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q_T Slicing with Multiple Jets

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Modern collider phenomenology requires unprecedented precision for the theoretical predictions, for which slicing techniques provide an essential tool at next-to-next-to-leading order (NNLO) in the strong coupling. The most popular slicing variable is based on the transverse momentum q_T of a color-singlet final state, but its generalization to final states with jets is known to be very difficult. Here we propose two generalizations of q_T that can be used for jet processes, providing proof of concept with an NLO slicing for $pp \to 2$ jets. We present factorization formulas that enable our approach to NNLO, calculate the NNLO collinear-soft function, and demonstrate slicing at this order for $e^+e^- \to 2$ jets. One of these generalizations of q_T only applies to planar Born processes, such as $pp \to 2$ jets, but offers a dramatic simplification of the soft function. We also discuss how our approach can directly be extended to obtain predictions for the fragmentation of hadrons. This presents a promising path for high-precision QCD calculations with multijet final states.

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Introduction—The excellent performance of the LHC experiments has increased the need for precise theoretical predictions. A crucial challenge towards precision is the perturbative corrections due to quantum chromodynamics (QCD), particularly for final states involving jets. Here much progress has been made in recent years extending to $pp \to 3$ jets at second order in perturbation theory [1–3] (which was already used in experimental measurements [4]) and even third order for some processes involving jets, e.g., [5–10]. The highest precision is typically achieved by individual calculations, while automated procedures are available primarily for leading order (LO) and next-to-leading order (NLO). Next-to-next-to-leading order (NNLO) differential cross sections by contrast are only available for relatively simple processes through public codes [11–16].

An important bottleneck is the handling of the cancellation of infrared (IR) divergences between real and virtual diagrams, which is guaranteed by the KLN theorem [17,18] but challenging to arrange in practice. A range of approaches have been developed [19–34], which can be roughly subdivided into local subtraction methods and slicing methods. While local subtractions are numerically more stable, slicing methods are easier to extend to new processes (illustrated by, e.g., the quick succession of results using 1-jettiness discussed below). A key challenge for slicing is the choice of a suitable resolution variable. In this Letter we introduce two novel transverse-momentum-based slicing variables for multijet final states.

For color-singlet production, its transverse momentum $q_T = |\vec{q}_T|$ serves as an effective slicing variable [20]: the virtual contribution occurs at $q_T = 0$ while real radiation leads to $q_T > 0$ [35]. Slicing capitalizes on this, by splitting the cross section into pieces without (unres.) and including (res.) additional resolved emissions. As unresolved real emissions are soft or collinear, the former contribution can then be approximated to leading power (LP) in the slicing cutoff:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}X} = \int_0^\delta \mathrm{d}q_T \frac{\mathrm{d}\sigma_{\text{unres.}}}{\mathrm{d}X\mathrm{d}q_T} + \int_\delta^\infty \mathrm{d}q_T \frac{\mathrm{d}\sigma_{\text{res.}}}{\mathrm{d}X\mathrm{d}q_T}$$
$$= \int_0^\delta \mathrm{d}q_T \frac{\mathrm{d}\sigma_{\text{LP}}}{\mathrm{d}X\mathrm{d}q_T}[1 + \mathcal{O}(\delta^p)] + \int_\delta^\infty \mathrm{d}q_T \frac{\mathrm{d}\sigma_{\text{res.}}}{\mathrm{d}X\mathrm{d}q_T}. \quad (1)$$

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Typically one uses a factorization formula for the cross section for $q_T < \delta$, obtained, e.g., using soft-collinear effective theory (SCET) [37–41], to approximate the cross section and handle the cancellation of IR divergences. The contribution for $q_T > \delta$ reduces to a simpler calculation, trading an α_s from a complicated loop calculation for an α_s for a simpler resolved additional parton in the final state. In Eq. (1), X denotes kinematics of the color-singlet final-state (e.g., its rapidity). This factorization is not exact, as indicated by the $\mathcal{O}(\delta^p)$ power corrections. Thus $\delta \ll 1$ is required, which, however, leads to large cancellations between the two terms in Eq. (1), affecting the numerical stability.

