Hybrid resonators for light trapping and emission control

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Chapter 4

Cavities as conjugate-matching networks for antennas at optical frequencies

This chapter discusses an electrical circuit analogue of a hybrid antenna-cavity system. First, two different electrical circuits that were proposed in literature to describe an optical antenna are discussed, and they are shown to be equivalent. We then apply the maximum power transfer theorem to find a fundamental bound for the radiation by a lossy nano-antenna, which is independent of its environment. This bound is reached when antenna and radiation load are conjugate-matched. Based on the derivation of the antenna circuit, we propose an equivalent circuit for a hybrid antenna-cavity system driven by an external field or a fluorescent emitter. We show how a cavity can be used to reach the conjugate-matching limit with an optically small nano-antenna, and elucidate the conditions for which this can be achieved. These results provide a useful analogy between nanophotonics and circuit theory, showing how cavities can be viewed as a matching network between an antenna and the radiation load and directly providing design rules for optimized scattering.
Cavities as conjugate-matching networks for antennas at optical frequencies

4.1 Introduction

Electromagnetic waves, whether at radio frequencies or optical frequencies, are all governed by the Maxwell equations. This fact has prompted many researches to make analogies between the older and more established field of radio-frequency engineering and the relatively new field of nanophotonics. Such analogies have aided research into several exciting phenomena in the optical domain, including strongly scattering [158], efficient [159] or directional [71, 160, 161] antennas, emission enhancement [162], plasmonic waveguides [163], optical cloaking [164, 165], epsilon-near-zero materials [166, 167], far-field super-resolution microscopy [168, 169] and optical metamaterials [170–172]. The connection with radio-frequencies has been useful also in a broader optics context — one of the works on compression of optical pulses that was awarded with the 2018 Nobel prize for physics was inspired by radar technology [173]. Among the efforts to connect radio-frequencies to plasmonics are several works that have described plasmonic nano-antennas by an equivalent electrical circuit containing lumped elements [160, 174–179]. This has led to novel design strategies for these antennas, including for example phased array nano-antennas [160], directive leaky-wave nano-antennas [161] or ‘antenna loading’ by dielectric spacers for tuning the scattering response [175, 176].

Inspired by the circuit analogy for a nano-antenna, in this chapter we present an equivalent circuit for a hybrid antenna-cavity system, and use it to show how cavities can be interpreted as a matching network between an antenna and the radiation load. Before we discuss this hybrid circuit, it is important to understand the equivalent circuit that describes a single nano-particle, as this will form the basis for our hybrid circuit. Therefore, in Section 4.2 we first discuss two nano-antenna circuits from the literature. While these circuits may appear completely different at first glance, we show that they can be made equivalent by a simple transformation. Section 4.3 then discusses the ‘conjugate-matching limit’ — a fundamental bound on antenna scattering imposed by the well-known maximum power transfer theorem applicable to circuits. The equivalent circuit for a hybrid system is introduced in Section 4.4, which can be used to study both scattering and local density of states. Using this circuit, in Section 4.5 we show how the cavity can strongly increase radiated power, even reaching the conjugate-matching limit if the antenna is sufficiently small. We discuss the two fundamental limits that govern scattering, and show how cavities act as a matching network between antennas and radiation. Such matching networks, which are an essential part of radio-frequency circuit design, are currently lacking at THz or optical frequencies due to a lack of high-quality lumped elements. Our results offer a perspective on matching networks at optical frequencies.
4.2 Equivalent circuits for an emitter coupled to a nano-antenna

Different circuit models have been proposed to describe a small nano-particle driven by an external field or by a fluorescent emitter. Here, we discuss two circuits proposed respectively by Krasnok et al. [179] and by Engheta et al. [174]. The first was proposed to study local density of states (LDOS) near an antenna, while the second has been used for example to study antenna resonance tuning [175, 176]. Although these circuits seem entirely different, we show that the two descriptions are equivalent.

