F-invariance and its application to the quantum Hall effect
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Chapter 4

The integer edge

4.1 Introduction

In this chapter we investigate edge excitations in the iqHe regime. We begin in section 4.2 by reviewing and extending the instanton vacuum approach, following the ordinary replica formalism for free electrons. We show that massless edge excitations are obtained from the fluctuations about integer quantised topological charge. We find a direct relationship between the instanton parameter $\sigma_{xy}$ and the phenomenon of inter-channel scattering at the edge. Here the number of edge channels equals the number of fully occupied Landau levels, and the inter-channel scattering is the result of a random short-ranged potential. In spite of the free boundary conditions on $Q$, it is still possible to formulate a topological principle for the quantisation of $\sigma_{xy}$. We use a background field calculation to show that spherical boundary conditions are dynamically generated.

In section 4.3 we embark on the full interaction problem. We first consider the case of a bulk d.o.s. gap and derive a complete chiral edge theory for the $Q$-field, the plasmon field and $A_\mu$. Then, using results from chapter 2, we extend the result to a theory for both bulk and edge and integrate out the plasmon field. As two limiting cases for zero edge d.o.s. and bulk d.o.s. we obtain effective actions $S_{eff}[Q, A_\mu]$ in the bulk and at the edge, respectively.

In section 4.4 we show that the effective action $S_{eff}[Q, A_\mu]$ at the edge is equivalent to a theory of chiral edge bosons. This demonstrates the connection between the instanton vacuum on the one hand and the pure Chern-Simons theory and chiral edge bosons on the other. Our chiral boson action represents the complete Luttinger liquid theory, in the sense that it includes Coulomb interactions and external electromagnetic fields. We use this theory to define the Hall conductance in a general, unambiguous way by expressing the appearance of a ‘chiral anomaly’ [5] in terms of Laughlin’s gauge argument [57].

In the derivations presented in sections 4.3 and 4.4 we make use of $\mathcal{F}$-invariance and the $\mathcal{F}$-algebra in an essential way. This shows that $\mathcal{F}$-invariance, which we put to the test in the weak coupling regime in chapter 3, retains its significance all the way to strong coupling.

In all the discussions so far, only white noise random potentials have been considered. This was always done for technical reasons alone. However, it is well known that in real quantum Hall devices slowly varying potentials are often present [57, 67]. These have always been difficult to handle. By defining our edge theory on spatially separated, independent channels which represent the equipotential contours, we obtain a description
that is able to treat the combination of interactions and smooth disorder.

In sections 4.5 and 4.6 we solve two problems where this combination gives rise to unexpected results. The first problem we address is that of electron tunnelling into the quantum Hall edge. We show that the Coulomb interactions between the edge and the ‘localised’ bulk orbits dramatically change the predictions of theories which are based on isolated edges alone. Tunnelling processes into the quantum Hall edge have, in fact, nothing to do with the quantisation of the Hall conductance or the ‘incompressibility’ statement which characterise the non-equilibrium properties of the electron gas. We find that the tunnelling d.o.s. near the edge can be understood in terms of an effective edge theory which describes the equilibrium properties of the combined edge and bulk degrees of freedom. The Luttinger liquid parameter \( g \) is related to the filling fraction \( \nu \) of the bulk. This leads to a tunnelling exponent which varies like \( 1/\nu \), in agreement with recent experimental data on the tunnelling current, taken from samples in the fractional quantum Hall regime \([27]\). This situation is dramatically different from what is expected by assuming an isolated edge, or in the case of short-ranged disorder which gives rise to scattering between different edge states \([36, 37]\).

In section 4.6 we embark on the problem of plateau transitions in the presence of smooth disorder. This we model as a percolating network of ‘edge states’ (equipotential contours) and widely separated ‘saddlepoints’. The system is then ‘mapped’ onto the nonlinear \( \sigma \) model and the main problem is to identify the length and energy scales of the ‘bare’ parameters or the mean field conductances which together determine the renormalisation starting point, i.e. the point where scaling occurs first. This starting point can involve, in principle, arbitrarily large distances and arbitrarily small energies, which obviously complicates the observability of the critical behaviour of the plateau transitions. We argue that Coulomb interaction effects lead to a modified mean field theory of transport. This, then, might conceivably be the explanation for the empirical fits of recent transport data \([71, 4]\). In section 4.6.5 the chiral boson theory is used to actually compute the inelastic relaxation rate of the conducting electrons in the saddlepoint network.

### 4.2 Edge excitations

#### 4.2.1 Sigma model

Let us recall the instanton vacuum theory \([58, 59, 63]\) for the iqHe, which is expressed in terms of the local field variables \( Q_{p p'}^{\alpha \beta} \), where \( \alpha, \beta = 1, \ldots, N_r \) are the replica indices and \( p, p' = \pm 1 \) are the indices denoting advanced/retarded components. They can be represented as

\[
Q = T^{-1} \Lambda T
\]

with

\[
\Lambda_{p p'}^{\alpha \beta} = \delta^{\alpha \beta} \delta_{p p'} \text{sgn}(p)
\]

and \( T \) a unitary matrix of size \( 2N_r \times 2N_r \). The action is given by

\[
S[Q] = -\frac{1}{2} \sigma_{xx}^0 \int d^2 x \text{ tr } (\nabla Q)^2 + \frac{1}{8} \sigma_{xy}^0 \int d^2 x \text{ tr } \epsilon_{ij} Q \partial_i Q \partial_j Q + \pi \rho \eta \int d^2 x \text{ tr } \Lambda Q. \tag{4.2.2}
\]

Here \( \sigma_{ij}^0 \) stands for the mean field conductances in units of \( e^2/h \) (see Fig. 1.2), \( \rho \) is the (exact) density of states at the Fermi energy and \( \eta \) is the frequency. The second term in (4.2.2), proportional to \( \sigma_{xy}^0 \), has remained one of the most difficult chapters in the theory of Anderson localisation in low dimensions. Most of the insight into the problem has
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come from weak coupling renormalisation theory, both perturbative and non-perturbative [58, 59]. In particular we mention the global scaling diagram of the conductances (Fig. 1.2) as well as the appearance of a critical fixed point in strong coupling regime [57]. This fixed point theory predicts a massless (metallic) phase at the Landau band center as well as the following scaling result for the conductances [62]

\[ \sigma_{ij}(L, B) = g_{ij}([L/\xi]^{1/\nu}) \quad ; \quad \xi = |B - B^*|^{-\nu}. \] (4.2.3)

Here, the function \( g_{ij} \) is a regular function of its argument, \( B^* \) is the critical magnetic field strength and \( \nu \) stand for the critical index for the localisation length \( \xi \). Following the experimental tests of (4.2.3) by H.P. Wei et al. [76], extensive numerical work on the free electron gas has been performed and the quoted best value for the critical index is \( \nu = 2.3 \) [33].

To date, no exact (conformal) scheme for the critical indices exists. All that one can say at this time is that the field of exactly solvable models is not sufficiently developed to be able to handle the subtleties of topology and replica field theory. These subtleties are all well understood in the framework of weak coupling expansion techniques [58, 59] and the results were used to predict the entire singularity structure of the theory, notably (4.2.3).

It was shown in [64] that the theories of free and interacting electrons share the same basic features such as asymptotic freedom, instantons etc. The same scaling diagram for the conductances was obtained, which means that (4.2.3) remains valid also when the Coulomb interactions are taken into account. This result was conjectured but not well understood at the time of the original experiments on criticality.

**Strong coupling**

In this section we address the subtleties of replica field theory in the exactly solvable limit where \( \rho \) and \( \sigma_n^\alpha \) vanish while the Hall conductance is integer quantised (\( \sigma_{xy} = m \)). Physically this happens when the Fermi energy is located in a d.o.s. gap between adjacent Landau bands. In this strong coupling limit, massless excitations exist at the edges of the system. Since several, basic aspects of the problem have previously gone unnoticed we will proceed first within the free electron formalism (4.2.2). We come back to the fermionic path integral in section 4.3.

For \( m \) completely filled Landau levels the action becomes simply

\[ S[Q] = \frac{m}{8} \int d^2 x \, \text{tr} \, \epsilon_{ij} \partial_\alpha Q \partial_\alpha \partial_\beta Q = \frac{m}{2} \oint d\vec{r} \cdot \text{tr} (\Lambda T \nabla T^{-1}) \] (4.2.4)

where the surface integral is taken over the sample's edge. Recall that (4.2.4) is quantised according to

\[ S[Q] = 2\pi im \cdot q[Q] \] (4.2.5)

with \( q \) the integer topological charge, provided that the \( T \)-matrix reduces to a \( U(N_r) \times U(N_r) \) gauge at the edge [63].

Under these circumstances the sample edge has been contracted to a single point (spherical boundary conditions) and (4.2.5) is a realisation of the homotopy theory result \( \pi_2[U(2N_r)] = \pi_2[U(N_r) \times U(N_r)] = \mathbb{Z} \) which states that the mapping of \( Q \) onto the 2D plane is described by a set of integers \( q \). It is natural to take the theory one step further and propose the quantisation of the charge \( q[Q] \) as the topological principle which forces the
Hall conductance itself to be integer quantised. The idea has indeed led to a consistent quantum theory of conductances that unifies a fundamental aspect of asymptotically free field theory (i.e. dynamic mass generation) with the quantum Hall effect [58, 59]. It says that the conductances in (4.2.3) always scale toward $a_{yx}=0, a_{xy}=m$ for $L$ large enough.

One can show [63] that the $U(N_r) \times U(N_r)$ gauge condition at the edge is the replica field theory version of a static $U(1)$ gauge acting on the physical edge states. Such a $U(1)$ gauge implies that an integer number of edge levels has crossed the Fermi level. This level-crossing is necessarily induced by the averaging procedure over random potentials, because different disorder configurations at fixed $\mu$ lead to different numbers of electrons in the sample.

Nevertheless, the topological invariant in (4.2.2), as it was discovered originally in a microscopic derivation, is truly defined with free boundary conditions and without any separation between edge and bulk degrees of freedom [58, 59].

4.2.2 Inter-channel edge scattering

In what follows, we show that the fluctuations about precisely quantised values for the topological charge represent the dynamics of massless edge excitations. In order to see this, we write $T$ as the product of a $U(N_r) \times U(N_r)$ gauge $U$ and a small fluctuation $t$,

$$T = Ut.$$  

(4.2.6)

The action now becomes

$$S[Q] = 2\pi im \cdot q[U] + \frac{m}{2} \int \text{d}x \cdot \text{tr} (\Lambda t \nabla t^{-1}) + \pi \rho_e \eta \int \text{d}x \text{ tr} \Lambda Q$$

(4.2.7)

where we have written $\rho_e$ for the density of edge states. One way of identifying (4.2.7) as the effective theory of disordered chiral edge states is to redo the derivation of the $Q$-field theory, but now for the 1D system with Hamiltonian

$$H_{\text{edge}} = -iv_d \partial_x + V(x)$$

(4.2.8)

where $v_d$ is the drift velocity of the edge electrons and $V(x)$ the random potential. It turns out that our initial guess (4.2.8) is correct only in the case $m=1$. This problem is easily resolved once one realizes that $m$ stands for the number of filled Landau levels and that (4.2.8) should be replaced by a Hamiltonian for a total of $m$ edge channels. Hence, an obvious second guess is

$$H_{\text{edge}} = \sum_{j=1}^{m} H^{(j)}_{\text{edge}}$$

(4.2.9)

where $H^{(j)}_{\text{edge}}$ is the same for all $j$, i.e. each of the $m$ eigenstates experiences the same white noise potential $V(x)$, just as it appears in the original problem in two spatial dimensions. This, however, is not correct and the theory with general $m$, (4.2.7), necessarily requires inter-channel scattering to take place. We have to start from a matrix Hamiltonian

$$H^{ij}_{\text{edge}} = -iv_d \delta_{ij} \partial_x + V_{ij}(x)$$

(4.2.10)

where $V$ is a Hermitian matrix. The matrix elements $V_{ij}$ connect the edge channels $j, j'$ and are distributed with a weight

$$P[V] \propto \exp\{-\frac{1}{2} \int \text{d}x \text{ tr} V^2\}.$$  

(4.2.11)
One can construct a generating function for the free particle Green's functions as usual, according to

\[
Z = \int \mathcal{D}\bar{\psi}\psi \int \mathcal{D}V \mathcal{P}[V] \exp \sum_{p,\alpha,\alpha'} \int dx \bar{\psi}_p^{\alpha,j} \left[(\mu + i\eta_\alpha)\delta_{\alpha\alpha'} - \mathcal{H}_{\text{edge}}^{j,j'}\right] \psi_p^{\alpha',j}.
\]  

(4.2.12)

In appendix 4C we show that (4.2.12) and (4.2.7) describe the same physics in the limit of large distances.

### 4.2.3 Criticality at the edge

We next point out that the results of the previous section provide an exact solution to our topological edge theory (4.2.4–4.2.7) for all values of \(N_r\). The simple but important observation to be made is that the random potential \(V_{jj'}(x)\) in (4.2.12) can be 'gauged away', i.e. absorbed in a redefinition of the fermion fields, and all that remains is the trivial theory of 'pure' chiral edge states,

\[
\exp \int dx \bar{\psi}_p^{\alpha,j} [-iv_d \partial_x + i\eta_\alpha] \psi_p^{\alpha,j}.
\]

(4.2.13)

Eq. (4.2.13) is just a formal way of saying that edge electrons do not Anderson localise, because chirality excludes backscattering processes on random impurities. However, the conclusions that are simply drawn from edge electrons (4.2.12, 4.2.13) become highly nontrivial statements when formulated in the language of Q-field theory. In this section we will use the simplicity of the former theory in order to derive explicit results for the latter. Write

\[
G^j_j(x,x') = \langle x, j | (-\mathcal{H}_{\text{edge}} \mp i\eta)^{-1} | x', j' \rangle
\]

(4.2.14)

and

\[
G^\pm(x,x') = \langle x | (iv_d \partial_x \mp i\eta)^{-1} | x' \rangle
\]

(4.2.15)

to represent the single particle propagator of the dirty edge (4.2.12) and clean edge (4.2.13) respectively. Some useful identities and relations are obtained as follows,

\[
\rho_e = \frac{1}{2\pi} \sum_j \left[ G^{jj}_+(x,x) - G^{jj}_-(x,x) \right] = \frac{m}{2\pi v_d} \left[ G_-(x,x) - G_+(x,x) \right].
\]

(4.2.16)

Here \(\rho_e\) denotes the density of edge states at the Fermi level which can be obtained explicitly from the r.h.s.,

\[
\rho_e(x) = \rho_e = m/2\pi v_d.
\]

(4.2.17)

Eq. (4.2.17) shows that the density of edge electrons is a constant, independent of \(x\) and disorder, as it should be. An important conclusion now follows for the theory of Q-fields (4.2.7), namely

\[
\langle Q \rangle = \Lambda
\]

(4.2.18)

(where the expectation is with respect to (4.2.7)), which holds for arbitrary \(N_r\). This result may be obtained e.g. by differentiating both theories (4.2.7) and (4.2.12, 4.2.13) with respect to \(\eta\). Notice that (4.2.18) can be regarded as the 'order parameter' (analogous to the magnetisation in the language of the Heisenberg ferromagnet) and one would naively expect this quantity to vanish in one spatial dimension. The result \(\langle Q \rangle = \Lambda\) indicates, however, that the continuous symmetry is permanently broken at the edge of the instanton.
vacuum for all numbers of field components \( N_r \). This apparent violation of the Mermin-Wagner-Coleman theorem is clearly due to the lack of positive definite Boltzmann weights in our problem that is described by an imaginary action (4.2.7). Eq. (4.2.18) also indicates that the edge of the topological vacuum is critical. The simplest way of demonstrating this is by employing the background field method. For example, the replacement \( t \to t \cdot t_0 \) in the second term of (4.2.7) can be written as

\[
\int d\vec{x} \cdot \text{tr} [\Lambda t \nabla t^{-1}] \to \int d\vec{x} \cdot \text{tr} [\Lambda t_0 \nabla (t_0^{-1}t^{-1})]
\]

\[
= \int d\vec{x} \cdot \text{tr} [\Lambda t \nabla t^{-1}] + \int d\vec{x} \cdot \text{tr} [\Lambda t_0 \nabla t_0^{-1}] .
\] (4.2.19)

Here, \( t_0 \) represents a fixed and slowly varying background field. We obtain an effective action for \( t_0 \) as follows,

\[
S_{\text{eff}}[t_0] = \frac{m}{2} \int d\vec{x} \cdot \text{tr} [t_0 \nabla t_0^{-1}(Q)] = \frac{m'}{2} \int d\vec{x} \cdot \text{tr} [\Lambda t_0 \nabla t_0^{-1}] .
\] (4.2.20)

Eq. (4.2.20) defines an ‘effective’ parameter \( m' = m \text{tr} \Lambda(Q)/2N_r \) which can be identified as the ‘Hall conductance’ and which provides information on the renormalisation of the theory at large distances \([66]\). Apparently we have \( m' = m \). The same conclusion can be drawn for the \( \eta \) parameter (i.e. \( \eta' = \eta \text{tr} \Lambda(Q)/2N_r = \eta \)) and hence we are dealing with a critical fixed point theory! The full significance of this result will become clear in section 4.4 where we make contact with the theory of chiral edge bosons.

