$$

In particular, R exists on $\mathbb{R}^n_{\geq 0}$. Furthermore, since R is a rational polynomial, the partial derivatives $D_d D_d \vec{R}(\vec{a})$ exist whenever $R(\vec{a})$ exists, and

$$\sum_{\vec{d} \in P_n} D_{\vec{d}} D_{\vec{d}} R(\vec{a}) \tag{7.47}$$

$$= \sum_{\vec{d} \in P_n} \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \left(R(\vec{a} + \epsilon\vec{d}) + R(\vec{a} - \epsilon\vec{d}) - 2R(\vec{a}) \right) \tag{7.48}$$

[17] R may be obtained from H_n by removing from the numerator and the denominator all terms of less than maximal total degree.

$$
\begin{aligned}
&= \sum_{\vec{d}\in P_n} \lim_{\epsilon\to 0}\lim_{\delta\to 0} \frac{\delta^2}{\epsilon^2}\left(H_n(\frac{\vec{a}}{\delta}+\frac{\epsilon}{\delta}\vec{d}) + H_n(\frac{\vec{a}}{\delta}-\frac{\epsilon}{\delta}\vec{d}) - 2H_n(\frac{\vec{a}}{\delta})\right) && (7.49)\\
&= \lim_{\delta\to 0}\sum_{\vec{d}\in P_n}\left(H_n(\frac{\vec{a}}{\delta}+\vec{d}) + H_n(\frac{\vec{a}}{\delta}-\vec{d}) - 2H_n(\frac{\vec{a}}{\delta})\right) && (7.50)\\
&= 0 && (7.51)
\end{aligned}
$$

Note that in the second-to-last equality, we are allowed to set $\epsilon = \delta$ and swap limits and sums, because the expression after the limit can be written as a rational polynomial in $a_1, \ldots, a_n$, δ and ϵ whose denominator is non-zero for $\delta = \epsilon = 0$ (provided the denominator of $R(\vec{a})$ is non-zero in the first place).
If we consider the simplex $S = \{\vec{a} \in I\!R^n : \sum_i a_i = 1, \forall i : a_i \geq 0\}$ in the hyperplane $V = \{\vec{a} \in I\!R^n : \sum_i a_i = 1\}$, then (after mapping V to $I\!R^{n-1}$) the function $h_n(\vec{a}) = R(\vec{a}) - 1/n$ satisfies the requirements, taking m to be the center of the simplex.

□

7.4.2. Remark. For $n = 2, 3$ we have

$$
\begin{aligned}
h_2(a_1, a_2) &= 1 \text{ for } a_1 + a_2 = 1 && (7.52)\\
h_3(a_1, a_2, a_3) &= a_1^2 + a_2^2 + a_3^2 + 6a_1a_2a_3 \text{ for } a_1 + a_2 + a_3 = 1 && (7.53)
\end{aligned}
$$