4.2.1 The circuit proposed by Krasnok et al.

Krasnok et al. [179] proposed a circuit model to describe the Purcell effect experienced by an emitter near a nano-antenna, based largely on a circuit proposed earlier by Greffet et al. [177]. For clarity, and because this forms the basis of the hybrid circuit discussed in Section 4.4, we briefly discuss the derivation of this circuit. Let us start with a sub-wavelength antenna driven by an unspecified external field $E_{\text{ext}}$. This induces a dipole moment $p = \alpha E_{\text{ext}}$ in the antenna, where $\alpha$ is the self-consistent antenna electric polarizability*

$$\alpha^{-1} = \alpha_0^{-1} - i \frac{k^3}{6\pi\epsilon_0 n^2}. \quad (4.1)$$

Here, $\alpha_0$ is the static electric polarizability and the second term on the right hand side in Eq. (4.1) represents radiation damping, with $k = \omega n/c$ the wavenumber and $n$ the index of the host material. For simplicity we restrict ourselves to particles in vacuum. To determine an equivalent circuit, we can define a driving voltage as $V_a = l_a E_{\text{ext}}$ and an induced current $I_a = \frac{1}{l_a} \frac{d}{dt} p = -\frac{i\omega p}{l_a}$, with $l_a$ some effective length of the antenna. The antenna impedance can then be recognized as

$$Z_a = \frac{V_a}{I_a} = -\frac{l_a^2}{i\omega \alpha}. \quad (4.2)$$

If $\alpha_0$ is assumed to be given by the classical Lorentz model (as also done in Section 2.2), we find

$$Z_a = -\frac{l_a^2}{i\omega} \left( \frac{\omega_0^2}{\beta} - \frac{\omega^2}{\beta} - \frac{i\omega\gamma}{\beta} - \frac{k^3}{6\pi\epsilon_0} \right) \quad (4.3)$$

*Following [179], we use scalar fields and dipole moments, which is valid for spherical particles or ellipsoidal particles driven along one of the main axes. To be consistent with the other chapters, however, we use a time convention $e^{-i\omega t}$, whereas the opposite convention is used in [179].
which maps onto the impedance of an inductor \( L_s = l_a^2/\beta \), a capacitor \( C_s = \beta/(l_a^2 \omega_0^2) \) and two resistors \( R_s = l_a^2 \gamma_i/\beta \) and \( R_r = \eta_0 k^2 l_a^2/(6\pi) \), with \( \eta_0 = (\epsilon_0 c)^{-1} \) the vacuum wave-impedance, in series connection. Here, \( \beta, \omega_0 \) and \( \gamma_i \) are the antenna oscillator strength, resonance frequency and ohmic damping rate, respectively. We can also write this as the sum of the quasi-static antenna impedance \( Z_s = R_s - i\omega L_s - (i\omega C_s)^{-1} \) and the radiation resistance \( R_r \). The corresponding equivalent circuit is thus composed of a voltage source driving a series connection of \( Z_s \) and \( R_r \), and is shown as the rightmost circuit in Fig. 4.1a.

To study Purcell enhancement, the external field is now assumed to come from a nearby emitter, which is modelled as a constant-current source [18, 179] with internal impedance \( Z_e + R_e \), where \( Z_e \) is entirely reactive and determines the emitter resonance frequency, while \( R_e = \eta_0 k^2 l_e^2/(6\pi) \) describes direct radiation to the background medium (vacuum).† Here, \( l_e \) is some effective length