For the remainder of this section we will elaborate on several other identities that will be used later on. The most important pair correlation of the \( Q \)-fields can be obtained as follows,

\[
N(x, x') = \pi^2 \rho_e^2 \langle Q_{+}^\alpha(x)Q_{-}^\beta(x') \rangle = \sum_{\alpha', \beta'} G_{\alpha \beta}^{\alpha', \beta'}(x, x')G_{\alpha', \beta'}^{\alpha, \beta}(x', x)
\]

\[
= m G_{-}(x, x')G_{+}(x', x) .
\] (4.2.21)

Here, the \( \alpha, \beta \) are fixed but arbitrary replica channels and

\[
G_{-}(x, x')G_{+}(x', x) = \frac{i}{v_d} \int \frac{dk}{2\pi} \frac{\exp[ik(x'-x)]}{vk + 2i\eta} = \frac{1}{v_d} \theta(x' - x) \exp \left[ -\frac{2\eta}{v_d} (x' - x) \right] .
\] (4.2.22)

The step function \( \theta \) shows that a chiral electron, being created at position \( x \) and drifting in the positive direction, can only be destroyed at a ‘later’ position \( x' > x \). Notice that we have the sum rule

\[
\int dx' N(x, x') = \pi \rho_e / \eta .
\] (4.2.23)

The other pair correlations of the \( Q \)-fields vanish identically. In particular, it is straightforward to show that

\[
\langle Q_{pp'}^{\alpha}(x)Q_{p'p}^{\beta}(x') \rangle_{\text{cum}} = 0
\]

for all \( p, p' = \pm \) and all replica channels \( \alpha, \beta, \gamma, \delta \). Finally we wish to clarify the significance of several \( Q \)-field operators that have appeared in different contexts before. First, there are the higher order corrections to the theory of (4.2.7). In appendix 4C we derive higher dimensional operators of the type

\[
\text{tr} \left[ \frac{m}{2} \partial_x + \pi \eta \rho_e \Lambda(Q) \right]^2 .
\] (4.2.25)
Secondly, we mention the bilinear combinations of the form
\[ A_1 \text{tr} \Lambda Q \text{tr} \Lambda Q + A_2 \text{tr} [\Lambda, Q][\Lambda, Q] \tag{4.2.26} \]
which are known to describe the anomalous fluctuations in the density at the quantum Hall transitions, as well as in the localisation problem in \( 2+\epsilon \) dimensions \[60\]. We have already seen, however, that the density of chiral electrons does not fluctuate and we therefore expect (4.2.26) to be irrelevant. A classification of these operators follows from the classical equations of motion of the topological action (4.2.7) which can be written as
\[ \left[ \frac{\partial}{\partial x} + \pi \eta \rho_c \Lambda, Q \right] = 0. \tag{4.2.27} \]
This immediately implies that the higher order generators (4.2.25) are, in fact, redundant (i.e. they can be ‘gauged’ away by a suitable transformation of \( Q \)). Next, from the identity
\[ \int dx' \text{tr} \Lambda Q(x') \left[ \frac{\partial}{\partial x'} + \pi \eta \rho_c \Lambda, Q(x') \right] \Lambda = 0 \tag{4.2.28} \]
it directly follows that the first term in (4.2.26) is also redundant. Finally, from (4.2.27) one obtains
\[ \int dx' \text{tr} \left[ \Lambda, Q(x') \right] \left[ \frac{\partial}{\partial x'} + \pi \eta \rho_{\text{edge}} \Lambda, \left[ \Lambda, Q(x') \right] \right] = 0, \tag{4.2.29} \]
and it is readily seen that the second operator in (4.2.26) is redundant as well.

### 4.2.4 Topological principle

Next there are the difficulties in establishing a topological principle for Hall quantisation, as mentioned in section 4.2.1. We have to see whether the instanton vacuum approach is free of ambiguities. For this purpose we will follow up on the background field method. We write
\[ \exp S_{\text{eff}}[t_0] := \int DQ \exp \left( S_0[t_0^{-1}Qt_0] + \pi \rho \text{Tr} \Lambda Q \right) \tag{4.2.30} \]
where
\[ S_0[Q] = -\frac{1}{8} \sigma_{\text{xy}} \text{Tr} (\nabla Q)^2 + \frac{1}{8} \sigma_{\text{xy}} \text{Tr} \epsilon_{ij} Q \partial_i Q \partial_j Q. \tag{4.2.31} \]
Eq. (4.2.30) defines an effective action for the fixed \( t_0 \) matrix field. By making use of the local \( U(N_c) \times U(N_v) \) gauge invariance, one can show that \( S_{\text{eff}} \) is of the same form as \( S_0 \), i.e.
\[ S_{\text{eff}}[t_0] = -\frac{1}{8} \sigma_{\text{xy}} \text{Tr} (\nabla Q_0)^2 + \frac{1}{8} \sigma_{\text{xy}} \text{Tr} \epsilon_{ij} Q_0 \partial_i Q_0 \partial_j Q_0 \tag{4.2.32} \]
with \( Q_0 = t_0^{-1}A t_0 \). The main problem next is to obtain explicit knowledge of the ‘effective’ parameters \( \sigma_{ij} \) in (4.2.32), which now represent the (exact) Kubo expressions for the conductances. As long as one works with spherical boundary conditions on the \( Q \)-field, the quantisation of the Hall conductance is readily established. All that one needs is in fact that the theory develops a mass gap in the limit of large distances. Under these circumstances the insertion of slowly varying, spherically symmetric background fields \( t_0 \) should be immaterial (as \( \eta \to 0 \)) and this then necessarily implies that \( \sigma_{\text{xy}} = 0 \) and \( \sigma_{\text{xy}} = \text{integer} \). For example, the explicit renormalisation group flows obtained from instanton calculations always indicate that the quantum Hall conditions \( \sigma_{\text{xy}} = 0, \sigma_{\text{xy}} = \text{integer} \) are the stable (massive) fixed points of the theory for arbitrary number of field components \( N_r \).
It is well known, however, that spherical boundary conditions are justified in weak coupling only, due to finite action requirements. Since there is no compelling argument which says that spherical boundary conditions should be retained all the way down to the strong coupling regime, one is clearly left with ambiguities in the instanton vacuum approach, at least as far as the quantisation of the Hall conductance is concerned.

Armed with the insight gained from edge excitations in the previous sections, we next allow free boundary conditions on $Q$, as it should be. For the special case where the Fermi energy lies in a d.o.s. gap, (4.2.30) has already been worked out in detail. A straightforward computation of $S_{\text{eff}}$ for general $N$, leads to

$$S_{\text{eff}}[t_0] = 2\pi i m \cdot g[Q_0] - \frac{m^2}{32 \pi \hbar c} \int \text{tr} (\partial_x Q_0)^2.$$  

(4.2.33)

The result naively diverges as $\eta \to 0$ and this clearly reflects the effect of the massless excitations in the problem. In the limit $\eta \to 0$, the field $t_0$ entering (4.2.33) is forced to obey the classical equations of motion (defined along the sample edge),

$$\partial_x \{ Q(\eta - \frac{1}{2} v_d \partial_x Q) \} = 0$$  

(4.2.34)

The solution $Q_0 = \text{constant}$ at the edge means that spherical boundary conditions are automatically generated by the background field procedure as $\eta \to 0$. The effect of $S_{\text{eff}}$ has now been reduced to a phase factor which is immaterial provided the Hall conductance precisely equals an integer. Physically, the phase factor describes an integer number of edge electrons which have crossed the Fermi level as a result of the background field insertion.

The same procedure can be repeated for the theory with $\sigma_{\text{d}}^{\text{eff}} \neq 0$, making use of the fact that a mass gap exists in the system of large wavelength excitations, i.e. a finite localisation length $\xi$. One expects (4.2.33) to be modified according to

$$S_{\text{eff}}[t_0] = -\frac{\sigma_{\text{d}}^{\text{eff}}}{8} \text{Tr} (\nabla Q_0)^2 + 2\pi i \sigma_{\text{xy}} q[Q_0] - g_m L_{\eta} \int \text{tr} (\partial_x Q_0)^2$$  

(4.2.35)

where the $\sigma_{ij}$ represent the ‘conductances’,

$$\sigma_{\text{xx}} = f_{\text{xx}} (\eta \xi^2) \approx O(\eta \xi^2); \quad \sigma_{\text{xy}} = f_{\text{xy}} (\eta \xi^2) \approx m + O(\eta^2 \xi^4).$$  

(4.2.36)

g_m = m/2 is the quantised 1D conductance of the chiral edge states and $L_{\eta} = m/16\pi \hbar c$ is the length scale induced by the frequency. In the limit $\eta \to 0$ the $Q_0$ is forced to obey not only the classical equations of motion on the edge (4.2.34), but also those arising from the bulk kinetic term in (4.2.35). The solutions are known as instantons and just as has happened before in the trivial example with a d.o.s. gap in the bulk, the effect of $S_{\text{eff}}$ reduces to that of a phase factor which is immaterial as long as $\sigma_{\text{xy}} = \text{integer}$. Therefore, the quantum Hall effect can be understood in terms of a continuous symmetry which is dynamically restored in the limit of large length scales.

4.3 Derivation of the full edge theory

In this section we derive the full $Q$-field theory on the edge in the presence of gauge fields and interactions.
4.3.1 Preliminaries

Our starting point is the truncated \(\tilde{Q}\)-field theory in the presence of the plasmon field \(\lambda\) and external potentials \(A_\mu\) (2.2.18),

\[
S[A, \tilde{Q}, \lambda] = -\frac{1}{2g} \text{Tr} \, \tilde{Q}^2 + \text{Tr} \ln[i\omega + i\dot{A}_0 + i\dot{\lambda} + \mu - \mathcal{H} + i\tilde{Q}]
-\frac{1}{2} \beta \int d^2xd^2x' \, \lambda'(\vec{x})U_0^{-1}(\vec{x} - \vec{x}')\lambda(\vec{x}').
\] (4.3.1)

In order to facilitate calculations we choose our sample to be the half plane \(y > 0\). The notation \(\int d^2x\) stands for integration over this half plane. Integration over the boundary \(y = 0\) is denoted by \(\oint dx\). Integration over the infinite plane will be written as \(\int_\infty d^2x\).

A theory for the edge is obtained by choosing the chemical potential \(\mu\) approximately halfway between Landau energies, where the bulk d.o.s. is virtually zero if the disorder is not too strong. The saddle point equation for \(Q\) is given by

\[
\tilde{Q}_{\text{sp}} \propto \rho_0 T^{-1} \Lambda T
\] (4.3.2)

where \(\rho_0\) is the d.o.s. Since we are interested in the limit \(\rho_0 \to 0\), we may replace the full expression for \(\tilde{Q}\) by a much simpler one,

\[
\tilde{Q} \to \varepsilon T^{-1} \Lambda T = \varepsilon Q, \quad \varepsilon \ll 1.
\] (4.3.3)

From [63] we know that the forms \(\tilde{Q} = T^{-1}PT\) and (4.3.3) give rise to identical results as long as the bulk d.o.s. \(\rho_0\) can be safely taken to zero. However, in order to deal with the complications of \(U(1)\) gauge invariance, there is considerable advantage in working with the simplified expression \(\varepsilon Q\), and we will refer to the details of more elaborate analyses only when necessary.

In order to facilitate the expansion of the Tr ln term in (4.3.1) we perform a gauge transformation that sets \(A_0 + \lambda = 0\) by acting with a suitable \(W\)-rotation. We introduce the following notation

\[
A_\mu^{(\lambda)} = A_\mu + \delta_\mu\lambda, \quad W = \exp \left[-i \sum_n \frac{\lambda^{(n)}_n}{\omega_n} a_n^{(n)} \right], \quad R = WQW^{-1}.
\] (4.3.4)

The \(\lambda\) is absorbed into \(A_0\), and the gauge invariant quantity \(R\) represents the \(Q\)-matrix in the gauge where \(A_0^{(\lambda)} = 0\). We will sometimes use notation with derivatives in the denominator. For any bosonic function \(f\) we define

\[
(f)_{n} := f_n \omega_n, \quad n \neq 0
\] (4.3.5)

For functions without a zero frequency component (which is the only kind we are dealing with) it is easily checked that \(\partial_0 f = \delta_0 \frac{\partial f}{\partial \lambda} = f\). The vector potential \(\vec{A}\) in the gauge \(A_0^{(\lambda)} = 0\) is denoted as \(\vec{z}\)

\[
\vec{z} = \vec{A} - \nabla \lambda^{(\lambda)} = \vec{E} - \nabla \lambda,
\] (4.3.6)

where \(\vec{E} = \partial_0 \vec{A} - \nabla A_0\). The last expression shows that \(\vec{z}\) is gauge invariant. Using the notation of (4.3.4)-(4.3.6) the action (4.3.1) becomes

\[
S[Q, \lambda, A] = -\frac{1}{2} \beta \int d^2xd^2x' \, \lambda'(\vec{x})U_0^{-1}(\vec{x} - \vec{x}')\lambda(\vec{x}')
+ \text{Tr} \ln \left[i\omega + \mu - \frac{1}{2m_e} (\frac{\vec{E}}{\varepsilon} - \vec{z}) \cdot (\frac{\vec{E}}{\varepsilon} - \vec{z}) + i\varepsilon R \right],
\] (4.3.7)
with $\bar{n}^{st}$ the zero frequency part of $\bar{n}$ (2.2.6),

$$
\bar{n}^{st} = -i\nabla - \tilde{A}^{st}
$$

(4.3.8)

where $\nabla \times \tilde{A}^{st}$ is the static magnetic field. The superscript ‘st’ on $\bar{n}$ will be dropped from this point on.

### 4.3.2 Expansion of the Tr In

Let us take a closer look at the last term in (4.3.7), $X := \text{Tr} \ln[i\omega + \mu - \mathcal{H}_N + i\varepsilon R]$. Introducing the notation

$$
D_\omega = TW^{-1}\omega WT^{-1} \quad ; \quad D_\pi = TW^{-1}(\frac{i}{\omega} \nabla - \tilde{A})WT^{-1}
$$

(4.3.9)

(where $D_\pi$ is not a differential operator) we can write

$$
X = \text{Tr} \ln[iD_\omega + \mu + i\varepsilon \Lambda - \frac{1}{2m_e}(\bar{\pi} \cdot \bar{\pi} + \bar{\pi} \cdot D_\pi + D_\pi \cdot \bar{\pi} + D_\pi^2)].
$$

(4.3.10)

Expansion to first order in $D_\omega$ and $D_\pi$ yields

$$
X \approx \text{Tr} \ln G_0^{-1} + i\text{Tr} G_0 D_\omega - \frac{1}{2m_e} \text{Tr} [G_0 \bar{\pi} \cdot D_\pi + G_0 D_\pi \cdot \bar{\pi}]
$$

(4.3.11)

where $G_0$ is the bare Green's function $[\mu - \mathcal{H}_0 + i\varepsilon \Lambda]^{-1}$. The Green's function can be expressed in terms of the eigenfunctions $\varphi_{nj}$ of the bare Hamiltonian $\mathcal{H}_0 = \frac{1}{2m_e} \pi \cdot \bar{\pi}$.

$$
\langle x | G_0 | x \rangle = \sum_{nj} \frac{|\varphi_{nj}(x)|^2}{\mu - E_{nj} + i\varepsilon \Lambda}
$$

(4.3.12)

$$
\langle x | \frac{G_0 \bar{\pi} + \bar{\pi} G_0}{2m_e} | x \rangle = \sum_{nj} \frac{\varphi_{nj}^* \nabla \varphi_{nj} - \varphi_{nj} \nabla \varphi_{nj}^* - 2\varphi_{nj}^* \varphi_{nj} \tilde{A}^{st}}{2m_e(\mu - E_{nj} + i\varepsilon \Lambda)}.
$$

(4.3.13)

Using the general relation $\rho(x) = -\frac{1}{\pi} \text{Im} G^+(x, x)$ for the d.o.s. at the Fermi energy $\mu$, we get

$$
\langle x | G_0 | x \rangle = -i\pi \rho(x) \Lambda + c(x) \mathbb{1} \quad ; \quad \langle x | \frac{G_0 \bar{\pi} + \bar{\pi} G_0}{2m_e} | x \rangle = -i\pi j(x) \Lambda + \tilde{c}(x) \mathbb{1},
$$

(4.3.14)

where $j(x)$ is the current density per energy interval at the Fermi energy. The $c$ and $\tilde{c}$ are real functions that disappear from the last two traces in (4.3.11). We can now write $X$ in the form

$$
X \approx \text{Tr} \ln G_0^{-1} + \pi \int d^2 x \rho(x) \text{tr} \Lambda D_\omega + i\pi \int d^2 x \tilde{j}(x) \cdot \text{tr} \Lambda D_\pi
$$

(4.3.15)

$$
= \text{Tr} \ln G_0^{-1} + \pi \int d^2 x \rho(x) \text{tr} \omega R + i\pi \int d^2 x \tilde{j}(x) \cdot \text{tr} [\frac{1}{i} \Lambda W^{-1} \nabla (T^{-1} W) - \tilde{\varepsilon} R].
$$

Since $\mu$ lies in a gap, the d.o.s. and the current density are nonzero only at the edge. This means that the 2D integral becomes a line integral. If we assume constant $\rho_e$ and $\tilde{j}$ on the edge, the resulting expression for $X$ is

$$
X \approx \text{Tr} \ln G_0^{-1} + \pi \rho_e \int dx \text{tr} \omega R - \frac{i}{2} \int d\tilde{x} \cdot \text{tr} \tilde{\varepsilon} R + m S_{\text{top}}[R]
$$