For processes with jets, such as $pp \to Z + \text{jet}$ or $pp \to 2$ jets, q_T *without modification* is unsuitable as a slicing variable because radiation emitted inside a jet leaves $q_T = 0$. As an alternative, the N-jettiness variable [42] has been proposed [25,26], and successfully applied to processes involving a color-singlet plus one jet in the final state [43–49]. One challenge in extending this to multijet final states is the complicated form of the soft function at NNLO [50,51] that enters the factorized cross section σ_{LP}. For color-singlet processes, q_T performs better than 0-jettiness as slicing variable [52], motivating the search for an extension of q_T to processes with jets. In this context, k_T-ness was recently introduced [53,54], but implementing it at NNLO will be very challenging in the absence of a factorization formula.

In this Letter we propose two ways of extending q_T as slicing variable, to processes with jets, using $pp \to 2$ jets as example. The first extension has a very simple factorization, but is restricted to processes that are planar at LO, while the second can be used in general and converges faster, at the price of a more complicated soft function. The key ingredient is the use of a recoil-free jet axis, e.g., by employing the winner-take-all (WTA) recombination scheme [55,56]. This seemingly subtle change enables slicing because the jet momentum is now deflected by radiation inside the jet, leading to a nonvanishing q_T. The transverse momentum decorrelation q_T we consider here is the vector sum of the momenta of all identified color singlets and reconstructed jets in the event [57]. Note that we only need to use this different recombination scheme to obtain our slicing variable q_T, and that the jets can be defined in the standard way. As proof of concept, we demonstrate our new slicing at NLO for $pp \to 2$ jets and NNLO for $e^+e^- \to 2$ jets. We present factorization formulas that can readily be used to extend $pp \to$ multiple jets to NNLO, discuss the ingredients that enter in it, and calculate the NNLO collinear-soft function (discussed in the Supplemental Material [58]).

q_x with jets: The planar case—For planar Born processes, such as $pp \to H + \text{jet}$ or $pp \to 2$ jets, we can use the transverse momentum component q_x *perpendicular* to this scattering plane as slicing variable, or, equivalently, the azimuthal decorrelation $\delta\phi$, see Fig. 1. By using the WTA scheme, the jet axis is also affected by radiation inside the jet leading to a nonzero q_x (one of the components of $\vec{q}_T$), making it suitable as slicing variable.

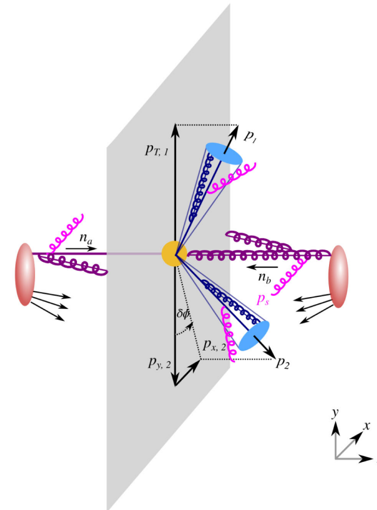

FIG. 1. The $pp \to 2$ jet process. By using the WTA scheme, the transverse momentum perpendicular to the scattering plane q_x (equal to $p_{x,2}$ in the above coordinates), or, equivalently, the azimuthal decorrelation $\delta\phi$, is a suitable slicing variable. The ingredients in the corresponding factorization are the hard scattering (yellow), collinear initial- (purple) and final-state (blue) radiation, and soft radiation (pink).

The key selling point of q_x as slicing variable is that we have a factorization formula with particularly simple ingredients. Denoting the transverse momenta and rapidities of the jets with $p_{T,1}, \eta_1, \eta_2$, and building on [70,71], the cross section for small q_x factorizes as

$$\frac{\mathrm{d}\sigma_{\text{LP}}}{\mathrm{d}p_{T,1}\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}q_x} = \int \frac{\mathrm{d}b_x}{2\pi} e^{\mathrm{i}q_x b_x} \sum_{i,j,k,\ell} B_i(x_a, b_x) B_j(x_b, b_x) \mathcal{J}_k(b_x) \mathcal{J}_\ell(b_x) \times \mathrm{tr}[\hat{\mathcal{H}}_{ij\to k\ell}(p_{T,1}, \eta_1 - \eta_2) \hat{S}_{ijk\ell}(b_x, \eta_1, \eta_2)]. \quad (2)$$