†Note that, since the emitter is taken as a constant current source, the emitter reactance \( Z_e \) has no effect on the Purcell effect and could also be omitted here. However, it can be used for example to find the eigenfrequencies of the coupled emitter-antenna system [177], for which the current source is removed.
4.2 Equivalent circuits for an emitter coupled to a nano-antenna

of the emitter and the current of this source is \( I_e = -i\omega p_e / l_e \), with \( p_e \) the emitter dipole moment. The corresponding emitter circuit is shown as the left circuit in Fig. 4.1. Note that this method of retrieving the Purcell effect is completely analogous to that used in Section 2.4. The emitter at location \( r_e \) creates a field at the antenna location \( r_a \) given by the Green’s function of the background medium \( G_{bg} = G_{bg}(r_e, r_a) \). In circuit terms, this is included as an induced electromotive force (IEMF) \( \varepsilon_{a-e} = G_{bg} p_e \) in the antenna circuit, which acts as its voltage source. The antenna, in turn, produces a field at the emitter location which is described by the same Green’s function due to reciprocity. This is similarly included as an IEMF \( \varepsilon_{e-a} \) placed in the emitter circuit, which is opposite to the driving current \( I_e \). The effect of this IEMF on the emitter can be captured by a mutual impedance \( Z_m = -\varepsilon_{a-e} / I_e \) placed in the emitter circuit, which accounts for the contribution of the antenna to the power emitted by the source. Inserting the expressions for the IEMFs and for the antenna dipole moment, one finds

\[
Z_m = \frac{l_e^2 l_a G_{bg}^2}{\omega^2 Z_a} = \frac{1}{(-i\omega L_s')^{-1} - i\omega C_s' + (R_s')^{-1} + (R_r')^{-1}}. \tag{4.4}
\]

This describes the impedance of a parallel circuit of an inductor \( L_s' = N^2 C_s \eta_0^2 \), capacitor \( C_s' = L_s / (\eta_0 N^2) \), and resistors \( R_s' = N^2 \eta_0^2 / R_s \) and \( R_r' = N^2 \eta_0^2 / R_r \), where \( N = l_e l_a G_{bg} / (\omega \eta_0) \) is a dimensionless transformer parameter. Therefore, one may draw a single equivalent circuit for the total system as shown in Fig. 4.1b, where the antenna is included in the emitter circuit as this parallel circuit. We may again separate the quasi-static antenna impedance \( Z_s' \) and the radiation resistance \( R_r' \). The total impedance in the circuit is \( Z_{tot} = Z_e + Z_m + R_e \). The Purcell enhancement or LDOS relative to vacuum can be obtained by the ratio of the total power \( P = \frac{1}{2} |I_e|^2 \text{Re} \{Z_{tot} \} \) delivered by the emitter coupled to the antenna, to that of the emitter in vacuum. This leads to exactly the same expression as found from our coupled-oscillator model, in absence of the cavity (Eq. (2.47)).

4.2.2 The circuit proposed by Engheta et al.

In their seminal work — one of the first to discuss the analogy between plasmonic antennas and circuit theory — Engheta et al. propose a different circuit to describe a nano-antenna driven by an external field \( E_{ext} \) [174]. This circuit has later proven instrumental in explaining, for example, the effect of dielectric perturbation on an antenna in terms of the well-known concept of antenna loading [175, 176]. The existence of a second circuit for the same physical problem may seem confusing, as one would expect such a circuit to be uniquely defined. Here we therefore briefly discuss this circuit, and show that it is, in fact, equivalent to the circuit proposed by Krasnok.

For the case of an illuminated, optically small nano-sphere of relative permittivity \( \varepsilon \) and radius \( a \) in vacuum, Engheta et al. examine the solution for
Cavities as conjugate-matching networks for antennas at optical frequencies

the various field components inside and outside the particle [174]. Employing
the boundary condition on the surface of the particle and neglecting radiation,
this approach leads to the following equation for the displacement currents in
the particle

\[-i \omega \epsilon_0 (\epsilon - 1) \pi a^2 E_{\text{ext}} = -i \omega \epsilon_0 \epsilon \pi a^2 \frac{\epsilon - 1}{\epsilon + 2} E_{\text{ext}} - i \omega \epsilon_0 2 \pi a^2 \frac{\epsilon - 1}{\epsilon + 2} E_{\text{ext}}. \tag{4.5}\]

