In the orthodox random walk, the probability of traversing a specific street from a specific intersection is always the same, unaffected by anything that has gone before. In this chapter, we will study reinforced random walks, where the walk is given a particular kind of feedback such that edges already traversed are either more or less likely to be traversed in the future. In terms of the Drunkard's Walk example, the drunkard vaguely recognizes streets he has walked before, and is either more likely to traverse them (as he considers them safe) or less likely (as he considers them boring), depending on the conditions of the reinforcement.

Reinforced random walks were first introduced by Diaconis and Coppersmith[22], and generalized later by B. Davis[23] and Penantle[29]. They were originally presented as an alternative to Pólya's urn as a simplified model of a self-organizing system, i.e. a system whose basic parameters are very simple, and whose behavior 'evolves' to approach a (possibly random) limit. Such systems occur naturally, for instance in the formation of stalactites and stalagmites. For another example, consider a man who has just moved to a new city: if he does not know the shops, he will start out by visiting shops at random, but after a while he will develop preferences and habits.

8.1 General Reinforced Random Walks

First we will compare reinforced walks with non-reinforced random walks, give analogues of results and techniques from Chapter 6, and show that under some

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18 Pólya's urn is one of the simplest (and oldest) processes with reinforcement. In this model, there is an urn containing red and blue balls. At time $t = 0$, the urn contains one red and one ball. At each time $t > 0$, a ball is chosen uniformly from the contents of the urn, and is put back into the urn along with another ball of the same color. Eggenberger and Pólya[30] showed that the proportion of red balls converges almost surely, and that the limit is random with uniform distribution on $[0,1]$. 
very general conditions reinforced random walks on trees are almost surely recurrent. Then we will give a sufficient condition for recurrence of reinforced random walks on general graphs, which we will use in later sections.

In a reinforced random walk, when an edge has been traversed we change the probability that it will be traversed again, by increasing or decreasing the weight of the edge. In general reinforced random walks, the new weight may depend on many things, such as the edge in question, the number of times it has been traversed before, the time of traversal and the pattern formed by edges traversed at previous times, etc. etc. B. Davis [23] defines the category of reinforced random walks of matrix type, where for each edge $vu$, the current weight of $vu$ is determined solely by the number of times $k_t(vu)$ it has been traversed up to then, and is not influenced by anything that has happened to any other edge. Note that in general walks of matrix type, the relationship between current weight and number of traversals may be different for each edge. In this chapter and the next we concern ourselves with a specific subclass of walks of matrix type, where a sequence $(\delta_k)_{k \in \mathbb{N}}$ is given which is the same for all edges, and the current weight of an edge at any given time is determined by multiplying its original weight by $\delta_{k_t(vu)}$. A formal definition:

**8.1.1. Definition.** Let $(\delta_k)_{k \in \mathbb{N}}$ be a sequence of strictly positive real numbers. Set the weight of $vu$ at time $t$ to

$$w_t(vu) = \delta_{k_t(vu)}w(vu)$$

where $k_t(vu)$ denotes the number of traversals of $vu$ up to time $t$, i.e.

$$k_t(vu) = \# \{ t' < t \mid v_{t'}v_{t+1} = vu \}$$

A reinforced random walk on a graph $G = (V, E, w)$ with reinforcement sequence $(\delta_k)_{k \in \mathbb{N}}$ is a series of stochastic variables $v_0, v_1, \ldots \in V$ such that for all $t \in \mathbb{N}$,

$$P(v_{t+1} = u \mid \mathcal{F}_t) = \begin{cases} \frac{w_t(vu)}{w_t(v)} & \text{if } u \in \mathcal{N}(v_t) \\ 0 & \text{otherwise} \end{cases}$$

Recurrence and transience are defined in the same manner as before.

**8.1.2. Remark.** The random walks defined above are similar but not identical to B. Davis' random walks of sequence type, where the current weight of an edge $vu$ is defined as $w_t(vu) = w_0(vu) + \delta_{k_t(vu)}$ for some non-descending sequence $(\delta_k)_{k \in \mathbb{N}}$ [23]. Davis gave many results for walks of this type on the linear lattice $\mathbb{Z}$, most of which also hold for the random walks defined above. We will concern ourselves mainly with more general classes of graphs.

There are a number of differences between a reinforced and a non-reinforced random walk. For instance, a reinforced random walk is influenced by its history.
and hence we might want to consider random walks with initial states in which some edges are considered to have been traversed already. Another difference is that if the reinforcement increases sharply enough, the random walk might get ‘stuck’ on an edge:

8.1.3. **Theorem.** Let $G = (V, E, w)$ be a weighted graph, and let $(\delta_k)_{k \in \mathbb{N}}$ be such that

$$\sum_{k=0}^{\infty} \frac{1}{\delta_k} \text{ converges.} \quad (8.4)$$

Then for any edge $vu \in E$, and all random walks on $G$, there exists a $t_0 \geq 0$ such that

$$P(\forall t > t_0 : v \in \{v, u\}) > 0 \quad (8.5)$$

**Proof**

Since $G$ is connected, every point is reachable, and hence there exists a $t_0 \in \mathbb{N}$ such that with non-zero probability $v$ is visited at time $t_0$. Assume that it has. Then the probability that from time $t_0$ on, the random walk will keep traveling from $v$ to $u$ and back again, is

$$\prod_{i=0}^{\infty} \frac{w_{t_0+i}(vu)}{w_{t_0+i}(v_0+i)} \quad (8.6)$$

$$\geq \prod_{i=0}^{\infty} \frac{\delta_{k_0}(vu)\prod_{i=0}^{\infty} w(vu)}{c + \delta_{k_0}(vu)\prod_{i=0}^{\infty} w(vu)} \quad (8.7)$$

$$\geq \prod_{k=k_0(vu)}^{\infty} e^{-c/(\delta_k w(vu))} \quad (8.8)$$

$$= e^{-c/(\delta_{k_0(vu)}\sum_{k=k_0(vu)}^{\infty} 1/\delta_k)} \quad (8.9)$$

$$> 0 \quad (8.10)$$

where $c$ is the total weight assigned at time $t$ to edges other than $vu$ that are incident with $v$ or $u$.

