SUPPLEMENTAL MATERIAL

$\vec{q}_T$ soft function

The soft function in impact parameter space $\vec{b}_T$ is given by the following correlator

$$\begin{aligned}\hat{S}\big(\vec{b}_T,\eta_1,\eta_2,R\big) &= \sum\!\!\!\!\!\!\!\int_X \langle 0|\overline{\mathbf{T}}[\sum_n \Theta_n(X) O_s^\dagger(b_n)|X\rangle\langle X|\mathbf{T}[O_s(0)]|0\rangle \\ &= \hat{1} + \hat{S}^{(1)}\big(\vec{b}_T,\eta_1,\eta_2,R\big) + \mathcal{O}(\alpha_s^2)\end{aligned} \tag{S-1}$$

of soft Wilson lines S_{n_i} tracing the beams and jets

$$O_s(x^\mu) = \big[S_{n_2}S_{n_1}S_{n_b}S_{n_a}\big](x^\mu)\,. \tag{S-2}$$

The implicit dependence on the jet rapidities $\eta_{1,2}$ is encoded in the light-like vectors n_i along the beam ($i=a,b$) and jet ($i=1,2$) directions,

$$n_a^\mu = (1,0,0,1), \quad n_b^\mu = (1,0,0,-1), \quad n_1^\mu = (1,0,\operatorname{sech}\eta_1,\tanh\eta_1), \quad n_2^\mu = (1,0,-\operatorname{sech}\eta_2,\tanh\eta_2).$$

For definiteness, we choose the planar beam-jet system to align with the yz-plane. (We will comment on non-planar case below, and note that for each additional jets an extra Wilson line must be included.) The indices "$ijkl$" of the soft function appearing in the factorization formulae label the color representation $\mathbf{T}_i$ of the corresponding Wilson line. The beam Wilson lines S_{n_a} and S_{n_b} are defined as

$$S_{n_i}(x^\mu) = \mathbf{P}\exp\left[\mathrm{i}g\int_{-\infty}^0 \mathrm{d}t\; n_i\cdot A_s^c(x^\mu + t n_i^\mu)\,\mathbf{T}_i^c\right], \tag{S-3}$$

with corresponding expressions for the outgoing Wilson lines for the jets.

In Eq. (S-1), $\Theta_n(X)$ depends on the various possibilities of soft particles in X in/outside of the jet. At NLO, there is a single emission that is either in or outside the jet, with

$$\begin{aligned}\Theta_{\text{in}} &= \Theta(R-\Delta R_1) + \Theta(R-\Delta R_2) \\ &= \Theta\big(R - \sqrt{(\phi-\phi_1)^2 + (\eta-\eta_1)^2}\big) + \Theta\big(R - \sqrt{(\phi-\phi_2)^2+(\eta-\eta_2)^2}\big),\end{aligned} \tag{S-4}$$

where R represents the jet radius, and ϕ and η denote the azimuthal angle and pseudo-rapidity of the soft emission momentum, respectively. The azimuthal angles for the two jets are taken to be $\phi_1 = 0$ and $\phi_2 = \pi$. The out-of-jet constraint is complementary to the in-jet constraint: $\Theta_{\text{out}} = 1 - \Theta_{\text{in}}$. Thus at this order, $\sum_n \Theta_n(X) O_s^\dagger(b_n) = \Theta_{\text{in}}(X)O_s^\dagger(b_{\text{in}}) + \Theta_{\text{out}}(X) O_s^\dagger(b_{\text{out}})$, with

$$b_{\text{in}} = (0,b_x,0,0)\,, \qquad b_{\text{out}} = (0,b_x,b_y,0) = (0,\vec{b}_T,0)\,. \tag{S-5}$$

The peculiar form of b_{in} is due to the fact that only the component of the transverse momentum perpendicular to the jets, here q_x, is measured inside the jet. Below we will also employ a polar representation for $\vec{b}_T$ such that $b_x = b_T\sin\phi_b$ and $b_y = b_T\cos\phi_b$.

