Hybrid resonators for light trapping and emission control

Doeleman, H.M.

Citation for published version (APA):
In this chapter, we derive a simple, fully classical and intuitive model to describe a cavity-antenna hybrid system, based on coupled oscillators. The coupled equations of motion are derived, and we study local density of states in these systems by including a dipolar constant current source. This model is independent of cavity and antenna geometry, and provides a unified framework through which we can understand various physical effects in these systems, including cavity perturbation, enhancements or suppression of the local density of states and observables in taper-coupled measurements of the cavity. Moreover, the model serves as the basis for the theoretical results and interpretation of experimental data discussed in Chapters 3, 4, 6 and 7.
2.1 Introduction

The major part of this thesis is devoted to the study of hybrid cavity-antenna systems. Both optical cavities and plasmonic antennas have become ubiquitous instruments in the manipulation of light-matter interaction [7, 117]. This interaction strength, which can be quantified as an increased local density of states (LDOS), depends on the photon storage time and confinement volume. While optical cavities have developed increasingly sophisticated techniques to reach extremely long photon storage times (up to $8 \times 10^9$ oscillations [38]), plasmonic particles instead achieve strong interactions by concentrating the light to volumes far below the diffraction limit. Hybrid cavity-antenna systems (see Fig. 2.1) have been proposed recently as a platform in which to combine the favourable properties of cavities and antennas [104, 105, 110]. If indeed a symbiotic relationship between a cavity and an antenna could be established, this could have large implications for applications including single-photon sources for quantum information processing [29, 31], optical particle sensing [26] and nano-scale lasers [25]. Therefore, important questions when studying these systems are: Can hybrid systems support stronger light-matter interactions (higher LDOS) than their constituents? If so, what are the requirements for this symbiosis? Could we design a system with high LDOS at any desired bandwidth of operation (something which is not possible with plasmonics or cavities alone)? High LDOS means an increase of emission rate, so where does this emitted light go? Are there fundamental limits to LDOS in these systems?

Early theoretical works on hybrid systems have shown that LDOS as well as figures of merit for particle sensing or trapping can indeed be increased by combining antennas with cavities [87, 93, 104, 105, 118]. While this shows the promise of these structures, these studies have focused on particular cavity or antenna geometries and are thus unable to explain the general physical phenomena underlying this symbiosis. Another problem is that most studies have thus far used finite-element simulations, which are particularly challenging and time-consuming for hybrids due to the widely different element and domain sizes required for cavities and plasmonics.

In this chapter, we introduce a simple, intuitive coupled-oscillator model which is applicable to any cavity-antenna system. The only assumptions are that the antenna is dipolar and that there is no radiation overlap between cavity and antenna. Despite its simplicity, this model captures the essential physics of cavity-antenna interaction, which allows us to answer the questions above. In Section 2.2, we derive from first principles the coupled equations of motion for the hybrid. These equations can be solved to find the hybridized eigenmodes of the system, which is done in Section 2.3. To study LDOS, in Section 2.4 we let the system be driven by a fixed-current source dipole placed in close proximity to the antenna. This classical source models a fluorescent emitter in the weak-coupling limit, and the LDOS experienced by the emitter can be obtained from the power radiated by the source. Finally, in Section 2.5
2.1 Introduction

Figure 2.1: Example of a hybrid cavity-antenna system. The cavity is represented by a disk supporting a high quality factor whispering-gallery mode (WGM) shown in the cut-out, and the antenna by a gold ellipsoid. LDOS effects can be studied through the emission of a drive dipole (modelling a fluorescent emitter) placed in the system.

We derive expressions for the experimental observables in the specific case of antennas coupled to a degenerate pair of counter-propagating whispering-gallery modes, driven not by an emitter but through a waveguide coupled to the cavity modes.

This chapter provides a didactical introduction to the physics of coupled antenna-cavity systems, particularly suited for those who have never studied such systems before. It provides a single framework that connects many different facets of these systems, including mode hybridization, LDOS effects, radiative or collection efficiency, temporal coupled-mode theory for waveguide-coupled cavities, multiple-scattering theory and the famous Bethe-Schwinger cavity perturbation theory. The more experienced reader may find many things to be familiar, and may treat this chapter as a reference for the results presented in subsequent chapters, since this chapter lays the foundations for the results and interpretations presented in Chapters 3, 4, 6 and 7. The model for LDOS derived in Section 2.4 is used in Chapter 3, where we discuss the resulting LDOS spectra and provide answers to the questions above, for example under what conditions one can optimally harness the strengths of optical cavities and plasmonics. Also Chapter 4 builds upon this model to construct an equivalent electrical circuit that describes an emitter in a hybrid system, which can be used to derive a fundamental limit on the radiated power. We employ the results from Section 2.5 to compare to experimental resonance shifts and linewidths in Chapter 6, and in Chapter 7 we use the LDOS results from Section 2.4 to explain our experimental fluorescence spectra.
2.2 Equations of motion for a cavity-antenna system

Here we derive the equations of motion for a coupled cavity-antenna system. We model the antenna as a point dipole with the familiar Lorentzian polarizability, as found e.g. for the Fröhlich mode of a small metal sphere [119] in vacuum. Radiation damping is included to make the model self-consistent and adhering to the optical theorem [120]. Interaction with the cavity mode is explicitly separated out from this radiation damping due to other modes, and included in a second equation of motion. This equation of motion, describing a single cavity mode, is based on temporal coupled mode theory. Its derivation is analogous to deriving the classical equation of motion for an atom in a cavity, as given by Haroche [121], where in our work the ‘atom’ will be representing the antenna. This approach requires no assumptions on the type of cavity or antenna, other than that the antenna is dipolar and that there is no radiation overlap between cavity and antenna. Throughout this derivation, all quantities are in SI units.

2.2.1 A dipolar antenna

We consider a system of a small nano-antenna positioned in the field of a cavity at position \( r_0 \). The antenna is described as a point dipole with dipole moment \( \mathbf{p} = p \hat{\mathbf{p}} \), where \( \hat{\mathbf{p}} \) is the unit vector pointing along \( \mathbf{p} \), and we assume for simplicity that it is only polarizable along \( \hat{\mathbf{p}} \) [122]. This analysis can be easily extended to a tensor polarizability.

The antenna response is modelled as a harmonic oscillator of charge \( q \) and mass \( m \) with resonance frequency \( \omega_0 \) that suffers from intrinsic damping due to Ohmic heating described by an energy damping rate \( \gamma_i \). The equation of motion (EOM) that governs the time dependence of the (complex) dipole amplitude \( p(t) \) is that of a damped, driven harmonic oscillator:

\[
\ddot{p} + \gamma_i \dot{p} + \omega_0^2 p = \beta E,
\]  

(2.1)

where \( \beta = q^2/m \) is the oscillator strength and \( E = \mathbf{E}(r_0) \cdot \hat{\mathbf{p}} \) is the total electric field \( \mathbf{E} \) present at the antenna position, projected on \( \hat{\mathbf{p}} \). While in a Drude model for a metal sphere of volume \( V_{\text{ant}} \) in vacuum, \( \beta \) simply reads \( 3V_{\text{ant}} \epsilon_0 \omega_0^2 \), in general it may be found for any antenna by polarizability tensor retrieval from a full wave simulation [123, 124]. We now separate \( E \) in three contributions:

\[
E = E_c + E_p + E_{\text{ext}}.
\]  

(2.2)

Here, \( E_c \) is the field of the cavity mode of interest at \( r_0 \), along the dipole direction. The second term is the field at the antenna, caused by the antenna
2.2 Equations of motion for a cavity-antenna system

It can be formally written as

\[ E_p(t) = \int_{-\infty}^{\infty} dt' \mathcal{G}_{bg}(t-t')p(t') = (\mathcal{G}_{bg} \ast p)(t), \]  

(2.3)

where \( \mathcal{G}_{bg}(t-t') \) is a linear response function that describes the field at the position of the antenna at time \( t \) due to a delta function excitation at time \( t' \). Choosing \( \mathcal{G}_{bg}(t-t') \) as the retarded Green’s function ensures that the integrand is zero for \( t' > t \), thus respecting causality [125]. Its Fourier transform is \( \mathcal{G}_{bg}(\mathbf{r}_0, \mathbf{r}_0, \omega) \equiv \hat{\mathbf{p}} \cdot \mathbf{	ilde{G}}_{bg}(\mathbf{r}_0, \mathbf{r}_0, \omega) \cdot \hat{\mathbf{p}} \) — the projection along the antenna direction of the Green’s tensor \( \mathbf{G}_{bg} \) that describes the field \( E_p \) of the antenna in its environment via \( E_p(\mathbf{r}, \omega) = \mathbf{	ilde{G}}_{bg}(\mathbf{r}, \mathbf{r}_0, \omega) \cdot p(\omega) \).