This involves the *standard* TMD beam functions $B_{i,j}$, whose matching onto parton distribution functions is well known at NNLO [72–79], a soft function that can directly be obtained [80] from the *standard* TMD soft function at NNLO [81,82] [83], and TMD jet functions that are already partially known at NNLO [85,86] and will soon be available fully [87,88] [89]. In particular, the analytic dependence of the soft function for this observable on the jet kinematics [see Eq. (8) of [80]], which enters through a boost in the scattering plane to make two of the beams or jets back to back, should be contrasted with the much more complicated kinematic dependence of the 2-jettiness soft function that requires numerical methods

[50,51]. The reason why the soft function is so simple in this case, is that the recoil from soft radiation is independent of the region of phase space it is emitted into. Note that one additional advantage of recoil-free schemes is that they typically remove [90] so-called nonglobal logarithms [91] from the soft function, which here simplifies the structure of the factorization theorem [71,92].

The hard function $\hat{\mathcal{H}}$ in Eq. (2) describes the hard partonic scattering process $ij \to k\ell$, which for $pp \to 2$ jets has been obtained at NNLO [93] from color-decomposed helicity amplitudes [94–98]. $\hat{\mathcal{H}}$ and $\hat{S}$ are matrices in color space (as indicated by the hat) and the trace is over color. The momentum fractions $x_{a,b}$ in the beam functions can be expressed in terms of the jet transverse momenta and rapidities. Beyond NLO, linearly polarized contributions to the beam and jet functions must be included [70,99].

As proof of concept, we show the result of using q_x as a slicing variable to obtain the NLO correction to dijet cross section $\delta\sigma^{\rm NLO}$ in Fig. 2. We reiterate that the WTA scheme is only used in the definition of q_x, and that the jets themselves are defined in the standard scheme. As is clear

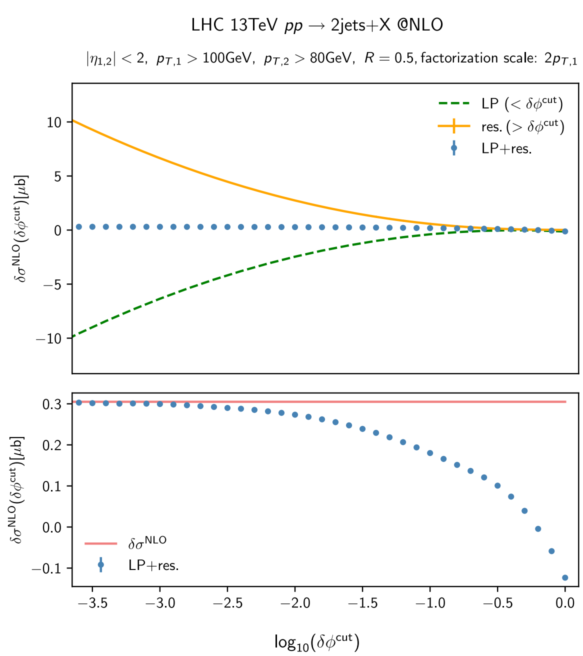

FIG. 2. In the lower panel the NLO correction $\delta\sigma^{\rm NLO}$ (blue dots) obtained using the slicing is plotted as a function of the cut on the azimuthal angle $\delta\phi^{\rm cut}$, showing that this converges for small $\delta\phi^{\rm cut}$ to the correct result (red line) obtained from NLOJET++ [100]. In the upper panel the individual terms (green dashed and yellow solid lines) in Eq. (1) are shown, of which the blue dots are the sum. Jets are defined using the anti-k_T algorithm [101] with radius $R = 0.5$, and subject to the following cuts on their transverse momentum and rapidity $p_{T,1} > 100$ GeV, $p_{T,2} > 80$ GeV, $|\eta_{1,2}| < 2$. The renormalization and factorization scales are set to $\mu_{R,F} = 2p_{T,1}$. Note the different ranges of the vertical axis for the two panels, and that the numerical uncertainties are smaller than the size of the markers.