Each of the three terms represents a current, which they respectively name
(from left to right in Eq. (4.5)) the 'impressed displacement current source'
\(I_{\text{imp}}\), the 'displacement current circulating in the nano-sphere' \(I_{\text{sph}}\), and the
'displacement current of the fringe field' \(I_{\text{fringe}}\), respectively. Since currents
are added, this effectively maps onto a parallel circuit as shown in Fig. 4.2a.
They define the voltage across the elements as the average potential difference
between the two hemispheres \(V = a(\epsilon - 1)E_{\text{ext}}/(\epsilon + 2)\), such that impedances
can be found as

\[Z_{\text{sph}} = -\frac{1}{i \omega \epsilon_0 \epsilon a}, \quad Z_{\text{fringe}} = -\frac{1}{i \omega \epsilon_0 2 \pi a}. \tag{4.6}\]

The fringe impedance \(Z_{\text{fringe}}\) is entirely reactive and describes a capacitor,
while the sphere impedance \(Z_{\text{sph}}\) can be partly resistive if \(\epsilon\) contains an im-
aginary part. If \(\text{Re}\{\epsilon\} < 0\), as is the case for example in noble metals at visible
frequencies [180], the sphere impedance can be interpreted as a parallel re-
sistor (representing ohmic losses) and an inductor which, combined with the
fringe capacitance, determines the resonance frequency \(\omega_0 = (L/C)^{-1/2}\).

The circuit in Fig. 4.2a can be compared to that proposed by Krasnok and
shown on the right in Fig. 4.1a, without the radiation resistance \(R_\text{r}\). The
two are very different — not just the impedances of the individual lumped
elements but even the manner in which they are connected, series or parallel,
differs. We thus have two completely different circuits describing the same problem. If both are correct, then how could we reconcile these two points of view? The difference between the circuits originates in different definitions of current and voltage. One has freedom in choosing these, provided that the dimensions remain correct. For example, the currents in Eq. (4.5) could have been multiplied by any dimensionless number, and the equation would still hold. However, a different choice of current and voltage can lead to different impedances and powers consumed by the circuit.

To reconcile the two pictures, we can redefine the current and voltage in the circuit proposed by Engheta, by multiplying each with a dimensionless quantity. We choose

\[ V' = V(\epsilon + 2)/(\epsilon - 1) = aE_{\text{ext}} \quad \text{and} \quad I'_{\text{imp}} = 4I_{\text{imp}}/(\epsilon + 2) = -i\omega \alpha_0 E_{\text{ext}}/a, \]

such that \( V' \) and \( I' \) now match the total voltage and current in the circuit proposed by Krasnok (with \( a \) taking the role of effective length \( l_a \)). The impedances of the elements should then be multiplied by \((\epsilon + 2)^2/(4(\epsilon - 1))\) to ensure \( Z' = V'/I' \). One then finds a total impedance of \( Z' = -a^2/(i\omega \alpha_0) \), matching the impedance of the circuit by Krasnok in absence of radiation (Eq. (4.2)). Finally, we can use the Thévenin equivalent generator theorem to replace the current source and parallel impedances by a voltage source, which supplies a voltage \( V' \), connected in series with a single impedance \( Z' \). This transformed circuit is shown in Fig. 4.2b, and is equal to the antenna circuit in Fig. 4.1a. This shows that the two circuits are in fact equivalent.

The redefinition of the currents and voltages affects not only impedances but also the power consumption, since \( P = VI \). It is easy to verify that expressions from the two original circuits for power absorption \( P_{\text{abs}} \) through Ohmic damping are not equal. After the redefinition of current and voltage in the circuit by Engheta, we find that, if radiation is neglected, both circuits produce the same absorbed power \( P_{\text{abs}} = |E_{\text{ext}}|^2\omega|\alpha_0|^2 \text{Im}\{-\alpha_0^{-1}\}/2 \). Similarly, radiated power can be found if we include the series-connected radiation resistance \( R_r \), which yields \( Pr = \frac{1}{2}|I_a|^2R_r = \frac{|E_{\text{ext}}|^2}{2\eta_0} \frac{k^4}{6\pi \epsilon_0} |\alpha|^2 \). By division over the irradiance of the external field \( |E_{\text{ext}}|^2/(2\eta_0) \) one obtains the familiar expression for the scattering cross-section of a dipolar particle [119].