Importantly, we need to explicitly omit the contribution of the cavity field in this response, since that will be accounted for in the next section. Instead, it is composed of the antenna radiation expanded in all modes except the cavity mode. It is for that reason that we use the subscript ‘bg’ to mean that only the dielectric ‘background’ contributes to \( \mathcal{G}_{bg} \). This dielectric background can in principle be inhomogeneous, and as such the response can be altered from that in a homogeneous medium due to the excitation of for example modes in a substrate or other cavity modes. If those contributions are negligible, the well-known expression for the Abraham-Lorentz force in a homogeneous medium can be used such that*

\[ E_p = \frac{\sqrt{\epsilon} \hat{p}}{6\pi\epsilon_0 c^3}, \]  

(2.4)

where \( \epsilon = \epsilon(r_0) \) is the relative permittivity of the medium, \( \epsilon_0 \) the vacuum permittivity and \( c \) the speed of light [12].

The final term \( E_{ext} \) in Eq. (2.2) is the external driving field, i.e. the electric field at the position of the antenna which does not find its origin in the antenna itself, and is distributed over other modes than the cavity mode. This can for example be the field due to an oscillating source dipole or an incident plane wave.

2.2.2 A cavity

Next, we seek to find a similar equation of motion for the cavity field \( E_c \). First, we must assume that the field can be expanded in orthogonal modes \( E_m \), of which the cavity mode \( E_c \) is just one. This is a standard approach to describe the physics of high-Q cavities in quantum optics. We note that for very open systems, there is currently a strong debate about quasi-normal modes appropriate for non-hermitian systems [62–64, 126]. We note that generalization of our formalism to deal with quasi-normal modes is outside the scope of this

*Note that this describes the imaginary part of \( E_p \), which governs e.g. radiated power. The real part of \( E_p \), which diverges in a homogeneous medium, is typically ignored [122].
A coupled-oscillator model for cavity-antenna systems

thesis. Such a generalization would also require to revisit the definitions of mode normalization, inner product, and energy density.

The assumed orthogonal eigenmodes satisfy the wave equation without sources, \[ \nabla \times \nabla \times E_m(r, t) + \frac{\epsilon(r)}{c^2} \frac{\partial^2}{\partial t^2} E_m(r, t) = 0, \] (2.5)

and can be factorized as

\[ E_m(r, t) = a_m(t) e_m(r), \] (2.6)

such that \( e_m \) satisfies both

\[ \nabla \times \nabla \times e_m(r) - \epsilon(r) \frac{\omega_m^2}{c^2} e_m(r) = 0 \] (2.7)

and

\[ \int \text{d}r \frac{1}{2} \epsilon_0 \epsilon(r) e_m^*(r) \cdot e_n(r) = \delta_{mn}, \] (2.8)

where we have assumed harmonic time dependence \( e^{-i\omega t} \) of the modes with eigenfrequencies \( \omega_m \).

The time dependence of each mode is captured in the (complex) mode amplitude \( a_m \), and the orthonormality condition defined in Eq. (2.8) ensures that

\[ |a_m|^2 = \int \text{d}r \frac{1}{2} \epsilon_0 \epsilon(r) |E_m(r, t)|^2 = U_m \] (2.9)

is the total energy in mode \( m \).

The total field is

\[ E(r, t) = \sum_m a_m(t) e_m(r) \] (2.10)

and hence

\[ a_m = \int \text{d}r \frac{1}{2} \epsilon_0 \epsilon(r) E(r, t) \cdot e_m^*(r). \] (2.11)

At this point we may introduce the antenna (or any other dipole) by including it as a dipolar 'source' in the wave equation. Note that the antenna is not actually a source but instead a polarizable dipolar particle which does not produce energy. In contrast, a real source like e.g. a fluorescent molecule is modelled as a dipole oscillating at a fixed amplitude, as we will discuss in Section 2.4.1. Nevertheless, in the wave equation both antenna and source are inserted in the same position. The total field then obeys the wave equation including this dipolar 'source' term

\[ \nabla \times \nabla \times E(r, t) + \frac{\epsilon(r)}{c^2} \frac{\partial^2}{\partial t^2} E(r, t) = -\frac{1}{\epsilon_0 c^2} \frac{\partial^2}{\partial t^2} P(r, t), \] (2.12)
where we will write the polarization by the antenna:

\[
P(r, t) = p(t) \delta(r - r_0) \hat{p}.
\] (2.13)

Starting with the wave equation (Eq. (2.12)), taking the product of both sides with \( \frac{1}{2} \epsilon_0 e_c^*(r) \) (where \( e_c(r) \) is the field profile of the cavity mode), making use of Eq. (2.6) and Eq. (2.7), and finally integrating over all space yields

\[
\sum_m \int dr \frac{1}{2c^2} \epsilon_0(r) e_m(r) \cdot e_c^*(r) (\omega_m^2 a_m + \dot{a}_m) = - \int dr \frac{\delta(r - r_0)}{2c^2} \dot{\hat{p}} \cdot e_c^*(r) \hat{p},
\] (2.14)

which results in

\[
\ddot{a} + \omega_c^2 a = - \frac{1}{2} \dot{\hat{p}} \cdot e_c^*(r_0) \hat{p},
\] (2.15)

where, for simplicity, we have replaced the mode amplitude \( a_c \) by the symbol \( a \). Eq. (2.15) represent a lossless cavity. We can introduce a phenomenological damping rate \( \kappa \) that describes cavity losses (excluding those related to the dipole), as well as a driving term \([127]\), to arrive at

\[
\ddot{a} + \kappa \dot{a} + \omega_c^2 a = - \frac{1}{2} \dot{\hat{p}} \cdot e_c^*(r_0) \hat{p} + 2 \sqrt{\kappa_{ex}} s_{in}.
\] (2.16)

Here, \( \kappa = \kappa_i + \kappa_{ex} \) includes both an intrinsic loss rate \( \kappa_i \) and coupling losses \( \kappa_{ex} \) due to coupling to the feeding channel (a waveguide, for example). This assumes ideal coupling, i.e. the waveguide does not induce any other losses to the cavity than those due to coupling to the input-output waveguide mode \([128, 129]\). It is important to realize that we can only include damping in this manner if we explicitly assume that there is no radiation overlap between the bare cavity and antenna. If that would be the case, neither cavity nor antenna loss rate could be assumed to be constants, as interference would make both depend on each other and on \( a \) and \( p \) \([130]\). The last term on the right hand side describes driving through the waveguide coupled to the cavity mode with a coupling rate \( \kappa_{ex} \). We normalize \( s_{in} \) such that \( |s_{in}| \) is the input power in this channel. In cavity literature, a different version of Eq. (2.16) is often used, namely \([127, 131, 132]\)

\[
\dot{a}(t) = \left(i \Delta - \frac{\kappa}{2}\right) a(t) + \sqrt{\kappa_{ex}} s_{in}(t),
\] (2.17)

with \( \Delta = \omega - \omega_c \). The difference, beside the absence of the antenna term, stems from the fact that we have not made the slowly varying envelope approximation. This assumes that cavity and driving oscillate at some carrier frequency \( \omega \), and amplitudes vary much more slowly than \( \omega \), which typically holds for narrowband driving. It can be easily verified that this approximation leads to the correct expression. For this, one transforms to a rotating frame by
A coupled-oscillator model for cavity-antenna systems

writing \( a(t) = e^{-i\omega t} \tilde{a}(t) \), \( p(t) = e^{-i\omega t} \tilde{p}(t) \) and \( s_{\text{in}}(t) = e^{-i\omega t} \tilde{s}_{\text{in}}(t) \), with \( \omega \)
the carrier frequency and \( \tilde{a}(t) \), \( \tilde{p}(t) \) and \( \tilde{s}_{\text{in}}(t) \) the envelope functions, which
are assumed to vary slowly such that \( \dot{\tilde{a}}(t) \ll \omega \tilde{a}(t) \) (and similarly for \( \tilde{p}(t) \) and \( \tilde{s}_{\text{in}}(t) \)). Furthermore assuming that we have a good cavity (\( \kappa \ll \omega_c \)) and
evaluating near the cavity resonance frequency (\( |\Delta| \ll \omega_c \)), we can rewrite
Eq. (2.16) as

\[
\dot{\tilde{a}}(t) = \left( i\Delta - \frac{\kappa}{2} \right) \tilde{a}(t) + \frac{i\omega^2}{4} \hat{\mathbf{p}} \cdot \mathbf{e}_c^*(r_0) \tilde{p}(t) + \sqrt{\kappa_{\text{ex}} \kappa_{\text{in}}} \tilde{s}_{\text{in}}(t) \tag{2.18}
\]

Except for the antenna term, which is new, this matches Eq. (2.17) exactly.

