Games, walks and grammars: Problems I've worked on

Vervoort, M.R.

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Chapter 9

Nonstandard Analysis

In Chapter 10, we will consider once-reinforced random walks whose reinforcement factor is near-infinite. To do that, we make extensive use of nonstandard analysis (NSA), the extension of real analysis with infinitesimals. Although it is known that anything that can be proven with NSA can also be proven without it, NSA allows for a much more intuitive treatment of concepts such as 'sufficiently large' and 'may be safely ignored', and hence is a very useful tool in this context.

This chapter aims to give a brief introduction to NSA, and an overview of its basic principles and techniques. A full treatment of NSA falls outside the scope of this chapter, but interested readers can find more material in [21], [27] and [28].

9.1 Introduction

The idea of using infinitesimals is nothing new. Newton used infinitesimals to define the derivative of a function, and in an old proof relating the area of a circle to its circumference the circle is treated as an infinity-sided polygon, and as a composition of triangles with infinitesimal bases. However, careless usage of infinitesimals and infinities can easily lead to contradictions, and hence the technique always was considered to be suspect. Eventually, the use of infinitesimals was discarded in favor of limit constructions.

The disadvantage of using limit constructions is that they are considerably less intuitive than infinitesimals. When we see the expression $\delta y/\delta x$, we may *define* it in terms of limits, but we *visualize* it as 'the rate of change over an infinitesimal interval'. For this reason, the study of infinitesimals was never wholly abandoned. And in 1961, these efforts finally bore fruit, as A. Robinson developed a consistent formalism for using infinitesimals, and founded the field of nonstandard analysis, abbreviated as NSA.

The nonstandard approach can also be applied to other fields of mathematics.

yielding concepts such as nonstandard ordinals, hyperfinitely-dimensional manifolds etc. In 1977, Nelson invented Internal Set Theory (abbreviated as IST) in an attempt to give a unified axiomatical background for nonstandard mathematics. IST extends ZFC, Zermelo-Fraenkel set theory, by adding a 'standardness' predicate, and three axioms, Idealization, Standardization and Transfer. In the next sections we will introduce and use this formalism.

Anything which holds in 'orthodox' mathematics also holds in nonstandard analysis. In a sense, the reverse holds as well: any statement that does not refer to any nonstandard concepts or constants, and that can be proven in nonstandard mathematics, can be proven in orthodox (non-nonstandard) mathematics. More formally:

9.1.1. THEOREM. *IST is a conservative extension of ZFC.*

Note that this implies that if ZFC is consistent, then so is IST. Nelson [28] gives an explicit algorithm to translate proofs to orthodox mathematics. So in a sense nonstandard analysis doesn't add anything new. However, proofs in nonstandard analysis are often much simpler.

9.2 The standardness predicate

Amusingly enough, the most important concept in nonstandard analysis is the concept of '*standard*'. The easiest way to introduce this concept is probably to consider infinitesimals, and what properties we *desire* them to have. For example, we want to be able to calculate with them: if δ is an infinitesimal, we want to be able to talk about 2δ, $1 + \delta$, $1/\delta$ etcetera. Furthermore, we want those 'nonstandard' numbers to obey the same rules that 'normal' numbers do.

Robinson's original approach was to take a model of the real line, and construct a *new* model of the axioms of the real numbers, by adding the infinitesimals (and related numbers) in such a manner that everything that held in the old model also held in the new one. The drawback of this approach is that it is not possible to talk about infinitesimals without in some way referring to the old model and how it differs from the new one. For instance, it is common practice to refer to those real numbers that already exist in the old model as being '*standard*'. Using this concept of standardness we can define infinitesimals as

> An infinitesimal is a real number whose absolute value is smaller than every standard positive real

However, it is *impossible* to define infinitesimals without either using the concept of standardness or referring to the old model in some other manner. The reason for this is that in the old model it was impossible to 'access' infinitesimals because they didn't exist, and (by our design) everything that held in the old model also holds in the new model.

Nelson's approach made it possible to bypass this need to refer to different models of the reals. Since from 'within' a model of the reals it is impossible to see in which model you are, it is consistent to assume that the real line that you are using already contains infinitesimals. So Nelson simply *postulated* that they exist, and introduced a standardness predicate 'st(x)', and some axioms to describe its properties, that allowed us to access them. In this perspective, rather than adding or creating infinitesimals and other 'nonstandard' numbers, we *discover* them using the new predicate and axioms.

The two approaches are basically two different perspectives on the same concept. In the first perspective there are two worlds, an 'old' one and a 'new' one containing extra elements. In this second perspective there is no new 'world', merely aspects of the old 'world' that existed before but couldn't be seen. The difference between the two perspectives is more or less a matter of personal taste. In these pages we will keep to the second perspective.