(4.3.16)
where we have used that \( \partial I_{edge}/\partial \mu = m/2\pi \) with the filling fraction \( m \) integer valued. Eq. (4.3.16), however, is not yet the complete answer. This can be seen from a different expansion procedure which can be followed in the special case where \( T = \mathbb{1} \) and \( W = \mathbb{1} \). In this case we have, instead of (4.3.10),

\[
X_2 = \text{Tr} \ln[i\omega + \mu - \mathcal{H}_e + i\varepsilon \Lambda] \approx \text{Tr} \ln G^{-1} - \frac{1}{2} \text{Tr} \left[ \frac{G(\pi \cdot \tilde{z} + \tilde{z} \cdot \pi)}{2m_e} \right]^2 - \frac{1}{2m_e} \text{Tr} \tilde{z}^2 G
\]

\[
G_{nm} = G_n \delta_{nm} = \delta_{nm}[i\omega_n + \mu - \frac{1}{2m_e} \pi \cdot \pi]^{-1}.
\]

This expression can be written as

\[
X_2 \approx \text{Tr} \ln G^{-1} - \frac{1}{2} \sum_{ij} \sum_{nm} \int d^2x d^2x' \langle z_i \rangle^a_n(x) \langle z_j \rangle^a_n(x') (\Pi_{ij})^a_n(x, x')
\]

\[\quad - \frac{1}{2m_e} \int d^2z \tilde{z}^a \cdot \tilde{z} \text{tr} G(x, x).\]

The 'polarisation operator' \( \Pi_{ij} \) is given by

\[
(\Pi_{ij})^a_n(x, x') = \left( \frac{1}{2m_e} \right)^2 \text{tr} [G(x, x')(\tilde{\pi}_i + \tilde{\pi}_i)I^a_n G(x', x)(\tilde{\pi}_j + \tilde{\pi}_j)I^a_n]
\]

\[= \left( \frac{1}{2m_e} \right)^2 \sum_k G_{k+n}(x, x')(\tilde{\pi}_i + \tilde{\pi}_i)G_k(x', x)(\tilde{\pi}_j + \tilde{\pi}_j)\]

(4.3.17)

The frequency sum can be split into two parts: (I) \( k \) and \( k+n \) have the same sign; (II) \( k \) and \( k+n \) have opposite signs. Case II has been done in great detail in the context of the SCBA. The conclusion is that (II) does not contribute either to \( \sigma_{xx} \) or \( \sigma_{xy} \) when \( \mu \) is in a d.o.s. gap. Case I for \( i \neq j \), using the relation \( \tilde{\pi} + \tilde{\pi} = -i2m_e[G^{-1}, \hat{x}] \), gives rise to the familiar 'Stêada' form for \( \sigma_{xy} \). For \( i = j \) the two contributions in (4.3.18) cancel each other. We arrive at the following expression,

\[
X_2 \approx \text{Tr} \ln G^{-1} + \frac{1}{2m} \sum_{nm} \int d^2x \int d^2x \ ar{z}_n^a \times \bar{z}_n^a.
\]

(4.3.19)

Now we have to find a match between the first order result (4.3.16) for \( T = 1, W = 1 \) and the second order result (4.3.20) for \( T = 1, W = 1 \). Up to a constant arising from the difference between \( G_0 \) and \( G \), this match is given by

\[
\text{Tr} \ln G^{-1} + \pi \rho_e \int dx \text{tr} \omega R + m \left( \frac{1}{8} \varepsilon^{ij} \text{Tr} R[R_i, R][D_j, R] - \frac{1}{2} \text{Tr} R \mathbf{V} \times \tilde{z} \right)
\]

\[= \text{Tr} \ln G^{-1} + \frac{m}{2v_d} \int dx \text{tr} R(\omega - iv_d \tilde{z}_x) + m S_{top}[R] - \frac{in\alpha}{4m} \int d^2x \tilde{z} \times \partial_0 \tilde{z} \]

(4.3.21)

with \( v_d \) the electron drift velocity at the edge,

\[v_d = m/2\pi \rho_e.\]

(4.3.22)

Now we wish to write the result (4.3.21) in terms of \( Q \) and \( A_\mu \). Using the \( \mathcal{F} \)-algebra relations presented in section 2.2.4, we find that the various expressions in terms of \( R \) and \( \tilde{z} \) take the following form

...
\[ \text{tr } \omega R = \text{tr } \omega Q + \text{tr } \hat{A}_0^{(A)} Q - \frac{\beta}{2\pi}[A_0^{(A)}]^\dagger A_0^{(A)} \]

\[ \text{tr } R \hat{A}_x = \text{tr } Q \hat{A}_x - \text{tr } Q \partial_x \hat{A}_0^{(A)} - \frac{\beta}{\pi} A_0^{(A)} \partial_x \hat{A}_0^{(A)} + \frac{\beta}{\pi}[A_0^{(A)}]^\dagger \partial_x \hat{A}_0^{(A)} \]

\[ S_{\text{top}}[\mathcal{R}] = S_{\text{top}}[Q] - \frac{i}{2} \oint \text{d}x \text{ tr } Q \partial_x \hat{A}_0^{(A)} + \frac{i\beta}{4\pi} \oint \text{d}x \partial_x \hat{A}_0^{(A)} \partial_x \hat{A}_0^{(A)} \]  

\( \beta \int d^2x \hat{z} \times \partial_0 \hat{z} = - \sum \int[A^{(A)}]_\alpha \wedge d[A^{(A)}]^\alpha - \beta \oint \text{d}x [\partial_0 A_x - \partial_x A_0^{(A)}] \partial_x \hat{A}_0^{(A)} \hat{A}_0^{(A)} \).  

Substitution of these expressions into (4.3.21) and inclusion of the plasmon term \( \propto \int U_0^{-1} \lambda^2 \lambda \) yields the gauge and \( F \)-invariant effective edge action \( S[Q, A, \lambda] \)

\[ S_{\text{eff}}^{\text{edge}}[Q, A, \lambda] = S_c[\lambda] + S_b[\lambda, A] + S_Q[Q, \lambda, A] \]  

\( S_c = -\frac{1}{2} \beta \int d^2x d^2x' \lambda^4(x) U_0^{-1}(x - x') \lambda(x') \)

\[ S_b = \frac{i}{4\pi} \sum \int[A^{(A)}]_\alpha \wedge d[A^{(A)}]^\alpha - \int \text{d}x [A_0^{(A)}]^\dagger \hat{A}_0^{(A)} \]

\[ S_Q = \frac{m}{2e} \oint \text{d}x Q(\omega + \hat{A}_0^{(A)}) + m S_{\text{top}}[Q]. \]

The first term is the Coulomb energy contribution from the plasmon field; the \( S_b \) is a ‘boson’ action (this adjective will become clear later on); the last expression, \( S_Q \), contains the action for the \( Q \) field and the coupling of \( Q \) with \( \lambda \) and \( A^\mu \). We have defined a ‘minus’ direction as follows,

\[ \partial_- = \partial_0 - i\nu_0 \partial_x \quad ; \quad A_-^{(A)} = A_0^{(A)} - i\nu_0 A_x \]  

reflecting the chirality inherent in the problem.

### 4.3.3 Combined bulk and edge theory

Eq. (4.3.24) shows us how to treat edge states and can be used to obtain a generalised action \( S[Q, A] \) for the case where both bulk and edge d.o.s. are nonzero. In the limiting case of zero edge d.o.s. the bulk theory (2.3.18) is obtained, and we derive an effective theory on the edge in the opposite limit of zero bulk d.o.s. We make contact between the edge action (4.3.24) and the bulk action (2B.13) by writing for the density of states \( \rho(x) = \rho_0 + \rho_e \delta(y) \).

Outside the sample, at \( y < 0 \), the d.o.s. isn’t of course \( \rho_0 \) but zero; this fact will be encoded in the fields \( Q \) and \( A^\mu \), which will be considered to vanish outside the sample. The reason for this construction is that we wish to write everything in momentum representation without encountering the complications that step functions in \( \rho(x) \) or in the integrand of \( \int d^2x \) would cause. We define the screened Coulomb interaction in the bulk in the usual way,

\[ U_{\text{scr}}(\vec{k}) = U_0(\vec{k})/[1 + 2\pi \rho_0 U_0(\vec{k})]. \]

A quantity that will often be encountered is the Fourier transform of a function \( f(\vec{x}) \) at \( y = 0 \). We write this transform as \( f(k_x) \),

\[ f(\vec{x})|_{y=0} \leftrightarrow f(k_x) := (2\pi)^{-1/2} \int \text{d}k_x f(\vec{k}) \]
Using this notation, the generalisation of (4.3.24) for nonzero $\sigma_{xy}$ and bulk d.o.s. can be written as

\[
S[Q, A, \lambda] = S_0[Q, A] + S[\lambda, Q, A]
\]

\[
S_0[Q, A] = -\frac{1}{4\pi} \sigma_{xy} \text{Tr} [\tilde{D}, Q]^2
\]

\[
+ \sigma_{xy}^0 S_{\text{top}} - \frac{i}{4\pi} \sigma_{xy}^0 \int d\gamma \text{tr} \tilde{A}_\gamma Q + \frac{i}{4\pi} \sigma_{xy}^0 \left[ \sum_\alpha \int A^\alpha \wedge dA^\alpha + \beta \int d\gamma A^\dagger_\gamma A_\gamma \right]
\]

\[
+ \frac{1}{2} \sigma_{xy}^{11} \sum_{n=0}^\infty \int d^2 x \left[ B^\dagger L_n + 2\pi \int d^2 x \rho(\bar{x}) \left[ \frac{1}{\text{tr} \eta Q} \right] \right] + \frac{1}{2} \int d^2 x \lambda^1(-\bar{q}) \lambda(\bar{q}) \frac{1}{\text{tr} \eta Q}
\]

\[
S[\lambda, Q, A] = -\frac{\beta}{4\pi} \int d^2 p d^2 q \lambda^1(-\bar{p}) \lambda(\bar{q}) \left[ 2\pi \rho_0 L_1(\bar{q}) + \rho_e \sqrt{2\pi} L_2(\bar{q}) \right].
\]

Here we have defined the gauge invariant quantities $L_1$ in the bulk and $L_2$ at the edge according to

\[
(L_1)_n = \text{tr} \Gamma_n Q - \frac{\beta}{\pi}(A_0)_n + \frac{\beta}{2\pi\rho_0} \sigma_{xy} B_n^\alpha ; \quad (L_2)_n = \text{tr} \Gamma_n Q - \frac{\beta}{\pi}(A_0)_n
\]

The $\sigma_{xy}^{11}$ goes to the integer value $m$ in the limit of $\rho_0 \to 0$. From the quadratic part in $\lambda$ we find the following propagator in 2D momentum space

\[
G(\tilde{k}, \tilde{p}) = U_{\text{scr}}(\tilde{k}) \delta(k_x - p_x) \left[ \delta(k_y - p_y) - \rho_e U_{\text{scr}}(\tilde{p}) \left( 1 + \rho_e \sqrt{2\pi} U_{\text{scr}}(k_x) \right) \right]^{-1}
\]

Integrating out $\lambda$ yields the following quadratic expression in $L_1$ and $L_2$

\[
\frac{\beta}{4\pi} \int d^2 k d^2 p \left[ 2\pi \rho_0 L_1(\tilde{k}) - \rho_e \sqrt{2\pi} L_2(\tilde{k}) \right] \left[ 2\pi \rho_0 L_1(\tilde{p}) + \rho_e \sqrt{2\pi} L_2(\tilde{p}) \right].
\]

Substitution of (4.3.33) into (4.3.34) yields four terms which can be organised in powers of $L_1$ and $L_2$. The quadratic term in $L_1$ consists of a normal bulk term and a complicated expression at the edge,

\[
\frac{\beta}{4\pi} \int d^2 k U_{\text{scr}} L_1(\tilde{k}) \left[ \delta(k_x - p_x) \right] \left[ 1 + \rho_e \sqrt{2\pi} U_{\text{scr}}(k_x) \right]^{-1} \int d k_y U_{\text{scr}}(\tilde{k}) L_1(\tilde{k})
\]

The expression $\left\{ \cdots \right\}^{-1}$ is analogous to $Y(\tilde{p})$ (2.3.7), with $\rho_0$ replaced by $\rho_e$ and $U$ by $U_{\text{scr}}$. In the following we will denote it by $Y_{\text{edge}}(k_x)$. The quantity involving $\int d k_y$ has the following form in coordinate space

\[
\sqrt{2\pi} \int d k_y U_{\text{scr}}(\tilde{k}) L_1(\tilde{k}) \leftrightarrow \int d^2 r U_{\text{scr}}(x_0, r) L_1(r),
\]

i.e. it is the potential at a point $x_0$ at the edge due to the screened interaction with $L_1$ everywhere in the bulk. The cross term $L_1 \cdot L_2$ is given by

\[
\frac{\beta}{4\pi} \rho_0 \rho_e \sqrt{2\pi} \int d k_x Y_{\text{edge}}(k_x) L_2^\dagger(-k_x) \int d k_y U_{\text{scr}}(\tilde{k}) L_1(\tilde{k})
\]
The quadratic term in $L_2$ is ‘doubly’ screened, once by $p_0$ and then additionally by $p_e$ on the edge,

$$\frac{\pi^2 p_e^2 (2\pi)^{3/2}}{4\beta} \int dk_x \frac{U_{\text{scr}}(k_x)}{1 + \rho_e \sqrt{2\pi U_{\text{scr}}(k_x)}} L_1'(k_x) M(k_x).$$  \hspace{1cm} (4.3.38)

The contributions which are hardest to interpret are (4.3.37) and the second term in (4.3.35). We expect the Coulomb interactions to cause a shift in the scalar potential $A_0$ of the form $\int U_{\text{scr}} L_1$. Since the free action (4.3.30) contains a term quadratic in $A_0$, we expect to see terms where the shift occurs squared or coupled to $A_0$. These are given by (4.3.37) and the second term in (4.3.35) respectively. The complete expression in $L_1$ and $L_2$ can also be written in the following form in coordinate space

$$S[L_1, L_2] = \frac{\pi^2 p_e^2}{2\beta} \int d^2x d^2x' U_{\text{scr}}(\vec{x} - \vec{x}') L_1'(\vec{x}) L_1(\vec{x'}) + \frac{\pi^2}{23} \rho_e \int d^2x L_2^1 L_2$$

$$- \frac{\pi^2}{23} \rho_e \int d^2x d^2x' U_{\text{scr}}(\vec{x} - \vec{x}') Y_{\text{edge}}(x - x') \left[ L_2(x) - \rho_0 \int d^2r U_{\text{scr}}(x, r) L_1(r) \right]$$

$$\times \left[ L_2(x') - \rho_0 \int d^2r U_{\text{scr}}(x', r) L_1(r) \right].$$  \hspace{1cm} (4.3.39)

This expression describes a quite nontrivial interplay between bulk and edge and should be able to shed some light on the full interaction problem. In this thesis, however, only the limiting scenarios of zero $p_e$ or $p_0$ will be considered.

### 4.3.4 Limiting cases

The two limits $p_e \to 0$ and $p_0 \to 0$ are easily taken. In the former case only the first term of (4.3.39) survives. Added to $S_0$ (4.3.30), this precisely gives the bulk action (2.3.18). The latter case is more complicated. In the limit $p_0 \to 0$, the first term in (4.3.39) vanishes. The expression $2\pi p_0 L_1$ does not vanish but becomes $\frac{1}{4} \sigma_{xy}^B$. At the same time we have $\sigma_{xy}^\Pi \to \sigma_{xy}^0 = m$, since we are talking about a d.o.s. gap. The screened interaction in the bulk $U_{\text{scr}}$ becomes $U_0$. Eq. (4.3.39) can then be written as

$$S[L_1, L_2] = \frac{\pi^2 p_e^2}{2\beta} \int d^2x L_1^1 L_2 - \frac{\pi^2}{23} \rho_e \sum_{mn} \int d^2x d^2x' Y_{\text{edge}}(x - x') \left[ \text{tr} \Gamma_n^0 Q - \frac{\beta}{\pi} (A_{\text{eff}}^0)_{\alpha n} \right] (x)$$

$$\times \left[ \text{tr} \Gamma_n^0 Q - \frac{\beta}{\pi} (A_{\text{eff}}^0)_{\alpha n} \right] (x')$$

$$\text{(4.3.40)}$$

where we have defined an effective scalar potential that is shifted due to the fluctuations of the magnetic field in the bulk. We will use the general notation $A_{\mu}^{\text{eff}}$, but only the scalar component is shifted,

$$A_{\text{eff}}^{\text{edge}} = \tilde{A}; \quad A_{\text{eff}}^0(\vec{x}) := A_0(\vec{x}) + \frac{im}{2\pi} \int d^2x' U_0(\vec{x} - \vec{x}') B(\vec{x'}).$$  \hspace{1cm} (4.3.41)

Adding (4.3.40) to the free action $S_0(\sigma_0 = 0, \rho_0 = 0, \sigma_{xy}^\Pi = \sigma_{xy}^0 = m)$ (4.3.30) yields the full effective edge action for $Q$ and $A_{\mu}$ in the presence of interactions. Here we present it in a form which most resembles the free particle result (4.3.24 with $\lambda = 0$),

$$S_{\text{eff}}[Q, A] = S_{\text{flux}}[A] + S_0[A] + S_Q[Q, A]$$

$$S_{\text{flux}} = \frac{\beta}{2} \frac{(m)^2}{2\pi} \int d^2x d^2x' B^1(\vec{x}) U_0(\vec{x} - \vec{x}') B(\vec{x}')$$

$$\text{(4.3.42)}$$

$$\text{(4.3.43)}$$
4.4 Chiral edge bosons

In this section we show that the iqHe edge action $S^{\text{edge}}[Q,A]$ is equivalent to a theory of chiral edge bosons. We proceed as follows. First we integrate out the $Q$-field and obtain an effective action $S^{\text{eff}}[A]$. This is an exact calculation, since we can use $W$-rotations to decouple $Q$ from $A_{\mu}$. Then we show that at $\sigma_{xy} = m$, the quadratic $A_{\mu}$ terms in $S^{\text{edge}}[A]$ can be decoupled by introducing bosonic Hubbard-Stratonovich fields $\varphi_1, \ldots, \varphi_m$. In other words, as far as the electromagnetic field is concerned, the $Q$-variable is equivalent to the fields $\varphi_i$. It turns out that the effective action $S[\varphi,A]$ describes chiral edge bosons.