2.2.3 Equations of motion in the Fourier domain

The obtained equations of motion Eqs. (2.1) and (2.16), are most easily solved
in the Fourier domain, where they result in\(^1\)

\[
\left( \omega_0^2 - \omega^2 - i\omega \gamma_i - \beta G_{bg}(r_0, r_0, \omega) \right) p - \beta \hat{\mathbf{p}} \cdot \mathbf{e}_c(r_0) a = \beta E_{\text{ext}}, \tag{2.19}
\]

\[
\left( \omega_c^2 - \omega^2 - i\omega \kappa \right) \mathbf{p} - \frac{\omega_c^2}{2} \hat{\mathbf{p}} \cdot \mathbf{e}_c^*(r_0) p = -2i\omega \sqrt{\kappa_{\text{ex}} \kappa_{\text{in}}}. \tag{2.20}
\]

Note that \( p, a, s_{\text{in}} \) and \( E_{\text{ext}} \) now represent the Fourier transforms of the corre-
sponding time-dependent quantities in Eqs. (2.1) and (2.16). We may absorb
the real part of \( G_{bg}(r_0, r_0, \omega) \) in \( \omega_0 \) and the imaginary part in the total antenna
damping rate \( \gamma \), such that \(^{[120]}\)

\[
\gamma = \gamma_i + \gamma_t = \gamma_i + \frac{\beta}{\omega} \text{Im} \{ G_{bg}(r_0, r_0, \omega) \} \tag{2.21}
\]

with \( \gamma_t \) denoting the radiative damping rate. Note that, using the relation
between \( \text{Im} \{ G_{bg}(r_0, r_0, \omega) \} \) and the partial local density of states of the back-
ground \( \rho_{bg} \), the radiative damping rate \( \gamma_t \) may also be expressed as \(^{[12]}\)

\[
\gamma_t = \frac{\beta \pi}{6 \varepsilon_0} \rho_{bg}, \tag{2.22}
\]

where, for example, \( \rho_{bg} = \omega^2 \sqrt{\varepsilon}/(\pi^2 c^3) \) for a homogeneous medium of per-
mittivity \( \varepsilon \). To further simplify Eqs. (2.19) and (2.20), we multiply Eq. (2.20)
with \( \hat{\mathbf{p}} \cdot \mathbf{e}_c(r_0) \) and introduce the effective mode volume

\[
V_{\text{eff}} = \frac{\int \! dr \, \varepsilon(r) |\mathbf{E}_c(r)|^2}{\varepsilon(r_0) |\hat{\mathbf{p}} \cdot \mathbf{E}_c(r_0)|^2} = \frac{2}{\varepsilon_0 \varepsilon(r_0) |\hat{\mathbf{p}} \cdot \mathbf{e}_c(r_0)|^2} \tag{2.23}
\]

which determines antenna-cavity coupling strength. Note that this is the effec-
tive mode volume as it is felt by the antenna at position \( r_0 \), and it is therefore

\(^{[1]}\)In this thesis, we always use the time convention in which \( e^{-i\omega t} \) describes the time-
dependence of a harmonically oscillating field. The Fourier transform of a function \( f(t) \) is defined
as \( f(\omega) = \int f(t) e^{i\omega t} \, dt \).
tunable by moving the dipole in the cavity mode. In that respect it differs from the more usual definition of a cavity mode volume \([14, 47]\) that uses the maximum field in the cavity mode instead. Also introducing the cavity field \(E_c = a \hat{p} \cdot e_c(r_0)\) projected on the antenna dipole axis, we finally obtain

\[
\left( \omega_0^2 - \omega^2 - i \omega \gamma \right) p - \beta E_c = \beta E_{ext},
\]

(2.24)

\[
\left( \omega_c^2 - \omega^2 - i \omega \kappa \right) E_c - \frac{\omega^2}{\epsilon_0 \epsilon V_{eff}} p = -2i\omega \sqrt{\kappa_{ex}} (\hat{p} \cdot e_c(r_0)) s_{in},
\]

(2.25)

where \(\epsilon = \epsilon(r_0)\).

Let us briefly interpret this result. In absence of the cavity, we recognize the bare antenna polarizability \(\alpha_{\text{hom}}\), defined through \(p = \alpha_{\text{hom}} E_{\text{ext}}\), as

\[
\alpha_{\text{hom}} = \frac{\beta}{\omega_0^2 - \omega^2 - i \omega \gamma},
\]

(2.26)

which is corrected for radiation damping through \(\gamma_r\). We see that the bare antenna shows a Lorentzian response, with a linewidth determined by \(\gamma\). Inclusion of \(\gamma_r\) ensures that our model is valid for both strongly and weakly scattering particles. Expressed in scattering terms, with radiation damping Eq. (2.24) represents the \(t\)-matrix of a dipolar scatterer with a consistent optical theorem for scattering, absorption and extinction [120]. Similar to the antenna, the cavity in absence of the antenna has a Lorentzian response with linewidth \(\kappa\), which is typically much smaller than the linewidth of the antenna. When the two components couple, we can expect that the system forms new, hybridized eigenmodes, and that response functions can be strongly affected. This will be discussed in the following section.

### 2.3 Hybridized eigenmodes

Eqs. (2.24) and (2.25) can be recognized as the equations of motion of two coupled, driven oscillators. In the absence of driving, they reduce to a quartic equation for \(\omega\) that can in principle be solved analytically. It will have two complex roots with negative imaginary parts, which correspond to the two eigenfrequencies of the coupled system. While the full solution is too lengthy to include here, we can consider an approximate solution if we assume that \(|\omega - \omega_c| \ll \{\gamma, \omega_c\}\). In other words, we are then looking for a solution close to the original cavity resonance frequency, with \(\kappa \ll \gamma\). We find in this case

\[
\omega = \omega_c - \frac{i \kappa}{2} \frac{\omega_c \alpha(\omega_c)}{2 \epsilon_0 \epsilon V_{eff}}.
\]

\[^4\text{Strictly speaking, the mathematical definition of a Lorentzian lineshape is } C \frac{\left(\omega - \omega_0\right)}{\left(\gamma/2\right)^2 - \left(\omega - \omega_0\right)^2}, \text{ with } C \text{ a pre-factor. Only in the ‘good cavity limit’ of small losses and evaluated near the resonance frequency } \omega_0, \text{ do } \text{Im} \{\alpha\} \text{ or } |\alpha|^2 \text{ assume this lineshape. In this thesis, however, we use the term ‘Lorentzian’ both for resonant lineshapes of the form in Eq. (2.26), and for resonances following the exact Lorentzian shape.}\]
A coupled-oscillator model for cavity-antenna systems

We can write this as

\[
\omega = (\omega_c + \delta \omega) - i \frac{\kappa + \delta \kappa}{2},
\]

(2.27)

where we defined the frequency shift \(\delta \omega\) and linewidth change \(\delta \kappa\) as

\[
\delta \omega = -\frac{\omega_c}{2\epsilon_0\epsilon V_{\text{eff}}} \text{Re} \{\alpha_{\text{hom}}(\omega_c)\},
\]

(2.28)

\[
\delta \kappa = \frac{\omega_c}{\epsilon_0\epsilon V_{\text{eff}}} \text{Im} \{\alpha_{\text{hom}}(\omega_c)\},
\]

(2.29)

respectively. Hence, the eigenfrequency of the coupled system is equal to the original complex cavity resonance frequency \((\omega_c - i\kappa/2)\) plus a complex frequency shift \(\delta \omega = \delta \omega_c - i\delta \kappa/2\). These expressions match exactly those found by Bethe-Schwinger cavity perturbation theory \([27, 133]\). This is consistent with the fact that we have explicitly assumed in the derivation of the equations of motion that there is no far-field interference between the antenna and the cavity radiation. Under these assumptions Bethe-Schwinger perturbation theory holds \([130]\). Note that the assumptions we have made do not permit us to find the complex frequency of the other mode of the system in this case, which is however expected to be close to the complex frequency of the isolated antenna \((\omega_0 - i\gamma/2)\).

2.4 Local density of states in a hybrid system

In Chapter 3 we will discuss the local density of states (LDOS)§ experienced by a fluorescent emitter coupled to a hybrid antenna-cavity system. In this section, we therefore extend the coupled-oscillator model to allow the study of LDOS. This requires the inclusion of a driving term associated to a fluorescent emitter, which will be derived in Section 2.4.1. We then continue to identify the response functions of the cavity and antenna, which hybridize when the two are coupled. The properties of these hybridized responses are discussed in Section 2.4.2. In Section 2.4.3 we obtain expressions for the total LDOS in a hybrid system. To study e.g. radiative efficiency or \(\beta\)-factor, knowledge is required of how the emitted light is distributed over different radiative and non-radiative decay channels. Expressions for the LDOS contribution of each decay channel are derived in Section 2.4.4. Finally, we test the consistency of our theory by verifying that the sum of these contributions equals the total LDOS.