So, nonstandard analysis contains all axioms of 'orthodox' real analysis, and hence all the usual theorems and tools are still valid. All the real numbers we knew before, such as 0, 1, e and π, still exist and have the same properties as before. But in addition they satisfy the standardness predicate, and we now can see that inside the gaps between standard reals exist *nonstandard* reals. However, the only abnormal property these reals have is that they do not satisfy the standardness predicate: they are not noticeable except for that.

9.2.1. REMARK. As a rule of thumb, anything that can be defined using only standard parameters, is itself standard. Conversely, anything that can be used to define something that is known to be nonstandard, is itself nonstandard.

The standardness predicate can also be applied to sets. Again, sets such as $\{0\}$, $\mathbb{R}$ and $[0, e]$ still exist, have the same properties as before, and additionally satisfy the standardness predicate. But now we have new sets, such as $\{\delta\}$ and $[0, \delta]$, which are nonstandard (if $\delta \neq 0$ is an infinitesimal). Note that it is not true that standard sets are sets containing standard elements. The correspondence *does* hold for finite sets. But the set $\mathbb{R}$, for example, is a standard set containing nonstandard elements, since by definition $\mathbb{R}$ contains *all* the reals, including the nonstandard ones.

In fact, it can be shown that every infinite set, whether standard or nonstandard, contains a nonstandard element. Infinite collections containing only standard elements are undefinable as a set. The reason for this is that such sets would contradict some of the laws for sets that already hold in orthodox real analysis. For example, since in orthodox real analysis every bound set has a greatest lower bound, the same should hold in nonstandard analysis. However, if we consider the 'set' of positive standard reals, every infinitesimal is a lower bound of this set. So the greatest lower bound cannot be 0, it obviously cannot be a positive standard real, and if it were an infinitesimal then twice that value would be a greater lower bound. Hence this 'set' would not have a greatest lower bound. The solution to this apparent paradox is to disallow the use of formulas containing the standardness predicate when defining a subset, i.e. when using the Separation Axiom. The same applies when using the Replacement Axiom. This ensures that collections such as the collection of standard positive reals are undefinable *as a set*.[23]

9.3 Basics of Nonstandard Real Analysis

In a sense, nonstandard real analysis is about infinitesimals. To prove the existence of infinitesimals, we need the axioms of Internal Set Theory. However, rather than immediately reviewing these axioms, we will first look at the infinitesimals themselves, consider some related concepts and their properties, and give some examples of their use in mathematics.
First, we will formalize the definition of infinitesimals given before:

9.3.1. DEFINITION. A real number $x \in I\!R$ is called *infinitesimal* if it satisfies

$$\forall^{\text{st}} \epsilon > 0 : |x| < \epsilon \tag{9.1}$$

Here the quantifier $\forall^{\text{st}} x$ is an abbreviation for $\forall x(\text{st}(x) \rightarrow \ldots)$, or 'for all standard x, ...'. Similarly, the quantifier $\exists^{\text{st}} \epsilon$ would be an abbreviation for $\exists x(\text{st}(x) \wedge \ldots)$, or 'there exists a standard x such that ...'. This definition properly captures the notion that infinitesimals are 'very, very small'. However, we also want to capture the notion that if δ is infinitesimal, then $1 + \delta$ is 'very, very close to 1':

9.3.2. DEFINITION. $x, y \in I\!R$ are called *infinitesimally close* (denoted $x \approx y$) if

$$\forall^{\text{st}} \epsilon > 0 : |x - y| < \epsilon \tag{9.2}$$

$x \in I\!R$ is called *nearstandard* if x is infinitesimally close to some standard real:

$$\exists^{\text{st}} y \in I\!R : y \approx x \tag{9.3}$$

This standard real is called the *standard part* of x, denoted ${}^{\circ}x$.

Now, for all $\delta > 0$, $1/\delta$ is a positive real. However, if δ is infinitesimal, then $1/\delta$ is very, very big. In fact, it is larger than every standard real. This is an example of a *hyperfinite* real:

[23] One of the few differences between the two perspectives we described before, is that in the first perspective, these collections are considered to exist as sets of elements of the model, or 'external sets'. What we consider to be a set, is called an 'internal set' in the first perspective, being a set of elements of the model that is also a set *in* the model. Since we keep to the second perspective, we only consider internal sets to be sets.

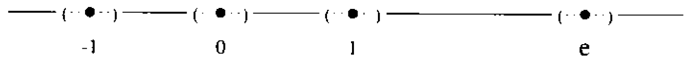

Figure 9.1: The standard reals −1, 0, 1 and e, each surrounded by a 'cloud' of infinitesimally close nonstandard reals.