The explicit form of the next-to-leading order soft function is given by the sum over color dipoles:

$$\begin{aligned}\hat{S}^{(1)}\big(\vec{b}_T,\eta_1,\eta_2,R,\mu,\nu\big) &= \sum_{i<j}\big(-\mathbf{T}_i\cdot\mathbf{T}_j\big)\frac{\alpha_s(\mu)}{4\pi}\; S_{ij}^{(1)}\big(\vec{b}_T,\eta_1,\eta_2,R,\mu,\nu\big) \\ &= \sum_{i<j}\big(-\mathbf{T}_i\cdot\mathbf{T}_j\big)\frac{\alpha_s(\mu)\mu^{2\epsilon}\pi^\epsilon e^{\gamma_E\epsilon}}{\pi^2}\int \mathrm{d}^d k\,\delta(k^2)\theta(k^0)\left(\frac{\nu}{2k^0}\right)^\eta \frac{n_i\cdot n_j}{n_i\cdot k\; n_j\cdot k} \\ &\quad\times\big[\Theta_{\text{out}}e^{\mathrm{i}\vec{k}_T\cdot\vec{b}_T} + \Theta_{\text{in}}e^{\mathrm{i}k_x b_x}\big].\end{aligned} \tag{S-6}$$

Using the relation $\Theta_{\rm out} = 1 - \Theta_{\rm in}$, the soft function can now be split into three components: a global soft function, as well as a 2-dimensional and 1-dimensional collinear-soft function,

$$\begin{aligned}\hat{S}^{(1)}\big(\vec{b}_T,\eta_1,\eta_2,R,\mu,\nu\big) &= \sum_{i<j}(-\mathbf{T}_i\cdot\mathbf{T}_j)\frac{\alpha_s(\mu)\mu^{2\epsilon}\pi^{\epsilon}e^{\gamma_E\epsilon}}{\pi^2}\int \mathrm{d}^d k\,\delta(k^2)\theta(k^0)\frac{n_i\cdot n_j}{n_i\cdot k\;n_j\cdot k}\\ &\quad\times\left[\left(\frac{\nu}{2k^0}\right)^{\eta}e^{i\vec{k}_T\cdot\vec{b}_T}-\Theta_{\rm in}e^{i\vec{k}_T\cdot\vec{b}_T}+\left(\frac{\nu}{2k^0}\right)^{\eta}\Theta_{\rm in}e^{ik_xb_x}\right]\\ &= \sum_{i<j}(-\mathbf{T}_i\cdot\mathbf{T}_j)\frac{\alpha_s(\mu)}{4\pi}\;\Big[S_{ij}^{\rm global}(\vec{b}_T,\eta_1,\eta_2,\mu,\nu)+S_{ij}^{cs,2d}(\vec{b}_T,R,\mu)+S_{ij}^{cs,1d}(b_x,\eta_1,\eta_2,R,\mu,\nu)\Big]\,.\end{aligned} \tag{S-7}$$

Intuitively, the global soft function measures the out-of-jet measurement everywhere, the 2d collinear-soft function subtracts the incorrect $\vec{q}_T$ measurement inside the jets, and the 1d collinear-soft function adds the correct q_x measurement. Note that depending on the dipole under consideration the global soft function may not always require the rapidity regulator. The dependence of the two collinear-soft functions on the jet radius can be obtained analytically, by expanding around $R=0$. We need to be mindful of the fact that we assumed for the jet function that $R \gg q_T/Q$, which means we typically need to include the subleading dependence in R in either the soft function (or the jet function instead). This allows us to derive analytic formulae, and is a mere computational tool: Our aim is not to investigate any R-dependency in logarithms, for us it is just a parameter.

In this Letter we are only interested in the magnitude of the transverse momentum imbalance q_T between the two jets, so we average the soft function's components over the azimuthal angle ϕ_b of $\vec{b}_T$, which we indicate with a bar.

The global soft function (for the 2-component vector $\vec{b}_T$) is known [1, 2], we thus merely have to compute the two collinear-soft functions explicitly. After Fourier transforming from $\vec{b}_T$ space to $\vec{q}_T$ space and integrating q_T up to $q_T^{\rm cut}$, we obtain:

$$\begin{aligned}\hat{\bar{S}}^{(1)}(q_T^{\rm cut},\eta_1,\eta_2,R,\mu,\nu) &= \int \mathrm{d}^2\vec{q}_T\,\Theta(q_T^{\rm cut}-q_T)\int\frac{\mathrm{d}^2\vec{b}_T}{(2\pi)^2}\,e^{-i\vec{b}_T\cdot\vec{q}_T}\,\hat{S}^{(1)}\big(\vec{b}_T,\eta_1,\eta_2,R,\mu,\nu\big)\\ &= q_T^{\rm cut}\int_0^\infty \mathrm{d}b_T\,J_1(b_Tq_T^{\rm cut})\,\hat{\bar{S}}^{(1)}(b_T,\eta_1,\eta_2,R,\mu,\nu),\end{aligned} \tag{S-8}$$

where J_1 is a Bessel function.