\section*{4.3 The conjugate-matching limit}

Having established an equivalent circuit for an emitter coupled to a dipolar nano-antenna, we now move on to use the circuit to derive a fundamental limit on the power scattered by the antenna. This limit is set by the particle geometry and material, and is independent of its environment. In that sense it forms a counterpart to the well-known unitary limit [118, 181], which is independent of particle geometry yet dependent on the environment. We note that this derivation is completely analogous to that given in chapter 3 of [158] for the maximum power scattered or absorbed by a nanoparticle. We merely repeat it here because it is important to understand the case of a
Cavities as conjugate-matching networks for antennas at optical frequencies

single antenna, before we include the cavity in Section 4.5. For simplicity, and because the phenomena we will discuss can be understood from the antenna circuit alone, we will restrict our discussion to the circuit describing a nano-antenna driven by an external field (rightmost circuit in Fig. 4.1a).

![Figure 4.3: A generator driving a load.](image)

The generator is a fixed voltage source of voltage $V_g$, with an internal impedance $Z_g$, while the load is represented by its complex impedance $Z_L$. The Thévenin equivalent generator theorem ensures that any generator circuit can be represented in this manner.

In circuit theory it is well known from the maximum power transfer theorem that the maximum power delivered by a generator to a load (see Fig. 4.3) is given by the 'conjugate-matching limit' — the power delivered to the load when load and generator impedances $Z_g$ and $Z_L$ are conjugate-matched, that is $Z_L = Z_g^*$ [182, 183]. This power is given as

$$ P_{cm} = \frac{|V_g|^2}{8R_g}, \quad (4.7) $$

where $R_g = \text{Re} \{Z_g\}$. For load resistance $R_L$ higher than $R_g$, the transfer efficiency $R_L / R_g$ may go up, however, the power $P_L$ delivered to the load decreases. Note the difference between conjugate matching and the well known impedance matching condition ($Z_L = Z_g^*$), which minimizes reflections instead.

Conjugate matching sets a limit on the scattered power by a lossy nano-antenna, as is evident from the antenna circuit in Fig. 4.1a. Scattering is given by the power $P_L$ consumed in the radiation load $Z_L = R_{\omega}$, and the rest of the circuit (that is, the driving voltage and the quasi-static antenna) can be interpreted as the generator. Figure 4.4a-c shows scattered power as well as load and generator impedances for the specific example of a spherical antenna of 5 nm radius in vacuum, with permittivity given by the Drude model for gold [184]. We see that scattering peaks at the resonance frequency $\omega_0$, which is given as the point where $Z_L = Z_g^*$, i.e. the imaginary parts of load and generator impedances are matched. Their real parts, however, are not matched for this particle, so scattering remains below the conjugate-matching limit. We can find the requirements for reaching this limit by setting $Z_L = Z_g^*$. For a particle described by a Lorentzian polarizability (Eq. (4.3)) — as is the case for a Drude
4.3 The conjugate-matching limit

Figure 4.4: Conjugate matching with a nano-antenna. We show scattered power (a,d,g), real (b,e,h) and imaginary (c,f,i) parts of the generator and load impedances \( Z_g = Z_s \) and \( Z_L = R_r \), respectively, for spherical gold antennas of 5 nm (a-c), 11 nm (d-f) and 21 nm (g-i) radius. The imaginary parts of the impedances are always conjugate-matched at the resonance frequency, yet only for an antenna of 11 nm radius the real parts also match, allowing this antenna to reach the conjugate-matching limit (solid grey line in (a,d,g)). For the largest antenna, scattering is limited by the unitary limit (dashed grey line (g)) more than by the conjugate-matching limit. Note that y-axes are different for each antenna, and that for the smaller antennas, the unitary limit lies far above the conjugate-matching limit. For calculating impedances, we set \( l_a = 1 \).