2.4.1 Driving by a dipolar source

A fluorescent emitter in the limit of weak coupling to the photon field can be modelled as a ‘constant current’ driving dipole, i.e. a small, non-polarizable,

§In this chapter, LDOS will always refer to the relative local density of states, i.e. compared to that in the surrounding homogeneous medium.
2.4 Local density of states in a hybrid system

dipole with harmonically oscillating dipole moment \( p_{dr} \hat{p}_{dr} \) at position \( r_{dr} \) [18]. This source dipole drives the antenna with a field

\[
E_{\text{ext}} = G_{bg}(r_0, r_{dr}, \omega) p_{dr} = G_{bg} p_{dr},
\]

where \( G_{bg}(r_0, r_{dr}, \omega) = \hat{p} \cdot \tilde{\mathcal{G}}_{bg}(r_0, r_{dr}, \omega) \cdot \hat{p}_{dr} \). The emitter being a point source, it will be able to drive all modes that have non-zero electric field at its position, including the cavity mode. If we redo the derivation of the cavity equation of motion Eq. (2.25), starting from Eq. (2.12) and including a drive dipole term in the polarization such that

\[
P(r, t) = p(t) \delta(r - r_0) \hat{p} + p_{dr}(t) \delta(r - r_{dr}) \hat{p}_{dr},
\]

we find

\[
(\omega_c^2 - \omega^2 - i\omega\kappa) E_c - \frac{\omega^2}{\epsilon_0 \epsilon_{\text{eff}}} p = \frac{\omega^2}{\epsilon_0 \epsilon_{\text{eff}}} \phi p_{dr},
\]

where we have set \( s_{\text{in}} \) to zero, since we study fluorescence and not waveguide driving. Here, \( \phi = (\hat{p}_{dr} \cdot \mathbf{e}_c^*(r_{dr})) / (\hat{p} \cdot \mathbf{e}_c^*(r_0)) \) is a complex factor accounting for a difference in orientation between \( \hat{p} \) and \( \hat{p}_{dr} \) and different cavity mode field at \( r_0 \) and \( r_{dr} \). In a scenario where spontaneous emission effects are desired, the source is typically placed very close to the antenna, where the antenna may create strong field enhancement. If we take the source to be polarized along the antenna axis and we assume that source and antenna are very close compared to the wavelength, we obtain \( \phi \approx 1 \). The EOMs including the dipolar driving terms then become

\[
(\omega_0^2 - \omega^2 - i\omega\gamma) p - \beta E_c = \beta G_{bg} p_{dr},
\]

\[
(\omega_c^2 - \omega^2 - i\omega\kappa) E_c - \frac{\omega^2}{\epsilon_0 \epsilon_{\text{eff}}} p = \frac{\omega^2}{\epsilon_0 \epsilon_{\text{eff}}} p_{dr}.
\]

2.4.2 The hybridized polarizability and cavity response

Before discussing the LDOS in our system, let us briefly consider the cavity and antenna response functions. As we will see in this chapter as well as Chapters 3 and 8, these quantities play a crucial role in LDOS as well as the response in a scattering measurement. If we consider first the uncoupled EOMs, we can recognize the bare antenna polarizability \( \alpha_{\text{hom}} \), given in Eq. (2.26), and bare cavity response \( \chi_{\text{hom}} \), defined through \( E_c = \chi_{\text{hom}} p_{dr} \) as

\[
\chi_{\text{hom}} = \frac{1}{\epsilon_0 \epsilon_{\text{eff}}} \frac{\omega^2}{\omega_c^2 - \omega^2 - i\omega\kappa}.
\]

When cavity and antenna are coupled, their own scattered fields act as additional driving terms, leading to the hybridized antenna polarizability \( \alpha_{\text{H}} \) and cavity response function \( \chi_{\text{H}} \), defined similarly as the responses of the antenna
A coupled-oscillator model for cavity-antenna systems

and cavity to any external field or dipole, respectively. Solving Eqs. (2.33) and (2.34) for \( p \) and \( E_c \), respectively, we find

\[
\begin{align*}
\alpha_H &= \alpha_{hom}^{-1} (1 - \alpha_{hom} \chi_{hom}), \\
\chi_H &= \chi_{hom}^{-1} (1 - \alpha_{hom} \chi_{hom}).
\end{align*}
\]

These expressions can be viewed as response functions dressed by an infinite series of cavity-antenna interactions, similar to a multiple-scattering series in a coupled point-scatterer model [122, 134]. As shown in Fig. 2.2, the hybridized cavity response \( \chi_H \) shows a Lorentzian lineshape that is shifted and broadened with respect to the bare cavity resonance. Using the same approximations as in Section 2.3, i.e evaluating near a narrow cavity resonance, it is straightforward to show that the shift \( \delta \omega_c \) and broadening \( \delta \kappa \) are equal to those in Eqs. (2.28) and (2.29) found for the hybrid eigenmode and predicted by Bethe-Schwinger cavity perturbation theory [27, 133]. The hybridized polarizability \( \alpha_H \), on the other hand, resembles the broad, Lorentzian lineshape of \( \alpha_{hom} \), yet with a sharp Fano-type resonance close to \( \omega_c \) [135]. This is similar to the polarizability discussed by Frimmer et al. [118]. In fact, we can strictly show that \( \alpha_H \) has a Fano lineshape by rewriting it as

\[
\alpha_H = \alpha_{hom} (1 + \alpha_{hom} \chi_H)
\]

Observables like the antenna scattering cross-section \( \sigma_s \), which is proportional to \( |\alpha_H|^2 \), are thus described (again, in the vicinity of the cavity resonance) by the familiar equation for a Fano lineshape [12]

\[
\sigma_s \propto |e^{i\theta} + E_1 \frac{\kappa'/2}{-i\Delta + \kappa'/2}|^2,
\]

with \( \kappa' = \kappa + \delta \kappa \), \( \theta = - \arg\{\alpha_{hom}(\omega_c)\} - \pi/2 \) and \( E_1 = |\alpha_{hom}(\omega_c)| \omega_c / (\epsilon_0 V_{eff} \kappa') \). The imaginary part of \( \alpha_H \) shows a very similar lineshape. The shape of the Fano resonance depends on Fano phase \( \theta \), which is determined by the phase of \( \alpha_{hom}(\omega_c) \). As such, the shape varies with antenna-cavity detuning, from a peak-dip structure at far red-detuning to the reverse at far blue detuning, with complete destructive interference (i.e. a dip) when antenna and cavity are on resonance (\( \theta = \pi \)). Increased radiation damping experienced by the antenna due to the cavity mode, as measured by Buchler et al. for a dipole near a mirror [136], is also captured in \( \alpha_H \). Recent experiments were even able to verify these Fano lineshapes in \( \alpha_H \), by measuring the absorption cross section of antennas coupled to microtoroid cavities [137, 138].

### 2.4.3 Total LDOS

The power emitted by the drive dipole is equal to the work done by its own field on itself, i.e. [12]

\[
P_{dr} = \frac{\omega}{2} \text{Im} \{p_{dr}^* E_{tot}\},
\]

where \( E_{tot} \) the total field at its position. Dividing \( P_{dr} \) by the power that the drive dipole emits in a homogeneous medium yields the local density of
optical states (LDOS) experienced by the drive dipole, relative to that of the surrounding medium [18]. In the context of emitters coupled to cavities, this relative LDOS evaluated at the cavity resonance is the Purcell factor.

To find $E_{\text{tot}}$, let us first use Eqs. (2.33) and (2.34) to express the antenna dipole moment $p$ in terms of the drive dipole amplitude $p_{\text{dr}}$ by eliminating the cavity field $E_c$ from the equations. We obtain

$$p = \alpha_H (G_{bg} + \chi_{\text{hom}}) p_{\text{dr}}.$$  \hspace{1cm} (2.40)

It can be seen that $p$ is polarized in response to both the direct excitation by the source $(G_{bg} p_{\text{dr}})$ and the cavity field $(\chi_{\text{hom}} p_{\text{dr}})$. However, it responds with a hybridized polarizability, due to coupling with the cavity. With $p$ known, we can then express the field scattered by the antenna at the position of the source dipole as:

$$E_s(r_{\text{dr}}) = G_{bg}(r_0, r_{\text{dr}}, \omega) p = G_{bg} p,$$  \hspace{1cm} (2.41)

where we have used reciprocity, i.e. $G_{bg}(r_0, r_{\text{dr}}, \omega) = G_{bg}(r_{\text{dr}}, r_0, \omega)$.
A coupled-oscillator model for cavity-antenna systems

Similarly, we can eliminate \( p \) from Eqs. (2.33) and (2.34) to express \( E_c \) as a function of \( p_{dr} \):

\[
E_c = \chi_H (1 + \alpha_{hom} G_{bg}) p_{dr}.
\]  

(2.42)

Similar to the situation in Eq. (2.40), we recognize that the cavity is excited by both the source and the induced dipole moment of the antenna, and responds with the hybridized cavity response \( \chi_H \). The cavity field returning at the source position is equal to \( E_c \). We can now express the total field at the location of the source as:

\[
E_{tot}(r_{dr}) = E_{hom}(r_{dr}) + E_s(r_{dr}) + E_c,
\]  

(2.43)

where \( E_{hom}(r_{dr}) = G_{bg}(r_{dr}, r_{dr}, \omega) p_{dr} \) is the field that has interaction with the homogeneous background medium only (i.e. the field responsible for Larmors expression for dipole radiation in a homogeneous medium [12]).