Renormalizing to subtract the regulator poles yields the following finite result, expanded around the small-R limit:

$$\begin{aligned}\hat{\bar{S}}^{(1)}_{\rm finite}(q_T^{\rm cut},\eta_1,\eta_2,R,\mu,\nu) = \frac{\alpha_s(\mu)}{4\pi}\Bigg\{&-4L_\mu^2\sum_i\mathbf{T}_i^2+L_\mu\bigg[4\ln\frac{\mu^2}{\nu^2}\sum_i\mathbf{T}_i^2+8\sum_{i<j}\mathbf{T}_i\cdot\mathbf{T}_j\ln\frac{n_i\cdot n_j}{2}\\ &-8\ln2\;(\mathbf{T}_a+\mathbf{T}_b)\cdot(\mathbf{T}_1+\mathbf{T}_2)-16\ln2\;\mathbf{T}_1\cdot\mathbf{T}_2\bigg]-\frac{\pi^2}{6}\sum_i\mathbf{T}_i^2\\ &+\left[(\mathbf{T}_a+\mathbf{T}_b)\cdot(\mathbf{T}_1+\mathbf{T}_2)+2\;\mathbf{T}_1\cdot\mathbf{T}_2\right]\left(4\ln2\ln\frac{\mu^2}{\nu^2}+\frac{\pi^2}{3}+4\ln^2\frac{R}{2}\right)\\ &+\sum_{j\in\rm jets}(\mathbf{T}_a+\mathbf{T}_b)\cdot\mathbf{T}_j\;8\ln2\ln\left(2\cosh\eta_j\right)\\ &+\mathbf{T}_1\cdot\mathbf{T}_2\left[8\ln2\ln(4\cosh\eta_1\cosh\eta_2)-2\ln^2(2+2\cosh(\eta_1-\eta_2))+2(\eta_1-\eta_2)^2\right]\\ &-\sum_{i<j}\mathbf{T}_i\cdot\mathbf{T}_j\,S_{ij}^{\rm corr}(\eta_1,\eta_2,R)\Bigg\}\end{aligned} \tag{S-9}$$

where $L_\mu = \ln(\mu/q_T^{\rm cut})$ and the sum on i,j run over the ordered set $\{a,b,1,2\}$, with a,b the beams, and $1,2$ the jets. The subleading R-correction terms for the various dipoles are given by:

$$S_{ab}^{\rm corr}(\eta_1,\eta_2,R) = 4R^2\left(1-2\ln\frac{R}{2}\right)+\frac{R^4}{3}+\mathcal{O}(R^6)\,, \tag{S-10}$$

$$\begin{aligned}S_{a1\text{ or }b2}^{\rm corr}(\eta_1,\eta_2,R) =& -R^2\ln\frac{R}{2}\left[1+\frac{4e^{2\eta_2}}{(e^{\eta_1}+e^{\eta_2})^2}\right]+R^2\left[\frac{7}{6}+\frac{2e^{2\eta_2}}{(e^{\eta_1}+e^{\eta_2})^2}\right]\\ &+R^4\left[\frac{49}{1440}+\frac{e^{2\eta_2}\left(10e^{2\eta_1}+e^{2\eta_2}-4e^{\eta_1+\eta_2}\right)}{6(e^{\eta_1}+e^{\eta_2})^4}-\ln\frac{R}{2}\left(\frac{1}{72}+\frac{2e^{2(\eta_1+\eta_2)}}{(e^{\eta_1}+e^{\eta_2})^4}\right)\right]+\mathcal{O}(R^6)\,,\end{aligned} \tag{S-11}$$