In an intermediate stage, we use the effective action for $A_{\mu}$ to study the bulk and edge currents in the presence of Coulomb interactions. We relate Laughlin’s gauge argument to the occurrence of a chiral anomaly at the edge whose amplitude is equal to the Hall conductance. The chiral anomaly persists when the bosons $\varphi_i$ are introduced.

4.4.1 The noninteracting case

Let us first examine the simple case of noninteracting electrons, described by the action (4.3.24 without $\lambda$). The only coupling term is the $\text{tr} \, Q \hat{A}_\sigma$. It is of course possible to
expand $Q$ in terms of $V$ as in chapter 3 and then integrate out $V$ in a loop expansion. However, a smarter way is possible which immediately yields the exact result. We always have the freedom to choose a certain gauge in which to perform the $Q$-integration. The result should be independent of the choice. A drastic simplification occurs when a gauge is chosen in which $Q$ and $A^\mu$ decouple, since the $Q$-integration then leads to an unimportant constant. The gauge in which the decoupling occurs is $A_\perp = 0$ and it is obtained by performing a gauge transformation on $A^\mu$ according to

$$A_\mu \rightarrow A_\mu - \partial_\mu \Omega \quad \partial_\perp \Omega = A_\perp \quad \Omega = \partial^{-1}_\perp A_\perp + \Omega^{\text{res}}.$$  \hspace{1cm} (4.4.1)$$

The inverse of $\partial_\perp$ has a natural definition in momentum space,

$$(\partial^{-1}_\perp F)(x, \tau) = \frac{1}{(2\pi)^d} \int dx' dr' F(x', \tau') \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega \exp[ik(x-x')-i\omega(\tau-\tau')] -i\omega + v_0 k$$

$$= \frac{1}{2\pi} \int dx' dr' F(x', \tau') \left[ \frac{\theta(\tau-\tau')}{v_d(\tau-\tau')-i(x-x')^2+\eta} + \frac{\theta(\tau'-\tau)}{v_d(\tau'-\tau')-i(x-x')^2-\eta} \right]$$

with $F$ an arbitrary function, $\theta$ the Heaviside step function and $\eta$ a regulator. The operation $\partial^{-1}$ does not commute with $\partial_\perp$. It is easily checked that

$$\partial_\perp (\partial^{-1}_\perp F) = F; \quad \partial^{-1}_\perp (\partial_\perp F) = F - F^{\text{res}}$$

(4.4.3)

with $F^{\text{res}}$ defined as that part of $F$ which satisfies $\partial_\perp F = 0$. Another property of this operation is

$$\int dx d\tau \ 1(x, \tau) F_1(\partial^{-1}_\perp F_2) = -\int dx d\tau \ (\partial^{-1}_\perp F_1) F_2.$$  \hspace{1cm} (4.4.4)$$

The transformation (4.4.1) substituted into the free action $S_{\text{eff}}[Q, A]$ yields the following effective action,

$$S[A] = \frac{m}{4\pi} \sum_\alpha \int A^\alpha \wedge dA^\alpha + \beta \oint dx \ E^\perp_x (\partial^{-1}_\perp A_\perp + \Omega^{\text{res}})$$.  \hspace{1cm} (4.4.5)$$

The presence of the ‘residual’ gauge freedom $\Omega^{\text{res}}$ in (4.4.5) is necessary for gauge invariance; under a transformation $A_\mu \rightarrow A_\mu + \partial_\mu \chi$ the $\partial^{-1}_\perp A_\perp$ is shifted by an amount $\chi - \chi^{\text{res}}$. The shift $\Omega^{\text{res}} \rightarrow \Omega^{\text{res}} + \chi^{\text{res}}$ precisely compensates for the $\chi^{\text{res}}$.

We can perform a check on the exact result (4.4.5) by comparing it to the result of a tree level calculation, where $Q$ is now integrated out without the use of $W$-rotations. We parametrise $T$ as follows

$$T = \exp i \begin{pmatrix} 0 & V \\ V^\dagger & 0 \end{pmatrix}. \hspace{1cm} (4.4.6)$$

(It can be checked that the Jacobian of this parametrisation in unity). This yields the tree level propagator

$$\langle \text{tr} [I^\beta_m Q(p)] \text{tr} [I^\alpha_n Q(-q)] \rangle = \delta^{\alpha\beta} \delta_{mn} \delta(p - q) \frac{\omega_n}{\omega_n + v_0 q}.$$  \hspace{1cm} (4.4.7)$$

This propagator is, in fact, exact, as can be checked by using the results of section 4.2.3, in particular (4.2.22) with the replacement $2n \rightarrow \omega_n$. Notice that the denominator $\omega_n + v_0 q$ is the Fourier transform of $\partial_\perp$. The expression for $S[A]$ which results from (4.4.7) is exactly
(4.4.5), but $\Omega^{\text{res}}$ now arises as a residual gauge freedom in $\text{tr} \, IQ$. The action (4.4.5) can also be written as a path integral over a total of $m$ charge 1 bosons (for simplicity we take $m > 0$),

$$S[A, \varphi_i] = \frac{i}{4\pi} \sum_{i=1}^{m} \left[ \sum_{\alpha} \int A^\alpha \wedge dA^\alpha - \beta \oint dx \left( D_\mu \varphi_i^\dagger D_\mu \varphi_i - E_\mu \varphi_i^\dagger \varphi_i \right) \right],$$

(4.4.8)

where the covariant derivative $D$ is defined as $D_\mu \varphi_i = \partial_\mu \varphi_i - A_\mu$. Under a general gauge transformation $A \to A + d\chi$ the boson fields change as $\varphi_i \to \varphi_i + \chi$. When (4.4.8) is integrated over the $\varphi_i$, (4.4.5) is obtained, where the $\Omega^{\text{res}}$ now has its origin in a gauge freedom of the boson fields, $\Omega^{\text{res}} = \frac{1}{m} \sum_i \varphi_i^{\text{res}}$. Notice that the zero-momentum part of each boson field has to be excluded from the path integral, since the action does not depend on it. We mention that (4.4.8) is completely equivalent to a Chern-Simons bulk theory with gauge fields $g_1, \ldots, g_m$ that represent potentials for the electron currents, coupled to the external potentials $A_\mu$,

$$S[A, g_i] = \frac{i}{4\pi} \sum_{\alpha} \sum_{i=1}^{m} \left[ - \int g_i^\alpha \wedge dg_i^\alpha + 2 \int g_i^\alpha \wedge dA^\alpha \right]$$

(4.4.9)

where the $g_i$ have the gauge fixing constraint $(g_i)_{\text{edge}} = 0$. In appendix 4B we explicitly show how integration over the potentials $g_i$ leads to the action (4.4.8).

### 4.4.2 The Coulomb case

Now that we have seen how edge bosons are related to the free $Q$-theory, let us investigate the effect of interactions. It is instructive to rewrite (4.3.42) in the following form

$$S_Q + S_b = \frac{im}{4\pi} \left[ \sum_{\alpha} \int (A^{\text{eff}})^\alpha \wedge d(A^{\text{eff}})^\alpha - \beta \oint dx \, A^\alpha_k A^\alpha_c \right] + m S_{\text{top}}[Q]$$

$$+ S_F^{\text{edge}}[Q] - \frac{m\pi}{4\beta} \sum_{n\alpha} \int dx \frac{1}{\xi_{\text{eff}}(k_x)} \left| \text{tr} \, \Gamma_n^\alpha Q - \frac{\beta}{2} (A^{\text{eff}})^\alpha_n \right|^2,$$

(4.4.10)

where $S_F^{\text{edge}}$ is the edge analogue of the Finkelstein action (2.3.5) in the bulk theory,

$$S_F^{\text{edge}}[Q] = \frac{\pi^2 \rho_c}{2\beta} \oint dx \left[ \sum_{n\alpha} \left| \text{tr} \, \Gamma_n^\alpha Q \right|^2 + 4 \text{tr} \, \eta Q \right].$$

(4.4.11)

In (4.4.10), as in the bulk limit of (2.3.18), the gauge field couples to $Q$ only via the $1/\xi_{\text{eff}}$ or ‘$Y$’ term. However, compared to the bulk case where the coupling results in the gauge invariant combination $(\text{tr} \, IQ - \frac{\beta}{2} A_0)$, the situation here is more subtle. The expression $(\text{tr} \, IQ - \frac{\beta}{2} A_0)$ appearing in (4.4.10) is, in fact, gauge dependent, but this gauge dependence is exactly what one needs to compensate for the edge contributions resulting from gauge transformations of the ‘boson’ action $S_b$ and the topological term. Therefore, the expression (4.4.10) is fully gauge invariant.

We now proceed as in section 4.4.1 and integrate out the $Q$ field. This is done in the same way as for the noninteracting case; either by doing it directly or by choosing a gauge $\Omega$ such that $Q$ decouples from $A_\mu$ in analogy with (4.4.1),

$$\partial_c \Omega = A_c^{\text{eff}}; \quad \Omega = \partial_c^{-1} A_c^{\text{eff}} + \Omega^{\text{res}}.$$

(4.4.12)
The only difference lies in the fact that we now work with ‘effective’ quantities. The arguments about the ‘residual’ part of the electric field can again be applied, but now for the effective quantities (4.3.41, 4.3.46, 4.3.47). We then get the effective action for the external field $A_\mu$ in the presence of Coulomb interactions,

$$S[A] = \frac{im}{4\pi} \left[ \sum_\alpha \int (A^{\text{eff}})^\alpha \wedge d(A^{\text{eff}})^\alpha + \beta \int dx \left( \partial_\nu A^{\text{eff}}_\nu + \Omega^{\nu \gamma} \right) E^{\text{eff}}_x \right]$$

$$+ \left( \frac{\beta}{2} \frac{m_v}{m} \right)^2 \int d^2x d^2x' B^i(x) U_0(x - x') B(x').$$

Again, the difference with the free particle case is the appearance of a flux-flux term and various replacements by effective quantities.

### 4.4.3 Edge currents and Laughlin’s gauge argument

The action (4.4.13) contains complete information on the response of the system to external electromagnetic fields. We define the current as $j^\mu(x) = \delta S/\delta A_\mu(x)$. In this way we find

$$j^0(x) = \frac{im}{2\pi} \left[ B - \delta(y) \partial_\nu E^{\text{eff}}_x \right]$$

$$j^1(x) = -\frac{im}{2\pi} \left[ E_x - \partial_x \int d^2x' U_0(x - x') j^0(x') \right] - \frac{mv}{2\pi} \delta(y) \partial_\nu E^{\text{eff}}_x$$

$$j^2(x) = \frac{im}{2\pi} \left[ E_x - \partial_x \int d^2x' U_0(x - x') j^0(x') \right].$$

It is easily verified that $\partial_\mu j^\mu = 0$. The edge currents are obtained by taking only those terms that possess a delta function. On the edge we get

$$j^0_{\text{edge}} = -\frac{im}{2\pi} \partial_\nu E^{\text{eff}}_x$$

$$j^1_{\text{edge}} = -i v A \cdot j^0_{\text{edge}}.$$

This yields for the edge anomaly

$$\partial_\mu j^\mu_{\text{edge}}(x) = -\frac{im}{2\pi} \left[ E_x - \partial_x \int d^2x' U_0(x, x') j^0(x') \right].$$

Notice that the expression between square brackets is the total electric field due to externally applied field and the charge accumulations on the edge. By applying Laughlin’s gauge argument [57] one can now directly relate the conductances defined by the bulk and edge theories. For example, let us do a linear response calculation for the case where $N$ flux quanta $h/e$ are created somewhere inside a hole in the sample. The charge $q$ flowing from one edge into the other is found using (4.4.19),

$$dq/d\tau = -i \int \partial_\mu j^\mu_{\text{edge}} = \frac{m_v}{2\pi} \Phi/d\tau$$

where $\Phi$ is the total flux $N \cdot h/e$ enclosed by the contour integral. This yields $q = m \cdot N$, as it should.
4.4.4 Interacting chiral bosons

As in the free electron situation, we can write the theory (4.4.13) in terms of edge bosons coupled to the external field, exactly of the form (4.4.8), but now with effective quantities and an extra flux-flux term,

\[
S[A, \varphi_i] = \frac{i}{4\pi} \sum_{i=1}^{m} \left[ \sum_{a} \int (A^{\alpha})^a \wedge d(A^{\alpha})^a - \beta \int d\mathbf{x} \left( D_x \varphi_i^\dagger D_x^{\alpha} \varphi_i - \varphi_i^\dagger E_x^{\alpha} \right) \right] + \frac{\beta}{2} (\frac{m}{2\pi})^2 \int d^2x d^2x' B^i(\vec{x}) U_0(\vec{x} - \vec{x}') B(\vec{x}').
\]

(4.4.21)

As in the noninteracting case, this result is equivalent to a Chern-Simons bulk theory of the form (4.4.9). In this case the action for the electron currents is given by

\[
S[A, g_i] = \frac{i}{4\pi} \sum_{a} \sum_{i=1}^{m} \left[ - \int g_i^a \wedge dg_i^a + 2 \int g_i^a \wedge d(A^{\alpha})^a \right] + \frac{\beta}{2} (\frac{m}{2\pi})^2 \int d^2x d^2x' B^i(\vec{x}) U_0(\vec{x} - \vec{x}') B(\vec{x}').
\]

(4.4.22)

with the gauge fixing conditions

\[
\left[ (g_i^a)_-(k_x) - i \frac{m}{\sqrt{2\pi}} U_0(k_x) \sum_{a=1}^{m} (g_i^a)_+(k_x) \right]_{\text{edge}} = 0.
\]

(4.4.23)

It is very instructive to write (4.4.21) also in the following way

\[
S = -\frac{\beta}{2} (\frac{1}{2\pi})^2 \sum_{i,j=1}^{m} \int d^2x d^2x' U_0(\vec{x} - \vec{x}') \nabla \times [\theta(y) \bar{D} \varphi_i(\vec{x})]_{\text{edge}} \times [\theta(y') \bar{D} \varphi_j(\vec{x}')]
\]

\[
+ \frac{i}{4\pi} \sum_{i=1}^{m} \left[ \sum_{a} \int A^{\alpha} \wedge dA^{\alpha} - \beta \int d\mathbf{x} \left( D_x \varphi_i^\dagger D_x^{\alpha} \varphi_i - E_x^{\alpha} \right) \right],
\]

(4.4.24)

where \( \theta \) is a step function. Notice that there are no effective quantities in this expression; the Coulomb interaction is completely contained in the first term. The charge density is given by \( \frac{m}{2\pi} [B + \delta(y) m^{-1} \sum_i D_x \varphi_i] \). Notice also that we have written a two-dimensional integral containing \( \varphi_i \), even though the boson fields only exist on the edge. This is not a problem, since the \( \varphi_i \) only get evaluated at the edge.