The converse implication, that if $\sum_{k \in \mathbb{N}} \delta_k$ diverges, the random walk will almost surely not get ‘stuck’, does not hold in general.\(^{19}\) However, it does hold for non-descending sequences, and for general sequences it is possible to come close, as the following analogues of Lemmas 6.1.4 and 6.1.5 show:

\(^{19}\)For instance, if $G$ is a tree with unit weights on which non-reinforced random walks are almost surely recurrent, then it can be shown that the reinforced random walk on $G$ with reinforcement sequence $(\delta_k)_{k \in \mathbb{N}} = (1.2.1.4.1.8.1.16.\ldots)$ starting from a vertex $v_0$ almost surely eventually stays within $\{v_0\} \cup N(v_0)$. 
8.1.4. **Lemma.** Let \( G = (V, E, w) \) be a weighted graph, and let \( (\delta_k)_{k \in \mathbb{N}} \) be such that

\[
\sum_{j=0}^{\infty} \frac{1}{\max(\delta_0, \delta_1, \ldots, \delta_j)} \quad \text{diverges.} \tag{8.11}
\]

Then a reinforced random walk on \( G \) starting from any initial state will almost surely visit infinitely many vertices, and

\[
P(\text{the walk is transient}) + P(\text{the walk is recurrent}) = 1 \tag{8.12}
\]

**Proof**

The first assertion follows from the second, since both transient and recurrent walks visit infinitely many vertices. To prove the second assertion it suffices to show that for all \( v, u \in V \)

\[
P(v \text{ is visited infinitely often and } u \text{ only finitely often}) = 0 \tag{8.13}
\]

If we can show that the above holds for vertices \( v, u \in V \) with \( vu \in E \), then the general result follows by induction on the distance \( d_G(v, u) \). So let \( v, u \in V \) with \( vu \in E \). Fix \( t_0 \in \mathbb{N} \), and suppose that \( u \) has not been visited since time \( t_0 \in \mathbb{N} \), and at some time \( t > t_0 \) \( v \) is visited again for the \( k \)-th time. Then \( w_t(v) \leq w(v) \max\{\delta_0, \delta_1, \ldots, \delta_{2k}\} \), and since \( vu \) has been traversed at most \( t_0 \) times, \( w_t(vu) \geq w(vu) \min\{\delta_0, \delta_1, \ldots, \delta_{t_0}\} \). Hence, the probability of not immediately traversing \( vu \) in this situation is at most

\[
1 - c/\max\{\delta_0, \delta_1, \ldots, \delta_{2k}\} < e^{-c/\max\{\delta_0, \delta_1, \ldots, \delta_{2k}\}} \tag{8.14}
\]

where \( c = w(vu) \min\{\delta_0, \delta_1, \ldots, \delta_{t_0}\} / w(v) \).

Therefore, applying induction on \( k \), we have that for all \( k \geq 1 \),

\[
P(u \text{ is not visited between } t_0 \text{ and the } k+1 \text{-th visit of } v) \leq \prod_{k'=1}^{k} e^{-c/\max(\delta_0, \ldots, \delta_{2k'})} \tag{8.15}
\]

\[
= e^{-c \sum_{k'=1}^{k} (1/\max(\delta_0, \ldots, \delta_{2k'}))} \tag{8.16}
\]

Consequently

\[
P(v \text{ is visited infinitely often and } u \text{ never after time } t_0) \leq e^{-c \sum_{\nu=1}^{\infty} (1/\max(\delta_0, \ldots, \delta_{2k}))} \tag{8.17}
\]

\[
= 0 \tag{8.18}
\]

Summing over all times \( t_0 \in \mathbb{N} \) gives the desired result.
8.1.5. **Lemma.** Let $G = (V, E, w)$ be a weighted graph, $F \subseteq V$ a finite set of vertices of $G$, and $v_0 \in V$. Let $(\delta_k)_{k \in \mathbb{N}}$ be such that equation (8.11) holds. Then for the reinforced random walk on $G$ starting from $v_0$, the following are equivalent:

(i) The reinforced random walk on $G$ starting from $v_0$ is almost surely recurrent.

(ii) For any $t_0 \in \mathbb{N}$, and any history up to time $t_0$, $F$ will be (re)visited at some time at or after time $t_0$ almost surely.

**Proof**

$(i) \implies (ii)$ is trivial. If $(ii)$ holds, then by applying it repeatedly we find that the reinforced random walk on $G$ starting from $v_0$ will almost surely visit $F$ infinitely often. Then the random walk is almost surely not transient, and by the previous Lemma, this implies it is almost surely recurrent.

8.1.6. **Remark.** In condition (ii) of Lemma 8.1.5, conceptually we restart the walk at time $t_0$, i.e. we look at a walk which starts at time $t_0$, with $t_0$ traversals part of a 'fixed' history up to time $t_0$ (as opposed to starting at time 0 with a blank initial state). If all such restarted walks can be shown to visit $F$ almost surely. Lemma 8.1.5 states that the original reinforced random walk is almost surely recurrent.

8.1.7. **Lemma.** For random walks on weighted trees, the direction in which an edge is traversed is the same at all odd-numbered traversals (and opposite to the direction of traversal at all even-numbered traversals). This allows us to replace, for reinforced random walks on weighted trees, the condition of Lemmas 8.1.4 and 8.1.5 by the condition that

$$\sum_{k=0}^{\infty} (1/\delta_{2k}) \text{ and } \sum_{k=0}^{\infty} (1/\delta_{2k+1}) \text{ both diverge.} \quad (8.20)$$

**Proof**

Consider a random walk on a weighted tree $G = (V, E, w)$, and assume that equation (8.20) holds. In order to show that the conclusions of Lemmas 8.1.4 and 8.1.5 hold, it suffices to show that for all vertices $v, u \in V$ with $vu \in E$,

$$P(v \text{ is visited infinitely often and } u \text{ only finitely often}) = 0 \quad (8.21)$$

So let $v \in V$, and let $u^0, u^1, \ldots, u^m$ be the neighbors of $v$ in $G$, with $u^0$ being the unique neighbor of $v$ that is on a path between $v$ and $v_0$ if $v \neq v_0$. Set, for $i \leq m, k \in \mathbb{N}$,

$$R'_{ki} = \delta_{2k+1} w(vu^i) \text{ if } i = 0 \text{ and } v \neq v_0. \quad R'_{ki} = \delta_{2k} w(vu^i) \text{ otherwise} \quad (8.22)$$
Then $R_k^i$ is the weight of the edge $vu'$ if $v$ is visited and the arc $vu'$ has been traversed (in that direction) $k$ times before.