$$S^{\text{corr}}_{a2\text{ or }b1}(\eta_1,\eta_2,R) = -R^2\ln\frac{R}{2}\left[1+\frac{4e^{2\eta_1}}{(e^{\eta_1}+e^{\eta_2})^2}\right] + R^2\left[\frac{7}{6}+\frac{2e^{2\eta_1}}{(e^{\eta_1}+e^{\eta_2})^2}\right]$$
$$+R^4\left[\frac{49}{1440}+\frac{e^{2\eta_1}\left(10e^{2\eta_2}+e^{2\eta_1}-4e^{\eta_1+\eta_2}\right)}{6(e^{\eta_1}+e^{\eta_2})^4} - \ln\frac{R}{2}\left(\frac{1}{72}+\frac{2e^{2(\eta_1+\eta_2)}}{(e^{\eta_1}+e^{\eta_2})^4}\right)\right] + \mathcal{O}(R^6)\,, \quad \text{(S-12)}$$

$$S^{\text{corr}}_{12}(\eta_1,\eta_2,R) = -2R^2\ln\frac{R}{2}\tanh^2\left(\frac{\eta_1-\eta_2}{2}\right) + R^2\left[\frac{7}{3}-\frac{6}{1+\cosh(\eta_1-\eta_2)}\right] \quad \text{(S-13)}$$
$$+R^4\left[\frac{49}{720}-\frac{e^{\eta_1+\eta_2}\left(3e^{2\eta_1}+3e^{2\eta_2}-8e^{\eta_1+\eta_2}\right)}{2(e^{\eta_1}+e^{\eta_2})^4} - \ln\left(\frac{R}{2}\right)\frac{\left(e^{2\eta_1}+e^{2\eta_2}-10e^{\eta_1+\eta_2}\right)^2}{36(e^{\eta_1}+e^{\eta_2})^4}\right] + \mathcal{O}(R^6)\,.$$

This soft function can be extended to the case in which the beams and jets do not form a common plane, or to cases with additional jets. In these situations some subtleties and changes arise, though the calculation of the various ingredients is only marginally modified at NLO:

- The leading beam-beam dipole contribution is unaffected, as its global soft contribution is insensitive to the jets, and no collinear-soft contribution is present.
- The beam-jet dipole global soft contributions are largely unaffected, as the impact of the azimuthal angle of the jet amounts to a simple rotation of the plane formed by beam and jet. This can be accounted for by shifting the azimuthal variable ϕ_b for the vector $\vec{b}_T$ accordingly. The same applies to the collinear-soft contribution.
- For a jet-jet dipole the collinear-soft contribution is the same as for a beam-jet dipole, just now for both jets, and can thus be easily adapted. The global soft contribution on the other hand picks up explicit dependence on the azimuthal separation of the two jets through the dipole kinematics, in addition to any potential global rotation accounted for by a shift in ϕ_b. This azimuthal separation dependence (which we assumed in this Letter to be equal to π) marginally modifies the appearing expressions, but does not add any difficulties.
- The subleading R-corrections in general exhibit a subtlety: They depend on the azimuthal orientation ϕ_b of the impact parameter $\vec{b}_T$, over which we average here, after which we expand in R. (Without expansion we were unable to find analytic expressions for the appearing terms.) This R-expansion however does not commute with the averaging, and more specifically does not commute with evaluations for certain special values of ϕ_b (with the R-expansion converging badly in their vicinity). If ϕ_b is not averaged the expansion should thus not be performed, which means the R-corrections must be evaluated numerically (the appearing integrals are either 1- or 2-dimensional).
- The R-corrections generally distinguish between active jets (those involved in the dipole, and thus already partially included via a collinear-soft function), and passive jets (those in the bulk). The beam-beam case only exhibits passive corrections, while for the jet-jet case the passive corrections can only be relevant if three or more jets are present. In general the passive jet corrections are more complicated than the active ones, as they encode more information about correlations between jets.

We have worked out the changes arising in non-planar and multi-jet situations, and while they are beyond the scope of this Letter, they can be made available upon request.

NNLO slicing in dijet production at lepton colliders

To demonstrate the effectiveness of our slicing variable at NNLO accuracy, we apply our q_T-slicing framework to the NNLO calculation of dijet production process in e^+e^- collisions. In this case, q_T is the momentum component of the subleading jet transverse to the leading jet (for which the factorization is equivalent to that of q_x in hadron collisions).