4.4.5 Tunnelling density of states

In the next chapter we are going to spend a lot of words on tunnelling exponents for ‘fractional’ edge quasiparticles. In order to set the stage for the discussion of tunnelling processes we now present the simple results for the iqHe. We are primarily interested in tunnelling from a Fermi liquid through ‘vacuum’ into the edge of a quantum Hall sample. The contribution to the tunnelling current from the Fermi liquid side is trivial. The quantity of interest at the sample’s edge is the tunnelling d.o.s., which is proportional to the one particle Green’s function \( G(\tau_2 - \tau_1, x_0) \).

\[
G(\tau_2 - \tau_1, x_0) = \langle \psi^\dagger(x_0, \tau_2) \psi^\alpha(x_0, \tau_1) \rangle.
\]

(4.4.25)
In terms of the $Q$-variable this expression is written as

$$\langle Q^{\alpha}(\tau_1, \tau_2, x_0) \rangle := \sum_n e^{i\omega_n(\tau_2 - \tau_1)} \langle Q^{\alpha}_n(x_0) \rangle,$$

(4.4.26)

where the expectation value is with respect to the action (4.3.42). The Green's function is a gauge dependent quantity and transforms as

$$e^{-i\int [\hat{Q}^{\alpha}(\tau_2, x_0) - \hat{Q}^{\alpha}(\tau_1, x_0)] (Q^{\alpha}(\tau_1, \tau_2, x_0))}.$$ (4.4.27)

Let us see what happens to (4.4.26) when the $Q$-field is integrated out. The gauge transformation (4.4.12) that decouples $Q$ from $A_\mu$ in (4.3.42) changes (4.4.26) into

$$\left\langle \exp -i \left\{ [\partial^{-1} A^{\text{eff}}_\mu]^{\alpha}(\tau_2, x_0) - [\partial^{-1} A^{\text{eff}}_\mu]^{\alpha}(\tau_1, x_0) \right\} \right\rangle$$ (4.4.28)

where the expectation value is now taken with respect to (4.4.13). When decoupling the quadratic edge term in $A_\mu$ (4.4.13) with the use of boson fields, this becomes

$$\left\langle \exp -i \int_{\eta_1}^{\tau_2} d\tau \partial_0 \varphi_j^0(\tau, x_0) \right\rangle, \ \ \ j = 1, \ldots, m$$ (4.4.29)

with respect to (4.4.21). The decoupling is not a unique procedure, since combinations of the boson fields $\varphi$ can be chosen other than (4.4.29). Any combination $\sum_i c_i \varphi_i$ will lead to the expression (4.4.28) after integration over the $\varphi$ fields as long as $\sum_i c_i = 1$. However, the form $c_i = \delta_{ij}$ (4.4.29) is the only one that yields the correct fermionic result $1/(\tau_2 - \tau_1)$.

$$\left\langle \exp -i \int_{\eta_1}^{\tau_2} d\tau \partial_0 \varphi^0_j(\tau, x_0) \right\rangle \propto (\tau_2 - \tau_1)^{-S} \ \ \ S = 1.$$ (4.4.30)

(See appendix 4A for the explicit calculation). Notice that we would have had a serious problem at this point if we had not excluded the zero-momentum components of the $\varphi^0_j$ when we introduced these auxiliary fields. A redefinition of the integration measure, $\int D\varphi \to \int D[\varphi + f]$, with $\partial_x f(x, \tau) = 0$, would yield a result depending on the arbitrary function $f$.

### 4.5 Long range disorder

#### 4.5.1 Separation of edge channels

Long range disorder can cause the edge states of different Landau levels to become spatially separated. A potential fluctuation at the edge can lift all states in such a way that new ‘edge’ states are created (see Fig. 4.1). If the chemical potential lies between the shifted and unshifted energy of a Landau level, the edge states of this Landau level will be shifted from the edge into the sample. If there are several potential jumps of this kind, all the edge channels can become separated. They can also start wandering into the interior of the sample. We propose that ‘edge channel separation’ is the dominant effect of smooth potential fluctuations as opposed to ‘inter-channel scattering’ which only occurs when the potential changes abruptly. In this section we wish to embark on the problem of smooth potential fluctuations in the presence of the Coulomb interactions.

In order to fix the thought we imagine a quantum Hall sample with filling fraction $\nu = 2 - \varepsilon$. Fig 4.2a illustrates the equipotential contours. We may distinguish between the
localised (closed) orbitals in the bulk of the sample and the extended edge states. This picture leads us to the idea of describing the chiral bosons by one field $\varphi(\vec{x})$ that lives on all the ‘edges’ instead of independent fields for every edge. The action (4.4.21) then becomes

$$S = \frac{i}{4\pi} \left[ \sum_a \int n A^a \wedge d A^a - \beta \sum_{a=1}^M s_a \int_{C_a} dx \left( D_x \varphi^\dagger [D_0 \varphi - is_a v_d D_x \varphi] - E^\dagger \varphi \right) \right]$$

$$- \beta/12 \int d^2 x d^2 x' U_0(\vec{x} - \vec{x}') \nabla \times [n(\vec{x}) \vec{B}(\vec{x})]' \nabla \times [n(\vec{x}') \vec{B}(\vec{x}')]. \quad (4.5.1)$$

The $n$ is a function of position labelling the ‘local’ filling fraction: outside the sample $n(\vec{x})$ is zero; going inward, it increases by one every time you cross an edge, until it reaches its bulk value $m$. At the bulk orbitals, $n(\vec{x})$ jumps again. (In the case $\nu = 2 - \varepsilon$, depicted in Fig. 4.2a, $n(\vec{x}) = 1$ inside the closed orbitals).

Each edge is described by a contour labelled $C_a$, with $a = 1, \ldots, m$ for the edge states and $a = m + 1, \ldots, M$ for the closed bulk orbitals. The coordinate $\vec{x}$, appearing in the edge terms, is defined on the contour and is taken in the positive (anticlockwise) direction. The symbol $s_a$,

$$s = (+1, \ldots, +1, -1, \ldots, -1) \quad (4.5.2)$$

incorporates the fact that the contours with $a \leq m$ and $a > m$ carry opposite current and charge densities. For simplicity we take the drift velocity $v_d$ the same for all edges. We can integrate out the boson field to obtain the generalisation of (4.4.13),

$$S[A] = \frac{i}{4\pi} \left[ \sum_a \int n(A^{\text{eff}})^a \wedge d(A^{\text{eff}})^a + \beta \sum_a s_a \int_{C_a} dx \left( \partial_x \varphi^\dagger A^{\text{eff}}_a - A^{\text{eff}}_a A^{\text{eff}}_a \right) \right]$$

$$+ \frac{\beta}{2(2\pi)^2} \int d^2 x d^2 x' n(\vec{x}) B^1(\vec{x}) U_0(\vec{x} - \vec{x}') n(\vec{x}') B(\vec{x}'). \quad (4.5.3)$$

The notation $\varphi_c$ (at contour $C_a$) now has the sign $s_a$ in front of the velocity and contains Coulomb interactions with all contours instead of just $C_a$ itself. The definition of the ‘effective’ potential $A^{\text{eff}}_0$ has also slightly changed,

$$A^{\text{eff}}_0(\vec{x}) = A_0(\vec{x}) + \frac{i}{2\pi} \int d^2 x' U_0(\vec{x} - \vec{x}') n(\vec{x}') B(\vec{x}'). \quad (4.5.4)$$
CHAPTER 4. THE INTEGER EDGE

Figure 4.2: (a) Equipotential contours corresponding to filling fraction $v = 2 - \varepsilon$. (b) Effective edge theory for filling fraction $v = 2 - \varepsilon$. The dashed line represents the (anti)chiral contribution from the bulk orbitals.

For completeness, in appendix 4D we also present the generalisation of the action $S[Q, A]$ (4.3.42) for the case of separated edge channels. Note that we are addressing the situation where the chemical potential is away from the narrow ‘percolation’ regime indicated by $W_0$ in Fig. 4.5.

4.5.2 Hall conductance

We will next exploit the simplicity of our model and demonstrate that the Hall conductance and the tunnelling density of edge states are fundamentally different quantities that correspond to completely different physical processes. First, it is straightforward to generalise the results of section 4.4.3 to include the separated edge channels and the bulk states into Laughlin’s flux argument. Differentiating the action (4.5.3) with respect to $A_\mu$, we obtain the generalised form of the currents (4.4.14-4.4.16),

$$ j^0(\vec{x}) = \frac{i}{2\pi} \left[ n(x)B - \sum_{a=1}^{M} s_a \delta \left( \vec{x} \text{ on } C_a \right) \frac{\partial^2}{\partial x^a} \right] $$

$$ j^j(\vec{x}) = -i \bar{n}(\vec{x}) \epsilon^{ij} \left[ E_j - \partial_j \int d^2 x' U_0(\vec{x} - \vec{x}') j^0(\vec{x}') \right] - \frac{s_a}{2\pi} \sum_{a=1}^{M} \delta(\vec{x} \text{ on } C_a) \partial^2 x' E^\text{eff}_x (\vec{e}_a) $$

where the vector $\vec{e}_a$ is tangent to the contour $C_a$ and points in the positive direction. Again it is easy to check that $\partial_\mu j^\mu = 0$, i.e. that current conservation is respected. The edge currents are given by

$$ j^0_{\text{edge}}(C_a) = -\frac{i}{2\pi} s_a \partial^2 E^\text{eff}_x $$

$$ j^j_{\text{edge}}(C_a) = -is_a v_a \cdot j^0_{\text{edge}}(C_a). $$

The edge anomaly applies to each bulk orbital and edge state separately,

$$ \partial_\mu j^\mu_{\text{edge}}(C_a) = -\frac{i}{2\pi} s_a \left[ E_x - \partial_x \int d^2 x' U_0(x, x') j^0(x') \right]. $$

As expected, the sign $s_a$ determines whether charge is transported into an edge or from an edge into the bulk. By repeating Laughlin’s flux argument it is now demonstrated explicitly that the localised bulk orbitals do not affect the transport of charge from one sample edge to the other, independent of the electron-electron interactions; taking (4.5.8)
and performing the contour integral over $C_a$ we obtain the charge transported per unit of
time from the $a$'th channel,

$$\frac{dQ_a}{d\tau} = -i \oint_{C_a} \partial_{\mu} j^\mu_{\text{edge}} = s a \frac{1}{2\pi} d\Phi_a/d\tau$$  \hspace{1cm} (4.5.9)$$

where $\Phi_a$ is the magnetic flux enclosed by $C_a$. For $a > m$ this flux is obviously zero, since
the localised bulk orbitals do not encircle the hole in the sample. This, then, shows that
the Hall conductance is quantised (equal to $m$) independent of $\epsilon$.

### 4.5.3 Continuous tunnelling exponent

Laughlin's flux argument for the Hall conductance expresses the quantum Hall state as
an exact 'excited' state of the system. Tunnelling processes into the edge, on the other
hand, are expressed in terms of eigenstates near the Fermi energy, i.e. the tunnelling
do.s., and due to the Coulomb interactions this quantity is sensitive to the presence of
bulk orbitals. We start our calculation of the tunnelling d.o.s. from the action (4.5.1),

Following section 4.4.5, Eq. (4.4.29), the one particle Green's function can be written as
follows

$$G(T_2 - t_1) = \exp \left\{-i [\varphi(T_2, x_0) - \varphi(T_1, x_0)] \right\}$$  \hspace{1cm} (4.5.11)$$

where $x_0$ denotes a point on the edge contour $C_1$. The presence of the Coulomb in­
teractions makes the computation of $G$ a complicated two dimensional problem. Some
procedure needs to be found that extracts the lowest energy excitations from (4.5.10). We
follow the strategy of interpreting the boson fields as one two dimensional field variable
and then collecting the terms with smallest momenta.

### Gradient expansion

The interaction term in (4.5.10) can be written as a sum over area integrals,

$$-\frac{1}{8\pi^2} \int d\tau \sum_{j,j'=1}^M \int_{C_j} d^2x \int_{C_{j'}} d^2x' \: s_j s_{j'} \: \partial_{ij} \varphi(\bar{x}) U_{ij}(\bar{x} - \bar{x}') \partial_{ij} \varphi(\bar{x}')$$  \hspace{1cm} (4.5.12)$$

with

$$U_{ij}(\bar{x} - \bar{x}') := \varepsilon_{abc} \varepsilon_{bcd} \frac{\partial}{\partial x_b} \frac{\partial}{\partial x_d} U_0(\bar{x} - \bar{x}')$$  \hspace{1cm} (4.5.13)$$
Since we are only interested in the $\varphi$ with the smallest momenta, we can make the replacement

$$\sum_{j=m+1}^{M} \int_{C_j} d^2x \to \Omega_f \int_{C_b} d^2x. \quad (4.5.14)$$

The $\Omega_f$ stands for the fraction of the total area that is enclosed by all the bulk orbitals together. The contour $C_b$ is not sharply defined and is located somewhere close to the edge (see Fig. 4.2b). It encloses the region within which the bulk orbitals are contained. The joint Coulomb effects of the bulk orbitals will effectively be comprised on this contour. For the terms in (4.5.10) containing $\partial_x \varphi \partial_0 \varphi$ we can write

$$\sum_{j>m} \int_{C_j} \partial_x \varphi \partial_0 \varphi = \sum_{j>m} \int_{C_j} d^2x \nabla \times (\nabla \varphi \partial_0 \varphi) \to \Omega_f \int_{C_b} d^2x \nabla \times (\nabla \varphi \partial_0 \varphi) = \Omega_f \int_{C_b} \partial_x \varphi \partial_0 \varphi. \quad (4.5.15)$$

The expression $\sum_{j>m} \int_{C_j} (\partial_x \varphi)^2$ averages out to $\kappa \int_{C_b} d^2x (\nabla \varphi)^2$ with $\kappa$ some positive constant related to the total length of all the bulk contours. If there are substantial stretches where a bulk orbital runs along the edge, interaction terms will arise, leading to a term $\int_{C_b} (\partial_x \varphi)^2$.

Note that in doing the replacement (4.5.14) in (4.5.12), one also needs to introduce correction terms that compensate for the errors made when the separation $|\vec{x} - \vec{x'}|$ is ‘small’ (of the order of the average size of the orbitals or less) and $U_{ab}$ does not vary slowly. These corrections are of the form $\int (\nabla \varphi)^2$. Then there are also extra correction terms that will arise if there are regions where a bulk orbital runs along the edge. This correction takes the form of a short-ranged interaction between $C_b$ and all the other contours (including $C_b$). Having done the replacement (4.5.14) and writing the interaction terms again as contour integrals, we have the following action,

$$S = -\frac{i}{4\pi} \int d\tau \left[ \sum_{j=1}^{m} \int_{C_j} \partial_x \varphi \partial_0 \varphi - \varepsilon \int_{C_b} \partial_x \varphi \partial_0 \varphi \right]$$

$$-\frac{1}{8\pi^2} \int d\tau \left[ \sum_{j'j=1}^{m} \int_{C_j} \int_{C_{j'}} dx dx' \partial_x \varphi U_{0b} \partial_x \varphi + \varepsilon^2 \int_{C_b} dx dx' \partial_x \varphi (U + V_b) \partial_x \varphi$$

$$-2\varepsilon \sum_{j=1}^{m} \int_{C_j} \int_{C_b} dx dx' \partial_x \varphi (U_0 + V_j) \partial_x \varphi \right] - g \int d\tau \int_{C_b} d^2x (\nabla \varphi)^2,$$

where $g$ is a positive constant. We have identified $\Omega_f$ with $\varepsilon$, since the fraction of the area occupied by bulk states is exactly the deviation from integer filling. We have written $V_j(x, x')$ for the short-ranged interaction between two points on $C_b$; The $V_j(x, x')$ denotes the short-ranged interaction between a point $x$ on $C_j$ and $x'$ on $C_b$. The precise expression for $V$ is unknown due to the fact that it has its origin in the twilight zone near the edge, where it is unclear whether a term contributes to the bulk or edge action.

Comparing this result (4.5.16) with (4.5.10), we see that the presence of the interacting bulk states effectively leads to the appearance of an additional (anti)chiral boson on the contour $C_b$, an extra short-ranged interaction with this contour, and a left over bulk term $\int (\nabla \varphi)^2$. 
Effect of the bulk term

In order to be able to calculate the tunnelling d.o.s. (4.5.11) we need an effective theory for the edge degrees of freedom, and therefore we have to understand how they are affected by the left over bulk term. To this end, we are going to split bulk and edge degrees of freedom. We write the bulk term as $\int_{C_b}(\nabla\Phi)^2$, where $\Phi$ represents the bulk degrees of freedom and is treated as an integration variable independent of $\varphi$. To reflect the fact that it is actually an extension of $\varphi$ into the bulk, we impose some boundary condition on $\Phi$, for instance $\Phi|_{\text{edge}} = \varphi$ or $\partial_\perp \Phi|_{\text{edge}} = \partial_\perp \varphi$ ($\partial_\perp$ is the derivative perpendicular to the contour). The effect of the bulk term on the edge theory is obtained by integrating out $\Phi$, which leads to an effective action for the boundary conditions. Let us consider a general scenario and impose the boundary conditions $\Phi|_{\text{edge}} = \psi_0$ and $\partial_\perp \Phi|_{\text{edge}} = \psi_1$, using constraint multipliers $k_0$ and $k_1$, respectively.

$$e^{S_{\text{eff}}[\psi_0(x),\psi_1(x)]} = \int \mathcal{D}\Phi(x) \mathcal{D}k_0(x) \mathcal{D}k_1(x) \times$$

$$\times \exp \left\{ i \int k_0(\Phi - \psi_0) + i \int k_1(\partial_\perp \Phi - \psi_1) - g \int d^2x (\nabla\Phi)^2 \right\}. \quad (4.5.17)$$

For notational simplicity we have omitted time dependence and the subscript $C_b$ under all the integrals. We first wish to integrate (4.5.17) over $\Phi(x)$ keeping $k_0$ and $k_1$ fixed. For this purpose we split $\Phi$, which has free boundary values, into a bulk and an edge part by writing

$$\Phi = \Phi_L + \hat{\Phi} \quad \partial_\perp \Phi_L|_{\text{edge}} = \partial_\perp \hat{\Phi}|_{\text{edge}} \quad \partial_\perp \hat{\Phi}|_{\text{edge}} = 0 \quad (4.5.18)$$

where $\Phi_L$ satisfies Laplace’s equation

$$\nabla^2 \Phi_L(x) = 0. \quad (4.5.19)$$

The $\Phi_L(x)$ is completely determined by $\partial_\perp \Phi_L$ on the edge, which we now take as an independent edge degree of freedom denoted by $E_1(x)$. Introducing the 2D Green’s function $G$,

$$G(x, x') = \frac{1}{2\pi} \ln |x - x'| \quad ; \quad \nabla^2 G(x, x') = \delta(x - x'), \quad (4.5.20)$$

and using Green’s theorem, we solve Laplace’s equation and obtain for $\Phi_L(x)$

$$\Phi_L(x) = -\int dx' \left[ G(x, x') E_1(x') - \Phi_L(x') \frac{\partial G}{\partial y}(x, y; x', 0) \right]. \quad (4.5.21)$$