A metal sphere in vacuum—we then find

\[
\omega = \omega_0 \quad \text{and} \quad a^3 = \frac{3\gamma_0 c^3}{2\omega_0^4}, \tag{4.8}
\]

which is reached for a ‘critical radius’ \( a_c \) of \( \sim 11 \) nm for Drude gold. Note that gold is no longer well described by a Drude model at the frequencies shown in Fig. 4.4. A critical radius can, however, still be found for realistic (tabulated) permittivities. Figure 4.4d-f show that indeed, this antenna reaches the conjugate-matching limit. This coincides with the radius for which antenna albedo \( A \) is exactly 50%, that is, half the total consumed power is radiated and the other half is absorbed in the antenna. Scattered power is then given by Eq. (4.7) as \( P_L = P_{cm} = |E_{ext}|^2/8\gamma_0 \). For antennas larger

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Note: The figure and text describe the behavior of nano-antennas, showing how they can reach the conjugate-matching limit, a condition under which the generator and load impedances are perfectly matched, allowing for maximum power transfer. The analysis involves the scattering of power and the calculation of impedances for different antenna sizes, with a focus on the critical radius and the unitary limit. The equations provided capture the theoretical framework for understanding these phenomena.
than this critical radius, $P_L$ falls below the conjugate-matching limit, as shown in Fig. 4.4g-i. Note, however, that in an absolute sense these particles do scatter more, as the conjugate-matching limit grows in proportion to volume ($\beta \propto a^3$). The conjugate-matching limit could therefore also be interpreted as a fundamental limit on the scattering per unit volume by a dipolar particle of a given lossy material. For particle size much beyond the critical radius (or a very low loss rate $g_i$), the conjugate-matching limit forms only a very loose constraint. Instead, these particles approach another fundamental limit — the unitary limit. As discussed in Section 3.3, this limit follows from energy conservation and bounds the extinction and scattering cross-sections to $\sigma_{ul} = \frac{3\lambda^2}{(2\pi n^2)}$ [118, 154, 155, 181]. Consequently, the scattered power is bound by $\sigma_{ul}$ times the driving beam irradiance. The fulfillment of this upper bound is guaranteed in the circuit model by the radiation resistance $R_r$, which is independent of the particle but dependent on the surrounding medium as $R_r \propto n$ (see Eq. (4.1)).

This analysis shows that there are two fundamental limits that govern the scattered power by a dipolar nano-antenna. The conjugate-matching limit depends only on the antenna and poses the strongest constraint for antennas small or lossy enough for dissipation to be the dominant loss source. Particles that are sufficiently big to be dominated by radiation, on the other hand, are constrained by the unitary limit, which depends not on the particle but on the environment. In general, this limit is inversely proportional to the local density of states of the antenna surrounding [118]. This implies that if one aims at designing an antenna for optimal scattering in a fixed environment, the unitary limit provides the ultimate bound. If, however, one aims at designing the environment instead (for example, in a Drexhage-type experiment [185]), the ultimate bound is set by the conjugate-matching limit. In Section 4.5, we will see how a cavity coupled to the antenna can be used to reach this limit.

### 4.4 An equivalent circuit for a hybrid system

Following a similar approach as used in Section 4.2.1 for an antenna, we can find an equivalent circuit for a hybrid antenna-cavity system. Based on the equations of motion for the coupled system (Eqs. (2.33) and (2.34)), this circuit would involve three separate circuits for emitter, antenna and cavity, which are all coupled to each other through induced electromotive forces or induced currents (for the emitter driving the cavity). For simplicity, however, we neglect direct cavity-emitter coupling, focusing only on the effect of the cavity on the antenna. Effectively, this corresponds to neglecting the cross-terms and cavity term responsible for the LDOS contributions shown in Fig. 3.2b and c, respectively. This approach is accurate if the antenna is simply driven by a plane wave, or for driving by an emitter if antenna-emitter coupling is much stronger than the cavity-emitter coupling. In this case, we can combine the
4.5 Conjugate matching in a hybrid system