The next part of the proof is based on a proof of H. Rubin concerning a generalized Pólya Urn problem [23]. Let $Y_k'$ be independent exponential random variables such that $E(Y_k') = 1/R_k^i$.\(^\text{20}\) and put

$$A' = \left\{ \sum_{k'=0}^{k} Y_k', k' \geq 0 \right\} \text{ for } i \leq m$$  \hspace{1cm} (8.23)

Define a sequence of edges $vu'$ by making the $k$-th element of the sequence $vu'$ if the $k$-th smallest element of $A_0 \cup \ldots \cup A_m$ is from $A_i$. Now since by equation (8.20)

$$\sum_{k=0}^{\infty} \frac{1}{R_k} \text{ diverges.} \hspace{1cm} (8.24)$$

we have that almost surely

$$\sum_{k=0}^{\infty} Y_k' \text{ diverges.} \hspace{1cm} (8.25)$$

and hence almost surely $vu'$ will appear infinitely often in the sequence for all $i \leq m$.

As it turns out, this sequence has exactly the same probability distribution as the sequence of edges traversed from $v$ in the reinforced random walk. In other words, we may decide that at visits to $v$ we traverse successive arcs of the sequence, without changing any probabilities. The proof of this relies on properties of exponential random variables, and is straightforward but cumbersome. Interested readers are referred to Rubin’s proof [23]. We conclude that equation (8.21) holds.

\[\square\]

### 8.2 A Martingale for Reinforced Random Walks

Now let us consider recurrence for reinforced random walks. The proofs given in Chapter 6 used the fact that, if a function $h$ on the vertex-set of a weighted graph $G$ is harmonic, then in a non-reinforced random walk, $h(v_i)$ behaves like a martingale. This does not in general hold for reinforced random walks. If $h$ is a harmonic function, then a vertex which has neighbors with higher $h$-values will also have neighbors with lower $h$-values, but the probabilities of the corresponding edges being traversed are not necessarily balanced, or even constant over time. In order to find an analogue of Lemma 6.2.6, we will need to compensate for the difference in probabilities.

\(^{20}\)i.e. the probability distribution of $Y_k'$ is given by $P(Y_k' > r) = e^{-rR_k^i}$ for all $r \in \mathbb{R}$
8.2. A Martingale for Reinforced Random Walks

8.2.1. Lemma. Let \( G = (V, E, w) \) be a weighted graph, and let \( h : V \rightarrow \mathbb{R} \) be a harmonic [superharmonic, subharmonic] function on a subset \( V' \subset V \). Consider the reinforced random walk with reinforcement sequence \((\delta_k)_{k \in \mathbb{N}}\) and define

\[
M_t = \sum_{t'=0}^{t} \left\{ \begin{array}{ll}
\frac{\Delta_h(v_{t'},v_{t'+1})}{\delta_k(v_{t'},v_{t'+1})} & \text{if } v_{t'} \in V' \\
0 & \text{otherwise}
\end{array} \right. \quad (8.26)
\]

for \( t \in \mathbb{N} \), where (as before) \( \Delta_h(\bar{v}u) \) denotes \( h(u) - h(v) \). Then \( M \) is a martingale [supermartingale, submartingale].

Proof

If \( v_t \in V - V' \), then \( M_{t+1} = M_t \), otherwise

\[
M_t = \left[ \gtrless \leq \right] M_t + \frac{1}{w_t(v_t)} \sum_{u \in N(v_t)} w(v_t u) \Delta_h(v_t u) \quad (8.27)
\]

\[
= M_t + \sum_{u \in N(v_t)} \frac{w_t(v_t u) \Delta_h(v_t u)}{w_t(v_t)} \delta_k(v_t u) \quad (8.28)
\]

\[
= M_t + \sum_{u \in N(v_t)} P(v_{t+1} = u \mid \mathcal{F}_t) \frac{\Delta_h(v_t u)}{\delta_k(v_t u)} \quad (8.29)
\]

\[
= E(M_{t+1} \mid \mathcal{F}_t) \quad (8.30)
\]

\[\square\]

As an application of the above martingale, we will show that if non-reinforced random walks on a weighted tree are almost surely recurrent, then for reinforced random walks on that tree, a very weak condition on the reinforcement sequence suffices to show recurrence.

8.2.2. Theorem. Let \( G = (V, E, w) \) be a weighted tree, with the property that non-reinforced random walks on \( G \) are almost surely recurrent. Let \((\delta_k)_{k \in \mathbb{N}}\) be a non-descending reinforcement sequence that satisfies the condition of Lemma 8.1.5 (or that of Lemma 8.1.7). Furthermore, assume either that \((\delta_k)_{k \in \mathbb{N}}\) is bounded, or that \(\delta_{k+1} > \delta_k\) for some even \(k \in \mathbb{N}\). Then the reinforced random walk with reinforcement sequence \((\delta_k)_{k \in \mathbb{N}}\) is almost surely recurrent.

Proof

Consider a reinforced random walk on \( G \) starting from some vertex \( v_0 \in V \). By Lemma 8.1.5 (or Lemma 8.1.7), to show recurrence, it suffices to show for all \( t_0 \in \mathbb{N} \), and any history up to time \( t_0 \), that \( v_0 \) will be revisited almost surely at some time at or after time \( t_0 \). So let \( t_0 \in \mathbb{N} \) and fix the history up to time \( t_0 \). First, we need a function \( h \) on \( V \) that is superharmonic on \( V - \{v_0\} \). Since non-reinforced random walks on \( G \) are almost surely recurrent, such a function \( h \) exists by Theorem 6.3.2. For \( r \in \mathbb{R} \), define the stopping time \( \tau_r \) as the first
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time \( t > t_0 \) at which \( v_t = v_0 \) or \( h(v_t) > r \). By Lemma 8.1.4, the random walk will almost surely leave the finite set of vertices \( \{ v \in V \mid h(v) \leq r \} \). Hence \( \tau_r < \infty \) almost surely.