This expression tells us that we need to now $\Phi_L$ on the edge in order to evaluate $\Phi_L$ in the bulk. Luckily, we do not need the full 2D $x$ dependence, since due to the splitting (4.5.18) $\Phi_L$ will get evaluated at the edge only. Using a special property of the Green’s function (4.5.20), namely $[\partial_y G](x, 0; x', 0) = 0$, we can explicitly write $\Phi_L$ on the edge as a function of $E_1$,

$$\Phi_L(x) = -\int dx' G(x, x') E_1(x'). \quad (4.5.22)$$

The action, written in terms of $\hat{\Phi}$ and $E_1$, is now given by

$$S = -g \int (\nabla \hat{\Phi})^2 - g \int E_1 G E_1 + \int E_1(2g\hat{\Phi} - igk_0) + i \int k_0(\hat{\Phi} - \psi_0) + i \int k_1(E_1 - \psi_1) \quad (4.5.23)$$
where we have used the shorthand notation $\int AGB$ for $\int dx \int dx' A(x)G(x, x')B(x')$. Integrating out $\Phi$ is now simply done by replacing $\Phi$ by its saddle point value. Varying the action with respect to $\Phi$, keeping $E_1$ fixed, we get the saddlepoint equation

$$\nabla^2 \Phi + \delta(y)[E_1 + \frac{i}{2g}k_0] = 0. \tag{4.5.24}$$

Using the Green’s function’s property $[\partial_{\nu'}G](x, 0; x', 0) = 0$ again, we find the following solution on the edge

$$\Phi(x) = -\int dx' G(x, x')[E_1 + \frac{i}{2g}k_0](x'). \tag{4.5.25}$$

In substituting this solution into (4.5.23) we do not need the full 2D $\vec{x}$-dependence of $\Phi(\vec{x})$, since we can write $\int (\nabla \Phi)^2 = -\int \Phi \nabla^2 \Phi$ and $\nabla^2 \Phi$ is an expression restricted to the edge. Substitution of (4.5.25) into (4.5.23) yields

$$S = -2g \int E_1 G E_1 + \frac{1}{4g} \int k_0 G k_0 - i \int k_0(\psi_0 + 2GE_1) + i \int k_1(E_1 - \psi_1). \tag{4.5.26}$$

Integrating out $k_0$ is straightforward and gives

$$S = g \int \left[ \psi_0 G^{-1} \psi_0 + 4\psi_0 E_1 + 2E_1 G E_1 \right] + i \int k_1(E_1 - \psi_1). \tag{4.5.27}$$

In the end we integrate out $k_1$, yielding the constraint $E_1 = \psi_1$. The final result for $S_{\text{eff}}[\psi_0, \psi_1]$ becomes

$$S_{\text{eff}}[\psi_0, \psi_1] = g \int \left[ \psi_0 G^{-1} \psi_0 + 4\psi_0 \psi_1 + 2\psi_1 G \psi_1 \right]. \tag{4.5.28}$$

We are going to put $\psi_0 = 0$ in order to avoid double counting of $(\partial_x \varphi)^2$ terms at the edge, and we put $\psi_1 = \partial_\perp \varphi$. The action (4.5.28) becomes

$$S[\partial_\perp \varphi] = 2g \int \partial_\perp \varphi \ G \partial_\perp \varphi. \tag{4.5.29}$$

This edge term, derived from the interaction with the bulk orbitals, is seriously going to affect the tunnelling exponent. A quick way to see this is as follows: on the contours $C_1, \ldots, C_b$, the field $\varphi(\vec{x})$ can be written as $\varphi(x, y$ on $C_1) +$ perpendicular derivatives. For the tunnelling exponent, only $\varphi|_{C_1}$ is needed, so we can integrate out the perpendicular derivatives in (4.5.16 minus bulk term+4.5.29) to obtain an effective action for $\varphi$ on $C_1$. The dominant part of the 1D propagator for $\partial_\perp \varphi$ is given by $G^{-1}(k) \propto |k|$, from which it follows that all terms introduced by the integration over $\partial_\perp \varphi$ are irrelevant. Higher powers of $\partial_\perp$ are even less relevant. Replacing all the $\varphi$ in (4.5.16) by $\varphi|_{C_1}$, we get a term $\nu \int \partial_\perp \varphi \partial_\perp \varphi$, leading to a tunnelling exponent $S = 1/\nu$ instead of the free particle result $S = 1$.

In what follows we are going to do this analysis more formally, based on a consideration of the neutral modes in the theory where the edge channels are not spatially separated.
Demise of the neutral modes

In the large wavelength limit, the contours $C_1, \cdots, C_b$ are lying so close together that we can effectively return to the picture where all the edge channels are sitting on top of each other. We label the channels $\varphi_1(x), \cdots, \varphi_m(x), \varphi_b(x)$. Let us for simplicity’s sake first consider the case $\nu = 1 - \varepsilon$, where we just have the two fields $\varphi_1$ and $\varphi_2$. In terms of these fields, the action (4.5.16), without the bulk term and the bulk effect (4.5.29), takes the form (again using abbreviated notation)

$$S_0[\varphi_1, \varphi_2] = -\frac{1}{4\pi} \oint \left[ \partial_x \varphi_1 \partial_0 \varphi_1 - \varepsilon \partial_x \varphi_b \partial_0 \varphi_b \right]$$

$$-\frac{1}{8\pi^2} \oint U_0 [\partial_x \varphi_1 - \varepsilon \partial_x \varphi_b]^2 - \frac{1}{4\pi^2} \sum_{k,l=1,2} \oint V_{kl} \partial_x \varphi_k \partial_x \varphi_l.$$  

We have put all the short range contributions into the $(m+1) \times (m+1)$ velocity matrix $V$. We next define a ‘charged mode’ $\Gamma$ and a ‘neutral mode’ $\gamma$ in such a way that only the charged mode feels the long-ranged part of the interaction,

$$\Gamma = \frac{1}{\nu}(\varphi_1 - \varepsilon \varphi_2) ; \quad \gamma = \varphi_1 - \varphi_2$$

$$\varphi_1 = \Gamma - \frac{\varepsilon}{\nu} \gamma ; \quad \varphi_2 = \Gamma - \frac{1}{\nu} \gamma.$$  

In the basis $(\Gamma, \gamma)$ the action (4.5.30) becomes

$$S_0[\Gamma, \gamma] = \frac{1}{4\pi} \oint \left[ \nu \partial_x \Gamma \partial_0 \gamma - \frac{\varepsilon}{\nu} \partial_x \gamma \partial_0 \gamma \right] - \frac{\varepsilon^2}{8\pi^2} \oint U_0 (\partial_x \Gamma)^2 - \frac{1}{8\pi^2} \oint \left[ \partial_x \Gamma \right] [\partial_x \gamma] \vec{V} \left[ \partial_x \Gamma \right]$$

where $\vec{V}$ is the velocity matrix in the new basis. The expression $\partial_+ \varphi$ is evidently equivalent to the neutral mode $\gamma \varphi_b - \varphi_1$. The leftover bulk contribution (4.5.29) therefore translates into an extra term involving the neutral mode,

$$S_{\text{bulk}}[\gamma] = \text{constant} \cdot \oint \gamma \, G \, \gamma.$$  

The tunnelling d.o.s. is now expressed as

$$\langle \exp -i \varphi_1 | r_1 \gamma \rangle \propto \int D\Gamma \, D\gamma \, \exp \left( -i(\Gamma - \frac{\varepsilon}{\nu} \gamma) \right) \frac{S_0[\Gamma, \gamma] + S_{\text{bulk}}[\gamma]}{\gamma^2}.$$  

If we perform the integration over $\gamma$ first, we see that the ‘bulk’ part of the action yields the following contribution to the inverse propagator: $G(k) \propto 1/k$, which is dominant at low momenta. The integration over $\gamma$ yields $\Gamma = \Gamma_0$ terms of order $k^2 \vec{V}(k)$. These are clearly irrelevant. For the tunnelling d.o.s. we can write

$$\langle \exp -i \varphi_1 | r_1 \gamma \rangle \propto \int D\Gamma \, \exp \left( -i \Gamma | r_1 \gamma \rangle \right) S_{\text{eff}}[\Gamma]$$

$$S_{\text{eff}}[\Gamma] = -\frac{\varepsilon^2}{4\pi} \oint \partial_x \Gamma \partial_0 \Gamma - \frac{\varepsilon^2}{8\pi^2} \oint \partial_x \Gamma (U_0 + \nu^{-2} \vec{V}(k)) \partial_x \Gamma.$$  

For small momenta the $\vec{V}$ essentially reduces to a constant and we can use the results of appendix 4A, obtaining

$$\langle \exp -i \varphi_1 | r_1 \gamma \rangle \propto (r_2 - r_1)^{-S} ; \quad S = 1/\nu.$$  

\[(4.5.35)\]
The general case $\nu = m - \varepsilon$

The results for $\nu = 1 - \varepsilon$ are easily generalised. From the ‘bulk’ channel $\varphi_b$ and the edge channels $\varphi_1, \ldots, \varphi_m$ we construct a charged mode $\gamma_0$ and $m$ neutral modes $\gamma_1, \ldots, \gamma_m$ as follows,

$$
\gamma_0 = \frac{1}{\nu} \left( \sum_{k=1}^{m} \varphi_k - \varepsilon \varphi_b \right)
$$

$$
\gamma_a = \frac{1}{\nu} \left( \sum_{k=1}^{a} \varphi_k - a \varphi_{a+1} \right) \quad a = 1, \ldots, m.
$$

(4.5.36)

where we define $\varphi_{m+1}$ as $\varphi_b$. The neutral modes $\gamma_1, \ldots, \gamma_{m-1}$ are the usual ones for a theory with $m$ edges. They are mutually perpendicular and normal to the charged mode. The additional $\gamma_m$ is normal to the other neutral modes but not to the charged mode. The $\varphi$’s are expressed in terms of the $\gamma$’s as follows

$$
\varphi_b = \gamma_0 - \frac{m}{\nu} \gamma_m
$$

$$
\varphi_k = \gamma_0 - \frac{\gamma_m}{\nu} (1 - \frac{1}{k}) \gamma_{k-1} + \sum_{a=k}^{m-1} \frac{1}{a+1} \gamma_a \quad k \leq m.
$$

(4.5.37)

Equation (4.5.30) is generalised to

$$
S[\varphi] = -\frac{1}{4\pi} \int \left[ \sum_{j=1}^{m} \partial_x \varphi_j \partial_0 \varphi_j - \varepsilon \partial_x \varphi_b \partial_0 \varphi_b \right] dx
$$

$$
-\frac{1}{8\pi^2} \int U_0 \left[ \sum_{j=1}^{m} \partial_x \varphi_j - \varepsilon \partial_x \varphi_b \right]^2 dx - \frac{1}{8\pi^2} \sum_{k,l=1}^{m+1} \int V_{kl} \partial_x \varphi_k \partial_x \varphi_l dx.
$$

(4.5.38)

Again, all the short-ranged contributions have been put into a velocity matrix $V$, which now has dimension $(m+1) \times (m+1)$. Writing (4.5.38) in terms of the $\gamma$-basis, we get

$$
S[\gamma] = -\frac{1}{4\pi} \int \left[ \nu \partial_x \gamma_0 \partial_0 \gamma_0 + \sum_{a=1}^{m-1} \frac{4}{a+1} \partial_x \gamma_a \partial_0 \gamma_a - m \varepsilon \partial_x \gamma_m \partial_0 \gamma_m \right] dx
$$

$$
-\frac{\nu^2}{8\pi^2} \int U_0 (\partial_x \gamma_0)^2 dx - \frac{1}{8\pi^2} \sum_{a,c=0}^{m} \int \tilde{V}_{ac} \partial_x \gamma_a \partial_x \gamma_c dx
$$

(4.5.39)

where $\tilde{V}$ is the velocity matrix in the basis of $\gamma$’s. The argument of (4.5.32 to 4.5.35) can be applied again, in a slightly modified form; the neutral modes are equivalent to $\partial_1 \varphi$ and higher derivatives. (A basis $\hat{\gamma}$ can be found for the neutral modes in which $\hat{\gamma}_n$ corresponds to the 1D lattice discretisation of $\partial_1^n \varphi$). On dimensional grounds the propagator for the $n$’th normal derivative of $\varphi$ has to be proportional to $k^{2n-1}$, leading to irrelevant contributions. A more concrete way of making this statement would be to generalise the analysis presented in (4.5.17 to 4.5.28), including boundary conditions for the higher normal derivatives. However, that would also require us to take into account higher order terms in the $\varphi$-theory (4.5.16). The resulting effective action for the charged mode $\gamma_0$ is of the form (4.5.34), with $\nu = m - \varepsilon$. 

We can summarise the results of section 4.5 as follows: We have seen that the Fermi liquid result $S = 1$ is obtained for the tunnelling d.o.s. (I) when the Coulomb interactions are omitted, or (II) when interactions are included but only short length scales are considered. An interacting theory for the lowest lying excitations, which are slowly varying field configurations, yields completely different results. The presence of bulk orbitals, interacting mutually and with the edge states, is effectively described by an extra edge channel with prefactor $-\varepsilon$ plus a remnant of the interactions in the bulk of the form $\int (\nabla \phi)^2$. The leftover bulk term serves to make all the neutral edge modes irrelevant, yielding an effective edge action for the one remaining, charged, mode. Due to the presence of the extra ‘bulk’ channel, the prefactor of this effective action $S[\Gamma]$ becomes $m - \varepsilon = \nu$, which is a continuous parameter in sharp contrast to the integer quantised $m$.

### 4.6 Plateau transitions

In this section we show how the notion of critical edge states can be used in order to gain insight into 'long range potential fluctuations'. This problem, which is very difficult to handle within the nonlinear sigma model method, plays an important role experimentally. For instance, it has been stressed that the plateau transitions as observed in the experiments of H.P. Wei et al. [76] are very difficult to observe in general, due to the presence of slowly varying potential fluctuations.

A slowly varying potential is the generic type of disorder in the standard GaAs hetero-structures, which has historically led to semiclassical considerations (percolation picture) of delocalisation near the Landau band center [57]. It is important to recognise that also our critical system (4.2.3) is very sensitive to the presence of smooth potentials (or ‘inhomogeneities’) in the sample. For example, the critical magnetic field $B^*$ may be slowly varying throughout the system due to inhomogeneities in the electron density. This means that the scaling result is valid only up to a certain fixed value for $L$. Beyond this value the remaining ‘extended’ states in the problem may be confined to the equipotential contours of the inhomogeneity potential, quite similar to the semiclassical picture of percolation.

It is generally difficult to obtain detailed knowledge on the various length- and energy scales that are involved in the cross-over problem between percolation and localisation. In what follows, we present the simplest possible scenario for crossover that enables us to deal simultaneously with interactions and such basic concepts as mean field theory and ‘universality’ of the plateau transition.