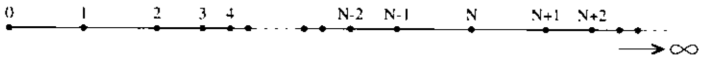

Figure 9.2: A hyperfinite number N

9.3.3. DEFINITION. $x \in I\!R$ is called *hyperfinite* if it satisfies

$$\forall^{st} r \in I\!R : |x| > r \tag{9.4}$$

The reciprocals of hyperfinite numbers are non-zero infinitesimals, and vica versa. Hyperfinite numbers can also be characterized as those numbers that are not nearstandard. Note that although a hyperfinite number is larger (in absolute value) than all standard real numbers, it is still a member of $I\!R$, and as such *not* infinite. Figure 9.2 sketches the position of a hyperfinite number N, relative to the standard natural numbers and to infinity.
Now with these concepts, we will give some examples of their use.

9.3.4. EXAMPLE. Let us imagine some object, say a bowling ball, falling from a large height with a constant acceleration $g = 9.8$. If the bowling ball starts out with speed $v(0) = 0$, its speed at time t satisfies

$$v(t) = gt \tag{9.5}$$

Now, let T be a standard time, and let N be a hyperfinite natural number. Then for any time t, $[t, t+T/N]$ is an infinitesimal time interval. The distance the ball travels in such an interval satisfies

$$gt(T/N) = v(t)(T/N) \leq s(t+T/N)-s(t) \leq v(t+T/N)(T/N) = g(t+T/N)(T/N) \tag{9.6}$$

We can divide the time interval $[0, T]$ into N such intervals and take the sum over these intervals. Since the speed of the ball increases steadily, this yields

$$\begin{aligned}
\frac{1}{2}gT^2 &\approx \frac{1}{2}g(T/N)^2(N-1)N && (9.7)\\
&= g(T/N)\sum_{i=0}^{N-1}(T/N)i && (9.8)\\
&\leq s(T) - s(0) && (9.9)\\
&\leq g(T/N)\sum_{i=0}^{N-1}(T/N)(i+1) && (9.10)
\end{aligned}$$

$$= \frac{1}{2}g(T/N)^2N(N+1) \tag{9.11}$$

$$\approx \frac{1}{2}gT^2 \tag{9.12}$$

If g, $v(0)$ and T are all standard, then so is $s(T) - s(0)$, and hence

$$s(T) - s(0) = \frac{1}{2}gT^2 \tag{9.13}$$

9.3.5. EXAMPLE. Consider a three-dimensional sphere S with center M and some standard radius $r > 0$. Let A denote the surface area of S, and V its volume. We can approximate this sphere by a polyhedron with hyperfinitely many faces, each of which is a triangle of infinitesimal dimensions. The surface area A' and volume V' of this polyhedron are infinitesimally close to A and V. Now, for each face DEF of the polyhedron, we can construct a tetrahedron $DEFM$. The volume of this tetrahedron is $r/3$ times the area of DEF. The polyhedron can be thought of as being composed of hyperfinitely many of these tetrahedrons, one for each face. Taking the sum over all these tetrahedrons, we get

$$V' = r/3 \cdot A' \tag{9.14}$$

Hence $V \approx r/3 \cdot A$. Since we are dealing with a sphere of standard radius, V and A are both standard, and hence

$$V = r/3 \cdot A \tag{9.15}$$

It can be shown (using the method of the previous example) that $V = 4/3\pi r^3$. Hence

$$A = 4\pi r^2 \tag{9.16}$$

9.4 Idealization, Standardization and Transfer

Nelson's Internal Set Theory extends ZFC with the standardness predicate and three axioms: Idealization, Standardization and Transfer. As stated before, the ZFC axiom schemas of Separation and Replacements are *not* extended to include formulae that use the standardness predicate. In this section we will review the three axioms and their common usage. We will formulate the axioms in terms of objects and sets rather than real numbers, in order to pave the way for the application of nonstandard analysis to graph theory. At the end of the section, we will give some examples of how the three axioms work together.

> **The Axiom of Idealization**: For there to exist an object which has a particular property relative to *all* standard objects, it suffices that there exist objects having that property relative to finitely many

standard objects at a time. Formally, for any formula $\phi(x, y)$ not containing the predicate st,

$$\left(\forall^{\mathrm{st\ fin}} F\, \exists x\, \forall y \in F\, \phi(x, y)\right) \;\leftrightarrow\; \left(\exists x \forall^{\mathrm{st}} y\, \phi(x, y)\right) \qquad (9.17)$$

$\phi(x, y)$ may contain standard or nonstandard constants, or free variables other than x and y.