Next, let \( M_t \) be the martingale of Lemma 8.2.1. For walks on weighted trees, the direction of traversal of an edge is the same for all odd-numbered traversals, and opposite to the direction for all even-numbered traversals. Furthermore, all odd-numbered traversals are traversals going from the lower to higher \( h \)-value, for otherwise it would be possible to construct an infinite sequence of vertices of decreasing \( h \)-value, which would contradict the fact that \( h \to \infty \) if \( v \to \infty \). Hence, an edge \( vu \) which has been traversed \( k \) times at time \( t \) contributes

\[
\Delta_h(v_0) \cdot \sum_{j=0}^{k-1} \begin{cases} 
1/\delta_j & \text{if } j \text{ is even and } vu \text{ is not incident with } v_0 \\
0 & \text{if } j \text{ is even and } vu \text{ is incident with } v_0 \\
-1/\delta_j & \text{if } j \text{ is odd}
\end{cases}
\] (8.31)

to the value of the martingale. Now by the conditions on \((\delta_k)_{k \in \mathbb{N}}\), there exists a \( c > 0 \) such that either \( 1/\delta_k > c \) for all \( k \in \mathbb{N} \) or \( 1/\delta_k - 1/\delta_{k+1} > c \) for some even \( k \in \mathbb{N} \). We can use either property, together with the monotonicity of \((\delta_k)_{k \in \mathbb{N}}\), to obtain the following lower bound on the above contribution:

\[
|\Delta_h(v_0)| \cdot \begin{cases} 
c & \text{if } k \text{ is odd} \\
0 & \text{if } k \text{ is even}
\end{cases} \quad (8.32)
\]

At any time \( t \), the edges of \( G \) that have been traversed an odd number of times are exactly the edges of the unique path in \( G \) between \( v_0 \) and \( v_t \). Furthermore, between times \( t_0 \) and \( \tau \), there will be no traversals of edges incident with \( v_0 \), except for a possible traversal to \( v_0 \) at time \( \tau \). Hence the martingale \( M_t \) satisfies

\[
M_t \geq c(h(v_t) - h(v_0)) - c'
\] (8.33)

where \( c' = \sum_{u \in N(v_0)} \Delta_h(v_0u)[k_0(v_0u)/2]/\delta_1 \). Now we can apply the Optional Stopping Times Theorem to obtain

\[
M_{t_0} \geq E(M_{\tau_r}) \geq (1 - P(v_{\tau_r} = v_0))c(r - h(v_0)) - c'
\] (8.34)

We conclude that \( P(v_{\tau_r} = v_0) \geq 1 - (M_{t_0} + c')/c(r - h(v_0)) \) for all \( r > h(v_0) \), and hence \( v_0 \) is almost surely revisited at some time after time \( t_0 \).

\( \square \)

8.2.3. REMARK. For the proof of the above theorem, we can weaken the conditions on the reinforcement sequence to the conditions of Lemma 8.1.7 and, for
some $c > 0$, the inequality
\[
\sum_{j=0}^{k-1} (-1)^j \delta_k > 0 \text{ for } k \text{ even }, > c \text{ for } k \text{ odd} \quad (8.35)
\]

### 8.3 Once-Reinforced Random Walks

In this section we will consider the once-reinforced random walk, where the weight of an edge only changes the first time it is traversed, and afterwards remains constant. For this walk, the martingale $M_t$ defined in the previous section can be expressed as $h(v_t)$ plus a certain (bounded) bias. If the expectation of the bias is small enough, we will be able to show recurrence in a similar manner as in Chapter 6.

#### 8.3.1 Definition. Let $\delta > 0$. The once-reinforced random walk with reinforcement factor $\delta$ is the reinforced random walk with reinforcement sequence
\[
(\delta_k)_{k=0}^{\infty} = (1, \delta, \delta, \delta, \ldots) \quad (8.36)
\]

#### 8.3.2 Definition. Define the stochastic variables $E_t$ and $A_t$, for $t \in \mathbb{N}$, by setting
\[
E_t = \{v_s v_{s+1} \mid s < t\} \quad (8.37)
\]
\[
A_t = \{v \delta \mid vu \in E_t, v \delta = \overrightarrow{v_s v_{s+1}} \text{ for } s = \min\{s' < t \mid v_s v_{s+1} = vu\}\} \quad (8.38)
\]
i.e. $E_t$ is an edge-set containing the edges that have been traversed up to time $t$, and $A_t$ is an arc-set obtained from $E_t$ by orienting each edge in the direction that it was first traversed.

#### 8.3.3 Lemma. In a once-reinforced random walk with reinforcement factor $\delta > 0$, let $t_0 \in \mathbb{N}$, and let $M_t$ be as in Lemma 8.2.1 for some function $h : V' \to \mathbb{R}$ which is (super/sub)harmonic on $V' \subset V$. Then for $t \geq t_0$,
\[
\delta(M_t - M_{t_0}) = h(v_t) - h(v_{t_0}) + (\delta - 1) \sum_{\overrightarrow{v \delta} \in A_t - A_{t_0}} \Delta_h(\overrightarrow{v \delta}) \quad (8.39)
\]
as long as $V - V'$ has not been visited at any time between $t_0$ and $t$ (including $t_0$ and excluding $t$).

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21 Regrettably, the constant $c > 0$ cannot be replaced by 0. We can find a counterexample on the linear lattice graph $G = (V,E,w)$ with $V = \{v^n \mid n \in \mathbb{N}\}$, $E = \{v^n v^{n+1} \mid n \in \mathbb{N}\}$, and $w(v^n v^{n+1}) = n + 1$. Non-reinforced random walks on this graph are recurrent, but the reinforced random walk with reinforcement sequence $(\delta_k)_{k \in \mathbb{N}} = (1,1,2,2,3,3,\ldots)$ starting in $v_0$ is almost surely transient.
Proof
At time \( t = t_0 \), the equality holds. If an arc \( \overrightarrow{v\theta} \) is traversed that has been traversed before, then \( M_t \) changes by \( \Delta_h(\overrightarrow{v\theta})/\delta \), \( h(v_t) \) changes by \( \Delta_h(\overrightarrow{v\theta}) \), and \( A_t \) does not change, so equality is preserved. If an arc \( v\theta \) is traversed that has not been traversed before, then \( M_t \) changes by \( \Delta_h(\overrightarrow{v\theta}) \), \( h(v_t) \) changes by \( \Delta_h(\overrightarrow{v\theta}) \), and \( \overrightarrow{v\theta} \) is added to \( A_t \), so equality is again preserved.