Since jets are clustered using the anti-k_T algorithm in hadron colliders, we adopt its generalization for electron-positron colliders, where the distances are defined as follows:

$$d_{ij} = \min(E_i^{-2}, E_j^{-2})\frac{1-\cos\theta_{ij}}{1-\cos R}, \qquad d_{iB} = E_i^{-2}. \quad \text{(S-14)}$$

For our calculation we choose $R = 0.5$, and the transverse momentum measurement is again based on a WTA recombination scheme to determine $q_T = |\vec{q}_T|$. The constant term of the WTA quark jet function at NNLO is taken from the calculation of [3], with the exception of the $C_F C_A$ color structure, for which the uncertainties from numerical integration are sizable. For this color structure we rely on the fit [4] using EVENT2 [5]. A fit of the total constant (i.e. summing over all color structures) was also presented in Ref. [6], but with larger uncertainties.

Following Ref. [4] we define our slicing variable θ as

$$\theta = \arctan\left(\frac{2q_T}{Q}\right) \approx \frac{2q_T}{Q}, \tag{S-15}$$

which is equal to π minus the angle between the leading and subleading jets.

The perturbative expansion for the inclusive total cross section σ of the $e^+e^- \to \gamma^* \to$ dijet process is given by:

$$\sigma = \sigma_0\left[1 + \frac{\alpha_s(Q)}{2\pi}K_1 + \left(\frac{\alpha_s(Q)}{2\pi}\right)^2 K_2 + \mathcal{O}(\alpha_s^3)\right], \tag{S-16}$$

with one- and two-loop coefficients [7, 8]

$$K_1 = 2, \quad K_2 = \frac{365}{6} - 44\zeta_3 + n_f\left(\frac{8}{3}\zeta_3 - \frac{11}{3}\right). \tag{S-17}$$

Here, σ_0 denotes the LO cross section, Q denotes the center-of-mass energy of the collision, and the number of (light) quark flavors is chosen as $n_f = 5$.

NNLO collinear-soft function

To highlight another advantage of our slicing scheme, namely its more readily available analytic control over relevant process parameters, we present the collinear-soft contribution at NNLO, for the full q_T, non-planar case. For a complete NNLO treatment of this slicing scheme this is on its own not sufficient, it must be combined with the global soft function at NNLO (which is independent of the jet recombination scheme and radius, and thus relevant to other processes, as well), the NNLO WTA jet functions, and subleading jet radius corrections in either the soft or jet sectors.

The explicit form of the bare collinear-soft function for the jet labeled by the final state parton j, taken to have azimuthal orientation $\phi = 0$, in the small-R limit and in two-dimensional Fourier space $\vec{b}_T$, is given by:

$$S_j^{cs}(b_\perp, b_-) = 1 + \frac{Z_\alpha \alpha_s}{4\pi} S_j^{\text{in}}(b_\perp) + \frac{Z_\alpha \alpha_s}{4\pi} S_j^{\text{out}}(b_-) + \left(\frac{Z_\alpha \alpha_s}{4\pi}\right)^2 S_j^{cs,(2)}(b_\perp, b_-) + \mathcal{O}(\alpha_s^3), \tag{S-18}$$

with $\alpha_s = \alpha_s(\mu)$, $b_\perp = b_x = b_T \sin\phi_b$ and $b_- = b \cdot n_j = -b_T \cos\phi_b \operatorname{sech}\eta_j$. Note that n_j is the full jet direction, unlike the normalized projection onto the transverse plane $\hat{n}_{T,j}$ in the main text. We use $\overline{\text{MS}}$ renormalization scheme,

$$g_{s,0}^2 = 4\pi\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \alpha_s Z_\alpha, \quad Z_\alpha = 1 - \frac{\alpha_s}{4\pi}\frac{\beta_0}{\epsilon} + \mathcal{O}\left(\alpha_s^2\right).$$

To simplify the variable dependence in the small R limit, we have suppressed in our notation non-critical variable dependence of the functions appearing here.