from the bottom panel of Fig. 2, for small values of q_x this reproduces the NLO cross section. However, as the top panel indicates, there is also a large cancellation between the two terms. This highlights the main bottleneck of slicing (in general), namely, the need for numerically precise results for the $\mathrm{d}\sigma_{\rm res}$. term in Eq. (1).

q_T with jets: The planar case—For processes that are nonplanar at leading order, such as $pp \to Z + 2$ jets or $pp \to 3$ jets, the azimuthal decorrelation cannot be used anymore, but the total transverse momentum $q_T = |\vec{q}_T|$ of the jets (and color singlet, if present) is still viable as a slicing variable, when using the WTA scheme. This is because additional radiation beyond the LO process again lifts the momentum balance of the final state jet axes, but in general there is no preferred scattering plane. To study the effects of this change, we first analyze the planar case again, now with q_T as slicing variable instead of q_x. For dijet production, employing q_T modifies the factorization in Eq. (2) to

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\rm LP}}{\mathrm{d}p_{T,1}\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}q_T} &= q_T \int \frac{\mathrm{d}^2\vec{b}_T}{2\pi} J_0(q_T|\vec{b}_T|) \\ &\quad\times \sum_{i,j,k,\ell} B_i(x_a,\vec{b}_T)B_j(x_b,\vec{b}_T)\mathcal{J}_k(b_x) \\ &\quad\times \mathcal{J}_\ell(b_x)\mathrm{tr}[\hat{\mathcal{H}}_{ij\to k\ell}(p_{T,1},\eta_1-\eta_2) \\ &\quad\times \hat{S}_{ijk\ell}(\vec{b}_T,\eta_1,\eta_2,R)]. \end{aligned} \tag{3}$$

Though we have switched from one to two components of the transverse momentum, the only major change compared to Eq. (2) is in the soft function.

Soft radiation always contributes through momentum conservation, recoiling the collinear radiation. However, if it is emitted into the jet it also contributes through the magnitude of the jet's transverse momentum, which for the WTA scheme is given by the scalar sum. We showed in [71] that to leading power these two contributions cancel each other for the transverse momentum component *parallel* to the jet's transverse momentum, while the *perpendicular* component is unaffected. Thus outside the jet (where the soft radiation *only* contributes via recoil) soft radiation contributes its full transverse momentum to $\vec{q}_T$, while inside the jet (where both magnitude and recoil effects are present) only the component perpendicular to the jet axis does. The corresponding soft function knows about the jet, in particular its radius, making it more complex than the R-independent soft function for q_x. A small computational trick allows us to derive analytic instead of numerical results nevertheless: In the limit of small R, it refactorizes into simpler global and collinear-soft contributions [102], presented in the Supplemental Material [58]. The leading term in the small R limit is sufficient to establish the consistency of the factorization in terms of the anomalous dimensions, and in our numerical results we add finite

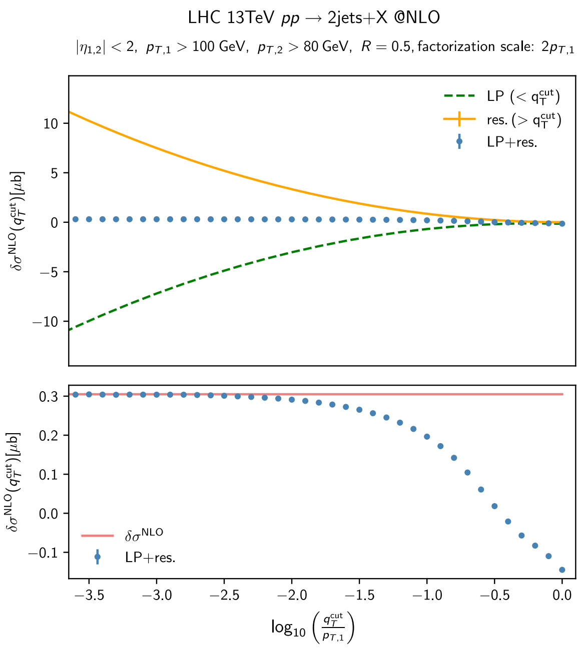

FIG. 3. Same as Fig. 2 but using instead a cut on the total transverse momentum $q_T^{\rm cut}$ (with the WTA scheme) for the slicing. This converges faster than $\delta\phi^{\rm cut}$ at the expense of a more complicated soft function, and can also be extended to nonplanar Born processes. Note again the different ranges of the vertical axis for the two panels.

terms up to $\mathcal{O}(R^4)$, achieving subpercent accuracy on the cross section for $R = 0.5$.