The equations of motion into one equation and lump the effect of the cavity into the hybridized antenna polarizability $\alpha_H$, given by Eq. (2.36) as

$$\alpha_H^{-1} = \alpha_0^{-1} - i \frac{k^2}{6\pi\epsilon_0 n^2} - \chi_{\text{hom}}, \quad (4.9)$$

with

$$\chi_{\text{hom}} = \frac{1}{\epsilon_0 \epsilon V_{\text{eff}} \omega_c^2 \omega - \omega^2 - i\omega\kappa}, \quad (4.10)$$

the bare cavity response function. The hybridized antenna impedance $Z_{a,\text{hyb}}$ is given by Eq. (4.2). If we assume again a Lorentz model for $\alpha_0$, its equivalent circuit contains the familiar antenna elements from Fig. 4.1a, connected in series with the cavity impedance $Z_c = -l_a^2 \chi_{\text{hom}}/(i\omega)$ representing a parallel RLC connection. This circuit is shown in Fig. 4.5a. Similar to the case of an antenna only, to study LDOS effects one can construct an equivalent circuit describing the interaction with the emitter by finding the emitter-hybridized-antenna mutual impedance $Z_{m,\text{hyb}}$. This yields

$$Z_{m,\text{hyb}}^{-1} = Z_m^{-1} + (Z_c')^{-1}, \quad Z_c' = i \frac{N^2}{l_a^2} \frac{\eta_0^2 \omega}{\chi_{\text{hom}}}. \quad (4.11)$$

Figure 4.5 shows this circuit. In the following section, we will proceed to analyse how the presence of the cavity affects the antenna radiation and LDOS in this system.

4.5 Conjugate matching in a hybrid system

Apart from the emitter radiation into the background medium, the power consumed by the circuit — and thus the LDOS — is entirely determined by the impedance $Z_{a,\text{hyb}}$ of the hybridized antenna, up to a pre-factor containing the emitter-antenna coupling. We can therefore simply analyse the hybridized antenna circuit on the right in Fig. 4.5a to learn how the cavity affects the LDOS and the radiation by the antenna.

The effect of the cavity is very different for small and large antennas. Figure 4.6 shows power consumption $P_L$ by the load, as well as impedances of the load and generator in the hybrid circuit from Fig. 4.5a. We define the load as the antenna radiation resistance $R_L$ and the cavity impedance $Z_c$ combined, while the generator impedance is set by the quasi-static antenna impedance $Z_a$, just as for a bare antenna. This ensures that the conjugate-matching limit is the same as for the bare antenna. Moreover, as cavities are typically made of lossless dielectrics, we can assume their losses to be radiative as well. For a small antenna (radius below the critical radius of 11 nm), Fig. 4.6a-d shows that a cavity can help the system reach the conjugate-matching limit. However, this is only reached for a specific cavity quality.
Cavities as conjugate-matching networks for antennas at optical frequencies

Figure 4.5: Equivalent circuits for a hybrid antenna-cavity system. (a) Two circuits describing emitter and hybridized antenna, coupled through IEMFs. The cavity is modelled by a parallel RLC connection in the antenna circuit. (b) Just as in Fig. 4.1, the coupled circuits in (a) can be replaced by a single circuit.