Inserting Eq. (2.43) into Eq. (2.39), we get

\[
P_{dr} = \frac{\omega}{2} |p_{dr}|^2 \text{Im}\left\{ G_{bg}(r_{dr}, r_{dr}, \omega) + \alpha H G_{bg}^2 + 2G_{bg} \alpha H \chi_{hom} + \chi_H \right\}.
\]  

(2.44)

Where we have used \( \alpha H \chi_{hom} = \alpha_{hom} \chi_H \). To arrive at LDOS, one should calculate the ratio of this power and that emitted by the same dipolar source in a homogeneous medium. The latter is given by Larmors formula [12] and is equal to the contribution of the first term in Eq. (2.44):

\[
P_{hom} = \frac{\omega^4 \sqrt{\varepsilon}}{12\pi \epsilon_0 c^3} |p_{dr}|^2.
\]  

(2.45)

The total LDOS experienced by the source dipole, normalized to LDOS in the embedding homogeneous medium, is thus

\[
\text{LDOS}_{tot} = \frac{P_{dr}}{P_{hom}} = 1 + \frac{6\pi \epsilon_0 c^3}{\omega^3 n} \text{Im}\left\{ \alpha H G_{bg}^2 + 2G_{bg} \alpha H \chi_{hom} + \chi_H \right\}.
\]  

(2.46)

Note that each of the terms in \( \text{LDOS}_{tot} \) corresponds to a multiple scattering path that radiation can take, departing from and returning to the source. The contributions of each path will be discussed in Chapter 3. Fig. 2.3 shows an example of a hybrid LDOS spectrum calculated using Eq. (2.46).

Finally, the normalized LDOS for a dipolar source coupled to a bare antenna or cavity, can, by taking respectively \( V_{eff} \rightarrow \infty \) or \( \beta \rightarrow 0 \), be straightforwardly found as

\[
\text{LDOS}_{p, tot} = 1 + \frac{6\pi \epsilon_0 c^3}{\omega^3 n} \text{Im}\left\{ \alpha_{hom} G_{bg}^2 \right\}.
\]  

(2.47)

\[
\text{LDOS}_{c, tot} = 1 + \frac{6\pi \epsilon_0 c^3}{\omega^3 n} \text{Im}\left\{ \chi_{hom} \right\}.
\]  

(2.48)
2.4 Local density of states in a hybrid system

![LDOS spectrum for a hybrid system](image)

Figure 2.3: LDOS spectrum for a hybrid system. We have taken the same antenna and cavity as used for the blue line in Fig. 2.2, i.e. a cavity red-detuned from the antenna resonance by $1\gamma$. The source dipole is placed 10 nm from the antenna surface and is aligned normal to this surface. The spectrum contains a narrow peak and a broad peak, corresponding to the ‘cavity-like’ and the ‘antenna-like’ eigenmode of the hybrid system, respectively. We recognize the characteristic Fano lineshape that was also visible for $\alpha_H$ in Fig. 2.2. For all figures in this chapter, LDOS is normalized to vacuum.

In Chapter 3, we will discuss in detail the implications of these results for the LDOS in a hybrid system. Amongst other things, it will be demonstrated that the LDOS in a hybrid system can be significantly higher than in the bare cavity or antenna, as one might already suspect from observing Fig. 2.3.

2.4.4 LDOS per loss channel

Having found an expression for the total LDOS, we may now proceed to derive the LDOS per loss channel in the system. This allows the study of e.g. radiative efficiency or extraction efficiency (i.e. $\beta$-factor [23]), which are crucial figures of merit for any single photon source. Moreover, we use these expressions in Chapter 3 to extract antenna and cavity parameters from finite-element simulations. In Chapter 7 they are used to calculate radiative LDOS, which governs our experimental emission spectra. The loss channels in an antenna-cavity hybrid are Ohmic absorption by the antenna, dipole radiation by source and antenna, and the cavity losses. A cartoon of these channels is shown in Fig. 2.4d.

**Antenna absorption**

The work done by any force $F$ on a particle moving a distance $dx$ is $Fdx$. From the equation of motion Eq. (2.1), we can recognize the ‘absorptive’ force, i.e. the force describing material absorption, working on the antenna as

$$F_{\text{abs}} = \text{Re} \left\{ m\gamma_i \frac{\dot{p}}{q} \right\}.$$  \hspace{1cm} (2.49)
Assuming harmonic time dependence, the power absorbed in the antenna is the oscillation frequency times the cycle-integrated work done by the absorptive force:

\[ P_{\text{abs}} = \frac{\omega}{2\pi} \int_0^\tau F_{\text{abs}} \frac{dx}{dt} dt \]  

with \( \tau = \frac{2\pi}{\omega} \) the cycle time. With \( \frac{dx}{dt} = \text{Re} \left\{ \dot{\mathbf{p}} \right\} \) we find

\[ P_{\text{abs}} = \frac{\omega^2}{2\beta} \gamma \mathbf{p}^2. \]  

Using Eq. (2.40) for the antenna dipole moment, we arrive at:

\[ P_{\text{abs}} = \frac{\omega^2}{2\beta} \gamma |\mathbf{p}|^2. \]  

We then divide by the homogeneous radiated power (Eq. (2.45)) to obtain normalized absorptive LDOS as

\[ \text{LDOS}_{\text{abs}} = \frac{6\pi \varepsilon_0 c^3}{\omega^2 n} \frac{\gamma}{\beta} |\alpha_H|^2 |G_{bg} + \chi_{\text{hom}}|^2 |\mathbf{p}_{\text{dr}}|^2 \]  

Dipole radiation by antenna and source

To calculate exactly the power radiated by the source and the antenna, one should calculate the overlap in their radiation patterns by integrating the Poynting flux of their added scattered fields over an enclosing surface. However, to first order we can assume that if the distance \( \delta r \) between source and antenna is sufficiently small (i.e. \( \delta r \ll \lambda \)), their radiation patterns overlap entirely. In that case, we may consider them as one effective dipole with total dipole moment \( p_{\text{tot}} = p_{\text{dr}} + p \) [140]. We can write down the radiation reaction force on a dipole \( p \) in analogy to the absorptive force in Eq. (2.49) as

\[ F_{\text{rad}} = \text{Re} \left\{ -m \gamma \frac{\dot{\mathbf{p}}}{q} \right\}. \]  

We may recognize that this equals the real part of the Abraham-Lorentz force (for harmonic time dependence) if we insert \( \gamma \) for a homogeneous medium [12]. A similar analysis as done for the antenna dissipation then leads to a radiated power by the antenna and source

\[ P_{p,\text{rad}} = \frac{\omega^2}{2\beta} \gamma |p_{\text{tot}}|^2 = \frac{\omega^2}{2\beta} \gamma |1 + \alpha_H (G_{bg} + \chi_{\text{hom}})|^2 |\mathbf{p}_{\text{dr}}|^2. \]
2.4 Local density of states in a hybrid system

If we assume that $\gamma_r$ is given by the radiative decay in a homogeneous medium (Eq. (2.22)), the corresponding normalized radiative LDOS then becomes:

$$\text{LDOS}_{p,\text{rad}} = |1 + \alpha_H (G_{bg} + \chi_{\text{hom}})|^2. \quad (2.56)$$

This answer is intuitive: the power radiated by a dipole is proportional to the square of the dipole moment, so we recognize that the radiative LDOS can be interpreted as an enhancement of the total dipole moment $p_{tot}$ with respect to that of the source, $p_{dr}$. In general, if $\gamma_r$ is not given by the radiative decay in a homogeneous medium, for example because the antenna and source are near an interface or inside a photonic bandgap medium [141], an additional factor $\rho_{bg}(\omega)/\rho_{\text{hom}}(\omega)$ appears in front of Eq. (2.56) which accounts for the difference in (radiative) local density of states between the background ($\rho_{bg}$) and a homogeneous medium ($\rho_{\text{hom}}$).

**Losses by the cavity**

The power $P_i$ emitted into the cavity intrinsic loss channel is simply the intrinsic cavity loss rate $\kappa_i$ times the energy in a cavity mode, i.e.

$$P_i = \kappa_i U_m = \kappa_i |a|^2$$

$$= \frac{\kappa_i}{2} \epsilon_0 \epsilon V_{\text{eff}} |E_c|^2, \quad (2.57)$$

where we have used Eqs. (2.9) and (2.23) to rewrite the mode energy $U_m$. We can use Eq. (2.42) for $E_c$, which leads to:

$$P_i = \frac{\kappa_i}{2} \epsilon_0 \epsilon V_{\text{eff}} |\chi_H (1 + \alpha_{\text{hom}} G_{bg})|^2 |p_{dr}|^2. \quad (2.58)$$

Division by the homogeneous radiated power gives the normalized LDOS in the cavity loss channel as

$$\text{LDOS}_i = \frac{6\pi \epsilon_0 c^3}{\omega^4 n} \kappa_i \epsilon_0 \epsilon V_{\text{eff}} |\chi_H (1 + \alpha_{\text{hom}} G_{bg})|^2. \quad (2.59)$$

In Section 3.6, we will discuss how the collection efficiency or $\beta$-factor of an on-chip single-photon source based on a hybrid system can be described as the ratio of light emitted into the cavity decay channel to the total emission, that is, as $\text{LDOS}_i/\text{LDOS}_{\text{tot}}$.