In Fig. 3 we show that using the total transverse momentum q_T of the two jets works well as a slicing variable. It converges faster than q_x shown in Fig. 2, but at the price of a more complicated soft function. This faster convergence is particularly clear when comparing the error on the slicing for small values of the slicing variable, shown in Fig. 4.

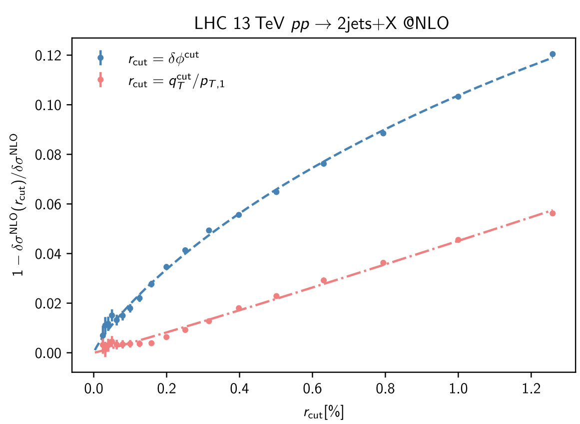

FIG. 4. A comparison of the precision of the slicing with $r_{\rm cut} = \delta\phi^{\rm cut}$ and $q_T^{\rm cut}/p_{T,1}$ for $pp \to 2$ jets with the same kinematics as in Figs. 2 and 3. Enlarging the region of small $r_{\rm cut}$, this clearly shows the faster convergence of q_T. The curves are obtained from fitting to $ar_{\rm cut} \ln r_{\rm cut} + br_{\rm cut}$.

q_T with jets: The nonplanar case—To illustrate that q_T can also be extended to the nonplanar case, we present the factorization formula for $pp \to 3$ jets:

$$
\begin{aligned}
&\frac{d\sigma_{\rm LP}^{pp\to 3\,{\rm jets}}}{dp_{T,1}dp_{T,2}d\eta_1 d\eta_2 d\eta_3 d\Phi dq_T} \\
&\quad = q_T \int \frac{d^2\vec{b}_T}{2\pi} J_0(q_T|\vec{b}_T|) \sum_{i,j,k,\ell,m} B_i(x_a,\vec{b}_T) B_j(x_b,\vec{b}_T) \\
&\qquad \times \mathcal{J}_k(b_{\perp,1}) \mathcal{J}_\ell(b_{\perp,2}) \mathcal{J}_m(b_{\perp,3}) \\
&\qquad \times {\rm tr}[\hat{\mathcal{H}}_{ij\to k\ell m}(\{p_{T,i}\},\Phi,\{\eta_i\}) \hat{S}_{ijk\ell m}(\vec{b}_T,\{\eta_i\},\Phi,R)],
\end{aligned}
\tag{4}
$$

which now also depends explicitly on the azimuthal angle Φ between two jets. We have verified the consistency of this factorization in terms of anomalous dimensions. The new challenge is that the jets are no longer in the same transverse plane, so the transverse momentum component perpendicular to the jet depends on the jet at hand. For the jet function this change is minor as only its argument is modified to $b_{\perp,i}$, which is the transverse component perpendicular to the ith jet direction $\hat{n}_{T,i}$, i.e. $\vec{b}_{\perp,i} = \vec{b}_T - \hat{n}_{T,i}(\hat{n}_{T,i} \cdot \vec{b}_T)$. In the soft function the modification is less trivial, particularly for the jet-jet dipoles, as discussed briefly in Supplemental Material [58].