Now, in our proof of the recurrence of non-reinforced random walks, a key point was that when we moved farther away from \( F \), the value of the martingale increased as well. Since the expectation of the martingale was bounded, this implied that the probability of reaching a border decreased if we moved the border further away. In order to use similar reasoning here, we will need the bias \((\delta - 1) \sum_{\overrightarrow{v\theta} \in A_t} \Delta_h(\overrightarrow{v\theta})\) to be positive in the long run, or at least not too negative.

8.3.4. Lemma. Let \( G = (V, E, w) \) be a weighted graph. Let \( h : V \to \mathbb{R} \) be a function satisfying

1. \( h \) is superharmonic everywhere except on a finite subset \( F \subset V \).
2. \( h \) goes to infinity if \( v \) goes to infinity.

Consider the once-reinforced random walk on \( G \) with reinforcement factor \( \delta \) starting at some vertex \( v_0 \). Suppose that for some \( \epsilon > 0 \), the following holds for any time \( t_0 \) and any history up to time \( t_0 \): There exists a \( c \in \mathbb{R} \) such that for all \( r_0 \in \mathbb{R} \) we can find \( r > r_0 \) with

\[
(\delta - 1) E \left( \sum_{\overrightarrow{v\theta} \in A_{r_0}} \Delta_h(\overrightarrow{v\theta}) \mid F_{t_0} \right) \geq -(1 - \epsilon)r - c \tag{8.40}
\]

(where the stopping time \( \tau_r \) is the first time at or after \( t_0 \) that \( F \) is visited or \( h(v_t) \geq r \)).

Then the once-reinforced random walk on \( G \) with reinforcement factor \( \delta \) starting at \( v_0 \) is almost surely recurrent.

Proof
Without loss of generality we may assume that \( h \geq 0 \). Note that the reinforcement sequence satisfies the condition of Lemmas 8.1.4 and 8.1.5. Therefore it suffices to show for all \( t_0 \in \mathbb{N} \), and any history up to time \( t_0 \), that \( F \) will be revisited almost surely at some time at or after time \( t_0 \). So let \( t_0 \in \mathbb{N} \), and fix the history up to time \( t_0 \).

Let \( M_t \) be the supermartingale of Lemma 8.2.1, and let \( r \in \mathbb{R} \). For any \( t \leq \tau_r \), the set \( A_t - A_{t_0} \) is contained in the finite set \( \{ \overrightarrow{v\theta} \in V \mid v \notin F \land h(v) < r \} \). So
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$M_t$ is bounded for $t \leq \tau_r$, and furthermore $\tau_r < \infty$ almost surely by Lemma 8.1.4. Hence we can apply the Optional Stopping Times Theorem to obtain $E(\delta M_{\tau_r}) \leq \delta M_0$, which by Lemma 8.3.3 is equivalent to

$$E(h(v_t)) \leq h(v_0) + (\delta - 1) \sum_{\vec{v} \in A_{r_0}} \Delta_h(\vec{v} \vec{\alpha}) - (\delta - 1)E \left( \sum_{\vec{v} \in A_{r_0}} \Delta_h(\vec{v} \vec{\alpha}) \right)$$

(8.41)

Combining this with the formula $E(h(v_t)) \geq (1 - P(v_r \in F | F_{t_0}))r$, we obtain

$$P(v_r \in F | F_{t_0}) \geq 1 - \frac{h(v_0)}{r} - \frac{\delta - 1}{r} \sum_{\vec{v} \in A_{r_0}} \Delta_h(\vec{v} \vec{\alpha}) + \frac{\delta - 1}{r}E \left( \sum_{\vec{v} \in A_{r_0}} \Delta_h(\vec{v} \vec{\alpha}) \right)$$

(8.42)

By assumption we can find $c, r \in \mathbb{R}$ such that

$$\frac{\epsilon}{2} r > c + h(v_0) + (\delta - 1) \sum_{\vec{v} \in A_{r_0}} \Delta_h(\vec{v} \vec{\alpha})$$

(8.43)

and (8.40) holds. Then

$$P(v_r \in F | F_{t_0}) \geq 1 - \frac{1 - \epsilon}{2} r - \frac{c - (1 - \epsilon)r + c}{r} = \epsilon/2$$

(8.44)

So there is at least a chance of $\epsilon/2$ of coming back to $F$ at time $t = \tau_r$. In the event that this does not happen, we repeat the entire process starting at time $\tau_r + 1$, and each time we have a chance of $\epsilon/2$ of visiting $F$. It follows that the random walk will visit $F$ almost surely.

Next are some applications of this lemma. We will write the bias as the sum of `local' biases in order to estimate it. The first application demonstrates how to use absolute bounds on $\sum_{\vec{v} \in A_{r_0}} \Delta_h(\vec{v} \vec{\alpha})$. to show recurrence for $\delta$ close to 1.

8.3.5. THEOREM. Let $n \geq 1$, and let $G = (V, E, w)$ be the square lattice graph on $\mathbb{Z} \times \{1, \ldots, n\}$ or on $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$. If $1 - \frac{1}{n} < \delta < 1 + \frac{1}{n^2}$ (for $n \geq 3$), or $1 - \frac{1}{n} < \delta$ (for $n = 1, 2$), then the once-reinforced random walk on $G$ with reinforcement factor $\delta$ is almost surely recurrent.

Proof

First assume that $G$ is the square lattice graph on $\mathbb{Z} \times \{1, \ldots, n\}$. With each vertex $v$ of $G$ we can associate coordinates $x_v, y_v$ with $x_v \in \mathbb{Z}, y_v \in \{1, \ldots, n\}$, in the obvious fashion. We may assume that the random walk starts at a point $v_0$ with $x_{v_0} = 0$. For our superharmonic function $h$ we will use $h(v) = |x_v|$, which

---

22Recurrence for $1 \leq \delta < 1 + \frac{1}{n^2}$ was first proven by Sellke in [31], using different methods.
is easily seen to be harmonic everywhere except on the finite set $F = \{ v \in V \mid x_v = 0 \}$.