To prepare for the upcoming Non-Abelian Exponentiation (NAE) and the subsequent ϕ_b-averaging procedure, we first present the ϕ_b-dependent NLO quark collinear-soft function, retaining the positive ϵ and η coefficients,

$$S_j^{\text{in}}(b_\perp) = C_F\left(\frac{\mu|b_\perp|}{b_0}\right)^{2\epsilon}\left(\frac{\nu|b_\perp|R_j}{b_0}\right)^{\eta} h_F^{\text{in}}, \qquad S_j^{\text{out}}(b_-) = C_F\left(\frac{i\, b_-\mu}{b_0 R_j}\right)^{2\epsilon} h_F^{\text{out}}, \tag{S-19}$$

where $b_0 = 2e^{-\gamma_E}$, $R_j = R/(2\cosh\eta_j)$ and

$$h_F^{\text{in}} = \frac{2}{\epsilon^2} - \frac{4}{\eta\epsilon} - \frac{\pi^2}{6} + \left(-\frac{\eta}{\epsilon^3} + \frac{\pi^2\eta}{12\epsilon} - \frac{\zeta_3}{3}\eta - \frac{\pi^2\epsilon}{3\eta} - \frac{4\zeta_3}{3}\epsilon - \frac{17\pi^4}{1440}\eta\epsilon - \frac{4\zeta_3\epsilon^2}{3\eta} - \frac{3\pi^4}{80}\epsilon^2 - \frac{\pi^4\epsilon^3}{40\eta}\right), \tag{S-20}$$

$$h_F^{\text{out}} = -\frac{2}{\epsilon^2} - \frac{\pi^2}{2} + \left(-\frac{14\zeta_3}{3}\epsilon - \frac{7\pi^4}{48}\epsilon^2\right). \tag{S-21}$$

This function is the non-averaged version of the NLO collinear-soft function at the beginning of the supplemental material.

Since the difference between final state quarks and gluons is completely encoded in the color representation of the corresponding Wilson lines, the NLO gluon collinear-soft function can be obtained by replacing the color factor C_F with C_A in Eq. (S-19). To obtain the NNLO gluon collinear-soft function from Eq. (S-23) below, we will only need to change the color factors C_F^2 and $C_F C_A$ to C_A^2 and $C_F n_f T_F$ to $C_A n_f T_F$.

For slicing at NNLO we only require the ϕ_b-averaged collinear-soft function, which we organize as follows:

$$\bar{S}_j^{cs,(2)}(b_T) = \int_0^{2\pi} \frac{\mathrm{d}\phi_b}{2\pi} S_j^{cs,(2)}(b_\perp, b_-) \tag{S-22}$$

$$\begin{aligned}
\bar{S}_j^{cs,(2)}(b_T) \;&= \Big(\frac{\mu b_T}{b_0}\Big)^{4\epsilon} \Bigg[\Big(\frac{\nu b_T R_j}{b_0}\Big)^{\eta} C_F C_A \bar{v}_A^{\rm in} + R^{-4\epsilon} C_F C_A \bar{v}_A^{\rm out} \\
&+ \Big(\frac{\nu b_T R_j}{b_0}\Big)^{\eta} \Big[\Big(\frac{\nu b_T R_j}{b_0}\Big)^{\eta} C_F^2 \bar{h}_{2F} + C_F C_A \bar{h}_A + C_F n_f T_F \bar{h}_f \Big] \\
&+ \Big(\frac{\nu b_T R_j}{b_0}\Big)^{\eta} R^{-2\epsilon} C_F^2 \bar{p}_{2F} + R^{-4\epsilon} (C_F C_A \bar{p}_A + C_F n_f T_F \bar{p}_f) + R^{-2\epsilon} C_F C_A \bar{p}_{\rm NGL} \\
&+ R^{-4\epsilon} \big(C_F^2 \bar{g}_{2F} + C_F C_A \bar{g}_A + C_F n_f T_F \bar{g}_f \big) \Bigg],
\end{aligned} \tag{S-23}$$

The first line of Eq. (S-23) represents the bare real-virtual correction, while the remaining three lines correspond to double real emissions: The second line describes the case where both emissions are inside the jet cone labeled by j. The third and fourth lines correspond to configurations with one radiation inside and one outside the jet cone, and with both emissions outside the jet cone, respectively. Note in particular that the dependence on R and η_j is fully analytic, the various auxiliary functions introduced here only contain numbers and regulator powers.