factor $Q$ or, equivalently, cavity Purcell factor $F_P$. Indeed, we see that only this cavity reaches perfect conjugate matching, while cavities with different $Q$ may match both real and imaginary parts of $Z_L$ and $Z_g$, yet not at the same frequency. This surprising result shows that there is an ‘optimal’ cavity for which power transfer to the load is maximized — a highly counter-intuitive result, considering that conventional cavity wisdom states that higher $Q$ is always better. It can be shown that such an optimal $Q$ can always be found for antennas with $R_t < R_s$ (that is, albedo < 50%). This behaviour is in stark contrast with that of a hybrid containing a large antenna (radius above the critical radius) shown Fig. 4.6e-h. This system cannot reach conjugate matching and is bound instead by the unitary limit. This can be seen from the impedance, as the cavity can only increase load resistance $\text{Re} \{Z_L\}$. Since this antenna has $\text{Re} \{Z_g\} < R_t$, the cavity thus cannot bring the system to perfect conjugate matching. Reactances can be matched, however, leading to a resonance peak. Here, there is no optimum and instead, peak scattering grows with $Q$, saturating for very high $Q$. We note that in reality, the unitary limit can be exceeded in a hybrid system, as discussed in Section 3.3. This is
Figure 4.6: Conjugate matching in a hybrid system. We show broadband spectra of load power $P_L$ (a,e) and narrowband spectra of load power (b,f), real (c,g) and imaginary (d,h) parts of the generator and load impedances $Z_g$ and $Z_L = R_r + Z_c$, respectively, for hybrid antenna-cavity systems with spherical gold antennas of 5 nm (a-d) and 21 nm (e-h) radius. Narrowband spectra are shown for three different cavity Q-factors. For the small antenna, the conjugate-matching limit (solid horizontal grey line in (a,b,e,f)) can be reached near the cavity resonance, yet only for a specific cavity $Q$. This can also be seen from $\text{Re} \{Z\}$ and $\text{Im} \{Z\}$, which are simultaneously matched only for this cavity $Q$. Conversely, for a large antenna conjugate matching cannot be reached, as $\text{Re} \{Z_g\} < R_r$ and the cavity can only increase $\text{Re} \{Z_L\}$. Dashed vertical lines show peak frequencies and the dashed grey curves in (e,f) shows the unitary limit. Again, y-axes are different for each antenna. We take cavities with $V_{\text{eff}} = 10\lambda^3$, red-detuned from the antenna by one antenna linewidth. We set $l_a = 1$. 
due to interference effects arising from the direct emitter-cavity coupling that was neglected in this analysis.

In analogy with circuit design, we may also interpret the cavity as a matching network between the generator (quasi-static antenna) and the load (radiation). Such matching networks are typically designed to maximize power transfer by bringing the generator and load impedance to conjugate matching. Matching networks can be easily designed at radio frequencies, where high-quality lumped circuit elements are available. However, at THz frequencies and above, such elements are lacking, making matching circuits very difficult to attain. Our results show that optical cavities may provide this functionality, offering perfect conjugate matching networks for small optical antennas.

4.6 Conclusion and outlook

We have compared two distinct circuit models from literature for a nano-antenna, and shown that the two models are equivalent. Using this circuit, we have discussed how the well-known maximum power transfer theorem sets a fundamental upper bound on the radiation by a lossy dipolar nano-antenna. This 'conjugate-matching limit' — which is reached if the generator (antenna) and load (radiation) impedances are conjugate-matched — is independent of the antenna environment and complements the unitary limit, which depends on the photonic environment yet is independent of antenna geometry.

In analogy with the antenna circuit, we then proposed an equivalent circuit for a hybrid antenna-cavity system, driven by an external field or by a fluorescent emitter. It was shown that the cavity can help the antenna reach the conjugate-matching limit, if the antenna is sufficiently small to have albedo below 50%. Surprisingly, we find an 'optimal’ cavity $Q$ for which this limit is reached. For antennas with albedo larger than 50%, we find that perfect conjugate matching cannot be reached by introducing a cavity, and scattering is bound instead by the unitary limit.

By making a connection between nanophotonics and the well-established field of electrical circuit theory, we have thus gained insight into the fundamental limits governing nano-antenna scattering. This directly provides a tool for the design of strongly scattering antennas or high Purcell factor systems, showing that cavities can be viewed as matching networks for optimizing the power transfer from optically small nano-antennas to radiation. We expect that the circuit analogy presented here can lead to further developments in unifying the fields of nanophotonics and electrical engineering, possibly leading to improved nanophotonic designs.