With this function $h$, the only edges that contribute to the bias are horizontal edges. For any $c \in \mathbb{Z}$, consider the column $C_c$ of $n$ horizontal edges connecting points $v$ with $x_v = c$ to points $u$ with $x_u = c + 1$. We need to estimate the number of edges of this column that, at first traversal, are traversed going from the lower to the higher $h$-value. This number is obviously at most $n$, and unless the column has not been traversed at all, it is at least 1 (since the random walk cannot reach the side of the column with higher $h$-values without crossing the column at least once). Similarly, the number of edges that, at first traversal, are traversed going from the higher to the lower $h$-value, is at least 0 and at most $n - 1$. So the contribution of the column to the bias satisfies

$$
(\delta - 1) \sum_{\vec{u} \in A_c, v \in C_c} \Delta_h (\vec{u}) \geq \left\{ \begin{array}{ll}
(\delta - 1) \max(0, n - 2) & \text{if } \delta \geq 1 \\
(1 - \delta)n & \text{if } \delta < 1
\end{array} \right. = -(1 - \epsilon) \quad (8.45)
$$

where $\epsilon = 1 - (\delta - 1)\max(0, n - 2) > 0$ if $1 \leq \delta < 1 + 1/\max(0, n - 2)$ and $\epsilon = 1 - (1 - \delta)n > 0$ if $1 - 1/n < \delta < 1$.

Now for any $t_0$ and any $r > t_0/\epsilon$, if $\tau_r$ is the first time at or after $t_0$ that $F$ is visited or $h(\tau_0) > r$, then the horizontal edges in $A_{\tau_r}$ are all contained in the $r + t_0$ columns with x-coordinates between $-t_0$ and $r$ (in the case that $x_{\tau_0} > 0$) or between $-r$ and $t_0$ (in the case that $x_{\tau_0} < 0$). Summing all columns, we obtain

$$
(\delta - 1) \sum_{\vec{u} \in A_{\tau}} \Delta_h (\vec{u}) \geq -(1 - \epsilon)r - (1 - \epsilon)t_0 \quad (8.46)
$$

Hence the conditions of Lemma 8.3.4 are satisfied, and the reinforced random walk is almost surely recurrent.

The proof for the square lattice graph on the cylinder $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$ is identical.

\[\square\]

8.3.6. REMARK. Of course, using the absolute bound on $\sum \{ \Delta_h (\vec{u}) \mid \vec{u} \in A_{\tau} \}$ is a very unsophisticated method of obtaining a bound on the expected value of
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Figure 8.2: A tree on which the once-reinforced random walk with $\delta < 1/4$ is not almost surely recurrent.

the bias. In the above case, we could improve the bounds on the expected value of the bias with a few simple probabilistic calculations, resulting in slight improvements to our bounds on $\delta$. This does not yield any substantial improvements, unfortunately.

In the next chapter a proof will be given of recurrence for large values of $\delta$. No proof is yet known for intermediate values of $\delta$. It is also not yet known whether the once-reinforced random walk on the square lattice graph on $\mathbb{Z}^2$ is recurrent for any reinforcement factor $\delta \neq 1$, although a related marginal result is given at the end of this chapter. The intuition, however, is that once-reinforced random walks on the square lattice graphs on $\mathbb{Z}^2$ and $\mathbb{Z} \times \{1, \ldots, n\}$ are recurrent for all $\delta \geq 1$.

For reinforced random walks on weighted trees, recurrence can be proven for all $\delta \geq 1$ (provided the tree is such that non-reinforced random walks are recurrent in the first place). This follows already from Theorem 8.2.2. To prove recurrence for all $\delta \geq 1$, in any graph on which non-reinforced random walks are recurrent, one could use something like

8.3.7. Proposition. For any edge $vu$ that is 'far away' from all edges traversed so far, if $v$ is closer than $u$ to the origin of the walk (in the sense that $h(v) < h(u)$), then $vu$ has at least as much chance of being traversed (the first time it is traversed) from $v$ to $u$ as it has of being traversed (the first time it is traversed) from $u$ to $v$.

This means, very loosely formulated, that closer vertices are visited earlier. Alas, so far this proposition has neither been proved nor refuted.

8.3.8. Example. For negative reinforcements, recurrence is not necessarily preserved. There are examples of cases where the non-reinforced random walk on a graph is recurrent, but for certain $\delta < 1$, the once-reinforced random walk with
reinforcement $\delta$ is not. Figure 8.2 is such a case: it can be shown that for $\delta < 1/4$, the once-reinforced random walk on this tree is not almost surely recurrent.

Next is an application of Lemma 8.3.4 that uses probabilistic methods rather than an absolute bound. Note that we actually use the generalization of Lemma 8.3.4 to graphs with parallel edges, rather than the lemma as written. As stated in Remark 6.1.1, we will simply postulate this generalization and proceed.

8.3.9. THEOREM. Let $G = (V,E,w)$ be a weighted graph with vertices $V = \{v^i \mid i \in \mathbb{Z}\}$ and for any $n$, a finite non-zero number of parallel edges between the vertices $v^n$ and $v^{n+1}$. If the non-reinforced random walk on $G$ is almost surely recurrent, then the reinforced random walk on $G$ is almost surely recurrent for any reinforcement factor $\delta > 0$.

Proof
If the non-reinforced random walk on $G$ is almost surely recurrent, then by Theorem 6.3.2 there exists a function $h : V \to \mathbb{R}$ such that $h$ is superharmonic on $V - \{v^0\}$ and $h(v^n) \to \infty$ if $n \to \infty$ or $n \to -\infty$. It is easily seen that $h(v^n) > h(v^m)$ if $n > m > 0$ or $n < m < 0$.

Now for $n \in \mathbb{Z}$, let $G^n = (V^n,E^n,w^n)$ be the subgraph induced by $\{v^n, v^{n+1}\}$ (i.e., $V^n = \{v^n, v^{n+1}\}$, $E^n$ is the set of edges between $v^n$ and $v^{n+1}$, and $w^n = w|_{E^n}$). Although events outside $G^n$ may effect whether and when an edge of $G^n$ is traversed, which edge of $G^n$ is traversed is only dependent on the relative current weights of the edges of $G^n$. So we can estimate the expected contribution to the bias of each set $E^n$ separately, and take the sum to arrive at an estimate for the total expected bias. The possibility that at some point the walk in $G$ will no longer return to $G_n$ can be simulated by a stopping time for the walk in $G_n$.