The right-to-left implication of this axiom is only used to show that finite sets are standard if and only if their elements are standard. The left-to-right implication is used to obtain nonstandard, 'idealized' objects. For instance, it is obvious that for any finite standard set of positive numbers F, there exists an $x \in \mathbb{R}$ such that $\forall y \in F : |x| < y$. Hence, if we apply Idealization to the formula $\phi(x, y) \equiv (y > 0 \rightarrow 0 < |x| < y)$, the left side of the equivalence holds, and we obtain the existence of $x \neq 0$ such that $\forall^{\mathrm{st}} y > 0 : |x| < y$, i.e. x is infinitesimal. Another form of (the left-to-right-implication of) the Idealization Axiom is that of the *principle of Overflow*, which states that any (definable) set containing arbitrarily large standard reals also contains a hyperfinite real, and the related *principle of Underflow*, which states that any (definable) set containing arbitrarily small positive hyperfinite reals also contains a nearstandard real.

The Axiom of Standardization: If S is an arbitrary set, then we can obtain a (unique) standard set $^{\circ}S$, the *standardization* of S, by changing just the nonstandard elements. Formally, for any sets S there exists a standard set $^{\circ}S$ such that

$$\forall^{\mathrm{st}} x (x \in {}^{\circ}S \leftrightarrow x \in S) \qquad (9.18)$$

Standardization is often used to allow us to 'ignore' infinitesimal discrepancies. Note that the Standardization Axiom does not necessarily remove the nonstandard elements of a set: it makes the set as a whole standard by making arbitrary changes in its nonstandard elements. Standardizing a standard set such as $\mathbb{R}$ will have no effect, and standardizing the set $\{0, 1, \ldots, N\}$, where N is a hyperfinite natural number, will actually *add* nonstandard numbers (resulting in the standard set $\mathbb{N}$).

Furthermore, although we can represent objects such as functions as sets in order to apply the Standardization Axiom, this will result in the standardization of *all* aspects of the object. In the case of a function, the domain may well change, for instance. And the Standardization Axiom cannot be applied to infinitely many objects at the same time: to standardize each object in a collection, we have to standardize the collection itself, including its index set. For these reasons, one has to take care to set up the right conditions before using Standardization.

The Axiom of Transfer: The Transfer axiom states, that if something holds for all standard objects, it holds for all objects, and conversely if there exists an object satisfying some condition, there exists a standard object satisfying that condition. The formula involved may not refer to standardness or to nonstandard constants. Formally, for each formula $\phi(x, \bar{y})$ not containing 'st', nonstandard constants or free variables other than x and $\bar{y}$,

$$\forall^{\text{st}}\bar{y}\,(\exists x \phi(x, \bar{y}) \leftrightarrow \exists^{\text{st}} x \phi(x, \bar{y})\,) \tag{9.19}$$

The Transfer Axiom can be used to drop a condition of the form 'let x be standard', and to translate results back into the language of Real Analysis. It can also be used to show that we may take some entity to be standard. For instance, if $F(\bar{y})$ is a function definable without using nonstandard constants or the predicate st, then by applying Transfer to the formula $\phi(\bar{y}) \equiv x = F(\bar{y})$ we obtain that for all standard parameters $\bar{y}$, if $F(\bar{y})$ exists it is standard. Hence objects that can be uniquely defined using only standard constants are standard (as we already stated in Remark 9.2.1).

9.4.1. EXAMPLE. The Idealization Axiom can be used to show that there exists a finite set containing all the standard elements. For it is obviously true that for any standard finite set F there exists a finite set x satisfying $y \in x$ for all $y \in F$: simply take F for x. Applying Idealization with $\phi(x, y) \equiv (x \text{ is finite}) \wedge (y \in x)$ yields the desired result. Note that the resulting set cannot be standard (else it would contain itself), and therefore must contain some nonstandard elements as well.

9.4.2. EXAMPLE. Standardization and Transfer can be used to obtain the standard part of a real $x \in I\!R$, provided x is not hyperfinite. Let $S = \{z \in I\!R \mid z \leq x\}$. Since x is not hyperfinite, $-C < x < C$ for some standard $C > 0$, so S has a standard element and a standard upper bound. Now consider the least upper bound of its standardization ${}^{\circ}S$. If y is a standard upper bound of S, then $y \geq z$ for all standard $z \in {}^{\circ}S$, and by Transfer $y \geq z$ for all $z \in {}^{\circ}S$. So S and ${}^{\circ}S$ have the same standard upper bounds. By the definition of ${}^{\circ}S$, they also have the same standard elements. It follows that the least upper bound of ${}^{\circ}S$ exists and is infinitesimally close to the least upper bound of S, i.e. to x. Furthermore, by Transfer the least upper bound of ${}^{\circ}S$ is standard. So the least upper bound is equal to the standard part ${}^{\circ}x$ of x.
Note that if we try to use this approach with a hyperfinite real $x \in I\!R$, then the resulting set ${}^{\circ}S$ turns out to be equal to $\emptyset$ or $I\!R$, depending on whether x is negative or positive. Hence the nearstandard reals are exactly those reals that are not hyperfinite.