**An example**

Fig. 2.4a-c show example spectra of the LDOS contributions by each loss channel. We notice that radiation and absorption both have a Fano lineshape, with shape (i.e. Fano phase) depending on cavity-antenna detuning. In fact, it can easily be shown that both $\text{LDOS}_{p,\text{rad}}$ and $\text{LDOS}_{\text{abs}}$ can be rewritten into the
A coupled-oscillator model for cavity-antenna systems

Figure 2.4: LDOS contributions of the loss channels in a hybrid system. (a-c) Spectra showing the LDOS contributions of dipole radiation (LDOS_{p, rad}, blue), absorption in the antenna (LDOS_{abs}, red) and the cavity loss channel (LDOS_{c}, green). We show spectra for cavities red-detuned by 3 (a) and 1 (b) antenna linewidth, and for a cavity at resonance with the antenna (c). Apart from this detuning, the same parameters were used as for Fig. 2.3. We notice that radiation and absorption show a Fano lineshape, while cavity losses always show a Lorentzian lineshape. Moreover, antenna losses become increasingly dominant as the cavity is tuned closer to resonance. Note that LDOS_{c} was multiplied by 10 in (c) for visibility. (d) Cartoon of a hybrid system and the emitted power flowing in the various loss channels. Note that the cavity losses are drawn here as radiation leaking through the mirrors, but they could also be e.g. absorption.

expression for a Fano resonance (Eq. (2.38)), as they depend on the square modulus of both resonant and non-resonant terms. The cavity losses, on the other hand, always show a Lorentzian lineshape, governed by the hybridized cavity response $\chi_H$. We also notice that antenna losses (radiation and absorption) are dominant when the cavity is near antenna resonance (e.g. in Fig. 2.4c), whereas far from resonance the cavity losses can become dominant. The ratio of radiation and absorption, which is mostly governed by the bare antenna albedo, also changes with cavity-antenna detuning. This is because the radiative rate $\gamma_r$ depends on frequency as $\omega^3$, causing lower albedo at lower frequencies.

2.4.5 Consistency check

The sum of the LDOS in separate loss channels should match our expressions (Eqs. (2.46) to (2.48)) for total LDOS. Indeed, in Fig. 2.5 we see that this is the case, both for the bare components and for the hybrid system. For a
bare cavity, there is perfect agreement. For a bare antenna, a small deviation remains, which can be assigned to the approximation made by assuming a 100% overlap between source and antenna radiation profiles. The deviation in LDOS is less than 0.5% of the total LDOS for this antenna-source geometry.

![Graph showing LDOS comparison](image)

Figure 2.5: Comparing total LDOS to LDOS per loss channel. (a) A bare antenna. We compare total ($LDOS_{p,\text{tot}}$) to radiative ($LDOS_{p,\text{rad}}$) and absorptive ($LDOS_{\text{abs}}$) LDOS. The sum of the latter two equals the total LDOS to within 0.5%. (b) A bare cavity. We see that total LDOS ($LDOS_{c,\text{tot}}$) equals 1 (for the radiation into the background medium) plus the LDOS in the cavity loss channel ($LDOS_{i}$). (c) A hybrid system, comparing $LDOS_{\text{tot}}$ to LDOS in the 3 possible loss channels: dipole radiation, absorption and the cavity loss channel. Again, the sum of these 3 channels equals the total LDOS to within 0.5%. In this figure, the same parameters were used as for Fig. 2.3.

### 2.5 Whispering-gallery modes and taper-coupled measurements

In the experimental studies of hybrids presented in this thesis, we make use of whispering-gallery-mode cavities. Here, each whispering-gallery mode occurs in (ideally) degenerate pairs: a clockwise and an anticlockwise mode. Moreover, these cavities are often studied using waveguide or tapered-fiber coupling. It is therefore useful to generalize the derived coupled equations of motion for a single cavity mode and a single antenna to the case of two counter-propagating cavity modes and $N$ antennas. Furthermore, we will derive expressions the observables in such a taper-coupled scattering experiment. Finally, we will discuss in more detail the case of a single antenna.
A coupled-oscillator model for cavity-antenna systems

on a taper-coupled whispering-gallery-mode cavity, which is relevant to the experiments in Chapter 6.

2.5.1 Equations of motion

![Diagram of a system of 2 whispering-gallery modes with N antennas coupled to them.](image)

Figure 2.6: A system of 2 whispering-gallery modes with N antennas coupled to them. The modes are coupled to a waveguide at rate $\kappa_{\text{ex}}$, which has input power $P_{\text{in}}$ flowing from left to right. $P_{\text{fw}}$ and $P_{\text{bw}}$ are powers flowing in the forward (i.e. transmission) and backward (i.e. reflection) direction in the waveguide.

We consider 2 counterpropagating, degenerate whispering-gallery modes, coupled to $N$ dipolar antennas, as sketched in Fig. 2.6. We assume for simplicity that there is no coupling between the antennas, other than through the two cavity modes. This is a reasonable assumption if the antennas are placed sufficiently far apart to avoid near-field coupling and if the cavity has a sufficiently high Purcell factor, such that antenna-cavity coupling is stronger than coupling of the antennas to other modes in the background environment.

Before discussing the equations of motion for this system, let us recast those for a single cavity and antenna (Eqs. (2.19) and (2.20)) in a simpler, more symmetric form. We can define a universal antenna-cavity coupling rate $\Omega$ as

$$\Omega = \sqrt{\frac{\beta}{2}} \hat{p} \cdot e_c(r_0),$$

(2.60)

which is generally a complex number. Furthermore, we replace the dipole moment $p$ by the new variable

$$b = \frac{\omega}{\sqrt{2\beta}} p,$$

(2.61)

Its square magnitude $|b|^2$ is an energy. We will see in Section 2.5.2 that we can think of this as the energy stored in the antenna. Together, this simplifies the
equations of motion for a single antenna and cavity mode to

\[
(\omega_0^2 - \omega^2 - i\omega\gamma) b - \omega \Omega a = \omega \sqrt{\frac{\beta}{2}} E_{\text{ext}}, \tag{2.62}
\]

\[
(\omega_c^2 - \omega^2 - i\omega\kappa) a - \omega \Omega b = -2i\omega \sqrt{\kappa_{\text{ex}} s_{\text{in}}}. \tag{2.63}
\]

Now let us consider the case of 2 WGMs coupled to N antennas. For each antenna, an equation of motion similar to Eq. (2.62) can be straightforwardly set up, now including two terms describing coupling to each of the cavity modes. The cavity equations of motion, analogous to Eq. (2.63), can be derived starting from Eq. (2.12) and including N different dipoles in the polarization such that

\[
P(r, t) = \sum_{m=1}^{N} p_m(t) \delta(r - r_m) \hat{p}_m. \tag{2.64}
\]

We then find the equations of motion for the cavity modes and for the m-th antenna as

\[
(\omega_c^2 - \omega^2 - i\omega\kappa) a_{\text{CW}} - \omega \sum_{m=1}^{N} \Omega_{\text{CW},m}^* b_m = -2i\omega \sqrt{\kappa_{\text{ex}} s_{\text{in}}}, \tag{2.65}
\]

\[
(\omega_c^2 - \omega^2 - i\omega\kappa) a_{\text{ACW}} - \omega \sum_{m=1}^{N} \Omega_{\text{ACW},m}^* b_m = 0, \tag{2.66}
\]

\[
(\omega_m^2 - \omega^2 - i\omega\gamma_m) b_m - \omega \Omega_{\text{CW},m} a_{\text{CW}} - \omega \Omega_{\text{ACW},m} a_{\text{ACW}} = \omega \sqrt{\frac{\beta_m}{2}} E_{m,\text{ext}}. \tag{2.67}
\]

This sets up a total of \( N + 2 \) coupled equations of motion. The clockwise and anticlockwise cavity modes are described by their mode amplitudes \( a_{\text{CW}} \) and \( a_{\text{ACW}} \), respectively, and each antenna by its mode amplitude \( b_m = \omega p_m / \sqrt{2\beta_m} \). We assumed that the waveguide only has power flowing in one direction, such that it only drives the clockwise mode. Note, however, that both cavity modes experience the same coupling losses, i.e. both have loss rate \( \kappa = \kappa_1 + \kappa_{\text{ex}} \). Each antenna may have different resonance frequency, linewidth and oscillator strength and may feel a different driving field \( E_{m,\text{ext}} \). The coupling rates of each antenna to the clockwise and anticlockwise modes are given as

\[
\Omega_{\text{CW},m} = \sqrt{\frac{\beta_m}{2}} \langle e_{\text{CW}}(r_m) \cdot \hat{p}_m \rangle, \tag{2.68}
\]

\[
\Omega_{\text{ACW},m} = \sqrt{\frac{\beta_m}{2}} \langle e_{\text{ACW}}(r_m) \cdot \hat{p}_m \rangle, \tag{2.69}
\]
respectively. At this point we can use a property of whispering-gallery modes in rotationally symmetric cavities, which is that the mode profiles of the CW and CCW modes must obey \[ e_{\text{CW}}(r) = e^*_{\text{CCW}}(r), \] \[ (2.70) \]
i.e. they are equal except for opposite rotation directions. This implies that \( \Omega_{\text{ACW},m} = \Omega^*_{\text{CW},m} \). Defining the coupling rate \( \Omega_m \equiv \Omega_{\text{CW},m} \), we can simplify the equations of motion to

\[ (\omega^2 - \omega^2 - i\omega\kappa) a_{\text{CW}} - \omega \sum_{m=1}^{N} \Omega^*_m b_m = -2i\omega \sqrt{\kappa_{\text{ex}}} s_{\text{in}}, \] \[ (2.71) \]

\[ (\omega^2 - \omega^2 - i\omega\kappa) a_{\text{ACW}} - \omega \sum_{m=1}^{N} \Omega_m b_m = 0, \] \[ (2.72) \]

\[ (\omega^2 - \omega^2 - i\omega\gamma_m) b_m - \omega \Omega_m a_{\text{CW}} - \omega \Omega^*_m a_{\text{ACW}} = \omega \sqrt{\beta_m / 2} E_{m,\text{ext}}. \] \[ (2.73) \]

### 2.5.2 Transmission, reflection and output powers

We can now write down expressions for the powers flowing into the different output channels of the system, as well as waveguide transmission and reflection, which are typically observables in a taper-coupled measurement.