#### 4.6.1 Percolation

In order to fix the thought, we imagine the equipotential contours near half filling to form a large cluster (Fig. 4.3) [13]. Since the disconnected, closed contours do not contribute to the transport, we focus our attention to an infinite backbone cluster which we take as a regular 2D array of ‘saddlepoints’ and we disregard all the loose hanging, finite pieces (Fig. 4.3b). The saddlepoints (the sites of the square lattice) are connected to one another by the disordered 1D chiral edge channels (links on the lattice). This network can alternatively be looked upon as a checkerboard with filling fractions alternating between the values $\nu = 0$ and $\nu = 1$. The kinetic part of the action for this system may be written in the form of (4.2.4)

$$S[Q] = \frac{1}{8} \int d^2 x \ m(\vec{x}) \text{tr} \ \epsilon_{ij} Q \partial_i Q \partial_j Q$$

(4.6.1)
with $m(x) = 0, 1$ (Fig. 4.3b). Using the parametrisation of (4.2.6), the action can also be written in the form (4.2.7) which is now solely defined on the links of the square lattice,

$$S[Q] = 2\pi i \cdot q[U] + \frac{1}{2} \sum_i \int dx \, \text{tr} (\Delta t_0 t_0^{-1}) + \pi \eta \rho_{\text{link}} \sum_i \int dx \, \text{tr} \, \Lambda Q.$$  

Here, the sum is over all the black squares and the integrals are over the contours of the black squares. Despite the fact that this action does not contain any dissipative ($\sigma_\text{xx}$) terms, it is easy enough to show that in the long distance limit, (4.6.1) reduces to the form of the sigma model action (4.2.2) with

$$\sigma_\text{xx}^0 = 1/2; \quad \sigma_\text{xy}^0 = 1/2$$  

The reason for this is contained in the fact that the saddlepoints act like scattering centers which render the system dissipative at large distances. In order to demonstrate this, all one needs to do is to follow up on (4.2.20) where the background field $t_0$ now represents the ‘slow modes’ that are kept. The $t$ field variables are the ‘fast modes’ which contain all the wavelengths smaller than the lattice constant, i.e. the average distance between the saddlepoints, and which are integrated out. This leads to an effective action for each link according to

$$S_{\text{link}}[t_0] = \frac{1}{2} \int_{\text{link}} dx \, \text{tr} ((Q) t_0 \partial_x t_0^{-1}) + \frac{1}{8} \left( \int_{\text{link}} dx \, \text{tr} (Q t_0 \partial_x t_0^{-1}) \right)^2_{\text{cum}} - \frac{1}{2} \int_{\text{link}} dx \, \text{tr} (\Delta t_0 \partial_x t_0^{-1}) - \frac{1}{8} \sigma_{\text{xx}}^0 \int_{\text{link}} dx \, \text{tr} (\partial_x Q_0)^2$$  

where $Q_0 = t_0^{-1} \Delta t_0$ and $\sigma_{\text{xx}}^0 = L_0/2$ is the 1D conductivity of a single channel of length $L_0$, a well-known result in the theory of pure metals. The expectation is with respect to the theory (4.2.7) with $m = 1$. The subscript ‘cum’ indicates that only connected diagrams are taken. Use has been made of (4.2.18), (4.2.21), (4.2.22) with $\eta = 0$ and (4.2.24). Next, by taking the sum over all links one can absorb the factor $L_0$ into the definition of a 2D integral,

$$-\frac{1}{16} \sum_{\text{links}} L_0 \int_{\text{link}} dx \, \text{tr} (\partial_x Q_0)^2 \longrightarrow -\frac{1}{16} \int d^2 x \, \text{tr} (\nabla Q_0)^2.$$  

Figure 4.3: Backbone cluster as a network of saddlepoints. Shaded areas have $\nu = 1$, white areas $\nu = 0$. The arrows indicate the direction of the currents. (a) Less than half filling; (b) Exactly half filling; (c) Filling fraction larger than one half.
4.6. PLATEAU TRANSITIONS

Here we only used the fact that the $Q_0$ field variable varies slowly over a distance $L_0$. The first term in (4.6.4) can be handled in a similar way. For instance, it can be rewritten in the form of (4.6.1) with $Q$ replaced by $Q_0$, which is then followed by taking the continuum limit according to

$$\frac{1}{2} \sum_{\text{links}} \int dx \, \text{tr} \,(\Lambda t_0 \partial_x t_0^{-1}) \rightarrow \frac{1}{8} \int d^2 x \, m(\bar{x}) \varepsilon_{ij} \text{tr} \, Q_0 \partial_i Q_0 \partial_j Q_0$$

$$\rightarrow \frac{1}{16} \int d^2 x \, \varepsilon_{ij} \text{tr} \, Q_0 \partial_i Q_0 \partial_j Q_0. \quad (4.6.6)$$

The result of (4.6.4–4.6.6) is identical to the statement made in (4.6.3). Notice that (4.6.3) is precisely the point where we expect the $\sigma$ model action (4.2.2) to have a critical phase. Hence, we have established a direct connection between critical 1D edge states on the one hand and the 2D delocalisation transition of the band center on the other. This connection has the following ingredients:

1. The infinite percolation cluster at the band center contains a finite density of saddlepoints. This translates into a finite density of scattering centers which, in turn, is responsible for making the sample diffusive (dissipative) at large distances.

2. The parameters $\sigma_{xx}^0$, $\sigma_{xy}^0$ (4.6.3) constitute a mean field theory of the conductances which is valid for length scales $L_0$.

Without going into further detail we mention that the analysis can easily be generalised to more complicated situations. For example, the links between the saddlepoints need not be straight lines. They can be taken as arbitrarily complex, non-intersecting paths reflecting the highly ramified percolation contours (Fig. 4.4). The same result (4.6.3) applies to all cases, indicating that the general result $\sigma_{xx}^0 = 1/2$ actually stands for the quantised conductance in one dimension.
4.6.2 Mean field theory

Next, we wish to extend our mean field analysis (4.6.3) to include also the energies away from the Landau band center. For this purpose we have to relate the range in energy $W_0$ within which the equipotential contours form an infinite saddlepoint cluster to the total bandwidth $W$ of the Landau band. It is understood that the phrase ‘saddlepoint’ actually stands for those special points where two equipotential contours approach each other at a distance of the order of the magnetic length $\ell_0$ or smaller. By assuming a simple quadratic form for the potential near saddle points we obtain the following estimate,

$$W_0 \approx (\ell_0/\lambda)^2 W$$

(4.6.7)

where $\lambda$ is the characteristic correlation length of the random potential, which we have taken to be much larger than $\ell_0$, and $W$ equals the amplitude of the potential fluctuations. The sigma model theory or, equivalently, the scaling theory of localisation only applies to the energy band $W_0$ about the band center. For energies just outside $W_0$ the network of saddlepoints is broken up into disconnected islands of size $L_0 \times L_0$ (Fig. 4.3a and c). The absence of any quantum tunnelling means that no correlation exists between the islands (they are represented by independent actions as long as one works within the free electron approach). In the language of the $\sigma$ model, the situation is represented by putting $\sigma_{xx} = 0$ but $\sigma_{xy} = m = $integer. The latter follows from the long-ranged correlations which still exist near the edge and which can be expressed in terms of an integer number $m$ of edge channels. In Fig. 4.5a we illustrate the behaviour of the d.o.s. $\rho$ and the conductances $\sigma^0_{ij}$ as a function of energy $\mu$ at zero temperature. The sigma model conductance parameters $\sigma^0_{ij}$ can be expressed as a function of the dimensionless quantity $\Delta \mu/W_0$

$$\sigma^0_{ij} = f_{ij}(\Delta \mu/W_0)$$

(4.6.8)
where $\Delta \mu$ is the energy relative to the Landau band center. The $f_{ij}$ are non-universal and generally depend on the microscopic details of the randomness. For comparison we have plotted the results of the more familiar theory of short-ranged scatterers (self-consistent Born approximation) in Fig. 4.5b. In this case, there is only a small difference between $W_0$ and $W$ due to the localised states in the Gaussian tails of the Landau band.

An estimate for $L_0$ can be obtained as follows. Let $|\Delta \mu| \approx W_0$ denote the energies where the saddlepoint network breaks up into disconnected equipotential contours of size $L_0 \times L_0$ (Fig. 4.3a,c). According to the semiclassical picture of percolation we can relate the typical cluster size $\xi_p$ to the energy $\Delta \mu$ according to

$$\xi_p \sim \lambda(\Delta \mu/W)^{-4/3} \quad (4.6.9)$$

where the critical index $4/3$ is the exponent for semiclassical localisation. By identifying the points $|\Delta \mu| = W_0$ and $\xi_p = L_0$ in (4.6.9) we obtain the estimate

$$L_0 \approx \ell_0(\lambda/\ell_0)^{11/3} \quad (\lambda \gg \ell_0) \quad (4.6.10)$$

or, more generally,

$$W_n \approx \frac{l_n}{\lambda} W \quad ; \quad L_n \approx \ell_n(\lambda/\ell_n)^{11/3} \quad (\lambda \gg \ell_n). \quad (4.6.11)$$

The $\lambda$ is an adjustable parameter in the theory and it ranges between microscopic distances ($\ell_0 \approx 100\,\text{Å}$) and infinity.

### 4.6.3 Interaction effects

It is quite possible that $L_0$ (4.6.10) is many times larger than the micron regime which is the typical scale for inelastic processes at low temperatures. This means that the critical behaviour (4.2.3) cannot be observed within the limitations of ordinary laboratory experiments. This, then, is the easiest and crudest explanation for the lack of scaling in many samples. As a first step toward a more quantitative understanding of transport at finite $T$, we come back to the distinction, made in the beginning, between the backbone cluster and the disconnected, ‘loose hanging’ pieces. Due to the electron-electron interactions, motion of the conducting electrons on the saddlepoint network is affected by the localised electrons. This may be expressed in terms of a relaxation time $\tau_n$ which is a characteristic time for equilibration between the conducting and localised electrons. Later on in this chapter (section 4.6.5) we will address the problem of interaction effects and show that

$$1/\tau_n = \beta_1 T + \beta_2 T^2 + \cdots \quad (4.6.12)$$

at low temperatures. This expression is determined by the collection of ‘nearly saddlepoints’ where quantum tunnelling is not possible but where the interactions between the conducting and localised particles are strongest nevertheless. The importance of ‘nearly saddlepoints’ can be seen by comparing the wavefunctions at different energies close to the Landau band center. What is a saddlepoint configuration at one energy may turn into a ‘nearly saddlepoint’ at another and vice versa. These abrupt changes in the configuration of the conducting network at slightly different energies blur the distinction between saddlepoints and ‘nearly saddlepoint’ configurations as far as finite temperatures are concerned. This means that the relaxation time $\tau_n$ (4.6.12) determines an effective bandwidth
$W_{\text{eff}} = W_0 + \tau_{\text{in}}^{-1}$ of states that contribute to the conduction at finite temperatures. Eq. (4.6.8) is replaced by the expression

$$\sigma_{ij}^0(T) = f_{ij}(\Delta \mu/W_{\text{eff}}) = f_{ij}(\Delta \mu/\left[W_0 + \tau_{\text{in}}^{-1}\right]).$$

This result is a characteristic feature of long-ranged potential fluctuations and it does not occur in the problem of short-ranged scatterers. To conclude this section, we next estimate the range of validity of the result (4.6.13). We write

$$v_d \tau_{\text{in}} = L_{\text{in}}, \quad v_d \approx 2\pi l_0^2 W/\lambda.$$

The $L_{\text{in}}$ is the mean free path for drifting along the links of the lattice. We mentioned earlier already that the actual path between two saddle points is arbitrarily convoluted and very long. Let $L_t$ denote the actual path length between saddle points, then the criterion for scaling is clearly given by

$$L_{\text{in}} > L_t.$$

Next we use the ramification hypothesis [57] in order to relate $L_t$ to the shortest distance between saddlepoints ($L_0$). We obtain

$$L_t \propto L_0^\sigma$$

with $\sigma$ somewhere between 1 and 2. The criterion for scaling (4.6.15) now implies

$$\tau_{\text{in}}^{-1} < (l_0/\lambda)^{8\sigma/3} W_0 \ll W_0.$$

This result indicates that (4.6.13) is very likely to be observed in samples with smooth disorder potentials.

The results of this section are consistent with the recently reported empirical fitting [71, 4] of the transport data in the quantum Hall regime. Since we are necessarily operating with an almost complete lack of knowledge on the microscopic details of sample disorder, it is conceivable that other types of inhomogeneity, especially those in low mobility samples, explain the same thing.

### 4.6.4 Independent Q-fields

The subjects of critical edge states and long-ranged disorder have left several conceptual questions that still need to be answered. For example, we have seen that short-ranged disorder causes inter-channel scattering between the chiral edge states. Since we do not expect inter-channel scattering to occur when the potential fluctuations are smooth (relative to the magnetic length), it is necessary to re-investigate the meaning of instanton vacuum theory for $\nu > 1$. Scattering between multiple edge states is avoided by writing, instead of (4.2.2)

$$S_{\text{eff}}[Q^{(n)}] = \sum_{n=0}^{\infty} \left[ -\frac{1}{8} \sigma_{xx}^{(n)} \int d^2 x \text{ tr } [\nabla Q^{(n)}]^2 + \frac{1}{8} \sigma_{xy}^{(n)} \int d^2 x \text{ tr } \varepsilon_{ij} Q^{(n)} \partial_i Q^{(n)} \partial_j Q^{(n)} \
+ \pi \rho^{(n)} \eta \int d^2 x \text{ tr } \Lambda Q^{(n)} \right].$$

(4.6.18)
where the sum runs over all the Landau levels $n$. The $Q^{(n)}$ stands for an independent field variable $Q$ for each Landau level separately. The $\sigma_{ij}^{(n)}$ are the $n$th Landau level contributions to the mean field conductances, which are now given by

$$\sigma_{ij} = \sum_{n=0}^{\infty} \sigma_{ij}^{(n)}. \quad (4.6.19)$$

The $\sigma_{ij}^{(n)}$ all have the same $\mu$-dependence (Fig. 4.5a) except for an appropriate shift in energy. Since $0 \leq \sigma_{ij}^{(n)} \leq 1$ for each $n$, it is clear that (4.6.18) is the appropriate generalisation of the theory (section 4.2.1) to include filling fractions larger than one. The theories of (4.6.18) and (4.2.2) are identical as far as the critical behaviour of the plateau transitions is concerned. Eq. (4.6.18) cannot, however, be used in the limit of small magnetic field, where the Landau levels partly or completely overlap. The details of crossover require a separate analysis.

### 4.6.5 Computation of $\tau_{in}$

We next return to the problem of the plateau transitions. Following section 4.6 we expect that the transport at high temperatures is dominated by interactions between the conducting electrons on the backbone saddlepoint network and those on the disconnected pieces or clusters.

The fundamental quantity to compute is the characteristic time $\tau_{in}$ that is needed for the backbone electrons to equilibrate with the rest of the network. In order to set up a theory for relaxation, we consider the 'nearly saddlepoints' in the network, where tunnelling is not possible but where the Coulomb forces nevertheless produce 'sudden
changes' in the motion of the conducting electrons. Fig. 4.6 illustrates the interaction of
the saddlepoint network with disconnected orbitals. The 'nearly saddlepoints' where the
Coulomb forces are most effective are indicated by the shaded areas. We can model the
situation by introducing a delta-function potential which acts in the small areas of the
nearly saddlepoints only. The action can be written as

\[ S_{\text{eff}}[\varphi] = S[\varphi_0] + \sum_i S[\varphi_i] - \sum_i \int d\tau \partial_\tau \varphi_0(\vec{a}_i) U_i \partial_\tau \varphi_i(\vec{a}_i) \]  \hfill (4.6.20)

where \( S[\varphi_0] \) is the action for the chiral boson field on a link of the saddlepoint network
that we denote as the contour \( C_0 \),

\[ S[\varphi_0] = -\frac{i}{4\pi} \int d\tau \int_{C_0} dx \partial_\tau \varphi_0 \partial_\tau \varphi_0. \]  \hfill (4.6.21)

This contour is taken to be very large or infinite. Similarly, we define chiral boson fields
\( \varphi_i \) on the disconnected but large contours \( C_i \),

\[ S[\varphi_i] = \frac{i}{4\pi} \int d\tau \int_{C_i} dx \partial_\tau \varphi_i \partial_\tau \varphi_i. \]  \hfill (4.6.22)

The sum in the interaction term in (4.6.20) is over the discrete set of nearly saddlepoints
\( \vec{a}_i \) along the contour \( C_0 \) where the fields \( \varphi_0 \) and \( \varphi_i \) interact with an appropriate, random
strength \( U_i \). This problem is in many ways quite similar to the problem of interacting
edge channels with a randomly varying separation between them. We proceed along the
same lines as in [54] and introduce a self energy \( \Sigma \) for the density-density correlation of the
field \( \varphi_0 \). If we denote the Fourier transforms of the propagators \((\partial_\tau \varphi_0(x, \tau) \partial_\tau \varphi_0(x', \tau'))\)
and \((\partial_\tau \varphi_i(x, \tau) \partial_\tau \varphi_i(x', \tau'))\) (with \( x, x' \) parametrising the positions on the contours \( C_0, C_j \)
respectively) as

\[ D_0(\omega, q) = \frac{-iq}{i\omega - v_a q} ; \quad D_j(\omega, q) = \frac{-iq}{i\omega + v_d q}, \]  \hfill (4.6.23)

then the introduction of a self energy takes the form

\[ D_0(\omega, q) \rightarrow -i \frac{q}{i\omega - (v_d + \Sigma)q}. \]  \hfill (4.6.24)

To lowest order in the interaction potential we may write

\[ \Sigma(\omega) = \frac{i}{2\pi} \bar{U}_j^2 \int dq D_j(\omega, q) = -\frac{i}{2\pi} \bar{U}_j^2 |\omega|. \]  \hfill (4.6.25)

Here, the bar stands for the average over the random positions \( \vec{a}_i \) along \( C_0 \) and \( z \) is the
linear density of saddlepoints. The result (4.6.25) can be used to obtain an expression
for \( 1/\tau_{in} \), i.e. the imaginary part of the self energy as it appears in the electron Green’s
function \( G(\omega, q) \), as follows

\[ 1/\tau_{in} = \int d\omega dq \Sigma(\omega) G(\omega + i0^+, q). \]  \hfill (4.6.26)

The \( \tau_{in} \) determines the rate at which the electrons on the backbone cluster equilibrate
with the rest of the electronic orbitals. We find \( \tau_{in}^{-1} \propto \varepsilon^2 \) or \( T^2 \) at finite temperatures.
This admittedly crude approach toward electron relaxation can be improved in several ways. For example, as the most important correction to the self energy (4.6.25) we find the self-interacting orbitals as depicted in Fig. 4.6b. These corrections replace the momentum integral in (4.6.25) in the following way (in space-time notation)

\[
\int dq \, D_j(\omega, q) = \int d\tau \, e^{-i\omega(\tau - \tau')} D_j(0, 0; \tau - \tau')
\]

\[
D_j(0, 0; \tau - \tau') \to D_j(0, 0; \tau - \tau') + \int d\tau_0 \int_0^L dx \int_{-\ell}^{\ell} dy \, D_j(0, x; \tau - \tau_0) \tilde{U}_j \, D_j(y, L; \tau_0 - \tau')
\]

where \(x, y\) are the positions of the nearly saddlepoint where the self-interaction takes place. The integrals stand for the averaging over positions and all dimensional factors are absorbed into \(\tilde{U}_j\). The length of the orbital is given by \(L\) and boundary conditions \(x = x + L\) and \(y = y + L\) are understood. Equation (4.6.27) can be rewritten as a shift in the chemical potential,

\[
\int dq \, D_j(\omega, q) \to \int dq \, \frac{-iq}{i\omega - \delta\mu + v_d q}
\]

\(\delta\mu = \tilde{U}_j\). (4.6.28)

This leads to a modified self energy according to

\[
\Sigma(\omega) \to -\frac{e^2}{v_d^2} \tilde{U}_j^2 (\omega + i\delta\mu) \text{sgn}(\omega).
\] (4.6.29)

The shift \(\delta\mu\) can be translated into a shift in the expression for \(\tau^{-1}_m\) following

\[
\tau^{-1}_m(\varepsilon, \delta\mu) = (1 + i\delta\mu \frac{\partial}{\partial\varepsilon}) \tau^{-1}_m(\varepsilon).
\] (4.6.30)

After the analytic continuation to real energies \((i\varepsilon \to \varepsilon)\) has been performed, we obtain the final result \(\tau^{-1}_m \propto \varepsilon\) or \(\tau^{-1}_m \propto T\) at finite temperatures. More generally, we expect the equilibration rate to be given by a regular series expansion in powers of \(T\) which is dominated by the lowest order \(\tau^{-1}_m \propto T\) as \(T\) approaches absolute zero.