So fix $n \in \mathbb{Z}$ and consider the reinforced random walk on the finite graph $G^n$, starting in $v^n$ if $n \geq 0$, and in $v^{n+1}$ otherwise. Note that in both cases the random walk will start at the vertex with the lower $h$-value and then alternate between the two vertices. Set $c = |h(v^{n+1}) - h(v^n)|$. If for $G^n$ we define $A^n_t$ and $E^n_t$ as usual, we have for any $t \in \mathbb{N}$

$$E\left(\sum_{\vec{a} \in A^n_{t+1}} \Delta_h(\vec{a}) \mid F_t\right) = \sum_{\vec{a} \in A^n_t} \Delta_h(\vec{a}) + (-1)^t \frac{w(E^n_t) - w(E^n_{t+1})}{w(E^n_t) + (\delta - 1)w(E^n_{t+1})} \cdot c \quad (8.47)$$

where $w(X)$ denotes $\sum_{e \in X} w(e)$. This implies that for any stopping time $\tau$

$$E\left(\sum_{\vec{a} \in A^n_2} \Delta_h(\vec{a})\right) = E\left(\sum_{t=0}^{\tau-1} (-1)^t \frac{w(E^n_t) - w(E^n_{t+1})}{w(E^n_t) + (\delta - 1)w(E^n_{t+1})} \cdot c \right) \quad (8.48)$$
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For all $t$, $w(E_t^n) \leq w(E_{t+1}^n) \leq w(E^n)$, and hence

$$\frac{w(E^n) - w(E_t^n)}{w(E^n) + (\delta - 1)w(E_t^n)} \leq \frac{w(E^n) - w(E_{t+1}^n)}{w(E^n) + (\delta - 1)w(E_{t+1}^n)} \tag{8.49}$$

We conclude that for any stopping time $\tau$,

$$0 \leq E\left(\sum_{\tilde{d} \in A^n} \Delta_{\tilde{d}}[\tilde{d}]\right) \leq c \tag{8.50}$$

Now let us return to the random walk on $G$. Fix $t_0 \in \mathbb{N}$ and the history up to time $t_0$. Then all the vertices that have been visited up to time $t_0$ have indices between $-t_0$ and $t_0$. Furthermore, all the vertices that can be visited after time $t_0$ are on the same side of $v^0$ until the first visit to $v^0$: without loss of generality we may assume that this is the side of the vertices with positive indices. If we transfer the results we obtained for the walks on the graphs $G^n$ to the random walk on the graph $G$ and take the sum of the inequalities over all edge-sets $E^n$ with $n \geq t_0$, then we obtain

$$-c' \leq E\left(\sum_{\tilde{d} \in A^n} \Delta_{\tilde{d}}[\tilde{d}] \mid F_{t_0}\right) \leq E(\max\{h(v_t) \mid t \leq \tau\}) - h(v^{t_0}) + c' \tag{8.51}$$

where $\tau$ is any stopping time such that the walk does not leave the set of vertices with positive indices, and $c' = \sum_{n=-t_0}^{t_0} (\#E^n)\|h(v_{n+1}) - h(v^n)\|$. This implies the condition of (the generalization of) Lemma 8.3.4 for all $\delta > 0$.

The third application of Lemma 8.3.4 yields an (admittedly rather marginal) result for a variant once-reinforced random walk on the square lattice graph with unit weights on $\mathbb{Z}^2$. In this variant once-reinforced random walk, the reinforcement factor is not constant, but is allowed to be different for each edge. It is not difficult to modify Definition 8.1.1 to allow this type of once-reinforced random walks, although we will encounter some hidden complications in modifying some of the Lemmas given in this chapter.

8.3.10. Theorem. Consider the variant once-reinforced random walk on the square lattice graph $G$ with unit weights on $\mathbb{Z}^2$, where the reinforcement factor is not constant but is, for each edge, reciprocal to the Euclidean distance of the edge from the origin $(0,0)$, i.e. for some $C > 0$,

$$w_r(vu) = \begin{cases} 1 & \text{if } vu \text{ has not yet been traversed} \\ 1 + C/(\max(||(x_v, y_v)||, ||(x_u, y_u)||)) & \text{if } vu \text{ has been traversed} \end{cases} \tag{8.52}$$

This random walk is recurrent for $C < 1/(2\sqrt{2\pi})$. 
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Proof
First we need to find analogues of Lemmas 8.2.1 and 8.3.3 for variant walks of this type. Unfortunately, the lack of a constant reinforcement factor makes it difficult to even formulate an analogue of Lemma 8.3.3, let alone prove it. This is not surprising: essential to the concept of Lemma 8.3.3 is that as long as the walk only traverses edges that have been traversed before, the change in the value of the martingale is reflected in the change of the current value of \( h(v_t) \). This no longer holds if we use the definition of Lemma 8.2.1 and the reinforcement factor is not constant.

However, there is a solution to this dilemma: we can modify both the nominal weight of the edges and the initial reinforcement factors (the multipliers that are applied to edges that have not been traversed before), in such a way that the reinforcement factors that are applied to traversed edges become constant. I.e., without changing the actual walk, we consider it to be on the square lattice graph \( G' \) on \( \mathbb{Z}^2 \) with weights

\[
w'(vu) = 1 + C/(\max(|(x_r, y_r)|, |(x_a, y_a)|)) \quad \text{for} \quad vu \in E
\]

and for each edge \( vu \in E \) a reinforcement sequence

\[
(\delta_k(vu))_{k=0}^\infty = \left\{ \frac{1}{1 + C/(\max(|(x_r, y_r)|, |(x_a, y_a)|))} \right\}_{1,1,1,\ldots}
\]

In the actual walk, this yields the same weights as before. Of course, this means that we need to select \( h \) to be a function with the right properties on \( G' \) rather than on \( G \). Fortunately, the weight of an edge of \( G' \) is never more than twice the weight of the corresponding edge of \( G \), so by Theorem 6.3.8 non-reinforced random walks are as recurrent on \( G' \) as they are on \( G \). Hence a function \( h \) with the necessary properties exists. As it turns out, the function

\[
h(x, y) = \begin{cases} 
\log(1/12) & \text{if } (x, y) = (0, 0) \\
\log(1/4) & \text{if } (x, y) = (0, \pm 1) \text{ or } (x, y) = (\pm 1, 0) \\
\log(x^2 + y^2 - 1) & \text{otherwise}
\end{cases}
\]

which we used in Example 6.3.3 for \( G \), also works for \( G' \).