The intrinsic cavity losses are given by Eq. (2.57), i.e. (for the clockwise mode)

\[ P_{i,\text{CW}} = \kappa_i U_{\text{CW}} = \kappa_i |a_{\text{CW}}|^2 \] \[ (2.74) \]

and similarly for the anticlockwise mode. The forward power flow in the waveguide \( P_{\text{fw}} \) (i.e. transmitted power) is the result of interference between the input field \( s_{\text{in}} \) and the field coupled out by the CW mode into the waveguide, i.e. \[ P_{\text{fw}} = |s_{\text{in}} + \sqrt{\kappa_{\text{ex}}} a_{\text{CW}}|^2. \] \[ (2.75) \]

As there is no input power in the backward direction, we can simply write

\[ P_{\text{bw}} = |\sqrt{\kappa_{\text{ex}}} a_{\text{CCW}}|^2 \] \[ (2.76) \]

for the power in this direction (i.e. the reflected power). Division by the input power \( P_{\text{in}} = |s_{\text{in}}|^2 \) gives the transmittance and reflectance,

\[ T = \frac{P_{\text{fw}}}{P_{\text{in}}} = \left| 1 + \frac{\sqrt{\kappa_{\text{ex}}} a_{\text{CW}}}{s_{\text{in}}} \right|^2, \] \[ (2.77) \]

\[ R = \frac{P_{\text{bw}}}{P_{\text{in}}} = \left| \frac{\sqrt{\kappa_{\text{ex}}} a_{\text{CCW}}}{s_{\text{in}}} \right|^2. \] \[ (2.78) \]
2.5 Whispering-gallery modes and taper-coupled measurements

Antenna absorption and radiation are derived analogously to Eqs. (2.51) and (2.55). The only difference is that, since there is no source dipole here, radiation is simply proportional to the dipole moment of each antenna alone. This assumes that there is no radiation overlap between the antennas (which is consistent with our assumption of uncoupled antennas, as radiation overlap would lead to complex coupling rates [130]). The absorbed and radiated powers by the $m$-th antenna are then simply given by

$$P_{\text{abs},m} = \gamma_{m,i}|b_m|^2,$$

$$P_{r,m} = \gamma_{m,r}|b_m|^2,$$

respectively, with $\gamma_{m,i}$ ($\gamma_{m,r}$) the ohmic loss rate (radiative loss rate) of this antenna. Here, we used Eq. (2.61) to express the powers in terms of $b_m$. The similarity between these elegantly simple expressions for $P_{\text{abs},m}$ and $P_{r,m}$ on the one hand and for $P_{r,\text{CW}}$ and $P_{r,\text{ACW}}$ on the other, show that we can interpret $|b_m|^2$ as the energy stored in the $m$-th antenna. This energy, multiplied by the energy decay rate into a particular channel, must be equal to the power flowing into that channel. Inserting $\gamma_{m,r}$ from Eq. (2.22) with the density of states for a homogeneous medium into Eq. (2.80) and expressing in terms of the dipole moment $p_m$, we find

$$P_{r,m} = \frac{\omega^4}{12\pi\epsilon_0 c^3} |p_m|^2,$$

which is exactly Larmor’s familiar expression for the radiation of a dipole in a homogeneous medium [12].

2.5.3 Special case of a single antenna, evaluated close to cavity resonance

Let us consider the special case of a waveguide-coupled WGM cavity containing only a single antenna. This simple example will help to gain intuition about these coupled systems. Moreover, in Chapter 6 we perform taper-coupled measurements of exactly such systems, so the results from this section can be directly applied to interpret those results.

We assume the system is excited only through the taper, such that the antenna is not directly driven ($E_{\text{ext}} = 0$). This corresponds to the experimental situation in Chapter 6. If we are interested only in its properties close to the (high-Q) cavity resonance, we can use $\kappa, \Delta \ll \omega_c$, where $\Delta = \omega - \omega_c$ is
A coupled-oscillator model for cavity-antenna systems

Figure 2.7: WGM field profiles. Sketch of intensity $|E|^2$ (a-d) and phase (e-h) for an example set of WGMs in a cylindrically symmetric cavity. All modes have intensity peaked at the edge of the disk. The basis of a CW (a,e) and ACW (b,f) eigenmode corresponds to running waves, whereas the basis of a symmetric (c,g) and an antisymmetric eigenmode (d,h) corresponds to standing waves, shifted from each other by a quarter period. The symmetric mode couples to the antenna, because it has a maximum at the antenna location. The antisymmetric mode has a node there, and therefore does not couple. The white lines indicate the edge of the cavity, and the antenna position is indicated by a white dot at the lower edge of the disk.

detuning, to simplify the three equations of motion to

\[
\begin{align*}
(-i\Delta + \kappa/2) a_{CW} - i\Omega b/2 &= \sqrt{\kappa} s_{in}, \\
(-i\Delta + \kappa/2) a_{ACW} - i\Omega b/2 &= 0, \\
(\omega_1^2 - \omega_c^2 - i\omega_c\gamma) b - \omega_c\Omega a_{CW} - \omega_c\Omega a_{ACW} &= 0.
\end{align*}
\]

Here we have dropped the subscript in $b_m$, $\gamma_m$ and $\Omega_m$, since there is only one antenna. Also, we have the freedom to define the phase of the mode $e_c$ at will, and we have fixed it such that $\hat{p} \cdot e_c(r_0) \in \mathbb{R}$ and thus $\Omega \in \mathbb{R}$. Note that we can then relate $\Omega$ to the effective mode volume through

\[
\Omega = \sqrt{\frac{\beta}{\epsilon_0 \epsilon(r_0) V_{\text{eff}}}}. \tag{2.85}
\]

We now transform to the new basis

\[
a_{as} = \frac{1}{\sqrt{2}} (a_{CW} + a_{ACW}), \tag{2.86}
\]

\[
a_{as} = \frac{1}{\sqrt{2}} (a_{CW} - a_{ACW}). \tag{2.87}
\]

\[\star\]Note that by making this approximation, Eq. (2.82) now contains the again the familiar waveguide coupling term, as also found by making the slowly varying envelope approximation (Eq. (2.18)) [132, 144].

46
of respectively the symmetric and antisymmetric modes (around the location of the antenna). These modes, together with the CW and ACW modes, are sketched in Fig. 2.7. By respectively adding and subtracting Eqs. (2.82) and (2.83), the EOMs for the symmetric and antisymmetric modes, as well as for the antenna, can be found as

\[
-i(\Delta + \kappa/2) a_s - i\Omega b/\sqrt{2} = \sqrt{\kappa_{ex}/2\sin},
\]

\[
-i(\Delta + \kappa/2) a_{as} = \sqrt{\kappa_{ex}/2\sin},
\]

\[
(\omega_1^2 - \omega_c^2 - i\omega_c\gamma_1) b - \sqrt{2}\omega_c\Omega a_s = 0.
\]

By making this basis transformation, we can see that we have gone from a system of 3 coupled EOMs to just 2 coupled and 1 independent EOM. This implies that the antenna is only coupled to the symmetric cavity mode, which has a maximum at the antenna position (see Fig. 2.7), and not to the antisymmetric mode, which has a node there [26]. Using Eq. (2.90) to eliminate \(b\) from Eq. (2.88) and defining the backscattering rate \(\gamma_{bs}\) and antenna-induced loss rate \(\gamma_l\) as

\[
\gamma_{bs} = 2\Omega^2 \text{Re} \left\{ \frac{\omega_c}{\omega_1^2 - \omega_c^2 - i\omega_c\gamma_1} \right\},
\]

\[
\gamma_l = 2\Omega^2 \text{Im} \left\{ \frac{\omega_c}{\omega_1^2 - \omega_c^2 - i\omega_c\gamma_1} \right\},
\]

we arrive at

\[
-i(\Delta_s + \kappa_s/2) a_s = \sqrt{\kappa_{ex}/2\sin},
\]

\[
-i(\Delta + \kappa/2) a_{as} = \sqrt{\kappa_{ex}/2\sin},
\]

where

\[
\Delta_s = \Delta + \gamma_{bs}/2,
\]

\[
\kappa_s = \kappa + \gamma_l
\]

are the detuning from resonance and total loss rate of the symmetric mode. These two uncoupled equations tell us that the antenna causes a shift and broadening of the symmetric mode with respect to the unperturbed, antisymmetric mode. These perturbations are related to the antenna properties and the antenna-cavity coupling rate. In fact, we can recognize that the shift \(\gamma_{bs}/2\) and broadening \(\gamma_l\) are the same as those of the hybridized eigenmode in Eqs. (2.28) and (2.29), except for a factor 2. In other words, the perturbed symmetric mode \(a_s\) is the eigenmode of the coupled antenna-cavity system, and we can think of bare cavity mode as the symmetric cavity mode in the absence of an antenna. The factor 2 difference comes from the fact that in Eqs. (2.91) and (2.92), \(\Omega = \Omega_{CW}\) which contains the effective mode volume of
A coupled-oscillator model for cavity-antenna systems

the *clockwise* mode, whereas Eqs. (2.28) and (2.29) contain the effective mode volume $V_{\text{eff},s}$ of the *symmetric* mode. Since $e_s = (e_{\text{CW}} + e_{\text{ACW}})/\sqrt{2}$, one finds $V_{\text{eff,CW}} = 2V_{\text{eff,s}}$, such that Eqs. (2.91) and (2.92) match exactly the shift and broadening in Eqs. (2.28) and (2.29).