The real-virtual contributions are of NLO form:

$$\begin{aligned}
\bar{v}_A^{\rm in} &= -\frac{1}{\epsilon^4} + \frac{4}{\eta\epsilon^3} - \frac{16\ln 2}{\eta\epsilon^2} + \frac{\frac{7\pi^2}{6} + 8\ln^2 2}{\epsilon^2} + \frac{2\left(\pi^2 + 16\ln^2 2\right)}{\eta\epsilon} - \frac{2\left(10\pi^2\ln 2 + 32\ln^3 2 + 34\zeta_3\right)}{3\epsilon} \\
&\quad - \frac{4\left(6\pi^2\ln 2 + 32\ln^3 2 + 46\zeta_3\right)}{3\eta} + \frac{1009\pi^4}{360} + \frac{52}{3}\pi^2\ln^2 2 + 32\ln^4 2 + 152\zeta_3\ln 2
\end{aligned} \tag{S-24}$$

$$\bar{v}_A^{\rm out} = \frac{1}{\epsilon^4} - \frac{\pi^2}{2\epsilon^2} + \frac{8\zeta_3}{3\epsilon} + \frac{\pi^4}{120} \tag{S-25}$$

The C_F^2 terms arise from uncorrelated double emissions, which can be derived directly from the NLO collinear-soft function via Non-Abelian Exponentiation (NAE), followed by averaging over ϕ_b,

$$\begin{aligned}
\bar{h}_{2F} &= \frac{6}{\epsilon^4} - \frac{8\ln 2}{\epsilon^3} - \frac{\pi^2}{3\epsilon^2} + \frac{4\pi^2\ln 2}{3\epsilon} + \frac{1}{\eta^2}\left(\frac{8}{\epsilon^2} - \frac{32\ln 2}{\epsilon} + 64\ln^2 2 + \frac{20\pi^2}{3}\right) \\
&\quad + \frac{1}{\eta}\left[-\frac{8}{\epsilon^3} + \frac{16\ln 2}{\epsilon^2} - \frac{4}{3}\left(10\pi^2\ln 2 + 32\ln^3 2 + 46\zeta_3\right)\right] + \frac{301\pi^4}{180} + \frac{32\pi^2\ln^2 2}{3} + \frac{4}{3}\left(16\ln^4 2 + 92\zeta_3\ln 2\right),
\end{aligned} \tag{S-26}$$

$$\begin{aligned}
\bar{p}_{2F} &= -\frac{4}{\epsilon^4} + \frac{4\left(\pi^2 + 6\ln^2 2\right)}{3\epsilon^2} - \frac{8\left(\pi^2\ln 2 + 4\ln^3 2 + 4\zeta_3\right)}{3\epsilon} \\
&\quad + \frac{1}{\eta}\left[\frac{8}{\epsilon^3} - \frac{16\ln 2}{\epsilon^2} + \frac{16\ln^2 2}{\epsilon} + \frac{16}{3}\left(-2\ln^3 2 + \zeta_3\right)\right] + \frac{16\pi^4}{45} + \frac{8}{3}\pi^2\ln^2 2 + 8\left(\ln^4 2 + 2\zeta_3\ln 2\right),
\end{aligned} \tag{S-27}$$

$$\bar{g}_{2F} = \frac{2}{\epsilon^4} - \frac{5\pi^2}{3\epsilon^2} - \frac{68\zeta_3}{3\epsilon} - \frac{101\pi^4}{180}. \tag{S-28}$$

We adopt the parametrization of Ref. [9] to compute the correlated double emissions pole by pole, and derive the following results for double in-cone radiation:

$$\begin{aligned}\bar{h}_A =& \frac{1}{\epsilon^4} + \frac{11}{6\epsilon^3} + \left(\frac{67}{18} - \frac{5\pi^2}{3} - 8\ln^2 2\right)\frac{1}{\epsilon^2} + \left(\frac{211}{27} - \frac{121\pi^2}{36} + 8\pi^2\ln 2 - \frac{44\ln^2 2}{3} + \frac{64\ln^3 2}{3} + \frac{35\zeta_3}{3}\right)\frac{1}{\epsilon} \\ &+ \left[-\frac{4}{\epsilon^3} + \left(-\frac{22}{3} + 16\ln 2\right)\frac{1}{\epsilon^2} + \left(-\frac{134}{9} - \frac{4\pi^2}{3} + \frac{88\ln 2}{3} - 32\ln^2 2\right)\frac{1}{\epsilon}\right. \\ &+ \left.\left(-\frac{808}{27} - \frac{55\pi^2}{9} + \frac{536\ln 2}{9} + \frac{16\pi^2\ln 2}{3} - \frac{176\ln^2 2}{3} + \frac{128\ln^3 2}{3} + \frac{268\zeta_3}{3}\right)\right]\frac{1}{\eta} - 365.293(48), \qquad \text{(S-29)} \\ \bar{h}_f =& -\frac{2}{3\epsilon^3} - \frac{10}{9\epsilon^2} + \left(-\frac{74}{27} + \frac{11\pi^2}{9} + \frac{16\ln^2 2}{3}\right)\frac{1}{\epsilon} + \left[\frac{8}{3\epsilon^2} + \left(\frac{40}{9} - \frac{32\ln 2}{3}\right)\frac{1}{\epsilon}\right. \\ &\left.\left(\frac{224}{27} + \frac{20\pi^2}{9} - \frac{160\ln 2}{9} + \frac{64\ln^2 2}{3}\right)\right]\frac{1}{\eta} - 40.8677(14) \qquad \text{(S-30)}\end{aligned}$$