### 4A One-dimensional propagator with Coulomb interaction

In this appendix we calculate the correlation function \(G(\tau, 0)\) for the charged boson fields \(\varphi_i\) (4.4.21),

\[
G(\tau, x) = \langle \varphi_i(\tau, x) \varphi_i(0, 0) \rangle \quad \tau > 0.
\] (4A.1)

In momentum and frequency space this correlator is given by (we omit the Landau level label \(i\) since it is of no consequence)

\[
\left\langle \varphi_a(k) \varphi_b(-k') \right\rangle = \frac{2\pi i}{\beta} \frac{\delta_{ab} \delta(k - k')}{k \left[ \omega_a + ik v_{\text{eff}}(k) \right]^2}.
\] (4A.2)

We write the 1D Coulomb interaction and the effective velocity \(v_{\text{eff}}\) in the following form

\[
U_0(k) = -c\sqrt{2\pi} \ln(k/\Lambda)^2 \quad ; \quad v_{\text{eff}}(k) = -mc\ln(k/\Lambda D)^2
\] (4A.3)
where $c$ is a positive constant indicating the strength of the Coulomb interaction, $\Lambda$ is an ultraviolet cutoff and $D = \exp(v_d/2mc)$. We will only consider low momenta $|k| < \lambda \Lambda$, with $\lambda < 1$, so that we are well away from the point where the Hamiltonian becomes negative. We take the Fourier transform of (4A.2) and change the frequency sum to an integral, writing $\sum_n \rightarrow \frac{\beta}{2\pi} \int d\omega$,

$$\partial_\tau G(\tau, 0) = \frac{i}{2\pi} \int_{-\lambda \Lambda}^{\lambda \Lambda} dk \ v^{\text{eff}}(k) \int_{-\infty}^{\infty} \frac{e^{i\omega T} d\omega}{\omega + i k v^{\text{eff}}(k)} = - \int_{-\lambda \Lambda}^{\lambda \Lambda} dk \ v^{\text{eff}}(k) \Theta(-kv^{\text{eff}}) e^{kv^{\text{eff}}(k)T}. \quad (4A.4)$$

The step function $\Theta(-kv^{\text{eff}})$ constrains the integration interval to $k < 0$. We can split the last expression in (4A.4) into two parts, using $\ln k dk = d(k \ln k - k)$, and get

$$\partial_\tau G(\tau, 0) = -\frac{i}{\tau} [1 - (\frac{\lambda}{\Lambda})^{2mc\tau\lambda^2}] - 2mc\Lambda D \int_0^{\lambda/D} du \exp[2mc\tau\lambda^2 D \cdot u \ln u]. \quad (4A.5)$$

The function $u \ln u$ is negative on the whole interval $(0, \lambda/D)$, since $\lambda/D < 1$. If we now send the cutoff $\Lambda$ to infinity, the term with the integral in (4A.5) will go to zero as $1/\ln \lambda$. The term $(\lambda/D)^{2mc\tau\lambda^2}$ also vanishes, yielding the free particle result

$$G(\tau, 0) = -\ln \tau + \text{constant}. \quad (4A.6)$$

### 4B Chern-Simons action for bulk currents

In this appendix we show that (4.4.8) is equivalent to the following bulk action:

$$S[A, g] = \frac{i}{4\pi} \sum_a \sum_{\alpha=1}^m \left[ - \int g^\alpha \wedge d g^\alpha + 2 \int g^\alpha \wedge d A^\alpha \right] \quad (4B.1)$$

with the condition $(g^\alpha)_- = 0$ on the edge. The $g_i$ are 2+1 dimensional potentials from which the electron current density $j$ for every Landau level can be found,

$$j^\mu_i \propto \varepsilon^{\mu\nu\lambda} \partial_\nu (g_i)_\lambda. \quad (4B.2)$$

Notice three important subtleties:

- The coupling of $g$ with the electromagnetic gauge field is of the form $g \wedge dA$ instead of the expected $A \wedge dg \propto j_\mu A^\mu$. These expressions differ by an edge term. The second form is not invariant under the gauge transformations $A \rightarrow A + d\chi$; the expression $dA$ on the other hand is manifestly gauge invariant.

- Putting an arbitrary spacetime component of $g$ zero on the edge ensures that the action is invariant under $g_\mu \rightarrow g_\mu + \partial_\mu \chi$, a gauge transformation that does not affect the current density. Without such a condition, gauge invariance is broken at the edge.

- Because of the invariance under $g_\mu \rightarrow g_\mu + \partial_\mu \chi$, a gauge fixing condition has to be specified for the path integration over $g$, for instance the Coulomb gauge $\nabla \cdot \vec{g} = 0$.
4C. INTER-CHANNEL SCATTERING AT THE EDGE

Let us now for simplicity drop the replica indices \( \alpha \) and the Landau level index \( i \) (effectively setting \( m = 1 \)). Having taken the condition \( g_-|_{\text{edge}} = 0 \), the component \( g_- \) in (4B.1) multiplies the following constraint:

\[
\nabla \times (\vec{g} - \vec{A}) = 0. \tag{4B.3}
\]

After integration over \( g_- \), what remains of the action is

\[
\frac{i}{4\pi} \int \! d^2x \left( -\vec{g} \times \partial_- \vec{g} + 2\vec{g} \times [\nabla A_- - \partial_- \vec{A}] \right) \tag{4B.4}
\]

subject to the constraint (4B.3). The general solution of (4B.3) is given by

\[
g_- = -\nabla \varphi \tag{4B.5}
\]

with \( \varphi(x) \) a real scalar field which is now the only integration variable that is left. Substitution into (4B.4) yields an action where \( \varphi \) features only on the edge,

\[
S[\varphi, A] = \frac{i}{4\pi} \left[ \int A \wedge dA - \oint \! dx \! d\tau \left( D_x \varphi D_\tau \varphi - \varphi E_\tau \right) \right]. \tag{4B.6}
\]

This is exactly of the form (4.4.8). One may worry that the path integration over \( \varphi \) is ill-defined, because of the bulk degrees of freedom of \( \varphi \), which do not appear in (4B.6). However, \( \varphi \) inherits something from the gauge fixing condition of \( g \). This is most easily seen in the case of the Coulomb gauge; here, \( \varphi \) has to satisfy \( \nabla^2 \varphi = 0 \). This means that the bulk degrees of freedom are completely determined by \( \varphi(x) \) at the edge (the well known case of Laplace's equation with Dirichlet boundary conditions) and therefore aren't independent integration variables.

One final remark on the boundary condition \( g_- = 0 \): The Hamiltonian (density) corresponding to (4B.6) is given by \( v_d (D_x \varphi)^2 \). One is not allowed to choose a velocity \( v_d < 0 \), since this would lead to energies that are unbounded from below. In general, the boundary condition has to be taken in such a way that the velocity of the chiral bosons has the same sign as the prefactor multiplying \( \frac{i}{4\pi} \) in (4B.1), otherwise the integration over \( g \) is ill-defined.

4C Inter-channel scattering at the edge

In this appendix we describe the various steps of the standard \( Q \)-field approach to edge disorder. For the general case of \( m \) chiral edge channels, one can differentiate between different types of disorder, depending on whether one allows inter-channel scattering or not. Although the different scattering potentials do not give rise to fundamentally different physical results, it is nevertheless important to define the ‘effective’ edge Hamiltonian (4.2.10) which gives rise to the same result (4.2.7) that was previously obtained for 2D electrons. Below we show that the following \( m \) channel model satisfies our requirements

\[
\mathcal{H}_{\text{edge}}^{kk'} = -iv_d \delta_{kk'} \partial_x + V_{kk'}(x), \tag{4C.1}
\]

where \( V \) is a hermitian random matrix and the elements \( V_{kk'} \) are distributed with a Gaussian weight

\[
P[V] = \exp\left\{-\frac{1}{g} \int \! dx \text{ tr } V^2\right\}. \tag{4C.2}
\]
The indices \( k, k' = 1, \ldots, m \) label the edge channels. The form (4C.1) implies that single potential scattering, as described by the 2D Hamiltonian
\[
\mathcal{H}_{2D} = \frac{1}{2m} (\hat{p} - \hat{A})^2 + V(x),
\]
does not naively translate into single potential scattering for the edge states as obtained by solving (4C.3) in the presence of an edge (infinite potential wall). Rather than that, one should allow for inter-channel scattering of the 'pure' eigenstates as in (4C.1) in order to reproduce the effect of dirt in the general 2D problem (4C.3). We start from the following generating function for the averaged free particle propagators
\[
Z = \int \mathcal{D}\tilde{\psi} \int \mathcal{D}V \{V\} \exp \left\{ \beta \sum_{p,\alpha,\beta} \int dx \tilde{\psi}^\alpha_p \mathcal{H}_{\text{edge}}^{\alpha\beta} \psi^\beta_p \right\}.
\]
Integration over randomness and introduction of the matrix field \( \tilde{Q} \) by performing the Hubbard-Stratonovich trick leads to
\[
Z = \int \mathcal{D}\tilde{Q} \exp \left\{ \frac{1}{g} \text{Tr} \tilde{Q}^2 + m \text{Tr} \ln[\mu - iv_d \partial_x + i\tilde{Q} + i\eta A] \right\}.
\]
Notice that the edge channel label is not present in the new field variable \( \tilde{Q} \), but it is simply contained in an overall factor \( m \).

We will next make use of the simple analytic properties of our 1D Hamiltonian and show that the saddlepoint technique yields, in fact, exact results for all \( m \) and that therefore there is no need to rely on \( m \) to be 'large'. The stationary point equation for \( \tilde{Q} \),
\[
\frac{i}{\hbar} \left[ \tilde{Q}^\beta_{\alpha p} \right] = \delta^{\alpha\beta} \delta_{pp'} [e_0 + (-1)^p i/2 \tau],
\]
can be written as
\[
\frac{2}{\hbar} (e_0 \pm i/2 \tau) = -m \int_{-\infty}^{\infty} \frac{dq}{2\pi} [\mu - v_d q + e_0 \pm i(1/2 \tau + \eta)]^{-1} = \pm im/2v_d
\]
with the simple solution \( e_0 = 0, \tau = 2v_d/mg \). One may next replace the original \( Q \)-field by the following change of variables,
\[
\tilde{Q} \rightarrow T^{-1} P T \rightarrow \frac{1}{2\tau} T^{-1} \Lambda T =: \frac{1}{2\tau} Q.
\]
Here, the \( T \in U(2N_r) \) are unitary rotations and the block-diagonal hermitian \( P^\alpha_p \) represent the longitudinal components. Replacing \( P \) by its saddlepoint value, as written in (4C.8), turns out to be an exact statement, valid for all \( m \). The reason is contained in the fact that the fluctuations in \( P \) are weighted by propagators with poles in either the positive or negative imaginary momentum plane. All the momentum integrals therefore sum up to zero, giving rise to a zero weight to all orders in the \( P \)-fluctuations. The replacement of (4C.8) is exact when inserted in the \( \text{Tr} \ln \). Eq. (4C.5) factorises into
\[
Z = Z_P \cdot Z_T
\]
\[
Z_P = \int \mathcal{D}P \{P\} \exp \left\{ -\frac{1}{g} \text{Tr} P^2 \right\}
\]
\[
Z_T = \int \mathcal{D}T \exp m \text{Tr} \ln[\mu + iv_d \partial_x + \frac{i}{2\tau} \Lambda + iB]
\]
where all $T$-dependence is contained in the quantity $B$ according to

$$B = v_d T \partial_x T^{-1} + \eta T \Lambda T^{-1} =: v_d T D_0 T^{-1}.$$  \hfill (4C.10)

Equation (4C.9) can be evaluated further, and to lowest few orders in an expansion in $B$
we obtain an effective action which can be written as

$$Z = \int D \Lambda \exp S_{\text{eff}}[\Lambda]$$

$$S_{\text{eff}}[\Lambda] = \frac{m}{2 v_d} \text{Tr} \Lambda B(x) - \frac{m \pi T}{8 v_d} \text{Tr} [B(x), \Lambda]^2 + \cdots$$ \hfill (4C.11)

$$= \frac{m}{2} \int dx \text{Tr} \Lambda T \partial_x T^{-1} + \frac{m \pi T}{8} \int dx \text{Tr} \Lambda Q - \frac{m \pi v_d}{8} \int dx \text{Tr} [D_0, Q]^2. \hfill (4C.12)$$

The coefficients appearing in (4C.12) all have a clear physical meaning in the context of disordered edge states (see also the main text). In particular, $m$ stands for the quantised Hall conductance $\sigma_{xy}$; $m/2 \pi v_d$ equals the total density of edge states $\rho_e$. The quantity $m \pi v_d$ that appears in the higher dimensional operators is the 1D conductivity $\sigma_\alpha$ of $m$ channel edge states. Here, $2 \pi v_d$ is the linear dimension which sets the smallest wavelength for the $Q$ field variables and $m/2$ is the (quantised) conductance ($g_m$) of the wire.

## 4D Action for $Q$ and $A_\mu$ on multiple edges

The generalisation of (4.3.42) is given by

$$S[Q, A] = \frac{\beta/2}{(2\pi)^2} \int d^2 x d^2 x' n(x') B^\dagger(x, x') U_0(x, x) n(x) B(x) + \sum_{a=1}^M s_a S^{(a)}_{\text{top}}[Q]$$

$$+ \frac{i}{4 \pi} \left[ \sum_a \int n(A^\text{eff}_a) \wedge d(A^\text{eff}_a) + \frac{M}{2} \sum_{a=1}^M s_a \int dx \left( A^\dagger_a A^\text{eff}_a - \frac{2 \pi \rho}{\beta} \text{tr} A_\alpha^\dagger Q \right) \right]$$

$$+ \sum_{a=1}^M S^{(a)}_{\text{F}}[Q] - \frac{\rho_0^2}{2 \beta} \sum_{\alpha=1}^M \sum_{a=1}^M \int C_a \frac{dk_x}{1 + \rho_e \sqrt{2 \pi} U_0(k_x)} \left| \text{tr} \Gamma_\alpha^\alpha Q(k_x) - \frac{\beta}{\pi} (A^\text{eff}_a)^\alpha_\alpha(k_x) \right|^2$$

$$+ \frac{1}{8 \pi^2} \sum_{a \neq b} s_a s_b \int \int C_a \frac{dx}{C_b} \int C_b' \sum_{\alpha=1}^M \left[ \text{tr} \Gamma_\alpha^\alpha Q - \frac{\beta}{\pi} (A^\text{eff}_a)^\alpha_\alpha(x) \right] \times$$

$$\times U_0(x, x') \left[ \text{tr} \Gamma_\alpha^\alpha Q - \frac{\beta}{\pi} (A^\text{eff}_a)^\alpha_\alpha(x') \right]$$ \hfill (4D.1)
CHAPTER 4. THE INTEGER EDGE

The indices $k, k' = 0, 1, \ldots, n$ label the edges of the graph. The edge $x = 0$ implies that single

particle scattering, as described by (4.3), is not possible.

However, we do not necessarily translate into single particle scattering as we described an edge obtained by walking (4.3) in the presence of an edge (finite potential wall). Rather than that,

each edge allows for interaction scattering of $Q$ and propagates energies according to (4.10)

due to the effect of the other in the exact 2D problem (4.3). We start from the

identifying identity of the vanishing edge (4.11) which includes the Q-field

\[ Z = \int [DQ] \exp \left( -\frac{i}{\hbar} \int \left( -\frac{1}{2} \nabla Q^2 + m^2 Q^2 + i \chi \cdot \nabla Q + i\chi \cdot \nabla \right) \right) \] (4.12)

Notice that the edge-induced labeling is not present in the new field variable $Q$, but it is simply contained as an overall factor in the energy of the system (4.3).

We will now make use of the simple analytic properties of our Hamiltonian to show that the problem is equivalent to writing the energy expressions in a way that therefore there is no need to worry on it to be large. The stationary point equations, which can be written as

\[ \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2 + \left( \frac{\partial}{\partial z} \right)^2 + M^2 + \lambda Q^2 = 0 \] (4.13)

with the simple solution of

\[ Q = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial z} \right) M \]

for which we obtain the following change of variables

\[ x = \left( \frac{\partial}{\partial x} \right) (0, x) \quad (x, y) = (x_1, x_2) \]

with $\lambda = 0$, and

\[ \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2 + \left( \frac{\partial}{\partial z} \right)^2 = 0 \]

the solution is

\[ M = 0 \]

This is the case in which the edge is not present or propagates with polar in the

null direction or in a parallel direction. All the momentum integrals should be able to do...