We may now derive explicit expressions for the power in the different output channels of the system. The power in the forward and backward direction can be derived from Eqs. (2.75) and (2.76) as

$$P_{\text{fw}} = \left| -s_{\text{in}} + \sqrt{\frac{\kappa_{\text{ex}}}{2}} (a_s + a_{\text{as}}) \right|^2$$

$$= P_{\text{in}} \left| -1 + \frac{\kappa_{\text{ex}}}{2} \left( \frac{1}{-i\Delta_s + \kappa_s/2} + \frac{1}{-i\Delta + \kappa/2} \right) \right|^2,$$

$$P_{\text{bw}} = \left| \sqrt{\frac{\kappa_{\text{ex}}}{2}} (a_s - a_{\text{as}}) \right|^2$$

$$= P_{\text{in}} \frac{\kappa_{\text{ex}}^2}{4} \left| -i\Delta_s + \kappa_s/2 - \frac{1}{-i\Delta + \kappa/2} \right|^2.$$ (2.97, 2.98)

In the limit of two strongly split modes, we can write for the transmission coefficient at resonance with one of the two modes (for example the unperturbed mode):

$$T \approx \left| 1 - \frac{\kappa_{\text{ex}}}{2} \frac{1}{\kappa/2} \right|^2$$

$$= \left| 1 - \frac{\kappa_{\text{ex}}}{\kappa_1 + \kappa_{\text{ex}}} \right|^2.$$ (2.99)

This shows that in this case, we can only reach critical coupling (*i.e.* $T=0$) if $\kappa_{\text{ex}} \to \infty$ (see Fig. 2.8). This is because the coupling to the symmetric mode is twice lower than to the clockwise mode, where one can reach critical coupling for $\kappa_{\text{ex}} = \kappa_1$ [143]. In reality, however, critical coupling can usually be reached either because of small splitting or because of slightly unequal coupling rates for the symmetric and the antisymmetric modes [144].

The intrinsic cavity losses are given through Eqs. (2.74) and (2.93) as the sum of the losses in the CW and ACW modes, or equivalently in the symmetric and antisymmetric modes

$$P_1 = \kappa_1 (|a_s|^2 + |a_{\text{as}}|^2) \frac{\kappa_1 \kappa_{\text{ex}}}{2} \left( \left| \frac{1}{-i\Delta_s + \kappa_s/2} \right|^2 + \left| \frac{1}{-i\Delta + \kappa/2} \right|^2 \right).$$ (2.100)

Using Eqs. (2.79), (2.80), (2.90) and (2.93) we can also express the scattered and
2.5 Whispering-gallery modes and taper-coupled measurements

absorbed power in the antenna as

\[ P_{\text{abs}} = \gamma_i |b|^2 = P_{\text{in}} \gamma_i \frac{\gamma_i^2}{2\Omega^2} \left| \frac{1}{-i\Delta_s + \kappa_s/2} \right|^2, \]  

(2.101)

\[ P_r = P_{\text{in}} \gamma_r \frac{\gamma_i^2}{2\Omega^2} \left| \frac{1}{-i\Delta_s + \kappa_s/2} \right|^2. \]  

(2.102)

Example spectra of the power in each of these different output channels are shown in Fig. 2.8.

Figure 2.8: Powers in the different output channels. Four example spectra, showing power (normalized to input power) in the various output channels during a taper-coupled experiment. We use a cavity with \( Q = 10^5 \), \( V_{\text{eff}} = 50\lambda^3 \) and an spherical gold antenna of 20 nm radius, resonant at 460 THz. Taper-cavity coupling rate \( \kappa_{\text{ex}} \) is chosen as \( \kappa_{\text{ex}} = \kappa_i \). (a) Without an antenna, we are critically coupled to the CW mode, causing transmission \( (P_{\text{fw}}) \) to go to zero. At resonance, all power flows into the cavity loss channel \( (\Pi) \). (b-d) Spectra for cavity modes detuned from the antenna by -1.5 (b), 0 (c) and 1.5 (d) antenna linewidths. Spectra show a narrow and a broad resonance, corresponding to the antisymmetric and the symmetric modes, respectively. At red-detuning (b), we notice a redshift of the symmetric mode \( (\gamma_{bs} > 0) \), whereas at blue-detuning (d), there is a blueshift \( (\gamma_{bs} < 0) \). At zero detuning (c), the modes are not split but the linewidth difference \( (i.e. \gamma_i) \) is maximal. The antisymmetric mode is no longer at critical coupling. Antenna scattering and absorption peak only at the symmetric mode.

Let us consider briefly the case of an antenna close to resonance, causing a strongly broadened mode. We assume weak coupling, such that \( \kappa_s \approx \gamma_i \) and
A coupled-oscillator model for cavity-antenna systems

driving at the perturbed resonance ($\Delta_s = 0$). It is then easy to show that

$$P_t \approx P_{in} \kappa_{ex} \frac{\gamma_r \gamma_2^2 + \gamma_1^2}{\Omega_1^2} = 2P_{in} \kappa_{ex} \epsilon_0 \epsilon(r_1) V_{eff} \frac{|\alpha|^2}{\beta (\text{Im}\{\alpha\})^2}$$  \hspace{1cm} (2.103)

where $V_{eff}$ is the effective mode volume of the symmetric mode and $\alpha$ the bare antenna polarizability. If the particle is close to resonance, $|\alpha| \approx \text{Im}\{\alpha\}$. Since $\gamma_r$ scales linearly with oscillator strength $\beta$, we can see that $P_t$ is independent of $\beta$. This implies that, contrary to intuition, bigger (i.e. larger $\beta$) particles do not scatter more strongly when excited through the taper! In other words, the increase in scattering rate that comes with an increase in size is countered by a decrease in energy circulating in the cavity due to the increased losses. Only if the antenna is far off-resonance do we see an increase of scattering with increasing particle size. We will see in Chapter 6 that this can pose some difficulties when measuring the induced cavity shift and broadening by a resonant antenna. It is convenient to use the scattered light for detecting the perturbed modes, but this scaling implies that on-resonant antennas are difficult to see.

2.6 Conclusion and discussion

We have developed a simple coupled-oscillator model to describe hybrid antenna-cavity systems. The model can predict both mode hybridization and familiar expressions for cavity perturbation. By including a source dipole, we show that we can calculate local density of states (LDOS) in these systems, as well as the fractions of light emitted into each of the system decay channels. Finally, explicit expressions are derived for the observables in a system of one or multiple antennas coupled to two counter-propagating whispering-gallery modes. Beside providing a general framework for understanding the physics of coupled antenna-cavity systems, this chapter provides the foundation for several of the following chapters. Chapters 3 and 4 use this model for theoretical studies of LDOS in hybrid systems, and in Chapters 6 and 7 expressions from this chapter are used to fit or compare to experimental data.

Our coupled-oscillator model is completely self-consistent, within the limits of the assumptions that were made. The first of these is that the antenna is dipolar. This holds only for antennas much smaller than the wavelength, and we shall see in Chapter 6 that a breakdown of the model can be observed for aluminium nano-rod antennas longer than 140 nm (at $\sim 780$ nm wavelength). The second important assumption is that there is no radiation overlap between the cavity and antenna, in which case also first-order cavity perturbation theory is valid [27]. While this holds for most geometries, recent experimental work has shown that a dramatic deviation from this perturbation theory is observed for arrays of antennas coupled to a microrotoroid cavity [130]. Through the use of quasi-normal modes [62, 63, 106], which correctly
describe the radiation properties of leaky resonators, this far-field interference between cavity and antenna can be included. For the LDOS studies, we make a third assumption, namely that the emitter-antenna coupling is described by the background Green’s function. This is not strictly valid for antennas with complex near-field patterns such as bow-tie [76] or nanoparticle-on-mirror geometries [68]. The model can be used in this case, however the Green’s function should then be interpreted as an effective parameter capturing this coupling.