The regulator poles in the equations above can be derived using consistency relations originating from anomalous dimensions, which we have used to replace our (compatible) numerical results with the corresponding analytic results. The finite matching correction is not subject to the consistency relations and thus remains numerical.

We can obtain the double out-cone results by transforming a same-hemisphere contribution of the NNLO hemisphere soft function in [10] from momentum space to Fourier $\vec{b}_T$ space,

$$\bar{g}_A = -\frac{1}{\epsilon^4} - \frac{11}{6\epsilon^3} + \frac{-67+6\pi^2}{18\epsilon^2} + \frac{-47840 - 2010\pi^2 + 1233\pi^4 - 75240\zeta_3}{3240} + \frac{-772 - 33\pi^2 + 468\zeta_3}{108\epsilon} \qquad \text{(S-31)}$$

$$\bar{g}_f = \frac{2}{3\epsilon^3} + \frac{10}{9\epsilon^2} + \frac{38+3\pi^2}{27\epsilon} + \frac{238 + 15\pi^2 + 684\zeta_3}{81}. \qquad \text{(S-32)}$$

The contribution from correlated one-in/one-out emissions can be computed using either a method-of-regions approach or a dedicated EFT treatment involving a coft mode, and yields

$$\bar{p}_f = \frac{4(3-2\pi^2)}{9\epsilon} - \frac{68}{9} + \frac{64\pi^2}{27} - \frac{16\zeta_3}{3}, \qquad \text{(S-33)}$$

$$\bar{p}_A = \left(-\frac{2}{3} + \frac{22\pi^2}{9} - 4\zeta_3\right)\frac{1}{\epsilon} + \frac{40}{9} - \frac{134\pi^2}{27} + \frac{44\zeta_3}{3} + \frac{8\pi^4}{45}, \qquad \text{(S-34)}$$

$$\bar{p}_{\rm NGL} = \frac{2\pi^2}{3\epsilon^2} + \left(8\zeta_3 - \frac{4}{3}\pi^2\ln 2\right)\frac{1}{\epsilon} + \left(\frac{13\pi^4}{45} + \frac{4}{3}\pi^2\ln^2 2 - 16\zeta_3\ln 2\right) \qquad \text{(S-35)}$$

The function $\bar{p}_{\rm NGL}$ here is an effective object that is useful solely for a slicing application. When we analyze our setup in a Method-of-Regions setting, $\bar{p}_{\rm NGL}$ encodes the contribution from a soft emission inside the jet, together with an emission outside the jet whose energy scale is suppressed by a power of R compared to the in-jet emission. In an EFT treatment [11–13] these emissions would be described by two separate field modes: A collinear-soft mode with standard soft virtuality, and a coft mode with similar rapidity but additional virtuality suppression[1]. In EFT-based factorization different field modes are modeled by different ingredient functions, our function $\bar{p}_{\rm NGL}$ is thus a simplified expression: It consists of the product of NLO collinear-soft and coft functions in the non-global factorization, fully integrated over the cross-talk between these functions. This is sufficient for fixed-order purposes, but would not be adequate if we wanted to resum the corresponding logarithms.

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[1] There would then also be a third mode, a global soft mode with the same virtuality as the collinear-soft mode, but isotropic in terms of rapidity, which neither the collinear-soft nor coft modes are. This mode then gives rise to the global soft function.

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