Extension to NNLO—While one crucial ingredient for WTA-q_T slicing at hadron colliders is missing, namely, the

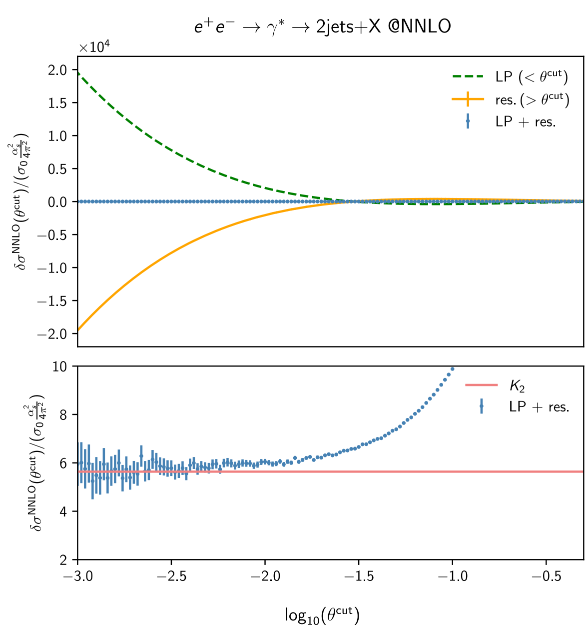

FIG. 5. In the lower panel the NNLO correction $\delta\sigma^{\rm NNLO}$ (blue dots with error bars) obtained using the slicing is plotted as a function of $\theta^{\rm cut}$. This converges for small $\theta^{\rm cut}$ to the correct result K_2 (red line), given in Eq. (S-17) of Supplemental Material [58]. In the upper panel the individual terms (green dashed and yellow solid curves) in Eq. (1) are shown, of which the blue dots are the sum.

gluon WTA jet function, all ingredients for an implementation at lepton colliders are available. Extending our slicing setup to calculate the inclusive dijet production cross section (for easy comparison to known results), we observe again the expected cancellation between LP and resolved emission contribution, as shown in Fig. 5. We relegate the details of this calculation (and in particular the differences to the hadronic setup) to Supplemental Material [58].

Extension to fragmentation—We can also use the same slicing variables based on transverse momentum to obtain predictions for the fragmentation of hadrons. For planar Born processes, the azimuthal decorrelation can be used again: One simply replaces one of the TMD jet functions in Eq. (2) with a TMD fragmentation function, whose matching onto fragmentation functions is known at NNLO [77–79,103]. [104]. If instead of q_x the full transverse momentum q_T is used, the soft function is not the same as in Eqs. (3) and (4), since there is no jet algorithm (or dependence on the jet radius) for hadrons.

Conclusions and outlook—In this Letter we have presented two generalizations of the total transverse momentum q_T, that are promising slicing variables for LHC processes with multiple jets. For planar Born processes, like $pp \to 2$ jets, the azimuthal decorrelation $\delta\phi$ can be used to handle the cancellation of IR divergences, while in general the total transverse momentum q_T can be employed. The key innovation that makes these suitable slicing variables is the use of a recoil-free axis, allowing us to also resolve emissions inside jets.

We demonstrated these slicing variables for $pp \to 2$ jets at NLO, and $e^+e^- \to 2$ jets at NNLO. We presented the factorization formulas that enable this for general processes with jets at NNLO. Most NNLO ingredients are already available, except for the constant in the gluon jet function, and the soft function for q_T. For $\delta\phi$ the NNLO soft function is known and clearly much simpler than for 2-jettiness. Though the power corrections are larger than for q_T, we expect that this simplicity should extend to the power corrections. For q_T, the NNLO soft function will be of comparable complexity as the N-jettiness soft function. However, we expect that the refactorization due to the expansion in R will lead to simplifications, and present the result for the NNLO collinear-soft function in Supplemental Material [58].

The factorization formulas for multijet final states can also be used as a baseline for studying factorization violating effects, where there has been significant interest in the transverse momentum of two back-to-back jets, see, e.g., Refs. [105,106].

Overall, this Letter outlines a promising pathway for achieving high-precision QCD calculations in multi-jet final states, paving the way for further advancements in theoretical and experimental high-energy physics.

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Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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