Now we can obtain, in sequence, analogues of Lemmas 8.2.1, 8.3.3 and 8.3.4. The proofs of the following statements is straightforward and quite similar to the proofs of the original lemmas, and therefore will be omitted. First, the stochastic process

\[
M_t = \sum_{t'=0}^t \left\{ \frac{\Delta h(v_{t'}v_{t'+1})}{\delta_{t'}(v_{t'}v_{t'+1})} \right\} \quad \text{if } v_{t'} \in 1' \\
0 \quad \text{otherwise}
\]

is a martingale. Next, for \( t \geq t_0 \geq 0 \) the equation

\[
M_t - M_{t_0} = h(v_t) - h(v_{t_0}) + \sum_{vu \in A_{t_0} - A_t} \left( \frac{1}{\delta_0(vu)} - 1 \right) \Delta h(v'v'')
\]
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holds, as long as $V - V'$ has not been visited at any time between $t_0$ and $t$. Finally, in order to show recurrence, it suffices to show that for some $\epsilon > 0$, for any time $t_0$, and for any history up to time $t_0$.

There exists a $c \in \mathbb{R}$ such that for all $r_0 \in \mathbb{R}$ we can find $r > r_0$ with

$$E \left( \sum_{\vec{v} \in A_r} \left( \frac{1}{\delta_0(\vec{v} \cdot u)} - 1 \right) \Delta_h(\vec{v} \cdot u) \mid \mathcal{F}_{t_0} \right) \geq -(1 - \epsilon)r - c \quad (8.58)$$

where the stopping time $\tau_r$ is the first time at or after $t_0$ that $(0, 0)$ is visited or $h(\tau_r) \geq r$.

So now the only thing left to do are a few calculations in the manner of Theorem 8.3.5. First note that for an edge $vu \in E$ with $1 < |(x_v, y_v)| \leq |(x_u, y_u)|$,

$$\left( \frac{1}{\delta_0(\vec{v} \cdot u)} - 1 \right) \Delta_h(\vec{v} \cdot u) = \frac{\log((x_v^2 + y_v^2) - 1) - \log((x_u^2 + y_u^2) - 1)}{|(x_u, y_u)|} \quad (8.59)$$

$$\leq \frac{x_u^2 + y_u^2 - x_v^2 - y_v^2}{(x_v^2 + y_v^2 - 1)|(x_u, y_u)|} \quad (8.60)$$

Now for most $v \in V$, there are two vertices $u \in N(v)$ with $|(x_u, y_u)| > |(x_v, y_v)|$, and

$$\sum_{u \in N(v), |(x_u, y_u)| > |(x_v, y_v)|} x_u^2 + y_u^2 - x_v^2 - y_v^2 \leq 2|x_v| + 2|y_v| + 2 \leq 2\sqrt{2}|(x_v, y_v)| + 2 \quad (8.61)$$

The exceptions are the vertices $v \in V$ with $x_v = 0$ or $y_v = 0$, and for those

$$\sum_{u \in N(v), |(x_u, y_u)| > |(x_v, y_v)|} x_u^2 + y_u^2 - x_v^2 - y_v^2 \leq 2|(x_v, y_v)| + 3 \leq 2\sqrt{2}|(x_v, y_v)| + 3 \quad (8.62)$$

Hence, for any $C'' > 2\sqrt{2}C'$, we can find an $R > 0$ such that for all $v \in V$ with $|(x_v, y_v)| > R$,

$$\sum_{u \in N(v), |(x_u, y_u)| > |(x_v, y_v)|} \left( \frac{1}{\delta_0(\vec{v} \cdot u)} - 1 \right) \Delta_h(\vec{v} \cdot u) \leq \frac{C''}{(x_v^2 + y_v^2)} \quad (8.63)$$

Furthermore, for any $C'' > C'$, we can find $r_0 \geq \log(R^2 - 1)$ such that for any $r > r_0$,

$$\sum_{\vec{v} \in A_r} \left( \frac{1}{\delta_0(\vec{v} \cdot u)} - 1 \right) \Delta_h(\vec{v} \cdot u) \leq \sum_{vu \in E, |(x_v, y_v)| < \sqrt{r^2 + 1}} \left( \frac{1}{\delta_0(\vec{v} \cdot u)} - 1 \right) \Delta_h(\vec{v} \cdot u) \quad (8.64)$$
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\[
\leq \sum_{\nu \in \mathcal{E}_r} \left| \frac{1}{\delta_0(vu) - 1} \right| \Delta_h(v\tilde{u}) + \sum_{\nu \in \mathcal{E}_r} \sum_{R < ||x_r, y_r|| < \sqrt{\epsilon^r + 1}} \frac{C'}{(x_r^2 + y_r^2)} \quad (8.65)
\]

\[
\leq \sum_{\nu \in \mathcal{E}_r} \left| \frac{1}{\delta_0(vu) - 1} \right| \Delta_h(v\tilde{u}) + \int_{D(O, R, \epsilon^r + 1)} \frac{C''}{(x_r^2 + y_r^2)} \quad (8.66)
\]

\[
\leq \sum_{\nu \in \mathcal{E}_r} \left| \frac{1}{\delta_0(vu) - 1} \right| \Delta_h(v\tilde{u}) + 2C'' \pi (\log(\sqrt{\epsilon^r + 1}) - \log(R)) \quad (8.67)
\]

\[
\leq \sum_{\nu \in \mathcal{E}_r} \left| \frac{1}{\delta_0(vu) - 1} \right| \Delta_h(v\tilde{u}) + C'' \pi \epsilon^r + 1 - 2C'' \pi \log(R) \quad (8.68)
\]

where \( D(O, R, \epsilon^r + 1) = \{(x, y) \in \mathbb{R}^2 \mid R < ||x, y|| < \sqrt{\epsilon^r + 1}\} \). For \( C < 1/(2\sqrt{2\pi}) \), we can select \( C' \) and \( C'' \) such that \( C'' < 1 \), and hence such that equation (8.58) is satisfied and the random walk is shown to be recurrent.

\( \square \)